On Distributions of Generalized Order Statistics from Kumaraswamy Distribution in Closed Forms

Dr. Samir K. Safi[1] **Dr. Rehab H. Al Sheikh Ahmed**[2]

Abstract

The Kumaraswamy distribution which introduced by Poondi Kumaraswamy (1980) is similar to the Beta distribution but has the key advantage of a closed-form cumulative distribution function. With its two non-negative shape parameters p and q, it was originally conceived to model hydrological phenomena, (See for example Mitnik, 2008). In this paper, we study some distributions of generalized order statistics on Kumaraswamy distribution. We have derived the joint probability density functions for generalized order statistics from Kumaraswamy distribution in closed-forms. In addition, the pdf of the conditional distribution for generalized order statistics from Kumaraswamy distribution is obtained. Furthermore, some special cases have been discussed.

Keywords: Kumaraswamy Distribution, Generalized Order Statistics, Random Variables, Conditional Distribution.

1. Introduction

Kumaraswamy (1980) was a leading Indian engineer and hydrologist, he introduced the distribution for variables that are lower and upper bounded. The Kumaraswamy distribution is a continuous probability distribution with double-bounded support, defined on the interval [0,1] differing in the values of their two non-negative shape parameters p and q. It is similar to the Beta distribution but has the key advantage of a closed form cumulative distribution function, Carrasco (2010).

The probability density function of the Kumaraswamy distribution is

$$f_Z(z)=\frac{1}{(b-c)}\,p\,q\left(\frac{z-c}{b-c}\right)^{p-1}\left[1-\left(\frac{z-c}{b-c}\right)^{p}\right]^{q-1} \quad ,c<z<b, \tag{1.1}$$

with shape parameters p>0 and q>0, and boundary parameters c and b. The standard form of the Kumaraswamy density function (c=0, b=1), kum(p, q) is given by

$$f_X\left(x;p,q\right)=p\,q\left(x\right)^{p-1}\left(1-x^{p}\right)^{q-1} \tag{1.2}$$

The closed form of the cumulative distribution function of the Kumaraswamy distribution is given by

$$F(x;p,q)=1-\left(1-x^{p}\right)^{q} \tag{1.3}$$

[1] Associate Professor in Statistics, The Islamic University of Gaza

[2] Statistician Consultant – Al-Azhar University - Gaza

The median of the Kumaraswamy distribution can be written as $md(x) = \left(1 - 0.5^{\frac{1}{q}}\right)^{\frac{1}{p}}$.

The expectation of the kum(p,q) is

$$E(x) = \frac{q\Gamma\left(1+\frac{1}{p}\right)\Gamma(q)}{\Gamma\left(1+q+\frac{1}{p}\right)} = qB\left(1+\frac{1}{p},q\right)$$

Where, $B\left(1+\frac{1}{p},q\right)$ is Beta distribution with parameters $1+\frac{1}{p}$ and q

The variance of the kum(p,q) is

$$Var(x) = \frac{q\Gamma\left(1+\frac{2}{p}\right)\Gamma(q)}{\Gamma\left(1+q+\frac{2}{p}\right)} - \left(\frac{q\Gamma\left(1+\frac{n}{p}\right)\Gamma(q)}{\Gamma\left(1+q+\frac{n}{p}\right)}\right)^2.$$

$$= qB\left(1+\frac{2}{p},q\right) - \left(qB\left(1+\frac{1}{p},q\right)\right)^2.$$

Relations between the Kumaraswamy Distribution and other Distributions

We derived some relations between the Kum(p,q) and other distributions.

For any non-negative shape parameters p and q (p>0 and q>0), we have derived the following propositions

1. $X \sim Beta(1,q) \Leftrightarrow X \sim kum(1,q)$.
2. $X \sim Beta(p,1) \Leftrightarrow X \sim kum(p,1)$.
3. $If\ Y = x^p\ then\ Y \sim Beta(1,q) \Leftrightarrow X \sim kum(p,q)$.
4. $X \sim Beta(1,q)$ then $X^{\frac{1}{p}} \sim kum(p,q)$.
5. $X \sim U(0,1) \Leftrightarrow X \sim kum(1,1)$.

