A RELATIVISTIC CLASSICAL SPINNING-PARTICLE AS SINGULAR LAGRANGIAN OF SECOND-ORDER

N. I. Farahat
Department of Physics
Islamic University of Gaza
P.O.Box 108, Gaza, Palestine
nfarahat@mail.iugaza.edu
and
Z. Nassar
Department of Physics
Islamic University of Gaza
P.O.Box 108, Gaza, Palestine
zahernassar@hotmail.com

Abstract

A relativistic classical spinning-point particle is studied as a second-order singular Lagrangian system using the canonical formulation. The equations of motion are total differential equations in many variables. These equations of motion are in exact agreement with those obtained using Dirac's method.

1 Introduction

The Hamiltonian formulation of singular Lagrangian systems was initiated by Dirac [1]. He obtained the equations of motion of a singular Lagrangian systems by using the consistency conditions. He also showed that the number of degrees of freedom of the mechanical system can be reduced.Recently, an alternative method is the Hamilton-Jacobi formulation, or the canonical method which is developed by Güler [2-4]. Güler used the Hamilton-Jacobi equations to obtain the equations of motion as total differential equations. Now, a brief review of this approach will be introduced. If the rank of the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\,, \qquad i, j = 1, ..., n \tag{1}$$

is $n-m$, $m < n$, then the system will be singular, otherwise it will be regular. The standard definitions of linear momenta are

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, ..., m \tag{2}$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = m + 1, ..., n. \tag{3}$$

In this formulation the usual Hamiltonian H_0 is defined as

$$H_0 = -L + p_a w_a - \dot{q}_\mu p_\mu. \tag{4}$$

Therefore, the equivalent Lagrangian method is used to obtain the set of Hamilton-Jacobi partial differential equations (HJPDEs) as

$$H'_\alpha(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_\beta}) = 0, \qquad \alpha, \beta = 0, 1, ..., m \tag{5}$$

where

$$H'_\alpha = H_\alpha + p_\alpha \tag{6}$$

The equations of motion are given as total differential equations in variables t_β,

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \qquad dp_\mu = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \tag{7}$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha, \qquad dz = (-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}) dt_\alpha, \tag{8}$$

where

$$z = S(t_\alpha, q_a). \tag{9}$$

In Ref.[4,5], a third method depends on Lagrangian formulation was developed to study the singular systems. here, the Euler-Lagrange equation for field system is used to obtain the equations of motion for the singular system. since there are additional constraints (Eqn.5) given in the phase space, they should also appear as constraints in the configuration space. The Euler-Lagrange equation for singular Lagrangian system are given in the form, [4,5]

$$\frac{\partial}{\partial x_\mu}(\frac{\partial L'}{\partial(\partial_\mu q_a)}) - \frac{\partial L'}{\partial q_a} = 0, \qquad \partial_\mu q_a = \frac{\partial q_a}{\partial x_\mu}, \tag{10}$$

with constraints

$$dG_\mu = -\frac{\partial L'}{\partial x_\mu}\,dt, \qquad dG_0 = -\frac{\partial L'}{\partial t}\,dt, \qquad \mu = 0, n-m+1, ..., n. \tag{11}$$

Here the modified Lagrangian L' is defined as

$$L'(x_\mu, \partial_\mu q_a, \dot{x}_\mu, q_a) \equiv L(x_\mu, q_a, \dot{q}_a = (\partial_\mu q_a)\dot{x}), \tag{12}$$

and

$$G\mu = H_(q_a, x_\mu, p_a = \frac{\partial L}{\partial q_a}). \tag{13}$$

The solution of equation (10) with the constraint equation (11) gives us the solution of the system. In this paper we will study a system of the interaction of a charged particle of charge e with electromagnetic field.

