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Nonstandard Topology on $\mathbb{R}$

Eissa D. Habil[1,*], Ahmed A. Ghneim[2]

[1]Department of Mathematics, Islamic University of Gaza, Gaza Strip, Palestine

[2]Department of Mathematics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia

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Abstract

Nonstandard topology on $\mathbb{R}$ is a kind of topology constructed by means of nonstandard analysis on $\mathbb{R}$. The ordered field of nonstandard real numbers (or simply hyperreals) $^*\mathbb{R}$ has been introduced. It extends the reals $\mathbb{R}$. The hyperreals properties and the main topological definitions for $\mathbb{R}$ with the standard topology have been presented in the nonstandard context. Nonstandard proofs of well-known theorems from the topology on $\mathbb{R}$ have been given and compared with the standard ones.

Keywords Ultrapower, Hyperreals, Transfer Principle, (Nonstandard) Topology on $\mathbb{R}$.

التبولوجى غير المألوف على الأعداد الحقيقية $\mathbb{R}$

ملخص

التبولوجى غير المألـــــوف على $\mathbb{R}$ هو نوع من التبولوجي الذي يتم بناؤه بوساطة التحليل غير المألوف على $\mathbb{R}$. لقد تـــم تقديم الحقل المرتب $^*\mathbb{R}$ المكون من الأعداد الحقيقية غير المألوفة والذي يعتبر توسعة للأعداد الحقيقية $\mathbb{R}$. كما تـــم عـــرض خواص الأعداد الحقيقية غير المألوفة والتعريفات الأساسية للتبولوجي الطبيعي (المألوف) على $\mathbb{R}$ من خـــلال الســـياق غيـــر المألوف، إضافة الى ذلك تم طرح براهين غير مألوفة لكثير من النظريات المشهورة الخاصة بالتبولوجي الطبيعي على $\mathbb{R}$ ثم مقارنتها بالبراهين المألوفة.

كلمات مفتاحية: فوق الأسيّة، الأعداد الحقيقية غير المألوفة، مبدأ التحويل، التبولوجى غير المألوف على $\mathbb{R}$.

*Corresponding author e-mail address: habil@iugaza.edu.ps

1. Introduction:

For over three hundred years, a basic question about calculus remained unanswered. Do the infinitesimals as conceptional understood by Leibniz and Newton, exist as formal mathematical objects? The question was answered affirmatively by Abraham Robinson [1] and the subject termed "Nonstandard Analysis" was introduced to the scientific world [2–6].

According to Machover and Herschfeld [6], the aim of Robinson's theory can perhaps be explained best by discussing an example from topology: A mapping f of a topological space X into a topological space Y is continuous at p if $f(x)$ is near $f(p)$ provided x is near p. Now a natural question to ask is: How this notion of "nearness" can be made precise? In particular, for the real line (with standard topology) this would mean giving a precise meaning to a number being "near" zero; i.e., "infinitesimal" [5, 7].

On the other hand, one cannot treat any of the already existing non-zero points of the real line as infinitesimals without getting an immediate contradiction. Now if one try to add the infinitesimals as new "ideal" elements of the real line, one would spoil its nice algebro-topological properties.

Robinson's theory solves this problem completely, by showing, e.g., how a topological space $\mathbb{R}$ can be imbedded in a "topological space" ${}^*\mathbb{R}$ such that:

(i) For any p in $\mathbb{R}$ the set $\{x : x \in \mathbb{R}$ and x is "near" $p\}$ can be defined rigorously and has all desirable properties.

(ii) For any mathematical property of $\mathbb{R}$, ${}^*\mathbb{R}$ has the "same" property.

