Scaling Rules for a Slab Waveguide Structure Comprising Nonlinear and Negative Index Materials

Ibrahim M. Qadoura, Sofyan A. Taya*, and Khitam Y. El-wasife

Physics Department, Islamic University of Gaza, P.O. Box 108, Gaza, Palestinian Authority.
Tel: 00972 8 2823311; Fax: 00972 8 2860800; E-mail: staya@iugaza.edu.ps

***Abstract*-We consider a Kerr-type nonlinear guiding layer surrounded by at least one medium made of a negative index material (NIM). Three different structures are investigated in this work. The first structure has a NIM cladding, the second has a NIM substrate, and the third has a NIM substrate and cladding. Three normalized parameters are used to study the dispersion characteristics of these waveguide structures. The dispersion curves are obtained and the effect of the NIM parameters on the guidance properties is presented. It is found that the dispersion characteristic of nonlinear waveguide structure with NIM cladding and dielectric substrate are similar to those for the same structure with NIM substrate and dielectric cladding. However, when both of them are NIMs new properties can be seen.**

***Index Terms*- slab waveguides, dispersion, negative index, nonlinear media.**

I. INTRODUCTION

In the past few years many scientists and engineers have been working on a novel material called negative index material (NIM) or metamaterial due to its unusual electromagnetic features [1-10]. The peculiar properties of NIMs were first theoretically studied by Veselago in 1968 [1]. He pointed out that the permittivity ε and permeability μ are simultaneously negative in such media. Moreover, he predicted a number of unusual properties of NIMs such as inverse refraction, negative radiation pressure, and inverse Doppler effect. Pendry et al. proposed an interesting theory [2, 3] of subwavelength imaging, which has led to a breakthrough in the field of NIMs. Developing the ideas of Pendry et al., Smith et al. [4] presented an evidence for a weakly dissipative composite medium displaying negative values for ε and μ. Recently, with the realization of microwave and optical structures having negative index of refraction, slab waveguides containing NIMs have been in deep concern. These structures may have prominent applications including optical waveguide sensors [6-9], antenna arrays [11,12], phase shifters [13], and filters [14].

Slab waveguides in which one of the media exhibiting Kerr-type nonlinearity have received increasing attention due to their potential applications in many optical fields [15-20]. For a nonlinear guiding layer sandwiched between two linear media, several forms of the characteristic equation have been obtained [20,21]. Kogelnik et al. presented a parameterization model for linear slab waveguides to obtain a universal description of different geometries of the waveguide [22]. The extension of this approach to a waveguide structure comprising a linear guiding layer and a Kerr-type nonlinear substrate was presented by Chelkowski et al. [23]. They defined a power-dependent parameter and were able to get a concise overview of the waveguide properties at a given power. In 1990, the scaling rules for a Kerr-type nonlinear guiding layer bounded by two linear media were presented by M. Fontaine [21].

In this work, we consider three nonlinear waveguide structures. In the first structure, a Kerr-type nonlinear guiding layer is bounded by a substrate and a NIM cladding. In the second structure, the nonlinear guiding layer is surrounded by a cladding and a NIM substrate. The third structure is assumed to comprise the nonlinear guiding layer sandwiched between a NIM substrate and cladding. The guiding properties of these structures are universally described in terms of three parameters: asymmetry coefficient, normalized film thickness and normalized guide index. The dispersion curves are obtained and the guidance properties are studied in terms of the negative parameters of the NIM.

II. THEORETICAL MODEL AND DISPERSION RELATION

The geometry of the asymmetric nonlinear waveguide under consideration is shown in Fig. 1. It consists of a Kerr-type nonlinear guiding layer of thickness d and nonlinear index of refraction $n_{nl}^2 = n_f^2 + a\, E_2^2$, where α is the nonlinearity constant, n_f is the linear part of the refractive index, and E_2 is the electric field in the guiding layer. The guiding layer is bounded by a semi-infinite substrate of parameters $\left(e_s, m_s\right)$ and a semi-infinite cladding of parameters $\left(e_c, m_c\right)$. The substrate and the cladding can be either NIMs or normal dielectrics with positive index of refraction. In this work, we restrict ourselves to s-polarized light (TE) and self-focusing nonlinear media for which α > 0.