2 Treatment of the interaction of a relativistic particle of charge e moving in an external electromagnetic field as a contentious system

We consider the singular Lagrangian of a complete system of particle and the electromagnetic field

$$L = -mc(\dot{q}_\mu \dot{q}^\mu)^{\frac{1}{2}} - \frac{e}{c}A_\mu \dot{q}^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{14}$$

This singular Lagrangian can be treated as continuous system in the form

$$q_1 = q_1(q_0, \tau),\ q_2 = q_2(q_0, \tau),\ q_3 = q_3(q_0, \tau). \tag{15}$$

The time derivatives of (15) takes the form

$$\dot{q}_1 = \frac{\partial q_1}{\partial \tau} + \frac{\partial q_1}{\partial q_0}\dot{q}_0,\ \dot{q}_2 = \frac{\partial q_2}{\partial \tau} + \frac{\partial q_2}{\partial q_0}\dot{q}_0,\ \dot{q}_3 = \frac{\partial q_3}{\partial \tau} + \frac{\partial q_3}{\partial q_0}\dot{q}_0. \tag{16}$$

The Euler-Lagrange equation of the continuous system is

$$\frac{\partial}{\partial x_\mu}[\frac{\partial L'}{\partial(\partial_\mu q_a)}] - \frac{\partial L'}{\partial q_a} = 0, \tag{17}$$

where $x_\mu \equiv q_0, \tau$ and $q_a \equiv q_1, q_2, q_3$. By substituting (16) into (14) the modified Lagrangian L' is

$$L' = -mc[\dot{q}_0^2 - (\frac{\partial q_a}{\partial \tau} + \frac{\partial q_a}{\partial q_0}\dot{q}_0)(\frac{\partial q_a}{\partial \tau} + \frac{\partial q_a}{\partial q_0}\dot{q}_0)]^{\frac{1}{2}} - \frac{e}{c}\,\dot{q}_0 A_0$$
$$+\frac{e}{c}\,(\frac{\partial q_a}{\partial \tau} + \frac{\partial q_a}{\partial q_0}\dot{q}_0)A_a - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}. \tag{18}$$

Euler-Lagrange equation (17) becomes

$$\frac{\partial}{\partial \tau}[\frac{\partial L'}{\partial(\frac{\partial q_a}{\partial \tau})}] + \frac{\partial}{\partial q_0}[\frac{\partial L'}{\partial(\frac{\partial q_a}{\partial q_0})}] - \frac{\partial L'}{\partial q_a} = 0. \tag{19}$$

Explicitly,

$$\frac{\partial}{\partial \tau}[mc(\frac{\partial q_a}{\partial \tau} + \frac{\partial q_a}{\partial q_0}\dot{q}_0)B + \frac{e}{c}A_a] + \frac{\partial}{\partial q_0}[mc(\frac{\partial q_a}{\partial \tau} + \frac{\partial q_a}{\partial q_0}\dot{q}_0)B + \frac{e}{c}A_a]\dot{q}_0$$
$$+\frac{e}{c}\dot{q}_0\frac{\partial A_0}{\partial q_b} - \frac{e}{c}[\frac{\partial q_b}{\partial \tau} + \frac{\partial q_b}{\partial q_0}\dot{q}_0]\frac{\partial A_b}{\partial q_a} + \frac{1}{4}\frac{\partial}{\partial q_a}(F_{\mu\nu}F^{\mu\nu}) = 0, \tag{20}$$

where

$$B = [\dot{q}_0^2 - (\frac{\partial q_a}{\partial \tau} + \frac{\partial q_a}{\partial q_0}\dot{q}_0)(\frac{\partial q_a}{\partial \tau} + \frac{\partial q_a}{\partial q_0}\dot{q}_0)]^{-\frac{1}{2}} \tag{21}$$

The constraints equations (10) read as

$$dG_\mu = -\frac{\partial L'}{\partial x_\mu}\,dt, \tag{22}$$

$$dG_0 = -\frac{\partial L'}{\partial \tau}\,dt = 0, \tag{23}$$

where $\frac{\partial L'}{\partial \tau} = 0$ because L' is a function of q_0, so q_a become function of q_0 only and one can treat q_0 as τ. Thus equation (20) takes the form

$$\frac{d}{dq_0}[mc(\frac{dq_a}{dq_0})B' + \frac{e}{c}A_a] + \frac{e}{c}\frac{\partial A_0}{\partial q_a} - \frac{e}{c}\frac{dq_b}{dq_0}\frac{\partial A_b}{\partial q_a} + \frac{1}{4}\frac{\partial}{\partial q_a}(F_{\mu\nu}F^{\mu\nu}) = 0, \tag{24}$$

where

$$B' = [1 - (\frac{dq_b}{dq_0})(\frac{dq_b}{dq_0})]^{-\frac{1}{2}} \tag{25}$$

In order to test the validity of the equations of motion (24), let us consider the following simple examples from the electrodynamics.