The reason for surrounding "topological space" and "same" by quotes is that ${}^*\mathbb{R}$ does not really have the same properties as $\mathbb{R}$, but only formally so. More precisely, given any mathematical property of $\mathbb{R}$, one writes down a sentence expressing the fact that this property holds for $\mathbb{R}$. Then one re-interprets this sentence (also in a way specified in advance) and under this new interpretation the sentence claims ${}^*\mathbb{R}$ has a certain property; moreover, ${}^*\mathbb{R}$ actually does have that property. Thus to every property of $\mathbb{R}$ there corresponds a property of ${}^*\mathbb{R}$ which is expressed by the same formula. It follows that formal reasoning and calculation can be performed for ${}^*\mathbb{R}$ in exactly the same way as for $\mathbb{R}$. It turns out that one can prove theorems about $\mathbb{R}$ by first "going cut" to ${}^*\mathbb{R}$ and later "coming back" to $\mathbb{R}$. This is the essence of nonstandard analysis.

In this paper, we will use a sequential approach presented in Tom Lindstrom [3] to construct ${}^*\mathbb{R}$. We will concentrate on the main definitions and applications of the topology on $\mathbb{R}$ using the nonstandard methods. Among other things, we shall concentrate on presenting nonstandard proofs of some well-known theorems and we shall compare these proofs with the standard ones. For more about nonstandard topology and its applications, we refer the reader to the references [3, 8–17].

This paper can be considered as a survey article on nonstandard topology on $\mathbb{R}$. It can be used as a teaching tool for beginners, and as a reference for a (graduate) student studying nonstandard analysis or nonstandard topology.

2. Ultrapower Construction of ${}^*\mathbb{R}$:

Depending on the sequential approach presented in Tom Lindstrom [3], we give here a construction of the nonstandard model ${}^*\mathbb{R}$ which is richer than the standard reals $\mathbb{R}$. This is an ordered field which extends the real numbers to include non-zero infinitesimals; that is, numbers the absolute value of which is smaller than any real number; and also positive and negative infinite numbers. There are several ways of constructing ${}^*\mathbb{R}$ [1–4, 6, 12, 13, 18, 19]. Here we use an ultrapower construction.

Definition 2.1. Let $\mathbb{R}^{\mathbb{N}}$ represents the set of all sequences with domain $\mathbb{N}$ and range values (images) in $\mathbb{R}$. Of course, sequences are functions, (maps, mappings, etc.). We define *binary operations* $+$ and $\cdot$, for sequences by simply taking any two A, $B \in \mathbb{R}^{\mathbb{N}}$ and defining $A + B = C$ to be the sequence C where the values of C are $C_n = A_n + B_n$ and $A \cdot B = D$ to be the sequence D where the values of D are $D_n = A_n \cdot B_n$ for each $n \in \mathbb{N}$. This forms what is called a ring with unity.

What we will do later is to show that there's a subset of $\mathbb{R}^{\mathbb{N}}$ that behaves like the real numbers, with respect to the defined operations, and we will use this subset as if it is the real numbers.

Definition 2.2 (Free Ultrafilter on $\mathbb{N}$)**.**
A *free ultrafilter* $\mathcal{U}$ on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ that is closed under finite intersections and supersets (i.e., $A \subseteq B$ and $A \in \mathcal{U}$ implies $B \in \mathcal{U}$), contains no finite sets and for every $A \subseteq \mathbb{N}$, either $A \in \mathcal{U}$ or $\mathbb{N} - A \in \mathcal{U}$.

In all that follows, $\mathcal{U}$ will always be a free ultrafilter on $\mathbb{N}$.

Definition 2.3 (Equality in $\mathcal{U}$)**.** Let $A, B \in \mathbb{R}^{\mathbb{N}}$. Define $A =_{\mathcal{U}} B$ iff $\{n : A_n = B_n\} = U \in \mathcal{U}$ (the set of all $n \in \mathbb{N}$ such that the values of the sequences A and B are equal.)

Theorem 2.4. [2] The relation $=_{\mathcal{U}}$ is an equivalence relation on $\mathbb{R}^{\mathbb{N}}$.