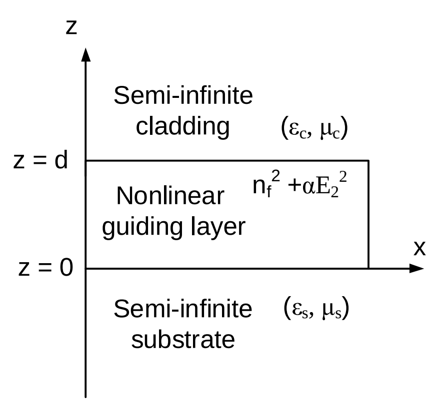

Fig. 1. Kerr-type nonlinear guiding layer bounded by a semi-infinite cladding (ε_c, μ_c) and semi-infinite substrate (ε_s, μ_s).

The guiding properties of any waveguide geometry can be described by three generalized parameters [10, 21-23]. These parameters are called normalized film thickness or normalized frequency (V), asymmetry coefficient (a), and normalized guide index (b) which are represented as

$$V = kd\sqrt{n_f^2 - e_s m_s}\,, \tag{1}$$

$$a = \frac{e_s m_s - e_c m_c}{n_f^2 - e_s m_s}, \tag{2}$$

and

$$b = \frac{N^2 - e_s m_s}{n_f^2 - e_s m_s}, \tag{3}$$

where k is the free space wavenumber and N is the modal index given by $N = \frac{b}{k}$ with b is the propagation constant.

The normalized film thickness and the asymmetry coefficient are two independent parameters. The normalized guide index is obtained from the solution of the dispersion relation of the waveguide structure. It was shown that the guiding properties of a linear guiding layer surrounded by a nonlinear substrate can be described by the three above mentioned parameters *(V, a, b)* and a new parameter denoted as b_1 given by [23]

$$b_1 = \frac{a E_0^2}{2\left(n_f^2 - e_s m_s\right)}, \tag{4}$$

where E_0 is the value of the field amplitude at the interface $z = 0$.

The guiding properties of nonlinear slab waveguides can be summarized in the universal dispersion curves $V\,(b, a, b_1)$.

The dispersion relation of the waveguide structure shown in Fig. 1 is obtained by matching the fields at the interfaces $z = 0$ and $z = d$. In the substrate and cladding, the electric field is given by

$$E_1 = E_0 e^{k_1 z}\,,\ z<0, \tag{5}$$

$$E_3 = E_b e^{k_3(d-z)}, z>d, \tag{6}$$

where, $k_1^2 = k^2(N^2 - e_s m_s)$, $k_3^2 = k^2(N^2 - e_c m_c)$, and E_b is the value of the field amplitude at $z = d$.
The solution of Helmholtz equation in the nonlinear guiding layer is one of the Jacobian elliptic functions [24]. There are twelve Jacobian functions and for the case when α >

0 and $n_f > \sqrt{e_s m_s}$ the solution of the field is given by the *cn* Jacobian function

$$E_2 = p\ cn[q(z+z_0)|m]. \qquad (7)$$

The parameters *q*, *p*, and *m* are given by

$$q = k(s_1)^{\frac{1}{2}}[s_2^2 + 4b_1 s_3]^{\frac{1}{4}}, \qquad (8)$$

$$p = s_4\left\{\left[s_2^2 + 4b_1 s_3\right]^{\frac{1}{2}} + s_2\right\}^{\frac{1}{2}}, \qquad (9)$$

and

$$m = \frac{1}{2}\left\{1 + \frac{s_2}{[s_2^2 + 4b_1 s_3]^{1/2}}\right\}. \qquad (10)$$

where $s_1 = \left(n_f^2 - e_s m_s\right)$, $s_2 = b - 1$, $s_3 = 1 + b_1$, and $s_4 = \left(\frac{E_0^2}{2b_1}\right)$

The parameter z_0 in Eq. (7) is a constant of integration. Using the mathematical properties of the *cn* function, Eq. (7) can be written in terms of two other Jacobi functions, *sn* and *dn*, without using z_0 [21].

