Explicit Equations for ACF in the Presence of Heteroscedasticity Disturbances in First-Order Autoregressive Models, $AR(1)$

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April 3, 2009

Abstract

The autocorrelation function, ACF, is an important guide to the properties of a time series. We derive explicit equations for ACF in the presence of heteroscedasticity disturbances in first-order autoregressive, $AR(1)$, models. We present two cases: (1) when the disturbance follows the general covariance matrix, $\mathbf{\Sigma}$, and (2) when the diagonal elements of $\mathbf{\Sigma}$ are not all identical but $\sigma_{i,j} = 0 \ \forall \ i \neq j$. In addition, we derive an equation to transform a model with heteroscedastic disturbances such that the model has homoscedastic disturbances.

Keywords: Heteroscedasticity, Homoscedasticity, Autocorrelation, Autoregressive, Covariance, Disturbance, Time Series.

1 Introduction

When the disturbance terms are identically distributed, it implies they have the same variance for all observations. This is known as homoscedasticity. If they are not, it causes serious problems for our estimates and must be corrected if we are to obtain reliable estimates. A sequence or a vector of random variables is heteroskedastic, or heteroscedastic, if the random variables have different variances. The term means "differing variance" and comes from the Greek "hetero" ('different') and "skedasis" ('dispersion'). Heteroscedasticity is a deviation from the identically distributed assumption because the variances are not the same for each value. Heteroscedasticity naturally arises when the observations are based on average data, and in a number of random coefficient models.

The econometrician Robert Engle won the 2003 Nobel Memorial Prize for Economics for his studies on regression analysis in the presence of heteroscedasticity, which led to his formulation of the ARCH (AutoRegressive Conditional Heteroscedasticity) modeling technique.

The consequences of Heteroscedasticity are serious. While parameter estimates remain unbiased, they are no longer efficient, i.e., no longer best linear unbiased estimator (BLUE). Since the estimated error's variance-covariance is not efficient, it invalidates the t-statistic and sometimes making insignificant variables appear to be statistically significant.

This paper is organized as follows. In section 2 we introduce the review of the literature. In Section 3 we derive explicit equations for ACF in the presence of heteroscedasticity disturbances in first-order autoregressive, $AR(1)$, models. In addition, we derive an equation to transform a model with heteroscedastic disturbances such that the model has homoscedastic disturbances. Section 4 summarizes the results and offers suggestions for future research on deriving explicit equations for ACF in the presence of heteroscedasticity disturbances.

2 Review of the Literature

The disturbance term in time series data is modeled under an assumption of constant variance and the assumption of heteroscedastic disturbances has traditionally been considered in the context of cross-sectional data. With time series data the disturbance term is modeled with some kind of stochastic process, and most of the conventional stochastic processes assume homoscedasticity (Judge et al., 1985). Studies of many econometric time series models for financial markets revealed that it is unreasonable to assume that conditional variance of the disturbance term is constant, as it for many stochastic processes. Two exceptions are the heteroscedastic stochastic processes proposed by Engle (1982) and Cragg (1982). Engle (1982), showed that, for many economic models, it is unreasonable to assume that the conditional forecast variance $var\left(y_t|y_{t-1}\right)$ is constant, and that is more realistic to assume that $var\left(y_t|y_{t-1}\right)$ depends on y_{t-1}.

Bumb, and Kelejian (1983) have studied the autocorrelated and heteroscedastic disturbances in linear regression analysis. They discussed various procedures to test for the possibility that the disturbance terms of a linear regression model are autocorrelated in a first order process with a constant autoregressive coefficient.

Heteroscedasticity is a problem often faced by statisticians and econometricians. There is a large literature on estimating and testing heteroscedasticity, see for example, Wallentin and Agren (2002), Kalirajan, K. P. (1989), Evans and King (1988) and Farebrother (1987). Praetz (2008) discussed the effect of autocorrelated disturbances when they are not modeled on the statistics used in drawing inferences in the multiple linear regression model. He derived biases for the F and R^2 statistics and evaluates them numerically for an example. He discussed the reflections for empirical research on the causes, detection and treatment of autocorrelation.

3 Autocorrelation Function (ACF)

The autocorrelation function, abbreviated ACF, is an important guide to the properties of a time series. It measures the correlation between observations at different distances apart. This behavior is a powerful tool to identify a preliminary model for the time series. The ACF gives a better understanding of correlation structure of the data, and, within the Box Jenkins framework, a rough idea of the order of the components to be used in any autoregressive model.

