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Path Integral Quantization of the Electromagnetic Field Coupled to A Spinor

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Abstract: The Hamilton-Jacobi approach is applied to the electromagnetic field coupled to a spinor. The integrability conditions are investigated and the path integral quantization is performed using the action given by Hamilton-Jacobi approach.

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1. Introduction

The most common method for investigating the Hamilton treatment of constrained systems was initiated by Dirac[1-4]. The main feature of his method is to consider primary constraints first. All constraints are obtained using consistency conditions. Hence, equations of motion are obtained in terms of arbitrary parameters.

The starting point of the Hamilton-Jacobi approach [5-10] is the variational principle. The Hamiltonian treatment of constrained systems leads us to total differential equations in many variables. The equations are integrable if the corresponding system of partial differential equations is a Jacobi system.

Path integral quantization based on Hamilton-Jacobi method is developed in references [11-15].

Our aim in this paper is to quantize a system of electromagnetic field coupled to a spinor. The paper is arrange as follows: In Sec.2 the Hamilton-Jacobi formulation is briefly described. In Sec.3 we present a system of the electromagnetic field coupled to a spinor, which is quantized using Hamilton-Jacobi formulation. Sec.4 is the conclusion.

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2. Hamilton-Jacobi Formulation

One starts from singular Lagrangian $L \equiv L(q_i, \dot{q}_i, t), \rangle i = 1, 2, \ldots, n$, with the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \rangle \partial \dot{q}_j} \rangle, \tag{1}$$

of rank $(n-r)$, $r < n$. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, 2, \ldots, n-r, \tag{2}$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}, \qquad \mu = n-r+1, \ldots, n. \tag{3}$$

where q_i are divided into two sets, q_a and x_μ. Since the rank of the Hessian matrix is $(n-r)$, one may solve Eq.(2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{x}_\mu, p_a; t). \tag{4}$$

Substituting Eq. (4), into Eq. (3), we get

$$p_\mu = -H_\mu(q_i, \dot{x}_\mu, p_a; t). \tag{5}$$

The canonical Hamiltonian H_0 reads as

$$H_0 = -L(q_i, \dot{x}_\nu, \dot{q}_a; t) + p_a \dot{q}_a - \dot{x}_\mu H_\mu, \qquad \nu = 1, 2, \ldots, r. \tag{6}$$

The set of Hamilton-Jacobi Partial Differential Equations is expressed as

$$H'_\alpha\left(x_\beta,\ q_\alpha, \frac{\partial S}{\partial q_\alpha}, \frac{\partial S}{\partial x_\beta}\right) = 0, \qquad \alpha, \beta = 0, 1, \ldots, r, \tag{7}$$

where

$$H'_0 = p_0 + H_0\ , \tag{8}$$

$$H'_\mu = p_\mu + H_\mu\ . \tag{9}$$

We define $p_\beta = \partial S[q_a; x_a]/\partial x_\beta$ and $p_a = \partial S[q_a; x_a]/\partial q_a$ with $x_0 = t$ and S being the action.

The equations of motion are written as total differential equations in the form

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dx_\alpha, \tag{10}$$

$$dp_a = \frac{\partial H'_\alpha}{\partial q_a} dx_\alpha, \tag{11}$$

$$dp_\beta = \frac{\partial H'_\alpha}{\partial t_\beta} dx_\alpha, \tag{12}$$

$$dz = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dx_\alpha, \tag{13}$$

where $Z = S(x_\alpha, q_a)$. These equations are integrable if and only if [12]

$$dH'_0 = 0, \tag{14}$$

and

$$dH'_\mu = 0, \qquad \mu = 1, 2, \ldots, r. \tag{15}$$

If conditions (14), and (15) are not satisfied identically, one considers them as a new constraints and a gain consider their variations. Thus, repeating this procedure, one may obtain a set of constraints such that all variations vanish. Simultaneous solutions of canonical equations with all these constraints provide the set of canonical phase space coordinates (q_a, p_a) as functions of t_a; the canonical action integral is obtained in terms of the canonical coordinates. H'_α can be interpreted as the infinitesimal generator of canonical transformations given by parameters t_α. In this case the path integral representation can be written as [14-16].

$$\langle Out \mid S \mid In \rangle = \int \prod_{a=1}^{n-p} dq^a dp^a \exp\left[i \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right], \tag{16}$$

$$a = 1, , \ldots, n-p, \qquad \alpha = 0, n-p+1, \ldots, n.$$

In fact, this path integral is an integration over the canonical phase space coordinates (q^a, p^a).