2. Joint Distribution of all Generalized Order Statistics

Definition 2.1 If the random variables $U(1,n,\tilde{m},k), U(2,n,\tilde{m},k),...,U(n,n,\tilde{m},k)$ possess a joint density function of the form $f^{U(1,n,\tilde{m},k),...,U(n,n,\tilde{m},k)}(u_1,u_2,...,u_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}(1-u_i)^{m_i}\right](1-u_n)^{k-1}$, on the cone $0 \le u_1 \le ... \le u_n \le 1$ of $\mathbb{R}^n$, then they are called Uniform Generalized Order Statistics.

Note that $m_1, m_2, ..., m_{n-1} \in R, K \ge 1, n \in N, 0 \le u_1 \le u_2 \le ... \le u_n \le 1, M_r = \sum_{j=r}^{n-1} m_j$ are parameters which satisfy $\gamma_r = k + n - r + M_r \ge 1$ for all values of $r \in \{1,2,\dots,n-1\}$, $\tilde{m} = (m_1, m_2, ..., m_{n-1})$. (Kamps, U. ,1995)

Definition 2.2 The random variables $X(r,n,\tilde{m},k) = F^{-1}(U(r,n,\tilde{m},k))$ are called generalized order statistics (based on F), which have a joint density function of the form

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}(1-F(x_i))^{m_i} f(x_i)\right](1-F(x_n))^{k-1} f(x_n),$$

$F^{-1}(0) < X_1 \le X_2 \le \dots \le X_n < F^{-1}(1)$. (Kamps, U. ,1995)

Special Case (The joint pdf of all the ordinary order statistics)
In Definition 2.2, if $k=1$ and $m_1 = m_2 = \dots = m_{n-1} = zero$, then

$$f^{X(1,n,0,1),\dots,X(n,n,0,1)}(x_1,\dots,x_n)\left[\prod_{j=1}^{n-1}\gamma_j\right]\left[\prod_{i=1}^{n-1}(1-F(x_i))^{0} f(x_i)\right](1-F(x_n))^{1-1} f(x_n).$$

$$\left[\prod_{j=1}^{n-1}\gamma_j\right]=\prod_{j=1}^{n-1}(k+n-j+M_j),\ M_j=\sum_{r=j}^{n-1}m_r = zero\ .$$

Then $\left[\prod_{j=1}^{n-1}\gamma_j\right]=\prod_{j=1}^{n-1}(1+n-j)=(1+n-1)(1+n-2)(1+n-3)\dots(1+n-n+1)$.

$$=n(n-1)(n-2)\dots 4.3.2.1=n!.$$

Therefore, $f^{X(1,n,0,1),\dots,X(n,n,0,1)}(x_1,\dots,x_n)=n!\prod_{i=1}^{n} f(x_i)$, *which is the well known pdf of all ordinary order statistics*.

Theorem 2.3 The joint pdf of $X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)$ for Kumaraswamy distribution is

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n)=$$

$$k\prod_{j=1}^{n-1}\gamma_j\prod_{i=1}^{n-1}\left[\left((1-x_i^{\,p})^{q}\right)^{m_i} p\,q\,x_i^{\,p-1}(1-x_i^{\,p})^{q-1}\right]\left(\left(1-x_n^{\,p}\right)^{q}\right)^{k-1} p\,q\,x_n^{\,p-1}(1-x_n^{\,p})^{q-1}$$

We discuss some special cases in Corollary 2.4.

Corollary 2.4

1. Let $p=q=1$, then

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n)=k\prod_{j=1}^{n-1}\gamma_j\left[\prod_{i=1}^{n-1}(1-x_i)^{m_i}\right](1-x_n)^{k-1}.$$

2. Let $p=1$, then

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n)=$$

$$k\prod_{j=1}^{n-1}\gamma_j\left[\prod_{i=1}^{n-1}\left((1-x_i)^{q}\right)^{m_i} q(1-x_i)^{q-1}\right]\left((1-x_n)^{q}\right)^{k-1} q(1-x_n)^{q-1}.$$