In order to find all the nonzero field components, we calculate H_x and H_z as

$$H_x^{(1)} = \frac{ik_1}{wm_1} E_0 e^{k_1 z}, \qquad (11)$$

$$H_x^{(2)} = -\frac{i}{wm_2} pq\ s_5 s_6, \qquad (12)$$

$$H_x^{(3)} = \frac{-ik_3}{wm_3} E_b e^{k_3(d-z)}, \qquad (13)$$

$$H_z^{(1)} = \frac{-b}{wm_1} E_0 e^{k_1 z}, \qquad (14)$$

$$H_z^{(2)} = \frac{-b}{wm_2} pcn[q(z+z_0|m], \qquad (15)$$

$$H_z^{(3)} = \frac{-b}{wm_3} E_b e^{k_3(d-z)}, \qquad (16)$$

where $s_5 = sn\left[q\left(z+z_0\right)\right]$ and $s_6 = dn\left[q\left(z+z_0\right)\right]$.

Matching the tangential fields at the interfaces *z = 0* and *z = d* generates the dispersion relation which can be written in a mathematical form without using the inverse Jacobi functions [20, 21]. In terms of *V*, *a*, *b* and b_1, the dispersion relation can be written as

$$cn[qd|m] = \frac{\left\{s_7\left(1-s_8\right)s_9\right\}}{\left\{1-m s_{10}\right\}}. \qquad (17)$$

where $s_7 = \left(\frac{E_0}{p}\right)\left(\frac{E_b}{p}\right)$,

$s_8 = \left[\frac{b\left(b+a\right)}{\left(b-1\right)^2 + 4b_1\left(1+b_1\right)}\right]^{\frac{1}{2}}$, $s_9 = \frac{m_2^2}{m_1 m_3}$

and $s_{10} = \left[1-\left(\frac{E_0}{p}\right)^2\right]\left[1-\left(\frac{E_b}{p}\right)^2\right]$

According to the definition of *p* in Eq. (9), the parameter $\frac{E_0}{p}$ is only function of *b* and b_1. Moreover, from Eqs. (1) and (8), we can also write the product *qd* as a function of *V*, *b* and b_1 as

$$qd = V\left[\left(s_2\right)^2 + 4b_1\left(s_3\right)\right]^{\frac{1}{4}}. \qquad (18)$$

In a similar manner to $\frac{E_0}{p}$, the ratio $\frac{E_b}{p}$ can take the form [21]

$$\frac{E_b}{p} = \left\{\frac{\left[s_{11}^2 + 4b_1 s_3\right]^{\frac{1}{2}} - s_{11}}{\left[s_2^2 + 4b_1 s_3\right]^{\frac{1}{2}} + s_2}\right\}^{\frac{1}{2}}. \qquad (19)$$

where $s_{11} = 1 + a$

The dispersion relation can then be written in a normalized form. This form indicates that the normalized guide index *b* depends only on three independent parameters *V*, *a*, and b_1.

III. RESULTS AND DISCUSSION

Simple numerical techniques can be applied to solve the normalized dispersion relation. We will consider three cases. In the first case, the cladding is assumed to be a NIM and the substrate is a positive index material (PIM). Therefore $\frac{m_f}{m_s} > 0$ and $\frac{m_f}{m_c} < 0$. In the second case, the substrate is taken to be NIM with $\frac{m_f}{m_s} < 0$ and the cladding is PIM with $\frac{m_f}{m_c} > 0$. The third case considers both the cladding and substrate are NIMs with $\frac{m_f}{m_c} < 0$ and $\frac{m_f}{m_s} < 0$.