3. Quantization of Electromagnetic Field Coupled to A Spinor

The Lagrangian of the electromagnetic field coupled to a spinor is given by[3,4].

$$L = -\frac{1}{4}\rangle F_{\mu\nu} F^{\mu\nu} + i\overline{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi, \tag{17}$$

where A_μ are even variables while ψ and $\overline{\psi}$ are odd ones. The electromagnetic tensor is defined as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and we are adopting the Minkowski metric $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$.
The Lagrangian function (17) is singular, since the rank of the Hess matrix (1) is three.

The momenta variables conjugated, respectively, to A_i, A_0, ψ and $\overline{\psi}$, are obtained as

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i} = -F^{0i}, \tag{18}$$

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0 = -H_1, \tag{19}$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = i\rangle\overline{\psi}\gamma^0 = -H_\psi\rangle, \tag{20}$$

$$p_{\overline{\psi}} = \frac{\partial L}{\partial \dot{\overline{\psi}}} = 0 = -H_{\overline{\psi}}\rangle, \tag{21}$$

where we must call attention to the necessity of being careful with the spinor indices. Considering, as usual, ψ as a column vector and $\overline{\psi}$ as a row vector implies that p_ψ will be a row vector while $p_{\overline{\psi}}$ will be a column vector.
With the aid of relation (18), the Lagrangian may be written as

$$L = -\frac{1}{2}\rangle \pi_i \pi^i - \frac{1}{4}\rangle F_{ij}F^{ij} + i\overline{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi, \tag{22}$$

then the canonical Hamiltonian takes the form

$$H_0 = \pi^i \dot{A}_i + \frac{1}{2}\rangle \pi_i \pi^i + \frac{1}{4}\rangle F^{ij}F_{ij} - i\overline{\psi}(\gamma^\mu ieA_\mu + \gamma^i\partial_i)\psi + m\overline{\psi}\psi. \tag{23}$$

The velocities $\dot{A}_i$ can be expressed in terms of the momenta π_i as

$$\dot{A}_i = -\pi_i + \partial_i A_0. \tag{24}$$

Therefore, the Hamiltonian is

$$H_0 = \frac{1}{4}\rangle F^{ij}F_{ij} - \frac{1}{2}\rangle \pi_i \pi^i + \pi^i \partial_i A_0 + \overline{\psi}\gamma^\mu eA_\mu \psi - \overline{\psi}(i\gamma^i\partial_i - m)\psi. \tag{25}$$

The set of Hamilton-Jacobi Partial Differential Equation (7) reads as

$$H_0' = p_0 + \frac{1}{4}\rangle F^{ij}F_{ij} - \frac{1}{2}\rangle \pi_i \pi^i + \pi^i \partial_i A_0 + \overline{\psi}\gamma^\mu eA_\mu \psi - \overline{\psi}(i\gamma^i\partial_i - m)\psi, \tag{26}$$

$$H_1' = \pi^0 + H_1 = \pi_0 = 0, \tag{27}$$

$$H_\psi' = p_\psi + H_\psi = p_\psi - i\overline{\psi}\,\gamma^0 = 0, \tag{28}$$

$$H_{\overline{\psi}}' = p_{\overline{\psi}} + H_{\overline{\psi}} = p_{\overline{\psi}} = 0. \tag{29}$$

Therefore, the total differential equations for the characteristic (9), (10) and (11), are obtained as

$$\begin{aligned} dA^i &= \frac{\partial H_0'}{\partial \pi_i}\rangle dt + \frac{\partial H_1'}{\partial \pi_i}\rangle dA^0 + \frac{\partial H_\psi'}{\partial \pi_i}\rangle d\psi + \frac{\partial H_{\overline{\psi}}'}{\partial \pi_i}\rangle d\overline{\psi}, \\ &= -(\pi^i + \partial_i A_0)\, dt, \end{aligned} \tag{30}$$