3. Let $q=1$, then

$$f^{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n)=k\prod_{j=1}^{n-1}\gamma_j\left[\prod_{i=1}^{n-1}(1-x_i^{\,p})^{m_i} p\,x_i^{\,p-1}\right](1-x_n^{\,p})^{k-1} p\ x_n^{\,p-1}.$$

3. Joint Distribution of Two Generalized Order Statistics

Definition 3.1 The joint pdf of i^{th} and j^{th} generalized order statistics, $X(i,n,\tilde{m},k)$ and $X(j,n,\tilde{m},k)$**,** is given by

$$f_{i,j,n,\tilde{m},k}\left(x_i,x_j\right)=\frac{c_j}{(i-1)!(j-i-1)!}\left[1-F\left(x_i\right)\right]^{m}\left[1-F\left(x_j\right)\right]^{\gamma_j-1}\left[g_m F\left(x_i\right)\right]^{i-1}\times$$

$$\left[g_m F\left(x_j\right)-g_m F\left(x_i\right)\right]^{j-i-1}f\left(x_i\right)f\left(x_j\right),\text{ for } 0<x_i<x_j<\infty, 1\le i<j\le n,$$

where, $c_r=\prod_{j=1}^{r}\gamma_j,\gamma_j=k+(n-j)(m+1)$ and

$$g_m(x)=\begin{cases}\dfrac{1}{m+1}\left[1-(1-x)^{m+1}\right], & m\neq -1.\\ -\ln(1-x) & ,m=-1, x\in(0,1)\end{cases}.$$

Since $\lim_{m\to -1}\frac{1}{m+1}\left[1-(1-x)^{m+1}\right]=-\ln(1-x)$, we shall write $g_m(x)=\frac{1}{m+1}\left[1-(1-x)^{m+1}\right]$ for all $x\in(0,1)$ and for all m with $g_{-1}(x)=\lim_{m\to -1}g_m(x)$. (Garg, M ., 2009)

Special Case (The joint pdf of two ordinary order statistics)

In Definition (3.1) , if $k=1$ and $m=0$, then

$$f_{i,j,n,0,1}\left(x_i,x_j\right)=\frac{c_j}{(i-1)!(j-i-1)!}\left[1-F\left(x_j\right)\right]^{\gamma_j-1}\left[g_m\left(F\left(x_i\right)\right)\right]^{i-1}\times$$
$$\left[g_m\left(F\left(x_j\right)\right)-g_m\left(F\left(x_i\right)\right)\right]^{j-i-1}f\left(x_i\right)f\left(x_j\right).$$

$$g_m\left(F\left(x_i\right)\right)=\frac{1}{0+1}\left[1-\left(1-F\left(x_i\right)\right)^{0+1}\right]=F\left(x_i\right),\ g_m\left(F\left(x_j\right)\right)=F\left(x_j\right).$$

$$\gamma_j=n-j+1,\ c_j=\prod_{t=1}^{j}\gamma_t=\left[1+(n-1)\right]\left[1+(n-2)\right]...\left[1+(n-j)\right]=\frac{n!}{(n-j)!}.$$

Then

$$f_{i,j,n,0,1}\left(x_i,x_j\right)=\frac{c_j}{(i-1)!(j-i-1)!}\left[1-F\left(x_j\right)\right]^{n-j}\left[F\left(x_i\right)\right]^{i-1}\times$$
$$\left[F\left(x_j\right)-F\left(x_i\right)\right]^{j-i-1}f\left(x_i\right)f\left(x_j\right),$$

which is the well known joint pdf of two ordinary order statistics x_i *and* x_j .

Theorem 3.2 The joint pdf of $X\left(1,n,\tilde{m},k\right)$ and $X\left(n,n,\tilde{m},k\right)$ for Kumaraswamy distribution is

$$f_{1,n,n,\tilde{m},k}\left(x_1,x_n\right)=\frac{c_n}{(n-2)!}\left[1-x_1^{\,p}\right]^{mq}\left[1-x_n^{\,p}\right]^{qk-q}\times$$

$$\left[\frac{1}{m+1}\left(\left(1-x_1^{\,p}\right)^{qm+q}-\left(1-x_n^{\,p}\right)^{qm+q}\right)\right]^{n-2}(pq)^2\left(x_1x_n\right)^{p-1}\left[\left(1-x_1^{\,p}\right)\left(1-x_n^{\,p}\right)\right]^{q-1}$$

Some special cases are obtained in Corollary 3.3.