Proof. Of course, properties of the $=$ for members of $\mathbb{R}$ are used. First, notice that $\{n : A_n = A_n\} = \mathbb{N} \in \mathcal{U}$ for any $A \in \mathbb{R}^{\mathbb{N}}$. Thus, the relation is reflexive. Clearly, for any $A, B \in \mathbb{R}^{\mathbb{N}}$, if $\{n : A_n = B_n\} \in \mathcal{U}$, then $\{n : B_n = A_n\} \in \mathcal{U}$. Thus, the relation is symmetric. Finally, suppose that $A, B, C \in \mathbb{R}^{\mathbb{N}}$ and $A =_{\mathcal{U}} B$ and $B =_{\mathcal{U}} C$. Then, $\{n : A_n = B_n\} \in \mathcal{U}$ and $\{n : B_n = C_n\} \in \mathcal{U}$. Since $\mathcal{U}$ is a filter, the word "and" implies

$$\{n : A_n = B_n\} \cap \{n : B_n = C_n\} \in \mathcal{U}.$$

Of course, this "intersection" need not give all the values of $\mathbb{N}$ that these three sequences have in common, but that does not matter since the "superset" property for a filter implies from the result $\{n : A_n = B_n\} \cap \{n : B_n = C_n\} \subseteq \{n : A_n = C_n\}$, that $\{n : A_n = C_n\} \in \mathcal{U}$. Thus, the relation is transitive. Hence $=_{\mathcal{U}}$ is an equivalence relation. □

Definition 2.5 (Equivalence Classes)**.**
We now use the relation $=_{\mathcal{U}}$ to define subsets of $\mathbb{R}^{\mathbb{N}}$. For each $A \in \mathbb{R}^{\mathbb{N}}$, let the set

$$[A] = \{x \in \mathbb{R}^{\mathbb{N}} : x =_{\mathcal{U}} A\}.$$

Note that for each $A, B \in \mathbb{R}^{\mathbb{N}}$, either $[A] = [B]$ or $[A] \cap [B] = \emptyset$ (The $=$ here is the set-theoretic equality). Denote the set of all of these equivalence classes by ${}^*\mathbb{R}$; i.e., ${}^*\mathbb{R} := \mathbb{R}^{\mathbb{N}}/{=_{\mathcal{U}}}$ and call this set the set of all *hyperreal numbers*. (The * is often translated as "hyper"). Consequently,

$${}^*\mathbb{R} = \{[A] : A \in \mathbb{R}^{\mathbb{N}}\} = \mathbb{R}^{\mathbb{N}}/ =_{\mathcal{U}} .$$

After various relations are defined on ${}^*\mathbb{R}$, the resulting "structure" is generally called an *ultrapower*.
The reals $\mathbb{R}$ are identified with the equivalence classes of constant sequences, so that ${}^*\mathbb{R}$ is then an extension of $\mathbb{R}$.

Definition 2.6 (Addition and Multiplication in ${}^*\mathbb{R}$)**.**
Consider any $a = [A]$, $b = [B]$, $c = [C] \in {}^*\mathbb{R}$. Define $a \,{}^*\!+ b := c$ iff $\{n : A_n + B_n = C_n\} \in \mathcal{U}$. And define $a \,{}^*\!\cdot b := c$ iff $\{n : A_n \cdot B_n = C_n\} \in \mathcal{U}$.

Theorem 2.7. The operations ${}^*+$ and ${}^*\cdot$ are well-defined.

Proof. Let $a, b \in {}^*\mathbb{R}$, and let $[A]$, $[D] \in a$, $[B]$, $[F] \in b$. Now, since $\{n : A_n = D_n\} \in \mathcal{U}$ and $\{n : B_n = F_n\} \in \mathcal{U}$, we have $\{n : A_n = D_n\} \cap \{n : B_n = F_n\} \in \mathcal{U}$, so that $\{n : A_n = D_n\} \cap \{n : B_n = F_n\} \subseteq \{n : A_n + B_n = D_n + F_n\}$, and by the superset property, we have

$$\{n : A_n + B_n = D_n + F_n\} \in \mathcal{U}.$$

Thus the ${}^*+$ is well-defined. In like manner for the ${}^*\cdot$. □

Theorem 2.8. [19] For the structure $\langle\ {}^*\mathbb{R}, {}^*+, {}^*\cdot\ \rangle$, the following holds:

(i) $[0]$ is the *additive identity*.