3.1 General Heteroscedastic Autocorrelation Function (GHACF)

In autoregressive models, the current value of the process is expressed as a finite, linear aggregate of previous values of the process and a noise $\mathbf{e}_t$. Let the values of a process at equally spaced times t, $t-1$, $t-2, \ldots$, denoted by $Z_t,\ Z_{t-1},\ Z_{t-2}, \ldots$. Then $Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + \mathbf{e}_t$ is called an autoregressive process of order $p,$ abbreviated $AR(p)$.

The p-th order autoregressive process may be written in terms of backward shift operator B

$$Z_t = \left(1 - \phi_1 B - \cdots - \phi_p B^p\right)^{-1} \mathbf{e}_t \tag{3.1}$$

The first-order autoregressive process, $AR(1)$, $Z_t = (1 - \phi B)^{-1} \mathbf{e}_t$ may be written as

$$Z_t = \sum_{j=0}^{\infty} \phi^j \mathbf{e}_{t-j}, \; t = 0, \pm 1, \pm 2, \ldots \tag{3.2}$$

$AR(1)$ is causal, i.e. $|\phi| < 1$. In this paper it will be assumed that the disturbance term has mean zero, $E(\mathbf{e}) = \mathbf{0}$, and the covariance matrix $Cov(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{\Sigma} = \sigma^2 \mathbf{\Psi}$ where

$$\mathbf{\Sigma} = \sigma^2 \mathbf{\Psi} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1t} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{t1} & \sigma_{t2} & \cdots & \sigma_{tt} \end{bmatrix} \tag{3.3}$$

Note $\mathbf{e}$ is a random vector with $E(\mathbf{e}) = \mathbf{0}$ and $E(\mathbf{e}\mathbf{e}') = \mathbf{\Sigma} = \sigma^2 \mathbf{\Psi}$, autocorrelation exists if the disturbance terms corresponding to different observations are correlated, that is, if $\mathbf{\Psi}$ is not diagonal.

Definition 3.1 *The covariance between Z_t and Z_{t+k}, separated by k intervals of time, which under the stationary assumption must be the same for all t, is called the autocovariance function at lag k, abbreviated ACVF, and is defined by*

$$\gamma_k = Cov(Z_t, Z_{t+k}) = E[(Z_t - \mu)(Z_{t+k} - \mu)] \tag{3.4}$$

In this paper we are assuming that Z_t has zero mean. We can always introduce a nonzero mean by replacing Z_t by $Z_t - \mu$ throughout our equations.

Definition 3.2 *The autocorrelation function at lag k, that is the correlation between Z_t and Z_{t+k} is defined by*

$$\rho_k = \frac{\gamma_k}{\gamma_0} \tag{3.5}$$

where $\gamma_0 = \sigma_Z^2$ is the same at time $t+k$ as at time t.

Lemma 3.1 *Consider a first-order autoregressive model with parameter ϕ, $Z_t = \phi Z_{t-1} + \mathbf{e}_t$, with $E\ (\mathbf{e}_t) = \mathbf{0}$, and $Cov\,(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{\Sigma}$, where $\mathbf{\Sigma}$ is given in (3.3). Then the autocovariance function at lag k is given by*

$$\gamma_k = \sum_{j=0}^{t-k-1} \sum_{i=0}^{t-1} \phi^{j+i} \sigma_{t-i,t-k-j} \tag{3.6}$$

Proof. Using Definition (3.2), the $AR(1)$ model can be written as

$$Z_t = (1 - \phi B)^{-1} \mathbf{e}_t = \sum_{i=0}^{t-1} \phi^i B^i \mathbf{e}_t$$

$$= \sum_{i=0}^{t-1} \phi^i \mathbf{e}_{t-i}$$

Then

$$Z_t Z_{t-k} = \left(\sum_{i=0}^{t-1} \phi^i \mathbf{e}_{t-i} \right) \left(\sum_{j=0}^{t-k-1} \phi^j \mathbf{e}_{t-k-i} \right)$$

$$= \sum_{j=0}^{t-k-1} \sum_{i=0}^{t-1} \phi^{j+i} \mathbf{e}_{t-i} \mathbf{e}_{t-k-i}$$

Using Definition (3.4), the ACVF at lag k is

$$\gamma_k = E\left[Z_t Z_{t-k}\right] = E\left[\sum_{j=0}^{t-k-1} \sum_{i=0}^{t-1} \phi^{j+i} \mathbf{e}_{t-i} \mathbf{e}_{t-k-i} \right]$$

$$= \sum_{j=0}^{t-k-1} \sum_{i=0}^{t-1} \phi^{j+i} E\left(\mathbf{e}_{t-i} \mathbf{e}_{t-k-i}\right) = \sum_{j=0}^{t-k-1} \sum_{i=0}^{t-1} \phi^{j+i} \sigma_{t-i,t-k-j} \ \blacksquare$$

The next theorem derives the GHACF at lag k when $\sigma_{i,j} \neq 0$ for all $i \neq j$ in $AR(1)$ models.