Corollary 3.3

1. Let $p=q=1$, then

$$f_{1,n,n,\tilde{m},k}\left(x_1,x_n\right)=\frac{k^n}{(n-2)!}\left[1-x_1\right]^{m}\left[1-x_n\right]^{k-1}\left[\frac{1}{m+1}\left(\left(1-x_1\right)^{m+1}-\left(1-x_n\right)^{m+1}\right)\right]^{n-2}.$$

2. Let $p=1$, then

$$f_{1,n,n,\tilde{m},k}\left(x_1,x_n\right)=\frac{k^n}{(n-2)!}\left[1-x_1\right]^{mq}\left[1-x_n\right]^{qk-q}\left[\frac{1}{m+1}\left(\left(1-x_1\right)^{qm+q}-\left(1-x_n\right)^{qm+q}\right)\right]^{n-2}\times$$

$$q^2\left[\left(1-x_1\right)\left(1-x_n\right)\right]^{q-1}\ .$$

3. Let $q=1$, then

$$f_{1,n,n,\tilde{m},k}\left(x_1,x_n\right)=$$

$$\frac{k^n}{(n-2)!}\left[1-x_1{}^p\right]^m\left[1-x_n{}^p\right]^{k-1}\left[\frac{1}{m+1}\left(\left(1-x_1{}^p\right)^{m+1}-\left(1-x_n{}^p\right)^{m+1}\right)\right]^{n-2}p^2\left(x_1x_n\right)^{p-1}$$

4. Distribution of Single Generalized Order Statistic

Definition 4.1 Further integrating out $x_1,\ldots,x_{r-1},x_{r+1},\ldots,x_n$ from Definition 2.2, we get the pdf $f_{r,n,\tilde{m},k}$ of $X\left(r,n,m,k\right)$ as

$$f_{r,n,\tilde{m},k}\left(x\right)=\frac{c_r}{(r-1)!}\left[1-F\left(x_r\right)\right]^{\gamma_r-1}g_m{}^{r-1}\left(F\left(x_r\right)\right)f\left(x_r\right)$$

Where, $c_r=\prod_{j=1}^{r}\gamma_j$, $\gamma_r=k+\left(n-r\right)\left(m+1\right)$ and $g_m\left(x\right)=\frac{1}{m+1}\left[1-\left(1-x\right)^{m+1}\right]$

for all x $\in$ (0,1) and for all m with $g_{-1}\left(x\right)=\lim_{x\to-1}g_m\left(x\right)$. (Garg, M. , 2009)

Special Case (The pdf of the single ordinary order statistic x_r **)**

Using Definition 4.1, if k=1 and m=0, then $\gamma_j=n-j+1\Rightarrow c_r=\prod_{j=1}^{r}n-j+1=\frac{n!}{\left(n-r\right)!}$

$$g_0\left(F\left(x_r\right)\right)=\left[1-\left(1-F\left(x_r\right)\right)\right]=F\left(x_r\right)\Rightarrow g^{-1}{}_0\left(F\left(x_r\right)\right)=\left[F\left(x_r\right)\right]^{r-1}.$$

$f_{r,n,0,1}\left(x\right)=\frac{n!}{\left(n-r\right)!(r-1)!}\left[1-F\left(x_r\right)\right]^{n-r}\left[F\left(x_r\right)\right]^{r-1}f\left(x_r\right)$, *which is the well known pdf of the single ordinary order statistic* x_r .

Lemma 4.2 The pdf of the minimum generalized order statistic is

$$f_{1,n,\tilde{m},k}\left(x\right)=\left[k+\left(n-1\right)\left(m+1\right)\right]\left[1-F\left(x_1\right)\right]^{k+(n-1)(m+1)-1}f\left(x_1\right) \qquad (4.1)$$

Proof:
Using Definition 4.1, let $r=1$, then $c_1=\gamma_1=k+\left(n-1\right)\left(m+1\right)$, we get (4.1) and that completes the proof.