First, we consider the case when the cladding layer is NIM and the substrate is PIM. For $\frac{m_f}{m_c} = -0.5$ and $\frac{m_f}{m_s} = 1$, we plot *b* as a function of *V* for different values of b_1 as shown in Fig. 2. We should note that when the value of b_1 equal to zero, we reproduce the results obtained for the linear guiding layer structure [10] in which the maximum value of *b* is 1.0. In the linear guiding layer structure, all the solutions correspond to guided modes for which ($N < n_f$). However, for the nonlinear case, solutions with b > 1.0 ($N > n_f$) could be found as shown in the figure. The figure shows *b* as a function of *V* for *a = 0* ($e_s m_s = e_c m_c$) which means $n_f^2 = n_c^2$, where n_i is the refractive index of layer *i*. Many interesting features can be seen in Fig. 2. First, the value of *b* exceeds 1.0. Following the classification presented by Boardman et al. in Ref 20, these solutions correspond to surface modes. Second, as b_1 increases (due to the increase in the nonlinearity constant) the range of *V* in which there are propagating modes is reduced. For example for b_1 = 0.3, the normalized film thickness has the range $0 < V < 0.66$, whereas it has the range $0 < V < 0.59$ for b_1 = 0.4. Third, there is no cutoff frequency which in contradiction with the behavior of the guided modes in a linear waveguide structure [10, 22]. Fourth, for small nonlinearity (small value of b_1), the dispersion curve looks like that of a linear waveguide structure. When $\frac{m_f}{m_c} = -0.5$, $\frac{m_f}{m_s} = 1$, and *a* = 1, the universal dispersion curves for the nonlinear thin film waveguide are shown in Fig. 3 for b_1 = 0.1, 0.2, 0.3, and 0.4. The figure shows that there is no solution for eq. (17) in the range $0 < b < 0.2$. Moreover, the range of *V* in which the solution exists is highly reduced for high b_1. For b_1 = 0.4, the range of *V* in which the solution exists is $0.45 < V < 0.65$.

A comparison between the dispersion curves for different values of the asymmetry coefficient *a* is shown in Fig. 4. For constant b_1, the shape of the dispersion curves for different values of *a* is almost the same with a shift towards higher values of *V* when *a* increases. For constant b_1 and *V*, the normalized guide index (*b*) decreases with increasing *a*.

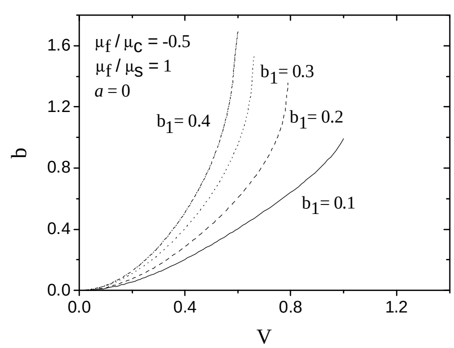

Fig. 2. Normalized dispersion curves for a nonlinear waveguide with NIM cladding and PIM substrate. ($m_f / m_c = -0.5$, $m_f / m_s = 1$, *a* = 0).

The dispersion characteristics of the proposed nonlinear waveguide structure is shown in Fig. 5 for b_1 = 0.1, *a* = 0, $\frac{m_f}{m_c} = -0.5$, and different values of $\frac{m_f}{m_s}$. As can be seen from the figure, increasing $\frac{m_f}{m_s}$ for constant a, $\frac{m_f}{m_c}$, b_1, and V leads to a considerable enhancement in b. Increasing b means that the modal index increases and therefore the confinement of the wave in the guiding layer is enhanced.

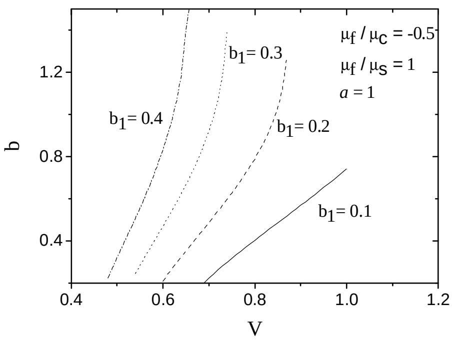

Fig. 3. Normalized dispersion curves for a nonlinear waveguide with NIM cladding and PIM substrate. ($m_f\ / m_c = -0.5$, $m_f\ / m_s = 1$, a = 1).