$$\begin{aligned} dA^0 &= \frac{\partial H_0'}{\partial \pi_0}\rangle dt + \frac{\partial H_1'}{\partial \pi_0}\rangle dA^0 + \frac{\partial H_\psi'}{\partial \pi_0}\rangle d\psi + \frac{\partial H_{\overline{\psi}}'}{\partial \pi_0}\rangle d\overline{\psi}, \\ &= dA^0, \end{aligned} \tag{31}$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H_0'}{\partial A_i}\rangle dt - \frac{\partial H_1'}{\partial A_i}\rangle dA^0 - \frac{\partial H_\psi'}{\partial A_i}\rangle d\psi - \frac{\partial H_{\overline{\psi}}'}{\partial A_i}\rangle d\overline{\psi}, \\ &= (\partial_l F^{li} - e\overline{\psi}\gamma^i\psi)\, dt, \end{aligned} \tag{32}$$

$$d\pi^0 = -\frac{\partial H_0'}{\partial A_0}\rangle dt - \frac{\partial H_1'}{\partial A_0}\rangle dA^0 - \frac{\partial H_\psi'}{\partial A_0}\rangle d\psi - \frac{\partial H_{\overline{\psi}}'}{\partial A_0}\rangle d\overline{\psi},$$

$$= (\partial_i \pi^i - e\overline{\psi}\gamma^0\psi)\, dt, \tag{33}$$

$$\begin{aligned} dp_\psi &= -\frac{\partial H_0'}{\partial \psi}\rangle dt - \frac{\partial H_1'}{\partial \psi}\rangle dA^0 + \frac{\partial H_\psi'}{\partial \psi}\rangle d\psi + \frac{\partial H_{\overline{\psi}}'}{\partial \psi}\rangle d\overline{\psi}, \\ &= -(i\gamma^i\partial_i + e\gamma^\mu A_\mu + m)\overline{\psi}\, dt, \end{aligned} \tag{34}$$

and

$$\begin{aligned} dp_{\overline{\psi}} &= -\frac{\partial H_0'}{\partial \overline{\psi}}\rangle dt - \frac{\partial H_1'}{\partial \overline{\psi}}\rangle dA^0 + \frac{\partial H_\psi'}{\partial \overline{\psi}}\rangle d\psi + \frac{\partial H_{\overline{\psi}}'}{\partial \overline{\psi}}\rangle d\overline{\psi}, \\ &= (-i\gamma^i\partial_i + e\gamma^\mu A_\mu + m)\psi\, dt - i\gamma^0 d\psi. \end{aligned} \tag{35}$$

The integrability condition ($dH_\alpha' = 0$) implies that the variation of the constraints H_1', H_ψ' and $H_{\overline{\psi}}'$ should be identically zero, that is

$$dH_1' = d\pi_0 = 0, \tag{36}$$

$$dH_\psi' = dp_\psi - i\gamma^0 d\overline{\psi} = 0, \tag{37}$$

$$dH_{\overline{\psi}}' = dp_{\overline{\psi}} = 0. \tag{38}$$

When we substituting from Eqs. (34) and (35) into Eqs.(37) and (38) respectively, the variation of

$$dH_\psi' = 0, \tag{39}$$

and

$$dH_{\overline{\psi}}' = 0, \tag{40}$$

are identically zero if the following relations are satisfied:

$$i\overline{\psi}\gamma^\mu(\overleftarrow{\partial}_\mu - ieA_\mu) + m\overline{\psi} = 0, \tag{41}$$

and

$$i(\partial_\mu + ieA_\mu)\gamma^\mu\psi - m\psi = 0, \tag{42}$$

Then the set of equations (30,32,33) is integrable and they are just ordinary differential equations which can be written in the forms

$$\dot{A}^i = -\pi^i - \partial_i A_0, \tag{43}$$

$$\dot{\pi}^i = \partial_l F^{li} - e\overline{\psi}\gamma^i\psi, \tag{44}$$

$$\dot{\pi}^0 = \partial_i \pi^i - e\overline{\psi}\gamma^0\psi. \tag{45}$$

These are the equations of motion with full gauge freedom. It can be seen, from Eq. (31), that A^0 is an arbitrary (gauge dependent) variable since its time derivative is arbitrary. Besides that, Eq. (43) shows the gauge dependence of A^i and it is clear that the curl of its vector form, leads to the well known Maxwell equations,

$$\frac{\partial \overrightarrow{A}}{\partial t} = -\overrightarrow{E} - \overrightarrow{\bigtriangledown}(A_0 - \alpha) \Rightarrow \frac{\partial \overrightarrow{B}}{\partial t} = -\overrightarrow{\bigtriangledown} \times \overrightarrow{E}. \tag{46}$$

Writing $j^\mu = e\overline{\psi}\gamma^\mu\psi$ we get, from Eq. (44), the inhomogeneous Maxwell equation

$$\frac{\partial \overrightarrow{E}}{\partial t} = \overrightarrow{\nabla} \times \overrightarrow{B} - \overrightarrow{j}, \tag{47}$$

while the other inhomogeneous equation

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = j^0, \tag{48}$$

follows from Eq. (45). Expressions (41) and (42) are the known equations for the spinor ψ and $\overline{\psi}$.