Corollary 4.3 (The pdf of the minimum ordinary order statistic)
Using (4.1), let $k=1$ and $m=0$ then, $f_{1,n,0,1}\left(x\right)=n\left[1-F\left(x_1\right)\right]^{n-1}f\left(x_1\right)$, *which is the well known pdf of the minimum ordinary order statistic* x_1 .

Lemma 4.4 The pdf of the maximum generalized order statistic is

$$f_{n,n,\tilde{m},k}(x)=\frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k\right]}{(n-1)!}\left[1-F(x_n)\right]^{k-1}\times$$

$$\left(\frac{1}{m+1}\left[1-\left(1-F(x_n)\right)^{m+1}\right]\right)^{n-1}f(x_n) \qquad (4.2)$$

Proof:

Using Definition (4.1), let $r=n$, then $g_m\left(F(x_n)\right)=\frac{1}{m+1}\left[1-\left(1-F(x_n)\right)^{m+1}\right]$ $,\gamma_n=k$

$c_n=\prod_{j=1}^{n}k+(n-j)(m+1)=\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k\right]$, we get (4.2) and that completes the proof.

Corollary 4.5 (The pdf of the maximum ordinary order statistic)

Using (4.2), let $k=1$ and $m=0$, then $f_{n,n,0,1}(x)=n\left[F(x_n)\right]^{n-1}f(x_n)$, *which is the well known pdf of the maximum ordinary order statistic* x_n .

Lemma 4.6 The pdf of the median generalized order statistic is

$$f_{t+1,n,\tilde{m},k}(\bar{x})=\frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k+t(m+1)\right]}{t!}$$

$$\left[1-F(\bar{x})\right]^{k+t(m+1)-1}\left(\frac{1}{m+1}\left[1-\left(1-F(\bar{x})\right)^{m+1}\right]\right)^{t}f(\bar{x}) \qquad (4.3)$$

Proof:

Using Definition (4.1), based on median property, let $r-1=t, n-r=t$, and $n=2t+1$, then $\gamma_r=k+t(m+1)$, $g_m^t\left(F(\bar{x})\right)=\left(\frac{1}{m+1}\left[1-\left(1-F(\bar{x})\right)^{m+1}\right]\right)^t$,

$c_r=\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k+t(m+1)\right]$, we get (4.3) and that completes the proof.

Corollary 4.7 (The pdf of the median ordinary order statistic)

Using (4.3), $k=1$ and $m=0$, then

$$f_{t+1,n,0,1}(\bar{x})=\frac{\left[1+(n-1)\right]\left[1+(n-2)\right]...\left[1+t\right]}{t!}\left[1-F(\bar{x})\right]^t\left(F(x)\right)^t f(\bar{x})$$

$$=\frac{(2t+1)!}{(t!)^2}\left[1-F(\bar{x})\right]^t\left(F(\bar{x})\right)^t f(\bar{x}),$$ *which is the pdf of the median ordinary order statistic.*

Lemma 4.8 The pdf of the rth generalized order statistic for Kumaraswamy distribution is

$$f_{r,n,\tilde{m},k}(x)=\frac{c_r}{(m+1)(r-1)!}\left[\left(1-x_r^{\,p}\right)^q\right]^{\gamma_r-1}\left[\frac{1}{m+1}\left(1-\left(1-x_r^{\,p}\right)^{q(m+1)}\right)\right]^{r-1}pq\left(x_r\right)^{p-1}\left(1-x_r^{\,p}\right)^{q-1} \qquad (4.4)$$

Some special cases are obtained in Corollary 4.9.