Theorem 3.1 *Suppose the disturbance term follows the general covariance matrix, i.e.* $Cov(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{\Sigma}$, *where* $\mathbf{\Sigma}$ *is given in (3.3) with* $\sigma_{i,j} \neq 0$ *for all* $i \neq j$. *Then the GHACF at lag* k *is given by*

$$\rho_k = \frac{\sum_{j=0}^{t-k-1} \sum_{i=0}^{j} \phi^j \ \sigma_{t-i,t-j+i-k} + \sum_{j=0}^{t-2} \sum_{i=1}^{t-j-1} \phi^{t+j-k} \ \sigma_{t-i-j,i}}{\sum_{j=0}^{t-1} \sum_{i=0}^{j} \phi^j \ \sigma_{t-i,t-j+i} + \sum_{j=0}^{t-2} \sum_{i=1}^{t-j-1} \phi^{i+j} \ \sigma_{t-i-j,i}} \tag{3.7}$$

Proof. Using (3.6), the ACVF at lag 0

$$\begin{aligned} & \sigma_{tt} + \phi\left(\sigma_{t,t-1} + \sigma_{t-1,t}\right) + \phi^2\left(\sigma_{t,t-2} + \sigma_{t-1,t-1} + \sigma_{t-2,t}\right) + \\ & \phi^3\left(\sigma_{t,t-3} + \sigma_{t-1,t-2} + \sigma_{t-2,t-1} + \sigma_{t-3,t}\right) + \cdots + \\ \gamma_0 = \ & \phi^{t-1}\left(\sigma_{t,1} + \sigma_{t-1,2} + \sigma_{t-2,3} + \cdots + \sigma_{1,t}\right) + \\ & \phi^{t}\left(\sigma_{t-1,1} + \sigma_{t-2,2} + \sigma_{t-3,3} + \cdots + \sigma_{1,t-1}\right) + \cdots + \\ & \phi^{2t-3}\left(\sigma_{12} + \sigma_{21}\right) + \phi^{2t-2}\sigma_{11} \end{aligned}$$

Collecting terms, we find that the ACVF at lag 0, i.e. the variance of the process is

$$\gamma_0 = \sum_{j=0}^{t-1} \sum_{i=0}^{j} \phi^j \ \sigma_{t-i,t-j+i} + \sum_{j=0}^{t-2} \sum_{i=1}^{t-j-1} \phi^{i+j} \ \sigma_{t-i-j,i} \tag{3.8}$$

$$
\gamma_1 = \begin{aligned}
&\sigma_{t,t-1} + \phi\left(\sigma_{t,t-2} + \sigma_{t-1,t-1}\right) + \phi^2\left(\sigma_{t,t-3} + \sigma_{t-1,t-2} + \sigma_{t-2,t-1}\right) + \\
&\phi^3\left(\sigma_{t,t-4} + \sigma_{t-1,t-3} + \sigma_{t-2,t-2} + \sigma_{t-3,t-1}\right) + \cdots + \\
&\phi^{t-1}\left(\sigma_{t-1,1} + \sigma_{t-2,2} + \sigma_{t-3,3} + \cdots + \sigma_{1,t-1}\right) + \\
&\phi^{t}\left(\sigma_{t-2,1} + \sigma_{t-3,2} + \sigma_{t-4,3} + \cdots + \sigma_{1,t-2}\right) + \cdots + \\
&\phi^{2t-4}\left(\sigma_{21} + \sigma_{12}\right) + \phi^{2t-3}\sigma_{11}
\end{aligned}
$$

Collecting terms, we find that the ACVF at lag 1 is

$$
\gamma_1 = \sum_{j=0}^{t-2}\sum_{i=0}^{j}\phi^j\ \sigma_{t-i,t-j+i-1} + \sum_{j=0}^{t-2}\sum_{i=1}^{t-j-1}\phi^{t+j-1}\ \sigma_{t-i-j,i} \tag{3.9}
$$

Similarly, the ACVF at lag k is

$$
\gamma_k = \sum_{j=0}^{t-k-1}\sum_{i=0}^{j}\phi^j\ \sigma_{t-i,t-j+i-k} + \sum_{j=0}^{t-2}\sum_{i=1}^{t-j-1}\phi^{t+j-k}\ \sigma_{t-i-j,i} \tag{3.10}
$$