Eqs. (12) and (26-29) lead us to the canonical action integral

$$Z = \int d^4x \Big(-\frac{1}{4}\rangle F^{ij}F_{ij} + \frac{1}{2}\rangle \pi_i\pi^i + \pi^i\dot{A}_i + \pi^i\partial_i A_0 + i\overline{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi \Big). \tag{49}$$

Making use of equations (16) and (49), we obtain the path integral as

$$\langle out|S|In\rangle = \int \prod_i dA^i\, d\pi^i\, d\psi\, d\overline{\psi}\rangle exp\Big[i\Big\{ \int d^4x \Big(-\frac{1}{4}\rangle F^{ij}F_{ij} + \frac{1}{2}\rangle \pi_i\pi^i + \pi^i\dot{A}_i + \pi^i\partial_i A_0 + i\overline{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi \Big)\Big\}\Big]. \tag{50}$$

Integration over π_i gives

$$\langle out|S|In\rangle = N \int \prod_i dA^i\, d\psi\, d\overline{\psi}\rangle exp\Big[i\Big\{ \int d^4x \Big(\frac{1}{2}(\dot{A}^i + \partial_i A_0)^2 - \frac{1}{4}\rangle F^{ij}F_{ij} + i\overline{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi \Big)\Big\}\Big], \tag{51}$$

which is an integration over the canonical phase space.

Conclusion

Path integral quantization of the electromagnetic field coupled to a spinor is obtained by the canonical path integral formalism based on Hamilton-Jacobi method [12-15]. The integrability conditions dH'_1, dH'_ψ and $dH'_{\overline{\psi}}$ are identically satisfied, and the system is integrable. Hence, the canonical phase space coordinates $(A^i, \pi^i), (\psi, p_\psi)$ and $(\overline{\psi}, p_{\overline{\psi}})$ are obtained in terms of the parameter τ. If the system is integrable, then one can construct the reduced canonical phase-space. The path integral is obtained as an integration over the canonical phase-space coordinates (A^i, π^i) and $(\psi, \overline{\psi})$ without using any gauge fixing condition. From the equations of motion of the system of electromagnetic field coupled to a spinor, we obtained the inhomogeneous Maxwell equations.

The advantage of this path integral formalism is that we have no need to enlarge the initial phase-space by introducing unphysical auxiliary field, no need to introduce Lagrange multipliers, no need to use delta functions in the measure as well as to use gauge fixing conditions; all that needed is the set of Hamilton-Jacobi partial differential equations and the equations of motion.

References

[1] P.A.M. Dirac, Lectures of Quantum Mechanics, Yeshiva University Press, New york (1964).

[2] P.A.M. Dirac, Can J. Math. 2, 129 (1950).

[3] D. M. Gitman and I. V. Tyutin, *Quantization of Fields with Constraints* Springer-Verlage (1990).

[4] K. Sundermeyer, *Lecture Notes in physics 169-constrained Dynamics,* Springer-Verlage (1982).

[5] Y. Güler, Nuovo Cimento **B107**, 1389 (1992).

[6] Y. Güler, Nuovo Cimento **B107**, 1143 (1992).

[7] E. Rabei and Y. Güler, Phys. Rev. **A76**, 3513 (1992).

[8] E. Rabei and Y. Güler, Tr. J. of Physics, 297 (1992).

[9] E. Rabei and Y. Güler, Nuovo Cimento **B110**, 893 (1995).

[10] N. Farahat and Y. Güler, Nuovo Cimento **B111**, 513 (1996).

[11] D. Baleanu and Y. Güler, Nuovo Cimento **B114**, 709, (1999).

[12] S. I. Muslih and Y. Güler, Nuovo Cimento **B113**, 277, (1998).

[13] S. I. Muslih, Nuovo Cimento **B115**, 1 (2000).

[14] S. I. Muslih, Nuovo Cimento **B115**, 7 (2000).

[15] S. I. Muslih, Nuovo Cimento **B117**, 4 (2002).