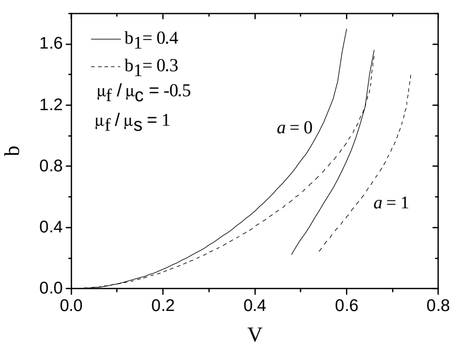

Fig. 4. Normalized dispersion curves for a nonlinear waveguide with NIM cladding and PIM substrate. ($m_f\ / m_c = -0.5$, $m_f\ / m_s = 1$, a = 0 and a = 1).

In Fig. 6, we investigate the dispersion curves of the nonlinear waveguide structure for constant b_1, a, and $\frac{m_f}{m_s}$ and different values of $\frac{m_f}{m_c}$. The figure shows dispersion characteristics different from those observed in Fig. 5. At constant $a, \frac{m_f}{m_s}$, b_1, and V, a significant enhancement in b is obtained when $\frac{m_f}{m_c}$ decreases to -0.8.

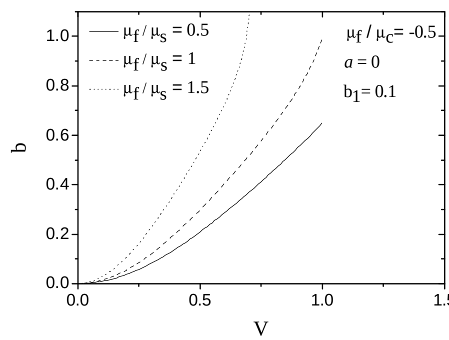

Fig. 5. Normalized dispersion curves for a nonlinear waveguide with NIM cladding and PIM substrate and film.
($m_f\ / m_c = -0.5$, $m_f\ / m_s = 0.5,1,1.5$, a = 0).

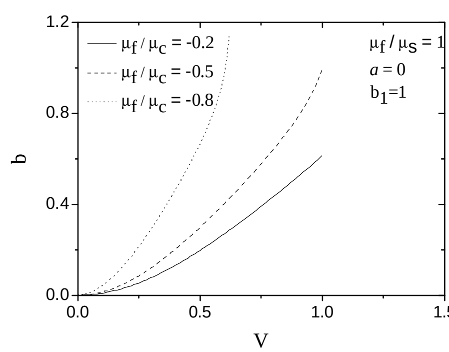

Fig. 6. Normalized dispersion curves for a nonlinear waveguide with NIM cladding and PIM substrate.
($m_f\ / m_c = -0.2, -0.5, -0.8$, $m_f\ / m_s = 1$, a = 0).

Second, we investigate the case when the substrate is NIM and the cladding is PIM.

Fig. 7 shows the dispersion curves of the nonlinear structure for $\frac{m_f}{m_c} = 0.5$, $\frac{m_f}{m_s} = -1$, and a = 0. The striking feature that can be seen in the figure is that there is no difference between the dispersion curves in Fig. 7 and Fig. 2 for the same value of b_1. This means that the dispersion characteristic of nonlinear waveguide structure with NIM cladding and PIM substrate are similar to those for the same structure with NIM substrate and PIM cladding. This is in contradiction with the dispersion properties of a linear waveguide structure in which the normalized dispersion curves are dependent on whether the NIM material is present in the cladding or in the substrate especially for the fundamental mode [10].

The dispersion curves of the proposed nonlinear waveguides is shown in Fig. 8 for a = 1 and different values of b_1. It is exactly similar to Fig. 3.

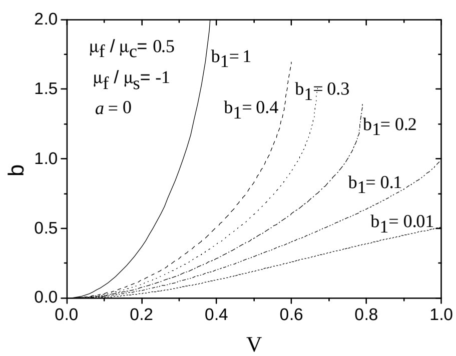

Fig. 7. Normalized dispersion curves for a nonlinear waveguide with NIM substrate and PIM cladding. ($m_f / m_c = 0.5$, $m_f / m_s = -1$, a = 0).