Dividing (3.10) by (3.8), we get (3.7), and that completes the proof. ■

3.2 Heteroscedastic Autocorrelation Function (HACF)

Heteroscedasticity exists if the diagonal elements of $\mathbf{\Sigma}$ in (3.3) are not all identical and the disturbance term is free from autocorrelation. In other words, the disturbances are pairwise uncorrelated. This assumption is likely to be realistic one when using cross-sectional data. In this case $\mathbf{\Sigma}$ can be written as a diagonal matrix with the ith diagonal element given by σ_{ii}. We assume $E\left(\mathbf{e}_t\right) = \mathbf{0}$, and $Cov\left(\mathbf{e}_i, \mathbf{e}_j\right) = \mathbf{\Sigma}$, where $\mathbf{\Sigma} = diag\left(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{tt}\right)$. Thus

$$Cov\left(\mathbf{e}_i, \mathbf{e}_j\right) = \mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & 0 & \dots & 0 \\ 0 & \sigma_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{tt} \end{bmatrix} \tag{3.11}$$

The next theorem derives the HACF, at lag k when $\sigma_{ij} = 0$ for all $i \neq j$, i.e. $\mathbf{\Sigma} = diag\left(\sigma_{11}, \sigma_{22}, \dots, \sigma_{tt}\right)$ in $AR(1)$ models.

Theorem 3.2 *Consider $AR(1)$ model with parameter ϕ, $Z_t = \phi Z_{t-1} + \mathbf{e}_t$, $E\left(\mathbf{e}_t\right) = \mathbf{0}$, and $Cov\left(\mathbf{e}_i, \mathbf{e}_j\right) = \mathbf{\Sigma}$, where $\mathbf{\Sigma} = diag\left(\sigma_{11}, \sigma_{22}, \dots, \sigma_{tt}\right)$ as given in (3.11). Then HACF at lag k is given by*

$$\rho_k = \frac{\sum_{i=k}^{t-1} \phi^{2i-k} \sigma_{t-i,t-i}}{\sum_{i=0}^{t-1} \phi^{2i} \sigma_{t-i,t-i}} \tag{3.12}$$

Proof. *Using (3.6) the ACVF at lag* 0,

$\gamma_0 = \sigma_{tt} + \phi^2 \sigma_{t-1,t-1} + \phi^4 \sigma_{t-2,t-2} + \cdots + \phi^{2t-2} \sigma_{11}$. *Then the ACVF at lag* 0, *i.e. the variance of the process is*

$$\gamma_0 = \sum_{i=0}^{t-1} \phi^{2i} \sigma_{t-i,t-i} \tag{3.13}$$

The ACVF at lag 1

$\gamma_1 = \phi \sigma_{t-1,t-1} + \phi^3 \sigma_{t-2,t-2} + \cdots + \phi^{2t-3} \sigma_{11}$. *Then the ACVF at lag* 1 *is*

$$\gamma_1 = \sum_{i=1}^{t-1} \phi^{2i-1} \sigma_{t-i,t-i} \tag{3.14}$$

Similarly, the ACVF at lag k is

$$\gamma_k = \sum_{i=k}^{t-1} \phi^{2i-k} \sigma_{t-i,t-i} \tag{3.15}$$

Dividing (3.15) by (3.13), we get (3.12), and that completes the proof. ■

Homoscedasticity exists if the diagonal elements of $\mathbf{\Sigma}$ in (3.3) are all identical and the disturbance term, $\mathbf{e}$, is free from autocorrelation, i.e. $\sigma_{ij} = 0$ for all $i \neq j$. In this case, the disturbance term is a sequence of independent, identically distributed random variables.

Corollary 3.1 *Suppose in the previous theorem $\sigma_{ij} = 0$ for all $i \neq j$, and $Var(\mathbf{e}_t) = \sigma^2$ for all t, by taking $t \to \infty$ in (3.13), (3.14), and (3.15), we get $\gamma_0 = \frac{\sigma^2}{1-\phi^2}$, $\gamma_1 = \phi\frac{\sigma^2}{1-\phi^2}$, and $\gamma_k = \phi^k \frac{\sigma^2}{1-\phi^2}$, respectively. Then the ACF at lag k is given by $\rho_k = \phi^k$, $k \geq 0$, which is the well known ACF for $AR(1)$ process.*

The main objective of the next theorem is to transform a model with heteroscedastic disturbances such that the model has homoscedastic disturbances. We start with $AR(1)$ process assuming that $\mathbf{e}_t$ has $N(0, \sigma^2 \mathbf{\Psi}_{tt})$, then we derive an equation which is a function of the autoregressive coefficient, ϕ, and the covariance matrix, $\mathbf{\Psi}_{tt}$, but with new disturbance term that follows $N(0, \sigma^2)$.