i- Motion in a uniform electric field

Consider the motion of a particle, initially at rest in a uniform electric field directed along the x-axis, that is

$$\mathbf{E} = E\hat{x}, \tag{26}$$

where E is constant. In this case the non-zero component of A^μ is $A^0 = \phi$. And $q_0 \equiv ct,\ q_1 \equiv x,\ q_2 \equiv y,\ q_3 \equiv z$. Thus the equation (24) becomes

$$\frac{d}{dt}\{m\frac{dx}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}}\} = -e\frac{\partial\phi}{\partial x} = eE, \tag{27}$$

$$\frac{d}{dt}\{m\frac{dy}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}}\} = 0, \tag{28}$$

and

$$\frac{d}{dt}\{m\frac{dz}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}}\} = 0. \tag{29}$$

Integrating equations (27), (28), and (29), one gets

$$m\frac{dx}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}} = eEt + c_1, \tag{30}$$

$$m\frac{dy}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}} = c_2, \tag{31}$$

and

$$m\frac{dz}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}} = c_3, \tag{32}$$

where $v^2 = \dot{x} + \dot{y} + \dot{z}$, and c_1, c_2, c_3 are constants. If we choose the initial conditions as $c_1 = 0$, $c_3 = 0$, the solutions of equations (30), (31), and (32) become

$$x = \frac{c^2}{eE}[(eEt)^2 + m^2c^2 + {c_2}^2]^{\frac{1}{2}} \tag{33}$$

$$y = \frac{c_2 c}{eE}\sinh^{-1}\ \frac{eEt}{\sqrt{m^2c^2 + c_2^2}}, \tag{34}$$

$$z = constant \tag{35}$$

ii- Motion in a uniform magnetic field

Consider a constant magnetic field $\mathbf{B} = \mathbf{B}\hat{\mathbf{z}}$ in the positive z-direction. therefore the components of the vector field $\mathbf{A}$ are

$$A_1 = A_x = -\frac{1}{2}By,\ A_2 = A_y = \frac{1}{2}Bx,\ A_3 = A_z = 0$$

with

$$q_1 \equiv x,\ q_2 \equiv y,\ q_3 \equiv z,\ q_0 = ct,\ A_0 = 0.$$

Thus, equations (24) read as

$$\frac{d}{dt}\{m\frac{dx}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}} - \frac{eB}{2c}\,y\} - \frac{eB}{2c}\frac{dy}{dt} = 0, \tag{36}$$

$$\frac{d}{dt}\{m\frac{dy}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}} + \frac{eB}{2c}\,x\} + \frac{eB}{2c}\frac{dx}{dt} = 0, \tag{37}$$

$$\frac{d}{dt}\{m\frac{dz}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}}\} = 0. \tag{38}$$

Integrating equations (36), (37), and (38) we get

$$m\frac{dx}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}} = \frac{eB}{c}\,y + c_1, \tag{39}$$

$$m\frac{dy}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}} = -\frac{eB}{c}\,x + c_2, \tag{40}$$

$$m\frac{dz}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}} = constant. \tag{41}$$

by letting $c_1, c_2 = 0$ and solving equation (39) together with (40) we get,

$$x^2 + y^2 = c, \tag{42}$$

where c is constant. From the above equations, the particle moves along a helix, [7].

iii- Motion in constant, uniform electric and magnetic fields

Finally, we consider the motion of a charge in the presence of both electric and magnetic fields, each of them is constant and uniform. Let us choose the direction of the magnetic field to be in the positive z-direction. Thus the components of the vector potential $\mathbf{A}$ are

$$A_x = -\frac{1}{2}By,\ A_y = \frac{1}{2}Bx,\ A_z = 0,\ A_0 = \phi(y,z) \tag{43}$$