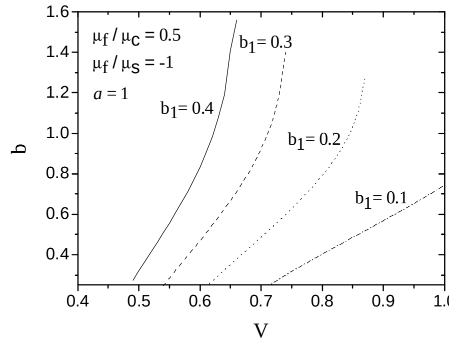

Fig. 8. Normalized dispersion curves for a nonlinear waveguide with NIM substrate and PIM cladding. ($m_f / m_c = 0.5$, $m_f / m_s = -1$, a = 1).

Third, we consider the case when both the cladding and substrate are NIMs. In this case, both $\frac{m_f}{m_s}$ and $\frac{m_f}{m_c}$ are negative. Fig. 9 shows the modal guide index (b) versus the normalized film thickness (V) for $\frac{m_f}{m_c} = -0.5$, $\frac{m_f}{m_s} = -1$, a = 0, and different values of b_1. A number of attractive features can be seen in the figure. Some of these features are analogous to those observed in Fig. 2 such as the nonexistence of the cut-off thickness and the enhancement of b with increasing b_1 for a given V. For waveguide structures with no cut-off thickness, the size of the guiding layer can theoretically go to zero. On the other hand, Fig. 9 has a distinguishable feature over Fig. 2 which is the independence of the range of V in which there is propagating modes on b_1. The range of V does not change with the increase in the nonlinearity constant of the guiding layer.

When $\frac{m_f}{m_c} = -0.5$ and $\frac{m_f}{m_s} = -1$, we plot b as a function of V for a = 1 in Fig. 10. In this case, there is a cut-off frequency at which the modal index is equal to that of the substrate, hence b = 0. The cut-off thickness is crucially dependent on the coefficient b_1. Therefore, for nonlinear waveguide with NIM substrate and cladding, there is a cut-off thickness for the guiding layer after which the wave propagation begins.

Finally, we investigate the dispersion characteristics for $\frac{m_f}{m_c} = -0.5$ and different values of $\frac{m_f}{m_s}$. Figure 11 shows b versus V for different values of $\frac{m_f}{m_s}$. For constant V, b is considerably enhanced with decreasing $\left|\frac{m_f}{m_s}\right|$ especially for large values of V. On the other hand, b versus V for different values of $\frac{m_f}{m_c}$ is shown in Fig. 12. The behavior of the dispersion curves for different $\frac{m_f}{m_c}$ is similar to that for different $\frac{m_f}{m_s}$. As $\left|\frac{m_f}{m_c}\right|$ increases, b is significantly reduced.

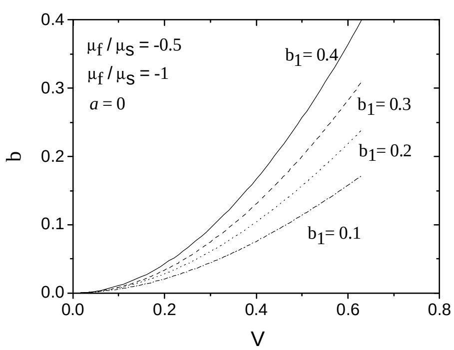

Fig. 9. Normalized dispersion curves for a nonlinear waveguide with NIM cladding and substrate. ($m_f / m_c = -0.5$, $m_f / m_s = -1$, $a = 0$).