Theorem 3.3 *Let* $Z_t = \phi Z_{t-1} + \mathbf{e}_t$ *where* $\mathbf{e}_t$ *has* $N(0, \sigma^2 \mathbf{\Psi}_{tt})$ *Define* $W_t = \frac{Z_t - Z_{t-1}}{\sqrt{\mathbf{\Psi}_{tt}}}$, *then* $W_t = (\phi - 1)\sqrt{\frac{\mathbf{\Psi}_{t-1,t-1}}{\mathbf{\Psi}_{tt}}} \sum_{j=0}^{t-1} W_{t-j-1} + \boldsymbol{\delta}_t$, *where* $\boldsymbol{\delta}_t = \frac{\mathbf{e}_t}{\sqrt{\mathbf{\Psi}_{tt}}}$ *has* $N(0, \sigma^2)$, *provided that* $\mathbf{\Psi}$ *is a full positive definite matrix, i.e.* $Z\mathbf{\Psi}Z' > 0$.

Proof. Recall $Z_t = \phi Z_{t-1} + \mathbf{e}_t$ and define $W_t = \frac{Z_t - Z_{t-1}}{\sqrt{\mathbf{\Psi}_{tt}}}$, then

$$W_t = \frac{\phi Z_{t-1} + \mathbf{e}_t - Z_{t-1}}{\sqrt{\mathbf{\Psi}_{tt}}}$$
$$= \frac{(\phi - 1) Z_{t-1}}{\sqrt{\mathbf{\Psi}_{tt}}} + \frac{\mathbf{e}_t}{\sqrt{\mathbf{\Psi}_{tt}}}$$

Then

$$W_t = \frac{(\phi - 1)}{\sqrt{\mathbf{\Psi}_{tt}}} Z_{t-1} + \boldsymbol{\delta}_t, \text{ where } \boldsymbol{\delta}_t = \frac{\mathbf{e}_t}{\sqrt{\mathbf{\Psi}_{tt}}} \sim N\left(0, \sigma^2\right) \tag{3.16}$$

Similarly,

$$W_{t-1} = \frac{Z_{t-1} - Z_{t-2}}{\sqrt{\mathbf{\Psi}_{t-1,t-1}}}$$
$$= \frac{Z_{t-1} - BZ_{t-1}}{\sqrt{\mathbf{\Psi}_{t-1,t-1}}}, \text{ where } B \text{ is backward shift operator, } BZ_{t-1} = Z_{t-2}$$
$$= \frac{(1 - B) Z_{t-1}}{\sqrt{\mathbf{\Psi}_{t-1,t-1}}}$$

Then

$$\sqrt{\mathbf{\Psi}_{t-1,t-1}} (1 - B)^{-1} W_{t-1} = Z_{t-1} \tag{3.17}$$

By substituting (3.17) in (3.16) we get

$$W_t = (\phi - 1)\sqrt{\frac{\mathbf{\Psi}_{t-1,t-1}}{\mathbf{\Psi}_{tt}}} (1 - B)^{-1} W_{t-1} + \boldsymbol{\delta}_t$$
$$= (\phi - 1)\sqrt{\frac{\mathbf{\Psi}_{t-1,t-1}}{\mathbf{\Psi}_{tt}}} \sum_{j=0}^{t-1} B^j W_{t-1} + \boldsymbol{\delta}_t, \qquad (1 - B)^{-1} = \sum_{j=0}^{t-1} B^j$$
$$= (\phi - 1)\sqrt{\frac{\mathbf{\Psi}_{t-1,t-1}}{\mathbf{\Psi}_{tt}}} \sum_{j=0}^{t-1} W_{t-j-1} + \boldsymbol{\delta}_t \quad \blacksquare$$

4 Summary and Future Research

This paper has investigated an important statistical problem concerning the autocorrelation function, ACF, in the presence of heteroscedasticity disturbances in first-order autoregressive, $AR(1)$, models. We have derived explicit equations for ACF when the disturbance follows the general covariance matrix, $\mathbf{\Sigma}$, and when the diagonal elements of $\mathbf{\Sigma}$ are not all identical but $\sigma_{i,j} = 0$ $\forall\ i \neq j$, i.e. $\mathbf{\Sigma} = diag\left(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{tt}\right)$. In addition, we have derived an equation to transform a model with heteroscedastic disturbances such that the model has homoscedastic disturbances.

The plan of the future research is to extend the explicit equations that we have derived in this paper for ACF in the presence of heteroscedasticity disturbances in the general form of the autoregressive models, i.e. in autoregressive models with order p, $AR(p)$.

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