(ii) For each $a = [A] \in {}^*\mathbb{R}$, $-a = [-A]$ is the *additive inverse*.

(iii) $[1]$ is the *multiplicative identity*.

(iv) If $a \neq [0]$, then there exists $b = [B] \in {}^*\mathbb{R}$ such that $a\ {}^*\cdot\ b = [1]$.

(v) For each $n \in \mathbb{N}$, if $D_n = A_n + B_n$ and $E_n = A_n \cdot B_n$, then $[A]\ {}^*+\ [B] = [D]$ and $[A]\ {}^*\cdot\ [B] = [E]$. That is, our definitions for addition and multiplication of sequences and the hyper operations ${}^*+$, ${}^*\cdot$ are compatible.

Proof. (i) Let $[A]\ {}^*+\ [0] = [C]$. Considering that $\{n : A_n + 0_n = C_n\} \in \mathcal{U}$ and $\{n : A_n + 0_n = C_n\} \subseteq \{n : A_n = C_n\} \in \mathcal{U}$, then $[A] = [C]$.
(ii) Since $\{n : A_n + (-A_n) = 0 = 0_n\} = \mathbb{N} \in \mathcal{U}$, we have $[A]\ {}^*+\ [-A] = [0]$.
(iii) Let $[A]\ {}^*\cdot\ [1] = [C]$. Considering that $\{n : A_n \cdot 1_n = C_n\} \in \mathcal{U}$ and $\{n : A_n \cdot 1_n = C_n\} \subseteq \{n : A_n = C_n\} \in \mathcal{U}$, we have $[A] = [C]$.
(iv) Let $[A] \neq [0]$. Then $\{n : A_n = 0 = 0_n\} = U \notin \mathcal{U}$. Hence, $\mathbb{N} - U = \{n : A_n \neq 0\} \in \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter. Define

$$B_n = \begin{cases} A_n^{-1} & \text{if } A_n \neq 0 \\ 0 & \text{if } A_n = 0. \end{cases}$$

Notice that $\{n : A_n \cdot B_n = 1 = 1_n\} = \{n : A_n \neq 0\} \in \mathcal{U}$. Hence $[A]\ {}^*\cdot\ [B] = [1]$.
(v) By definition, $[A]\ {}^*+\ [B] = [C]$ iff $\{n : A_n + B_n = C_n\} \in \mathcal{U}$. However, $\{n : A_n + B_n = D_n\} = \mathbb{N} \in \mathcal{U}$. Hence,
$\{n : A_n + B_n = C_n\} \cap \{n : A_n + B_n = D_n\} = \{n : C_n = D_n\} \in \mathcal{U}$. Thus $[C] = [D]$. In like manner, the result holds for multiplication. □

Definition 2.9 (Order)**.**
For each $a = [A]$, $b = [B] \in {}^*\mathbb{R}$ define $a\ {}^*\leq\ b$ iff $\{n : A_n \leq B_n\} \in \mathcal{U}$.

Theorem 2.10. [19] The structure $\langle\ {}^*\mathbb{R}, {}^*+, {}^*\cdot, {}^*\leq\ \rangle$ is a totally ordered field.

Proof. First, notice that $\{n : A_n \leq A_n\} = \mathbb{N} \in \mathcal{U}$. Thus, ${}^*\leq$ is reflexive. Next, this relation needs to be anti-symmetric. So, assume that $[A]\ {}^*\leq\ [B]$ and $[B]\ {}^*\leq\ [A]$. Then
$\{n : A_n \leq B_n\} \cap \{n : B_n \leq A_n\} \subseteq \{n : A_n = B_n\} \in \mathcal{U}$. Hence, $[A] = [B]$. For transitivity, consider $[A]\ {}^*\leq\ [B]$ and $[B]\ {}^*\leq\ [C]$. Then
$\{n : A_n \leq B_n\} \cap \{n : B_n \leq C_n\} \subseteq \{n : A_n \leq C_n\} \in \mathcal{U}$. Thus, $[A]\ {}^*\leq\ [C]$. It follows that $\langle\ {}^*\mathbb{R}, {}^*\leq\ \rangle$ is a partially ordered set. (Notice that the same processes seem to be used each time. That is because $\mathcal{U}$ is closed under finite intersections and supersets.)