Equation (24) becomes

$$\frac{d}{dt}\{m\frac{dx}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}} - \frac{eB}{2c}\,y\} - \frac{eB}{2c}\frac{dy}{dt} = 0, \tag{44}$$

$$\frac{d}{dt}\{m\frac{dy}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}}+\frac{eB}{2c}\,x\}+\frac{eB}{2c}\frac{dx}{dt}+e\frac{\partial\phi}{\partial y}=0, \tag{45}$$

$$\frac{d}{dt}\{m\frac{dz}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}}\}+e\frac{\partial\phi}{\partial z}=0, \tag{46}$$

where $\frac{\partial\phi}{\partial y}=-E_y$ and $\frac{\partial\phi}{\partial z}=-E_z$. The integration of equations (44), (45), and (46) respectively are

$$m\frac{dx}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}}-\frac{eB}{c}\,y=C_1, \tag{47}$$

$$m\frac{dy}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}}+\frac{eB}{c}\,x-eE_yt=C_2, \tag{48}$$

and

$$m\frac{dz}{dt}[1-\frac{v^2}{c^2}]^{-\frac{1}{2}}=eE_zt+C_3, \tag{49}$$

where C_1, C_2, and C_3 are constants. If the particle starts from rest at $t=0$, then the solution of (47) and (48) together is

$$x^2+y^2=(\frac{eE_yv_x}{m\omega})t^2, \tag{50}$$

where $\omega=\frac{qB}{mc}$. Here we take $v_x=\frac{dx}{dt}=constant$, where $E_x=0$. One observes that the particle moves in a helix with radius varies with time. Let us take a special case where the velocity of the charge is less than c, i.e $v<<c$. In this case equations (47), (48), and (49) become

$$\frac{dx}{dt}=\frac{eB}{mc}y\equiv\dot{x}, \tag{51}$$

$$\frac{dy}{dt}=-\frac{eB}{mc}\,x+\frac{eE_yt}{m}\equiv\dot{y}, \tag{52}$$

$$\frac{dz}{dt}=\frac{eE_z}{m}\,t+C_3\equiv\dot{z}. \tag{53}$$

Combining eqn. (51) with (52) to get

$$\frac{d}{dt}(\dot{x}+i\dot{y})+i\omega(\dot{x}+i\dot{y})=i\frac{eE_y}{m}. \tag{54}$$

Solving (54) one gets

$$\dot{x}=acos\omega t+\frac{qE_y}{m\omega}, \tag{55}$$

$$\dot{y}=-asin\omega t. \tag{56}$$

The components of velocity v_x, v_y are periodic functions of time. In addition we have

$$\dot{z} = \frac{eE_z}{m} t + C_3' = v_z. \quad (57)$$

That is, the z-component of velocity is increasing with time. Integrating equations (55), (56), and (57), with choosing the constants of integration such that at $t = 0$, $x = y = z = 0$, we obtain

$$x = \frac{qE_y}{m\omega} [t - sin\omega t] \quad (58)$$

$$y = \frac{qE_y}{m\omega} [1 - cos\omega t] \quad (59)$$

$$z = \frac{eE_z}{2m} t^2 + v_{0z} t. \quad (60)$$

We observed that the particle moves in a cycloid trajectory in the $x-y$ plane in the limit of $v << c$.

3 Conclusion

The complete system of a particle and electromagnetic field is discussed by using the canonical method, [2], [3]. The treatment of a singular Lagrangian as field or continuous system is an alternative formulation to study the singular Lagrangian systems.

Our proposal to treat singular Lagrangian system as field system leads us to the ordinary classical electromagnetic. This has been done for the relativistic case and the case where $v << c$. The Euler-Lagrange equation for field system is used to obtain the equations of motion for singular Lagrangian system.

Examples from classical electromagnetic are evaluated by using this treatment, where the final results are in exact agreement with those given in the canonical method.

References

[1] P.A.M. Dirac, Lectures on Quantum Mechanics, Yeshiva University, new York, 1964; Cand.J.Math., 2, 129(1950).
[2]Y.Güler, Nouva Cimento B, 107, 1143(1992).
[3]Y.Güler, Nuova Cimento B, 107, 1389(1992).
[4]Y.Güler, Nuova Cimento B, 113, 893(1998).

[5]N.I.Farahat and Y.Güler, Nuova Cimento B, 111, 513(1996).
[6]N.I.Farahat and Y.Güler, Phys. Rev. A51, 68(1995).
[7]N.I.Farahat and Kh.A.Ramadan, Hadronic Jornal, 25(2002)291.
[8]Reitz J.R., Milford F.J., and Chrity R.W., Foundation of electromagnetic theory, (Addison-Wesly)1979.