Corollary 4.9

1. Let $p = q = 1$, then

$$f_{r,n,\tilde{m},k}(x) = \frac{c_r}{(r-1)!}\left[(1-x_r)\right]^{\gamma_r-1}\left(\frac{1}{m+1}\left[1-(1-x_r)^{m+1}\right]\right)^{r-1}.$$

2. Let $p = 1$, then

$$f_{r,n,\tilde{m},k}(x) = \frac{c_r}{(r-1)!}\left[(1-x_r)^q\right]^{\gamma_r-1}\left(\frac{1}{m+1}\left[1-\left((1-x_r)^q\right)^{m+1}\right]\right)^{r-1} q\,(1-x_r)^{q-1}.$$

3. Let $q = 1$, then

$$f_{r,n,\tilde{m},k}(x) = \frac{c_r}{(r-1)!}\left[(1-x_r{}^p)\right]^{\gamma_r-1}\left(\frac{1}{m+1}\left[1-(1-x_r{}^p)^{m+1}\right]\right)^{r-1} p\,(x_r)^{p-1}.$$

Lemma 4.10 The pdf of the minimum generalized order statistic for Kumaraswamy distribution is

$$f_{1,n,\tilde{m},k}(x) = k+(n-1)(m+1)\left(1-x_1{}^p\right)^{q\left[k+(n-1)(m+1)-1\right]} p\,q\,x_1^{p-1}\left(1-x_1{}^p\right)^{q-1} \tag{4.5}$$

Proof:

Using Definition (4.1), let $r = 1$, then $c_1 = \prod_{j=1}^{1}\gamma_1$, $\gamma_1 = k+(n-1)(m+1)$, we get (4.5) and that completes the proof.

We discuss some special cases in Corollary 4.11.

Corollary 4.11

1. Let $p = q = 1$ then $f_{1,n,\tilde{m},k}(x) = k+(n-1)(m+1)\left[(1-x_1)\right]^{k+(n-1)(m+1)-1}$.
2. Let $p = 1$ then $f_{1,n,\tilde{m},k}(x) = k+(n-1)(m+1)\left[(1-x_1)^q\right]^{k+(n-1)(m+1)-1} q\,(1-x_1)^{q-1}$.
3. Let $q = 1$ then $f_{1,n,\tilde{m},k}(x) = k+(n-1)(m+1)\left[(1-x_1{}^p)\right]^{k+(n-1)(m+1)-1} p\,(x_1)^{p-1}$.

Lemma 4.12 The pdf of the maximum generalized order statistic for Kumaraswamy distribution is

$$f_{n,n,\tilde{m},k}(x) = \frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]\ldots\left[k\right]}{(n-1)!}\left[\left(1-x_n{}^p\right)^q\right]^{k-1}$$
$$\left(\frac{1}{m+1}\left[1-\left(\left(1-x_n{}^p\right)^q\right)^{m+1}\right]\right)^{n-1} p\,q\,(x_n)^{p-1}\left(1-x_n{}^p\right)^{q-1} \tag{4.6}$$

Proof:

Using Definition (4.1), let $r = n$ **,** then $c_n = \prod_{j=1}^{n}\gamma_j$, $\gamma_j = k+(n-j)(m+1)$, $\gamma_n = k$,

$c_n = \left[k + (n-1)(m+1)\right]\left[k + (n-2)(m+1)\right]...\left[k\right]$, and

$g_m\left(F(x_n)\right) = \frac{1}{m+1}\left[1-\left(1-F(x_n)\right)^{m+1}\right]$, we get (4.6) and that completes the proof.

Some special cases are obtained in Corollary 4.13.

Corollary 4.13

1. Let $p = q = 1$, then

$$f_{n,n,\tilde{m},k}(x) = \frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k\right]}{(n-1)!} \times \left[(1-x_n)\right]^{k-1}\left(\frac{1}{m+1}\left[1-(1-x_n)^{m+1}\right]\right)^{n-1}.$$

2. Let $p = 1$, then

$$f_{n,n,\tilde{m},k}(x) = \frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k\right]}{(n-1)!}\left[(1-x_n)^q\right]^{k-1} \times \left(\frac{1}{m+1}\left[1-\left((1-x_n)^q\right)^{m+1}\right]\right)^{n-1} q\,(1-x_n)^{q-1}.$$