Next to show that $\langle\ {}^*\mathbb{R}, {}^*\leq\ \rangle$ is totally ordered, let $[A]$, $[B] \in {}^*\mathbb{R}$. Then by trichotomy law for $\mathbb{R}$, we have $\{n : A_n < B_n\} \in \mathcal{U}$ or $\{n : A_n > B_n\} \in \mathcal{U}$ or $\{n : A_n = B_n\} \in \mathcal{U}$. Hence $[A]\ {}^*<\ [B]$ or $[A]\ {}^*>\ [B]$ or $[A]\ {}^*=\ [B]$. To show that $\langle\ {}^*\mathbb{R}, {}^*+, {}^*\cdot, {}^*\leq\ \rangle$ is a totally ordered field, all that's really needed is to show that it satisfies two properties related to this order and the ${}^*+$, ${}^*\cdot$ operations [23]. So, let $[A]$, $[B]$, $[C] \in {}^*\mathbb{R}$, and let $[A]\ {}^*\leq\ [B]$. Then

$$\{n : A_n \leq B_n\} \subseteq \{n : A_n + C_n \leq B_n + C_n\} \in \mathcal{U}.$$

Thus $[A]\ {}^*+\ [C]\ {}^*\leq\ [B]\ {}^*+\ [C]$. Now suppose that $[0]\ {}^*\leq\ [A], [B]$. Then
$\{n : 0 \leq A_n\} \cap \{n : 0 \leq B_n\} \subseteq \{n : 0 \leq A_n \cdot B_n\} \in \mathcal{U}$. Hence $[0]\ {}^*\leq\ [A]\ {}^*\cdot\ [B]\ {}^*=\ [AB]$. □

Definition 2.11 (Hyper * Extensions of Standard Objects [19])**.**
For any $C \subseteq \mathbb{R}$ (a $1-$ary relation), let $b = [B] \in {}^*C$, iff $\{n : B_n \in C\} \in \mathcal{U}$. Let Φ be any $k-$ary $(k > 1)$ relation. Then
$(a_1, \ldots, a_k) = ([A_1], \ldots, [A_k]) \in {}^*\Phi \Leftrightarrow \{n : (A_1(n), \ldots, A_k(n)) \in \Phi\} \in \mathcal{U}$. This extension process can be continued for other mathematical entities.

Theorem 2.12. [19] The hyper-extensions of standard objects are well-defined.

Proof. In general, for any $[B] \in {}^*\mathbb{R}$, let $[B] = [B']$. Then $\{n : B_n = B'_n\} \in \mathcal{U}$. That is, let $B' \in \mathbb{R}^{\mathbb{N}}$ be any other member of the equivalence class $[B]$. Let $C \subseteq \mathbb{R}$ be a 1-ary relation. Then
$\{n : B_n = B'_n\} \subseteq \{n : (B_n \in C) \Leftrightarrow (B'_n \in C)\} \in \mathcal{U}$,
$\{n : B_n \in C\} \cap \{n : (B_n \in C) \Leftrightarrow (B'_n \in C)\} \subseteq \{n : B'_n \in C\} \in \mathcal{U} \Rightarrow [B'] \in {}^*C$,
$\{n : B'_n \in C\} \cap \{n : (B_n \in C) \Leftrightarrow (B'_n \in C)\} \subseteq \{n : B_n \in C\} \in \mathcal{U} \Rightarrow [B] \in {}^*C$. Thus the 1-ary relation C is well defined. For the other $k-$ary relations, proceed as just done but alter the proof by starting with
$\{n : B_1(n) = B'_1(n)\} \cap \cdots \cap \{n : B_k(n) = B'_k(n)\} \subseteq \{n : (B_1(n), \ldots, B_k(n)) \in \Phi \Leftrightarrow (B'_1(n), \ldots B'_k(n)) \in \Phi\}$. Thus the $k-$ary relation Φ is well defined. □