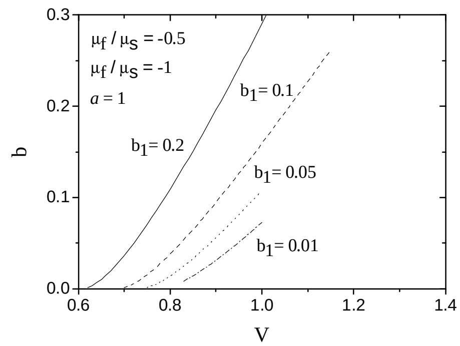

Fig. 10. Normalized dispersion curves for a nonlinear waveguide with NIM cladding and substrate. ($m_f / m_c = -0.5$, $m_f / m_s = -1$, $a = 1$).

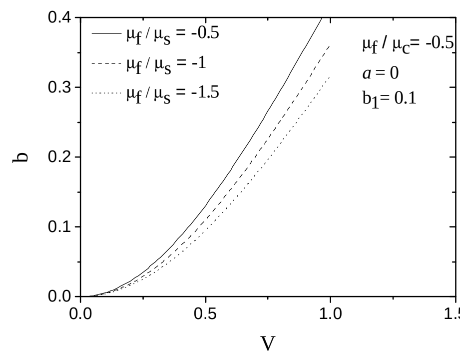

Fig. 11. Normalized dispersion curves for a nonlinear waveguide with NIM cladding and substrate.
($m_f / m_c = -0.5$, $m_f / m_s = -0.5, -1, -1.5$, $a = 0$).

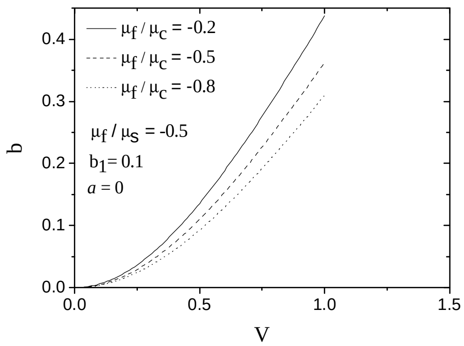

Fig. 12. Normalized dispersion curves for a nonlinear waveguide with NIM cladding and substrate. ($m_f / m_c = -0.2, -0.5, -0.8$, $m_f / m_s = -1$, a = 0).

IV. CONCLUSION

We presented three waveguide structures consisting of Kerr-type nonlinear guiding layer and at least one layer made of a negative index material. The dispersion characteristics of each structure were investigated. Many interesting features have been observed. When one of the surrounding media is made of a NIM, the value of the normalized guide index can exceed unity, a phenomenon that does not exist in conventional linear waveguides. Moreover, the range of the normalized film thickness in which there are propagating modes is dependent on the nonlinearity constant of the guiding layer. When both the cladding and substrate are NIMs, the cut-off thickness is found to have a significant dependence on the nonlinearity constant.

REFERENCES

[1] V.G. Veselago, "The electrodynamics of substances with simultaneously negative values of ε and μ", *Sov. Phys. Usp.*, Vol. 10, pp. 509-526, July 1968.

[2] J.B. Pendry, A.J. Holden, W.J. Stewart, and I. Youngs, "Extremely low frequency plasmons in metallic mesostructures", *Phys. Rev. Lett.*, Vol. 76, pp. 4773-4776, June 1996.

[3] J.B. Pendry, A.J. Holden, D.J. Robbins, and W.J. Stewart, "Magnetism from conductors and enhanced nonlinear phenomena", *IEEE Trans. Microwave Theory Tech.*, Vol. 47, pp. 2075-2084, Feb. 1999.

[4] D.R. Smith, W.J. Padilla, D.C. Vier, S.C. Nemat-Nasser, and S. Schultz, "Composite medium with simultaneously negative permeability and permittivity", *Phys. Rev. Lett.*, Vol. 84, pp. 4184-4187, May 2000.

[5] R.A. Shelby, D.R. Smith, and S. Schultz, "Experimental verification of a negative index of refraction", *Science*, Vol. 292, pp. 77-79, April 2001.

[6] S.A. Taya, M.M. Shabat, and H.M. Khalil, "Enhancement of Sensitivity in optical sensors using left-handed materials", *Optik*, Vol. 120, pp. 504-508, June 2009.