3. Let $q = 1$, then

$$f_{n,n,\tilde{m},k}(x) = \frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k\right]}{(n-1)!}\left[\left(1-x_n{}^p\right)\right]^{k-1} \times \left(\frac{1}{m+1}\left[1-\left(1-x_n{}^p\right)^{m+1}\right]\right)^{n-1} p\,(x_n)^{p-1}.$$

Lemma 4.14 The pdf of the median generalized order statistic for Kumaraswamy distribution is

$$f_{t+1,n,\tilde{m},k}(\bar{x}) = \frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k+t(m+1)\right]}{t!} \times \left[\left(1-\bar{x}^p\right)^q\right]^{k+t(m+1)-1}\left(\frac{1}{m+1}\left[1-\left(\left(1-\bar{x}^p\right)^q\right)^{m+1}\right]\right)^t p\,q\,(\bar{x})^{p-1}\left(1-\bar{x}^p\right)^{q-1} \quad (4.7)$$

Proof:

Using Definition (4.1), based on median property, let $r-1 = t, n-r = t$, and $n = 2t+1$, then

$c_r = \left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k+t(m+1)\right]$

$\gamma_r = k + (n-r)(m+1) = k + t(m+1)$, and $g_m{}^t\left[F(\bar{x})\right] = \left(\frac{1}{m+1}\left[1-\left(1-F(x)\right)^{m+1}\right]\right)^t$, we get (4.7) and that completes the proof.

Some special cases are obtained in Corollary 4.15.

Corollary 4.15

1. Let $p = q = 1$, then

$$f_{t+1,n,\tilde{m},k}(\bar{x}) = \frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k+t(m+1)\right]}{t!} \times$$
$$\left[(1-\bar{x})\right]^{k+t(m+1)-1}\left(\frac{1}{m+1}\left[1-\left((1-\bar{x})\right)^{m+1}\right]\right)^t .$$

2. Let $p = 1$, then

$$f_{t+1,n,\tilde{m},k}(\bar{x}) = \frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k+t(m+1)\right]}{t!} \times$$
$$\left[(1-\bar{x})^q\right]^{k+t(m+1)-1}\left(\frac{1}{m+1}\left[1-\left((1-\bar{x})^q\right)^{m+1}\right]\right)^t q(1-\bar{x})^{q-1} .$$

3. Let $q = 1$, then

$$f_{t+1,n,\tilde{m},k}(\bar{x}) = \frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]...\left[k+t(m+1)\right]}{t!}$$

5. Conditional Distribution of Generalized Order Statistics

Theorem 5.1 (Samuel, 2005)

Let $X_1,...,X_n$ be a random sample from a continuous population with cdf $F(x)$ and pdf $f(x)$. Let $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k),...,X(n,n,\tilde{m},k)$ denote the generalized order statistics obtained from this sample. Then the conditional distribution of $X(s,n,\tilde{m},k) \mid X(r,n,\tilde{m},k) = x$ for $r < s$ $\mid X(r,n,m,k) = x$ for $r < s$ is

$$h\left(X(s,n,\tilde{m},k) \mid X(r,n,\tilde{m},k)\right) =$$
$$\frac{c_s\left[1-F(x)\right]^m\left[1-F(y)\right]^{\gamma_s-1}\left[g_m(F(y))-g_m(F(x))\right]^{s-r-1} f(y)}{c_r(s-r-1)!\left[1-F(x)\right]^{\gamma_r-1}},$$

$F^{-1}(0) < x \le y < F^{-1}(1)$

Notes

$c_r = \prod_{j=1}^{r}\gamma_j$, $\gamma_j = k+(n-j)(m+1)$ and $g_m(x) = \frac{1}{m+1}\left[1-(1-x)^{m+1}\right]$ for all $x \in (0,1)$ and for all m with $g_{-1}(x) = \lim_{m\to -1} g_m(x)$.