Definition 2.13 (Standard Objects Operator [19]).
For each $x \in \mathbb{R}$, let ${}^*x := [X] \in {}^*\mathbb{R}$, where $\{n : X_n = x\} = \mathbb{N}$ (the constant sequence). Then for $X \subseteq \mathbb{R}$, let ${}^\sigma X := \{{}^*x : x \in X\} \subseteq {}^*\mathbb{R}$. For $n > 1$ and each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let ${}^*x = ({}^*x_1, \ldots, {}^*x_n) \in {}^*(\mathbb{R}^n)$. For $X \subseteq \mathbb{R}^n$ let ${}^\sigma X := \{{}^*x : x \in X\} \subseteq {}^*(\mathbb{R}^n)$. Each such ${}^*x$ and ${}^\sigma X$ is called a *standard object*. Thus, ${}^\sigma\mathbb{R}$ is the set of *embedded real numbers*.

Theorem 2.14. [19] Let infinite $X \subseteq \mathbb{N}$. Then there exists a free ultrafilter $\mathcal{U}$ such that $X \in \mathcal{U}$. Moreover, if $[A], [B] \in {}^*\mathbb{R}$, then $[A] = [B]$ for all free ultrafilters iff $\{n : A_n = B_n\} \in \mathcal{C}$, where $\mathcal{C} = \{x : (x \subseteq X) \wedge (X - x) \text{ is finite}\}$.

Theorem 2.15 (${}^*-$Algebra [19]).

(i) ${}^*\emptyset = \emptyset$.

(ii) If $X \subseteq \mathbb{R}$, then ${}^\sigma X \subseteq {}^*\mathbb{R}$

(iii) If $X \subseteq \mathbb{R}$, then ${}^*x \in {}^\sigma X$ iff $x \in X$ iff ${}^*x \in {}^*X$.

(iv) Let $X, Y \subseteq \mathbb{R}$. Then $X \subseteq Y$ iff ${}^*X \subseteq {}^*Y$.

(v) Let $X, Y \subseteq \mathbb{R}$. Then ${}^*(X - Y) = {}^*X - {}^*Y$.

(vi) Let $X, Y \subseteq \mathbb{R}$. Then ${}^*(X \cup Y) = {}^*X \cup {}^*Y$. Also ${}^*(X \cap Y) = {}^*X \cap {}^*Y$.

(vii) Let $\emptyset \neq X \subseteq \mathbb{R}$. Then X is finite iff ${}^*X = {}^\sigma X$.