[7] M. Abadla, S.A. Taya, and M.M. Shabat, "Four-layer slab waveguide sensors supported with left handed materials", *Sensor letters*, Vol. 9, pp. 1823-1829, Oct. 2011.

[8] S.A. Taya, T.M. El-Agez, H. Kullab, M.M. Abadla, and M.M. Shabat, "Theoretical study of slab waveguide optical sensor with left handed material as a core layer", *Optica applicata.*, Vol. 42, pp. 193-205, Sep. 2012.

[9] H.M. Kullab, S.A. Taya, and T.M. El-Agez, "Metal-clad waveguide sensor using a left-handed material as a core layer", *J. Opt. Soc. Am.*, Vol 29, pp. 959-964, May 2012.

[10] S.A. Taya and I.M. Qadoura, "Guided modes in slab waveguides with negative index cladding and substrate", *Optik*, http://dx.doi.org/10.1016/j.ijleo.2012.03.057, 2012.

[11] L. Liang, B. Li, S.-H. Liu, and C.-H. Liang, "A study of using the double negative structure to enhance the gain of rectangular waveguide antenna arrays", *Progress In Electromagnetics Research*, Vol. 65, pp. 275–286, 2006.

[12] A.-K. Hamid, "Axially slotted antenna on a circular or elliptic cylinder coated with metamaterials", *Progress In Electromagnetics Research*, Vol. 51, pp. 329–341, 2005.

[13] M.A. Antoniades, and G.V. Eleftheriades, "Compact linear lead/lag metamaterial phase shifters for broadband applications", *IEEE Antennas and Wireless Propagation Letters*, Vol. 2, pp. 103-106, 2003.

[14] T.-N. Kuo, S.-C. Lin, and C. H. Chen, "Compact ultra wideband band pass filters using composite microstrip-coplanar waveguide structure", *IEEE Transactions on Microwave Theory and Techniques*, Vol. 54, pp. 3772-3778, Oct. 2006.

[15] A.D. Boardman, M.M. Shabat, and R.F. Wallis "TE Waves at an interface between linear gyromagnetic and nonlinear dielectric media", *J. Phys. D: Appl. Phys.*, Vol. 24, pp. 1702-1707, 1991.

[16] S.A. Taya, M.M. Shabat, and H.M. Khalil, "Nonlinear Planar Asymmetrical Optical Waveguides for Sensing Applications", *Optik*, Vol. 121, pp. 860-865, Sep. 2008.

[17] S.A. Taya, M.M. Shabat, H.M. Khalil, and D. Jäger, "Theoretical Analysis of TM Nonlinear Asymmetrical Waveguide Optical sensors", *Sens. Actuat. A*, Vol. 147, pp. 137-141, May 2008.

[18] M.M. Shabat, H.M. Khalil., S.A. Taya, and M. Abadla, "Analysis of the sensitivity of self-

focused nonlinear optical evanescent waveguide sensors", *Int. J. Optomechatronics*, Vol. 1, pp. 284-296, July 2007..

[19] H.M. Khalil, M.M. Shabat, S.A. Taya, and M. Abadla, "Nonlinear optical waveguide structure for sensor application: TM case", *Int. J. Modern Phys. B*, Vol. 21(30), pp. 5075-5089, 2007..

[20] A.D. Boardman and P. Egan, "Optically Nonlinear waves in thin Films", *IEEE Journal of Quantum electronics*, Vol. QE-22, No.2, Feb. 1986.

[21] M Fontaine, "Scaling rules for nonlinear thin film optical waveguides", *Appl. Opt.*, Vol. 29, No 27, pp. 3891-3899, Sep. 1990.

[22] H. Kogelnik and V. Ramaswamy, "Scaling Rules for Thin –Film Optical Waveguides", *Appl. Opt.*, Vol. 13, pp. 1857-1862, Aug. 1974.

[23] S. Chelkowski and J. Chrostowski, "Scaling Rules for Slab Waveguides with Nonlinear Substrate ", *Appl. Opt.*, Vol. 26, pp. 3681-3686, Sep. 1987.

[24] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Washington, 1972.