Special Case (Conditional distribution of an ordinary order statistics)

Using Theorem (5.1), if $k = 1$ and $m = 0$, then $h\left(X(s,n,0,1) \mid X(r,n,0,1)\right) =$

$$\frac{\left[1-F(x)\right]^{0}\left[1-F(y)\right]^{\gamma_s-1}\left[g_0(F(y))-g_0(F(x))\right]^{s-r-1} f(y)c_s}{\left[1-F(x)\right]^{\gamma_r-1} c_r (s-r-1)!},\ x(r,n,0,1)\le y(s,n,0,1)<\infty$$

$\gamma_r = 1+(n-r)$, $\gamma_s = 1+(n-s)$, $\gamma_{r+1} = 1+(n-r-1) = n-r$. $g_0(F(y)) = F(y)$, $g_0(F(x)) = F(x)$. $c_s = n(n-1)(n-2)...(1+n-s) = \frac{n!}{(n-s)!}$, and

$c_r = n(n-1)(n-2)...(1+n-r) = \frac{n!}{(n-r)!}$. Then

$$h\left(X(s,n,0,1)\middle| X(r,n,0,1)\right) = \frac{\left[1-F(y)\right]^{n-s}\left[(F(y))-(F(x))\right]^{s-r-1} f(y)(n-r)!}{\left[1-F(x)\right]^{n-r}(n-s)!(s-r-1)!}.$$

Which is the well known conditional distribution of an ordinary order statistics $X(s,n,0,1)\,|$ $X(r,n,0,1)$.

Theorem 5.2 The conditional pdf of $X(s,n,\tilde{m},k)\,\big|\,X(r,n,\tilde{m},k)=x$ for Kumaraswamy distribution is

$$h\left(X(s,n,\tilde{m},k)\middle| X(r,n,\tilde{m},k)\right) =$$

$$\frac{c_s}{c_r(s-r-1)!}\left[1-x^p\right]^{q(m-\gamma_r+1)}\left[\frac{1}{m+1}\left[\left(1-x^p\right)^{q(m+1)}-\left(1-y^p\right)^{q(m+1)}\right]\right]^{s-r-1}\times$$

$$\left[1-y^p\right]^{q(\gamma_s-1)} pq(y)^{p-1}\left(1-y^p\right)^{q-1},\ F^{-1}(0)<x\le y<F^{-1}(1)$$

Some special cases are derived in Corollary 5.3.

Corollary 5.3

1. Let $p=q=1$, then

$$h\left(X(s,n,\tilde{m},k)\middle| X(r,n,\tilde{m},k)\right) =$$

$$\frac{c_s}{c_r(s-r-1)!}\left[1-x\right]^{(m-\gamma_r+1)}\left[\frac{1}{m+1}\left[\left(1-x\right)^{(m+1)}-\left(1-y\right)^{(m+1)}\right]\right]^{s-r-1}\left[1-y\right]^{(\gamma_s-1)}$$

2. Let $p=1$, then

$$h\left(X(s,n,\tilde{m},k)\middle| X(r,n,\tilde{m},k)\right) =$$

$$\frac{c_s}{c_r(s-r-1)!}\left[1-x\right]^{q(m-\gamma_r+1)}\left[\frac{1}{m+1}\left[\left(1-x\right)^{q(m+1)}-\left(1-y\right)^{q(m+1)}\right]\right]^{s-r-1}\left[1-y\right]^{q(\gamma_s-1)} q\left(1-y^p\right)^{q-1}$$

3. Let $q=1$, then

$$h\left(X(s,n,\tilde{m},k)\middle| X(r,n,\tilde{m},k)\right) =$$

$$\frac{c_s}{c_r(s-r-1)!}\left[1-x^p\right]^{(m-\gamma_r+1)}\left[\frac{1}{m+1}\left[\left(1-x^p\right)^{(m+1)}-\left(1-y^p\right)^{(m+1)}\right]\right]^{s-r-1}\left[1-y^p\right]^{(\gamma_s-1)} p\,(y)^{p-1}$$

6. Conclusion and Future Research

In this paper, we have derived the joint pdfs of generalized order statistics for different cases from Kumaraswamy distribution in closed-forms. In addition, the pdf of the conditional distribution of generalized order statistics from Kumaraswamy distribution is given. Furthermore, some special cases have been discussed.

Many opportunities of future research are available. The plan for the future research on generalized order Statistics from Kumaraswamy distribution can be split into two main areas. Estimation and hypothesis testing of Kumaraswamy parameters based on generalized order statistics.

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