Proof. (i) If $S = \emptyset$, then for any $a \in \mathbb{R}^{\mathbb{N}}$, $\{n : A_n \in S\} = \emptyset \notin \mathcal{U}$. Thus ${}^*\emptyset = \emptyset$.
(ii) This is simply a repeat of Definition (2.13).
(iii) By definition, ${}^*x \in {}^\sigma X$ iff $x \in X$. Now assume that $x \in X$. Then, by definition, ${}^*x = [A]$, $A_n = x$ for each $n \in N$. Hence, $\{n : A_n \in X\} = \mathbb{N} \in \mathcal{U}$. Thus, ${}^*x = [A] \in {}^*X$. Conversely, assume that ${}^*x = [A] \in {}^*X$. By definition, $A_n = x$ for all $n \in \mathbb{N}$. Thus $\{n : A_n = x\} = \mathbb{N} \in \mathcal{U}$. Hence $x \in X$.
(iv) Let $X \subseteq Y \subseteq \mathbb{R}$, and let $a = [A] \in {}^*X$. Then $\{n : A_n \in X\} \in \mathcal{U}$. But, $\{n : A_n \in X\} \subseteq \{n : A_n \in Y\}$. Thus, $\{n : A_n \in Y\} \in \mathcal{U}$. Hence $a \in {}^*Y$ and therefore ${}^*X \subseteq {}^*Y$. Conversely, assume that ${}^*X \subseteq {}^*Y$. Then for each $x \in X$, ${}^*x \in {}^*X$ by (iii). Thus ${}^*x \in {}^*Y$. Again by (iii) $x \in Y$. Thus $X \subseteq Y$.
(v) Let $X, Y \subseteq \mathbb{R}$, and let $a = [A] \in {}^*(X - Y)$. Then $\{n : A_n \in X - Y\} = U \in \mathcal{U}$. But this implies that $U \subseteq \{n : A_n \in X\} \in \mathcal{U}$ and $U \subseteq \{n : A_n \notin Y\} \in \mathcal{U}$. Hence $a \in {}^*X - {}^*Y$. Conversely, let $a = [A] \in {}^*X - {}^*Y$. Then $\{n : A_n \in X\} \in \mathcal{U}$ and $\{n : A_n \notin Y\} \in \mathcal{U}$. Thus $\{n : A_n \in X\} \cap \{n : A_n \notin Y\} \in \mathcal{U}$. And so $\{n : A_n \in (X - Y)\} \in \mathcal{U}$. Hence $a \in {}^*(X - Y)$.
(vi) Let $C = X \cap Y$. Then $C = X - (X - Y)$. Thus, by (v), ${}^*C = {}^*X - ({}^*X - {}^*Y) = {}^*X \cap {}^*Y$. Hence ${}^*(X \cap Y) = {}^*X \cap {}^*Y$. Now let $a = [A] \in {}^*(X \cup Y)$. Then $\{n : A_n \in (X \cup Y)\} \in \mathcal{U}$ and so $\{n : A_n \in X\} \cup \{n : A_n \in Y\} \in \mathcal{U}$. Hence either $\{n : A_n \in X\} \in \mathcal{U}$ or $\{n : A_n \in Y\} \in \mathcal{U}$, and so $a \in {}^*X$ or $a \in {}^*Y$. Thus ${}^*(X \cup Y) \subseteq {}^*X \cup {}^*Y$. Conversely, since $X \subseteq (X \cup Y)$ and $Y \subseteq (X \cup Y)$, it follows from (iv) that ${}^*X \cup {}^*Y \subseteq {}^*(X \cup Y)$ and the result follows.
(vii) The first part is established by induction.

Let $\{x\} \subseteq \mathbb{R}$. By definition $a = [A] \in {}^*\{x\}$ iff $\{n : A_n \in \{x\}\} = \{n : A_n = x\} \in \mathcal{U}$ iff $a \in \{{}^*x\}$. Assume the result holds for any set of k numbers. Then ${}^*\{x_i, \ldots, x_{k+1}\} = {}^*(\{x_1, \ldots, x_k\} \cup \{x_{k+1}\}) = {}^*\{x_1, \ldots, x_k\} \cup {}^*\{x_{k+1}\} = \{{}^*x_1, \ldots, {}^*x_{k+1}\}$ by the induction hypothesis and (v), and so the result holds for any $k \geq 1$.

Conversely, let $X \subseteq \mathbb{R}$ be infinite and assume that ${}^\sigma X = {}^*X$. Then there exists an injection $B : \mathbb{N} \to X$. So that $\{B_n : n \in \mathbb{N}\}$ is an infinite subset of X. Let ${}^*x = [A] \in {}^\sigma X$. Then $A_n = x \in X$ for each $n \in \mathbb{N}$. But $\{n : A_n = B_n\}$ is finite. Using Theorem (2.14), we have $[A] \neq [B]$ since $\{n : A_n \neq B_n\} \in \mathcal{C}$. Also $\{n : B_n \in X\} = \mathbb{N} \in \mathcal{U}$ implies that $b = [B] \in {}^*X$. Thus there is no $x \in X$ such that ${}^*x = b \in {}^*X$, which implies ${}^\sigma X \neq {}^*X$. □

3. The Hyperreals:

Definition 3.1. [7] Let $x, y \in {}^*\mathbb{R}$. We say that:

(1) x is *infinitesimal* if $|x| < \epsilon$, for any positive real number ϵ; we write $x \approx 0$, where $|.|$ is the extension of the modolus function to ${}^*\mathbb{R}$. This takes its values in ${}^*\mathbb{R}$, and is defined just as in $\mathbb{R}$, so that $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

(2) x is *finite* if, for some positive real number ϵ, $|x| < \epsilon$.

(3) x is *infinite* (or *infinitely large*) if it is not finite; i.e., $|x| > \epsilon$ for any positive real number ϵ; we write $x \approx \infty$.

(4) x, y are *infinitely close* if $x - y$ is infinitesimal; we write $x \approx y$.

Let $\mathcal{I}({}^*\mathbb{R})$, $\mathcal{F}({}^*\mathbb{R})$ and $\mathcal{L}({}^*\mathbb{R})$ denote the sets of the infinitesimals, finite and infinitely large numbers in ${}^*\mathbb{R}$, respectively. It can be easily shown (as in any totally ordered field – see [19]) that

$$\begin{aligned}
&{}^*\mathbb{R} = \mathcal{F}({}^*\mathbb{R}) \cup \mathcal{L}({}^*\mathbb{R}),\ \mathcal{F}({}^*\mathbb{R}) \cap \mathcal{L}({}^*\mathbb{R}) = \emptyset,\\
&\mathcal{I}({}^*\mathbb{R}) \subseteq \mathcal{F}({}^*\mathbb{R}),\ \mathbb{R} \subseteq \mathcal{F}({}^*\mathbb{R}),\\
&\mathbb{R} \cap \mathcal{I}({}^*\mathbb{R}) = \{0\},\\
&\mathcal{L}({}^*\mathbb{R}) = \{1/x : x \in \mathcal{I}({}^*\mathbb{R}),\ x \neq 0\}.
\end{aligned}$$

Figure 1 below illustrates the relationship between these sets.

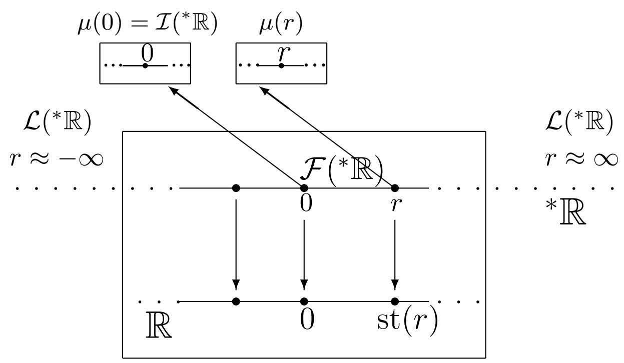

Figure 1 *The reals and the hyperreals*

Definition 3.2. For $x \in {}^*\mathbb{R}$, the *monad* of x is the subset of ${}^*\mathbb{R}$ given by:

$$\mu(x) := \{y \in {}^*\mathbb{R} : x \approx y\}.$$

Theorem 3.3 (Standard Part Theorem [2])**.** If $x \in {}^*\mathbb{R}$ is finite, then there is a unique $r \in \mathbb{R}$ such that $x \approx r$; that is, any finite hyperreal x is uniquely expressible as $x = r + \delta$ with r a standard real and δ an infinitesimal.

Proof. For the existence, let $r = \sup\{b \in \mathbb{R} : b < x\}$. Since x is finite, r exists. We must show that $x - r$ is infinitesimal. Assume not, then there is a real number k such that $0 < k < |x - r|$. If $x - r > 0$, this implies that $r + k < x$, contradicting the choice of r. If $x - r < 0$, we get $x < r - k$, also contradicting the choice of r. The uniqueness is obvious since if $x = r_1 + \delta_1 = r_2 + \delta_2$, then $r_1 - r_2 = \delta_2 - \delta_1$ is both real and infinitesimal, so it must be zero. □

Definition 3.4 (Standard Part)**.**
If x is a finite hyperreal, then the unique real

$r \approx x$ is called the *standard part* of x, and it is denoted by $\mathrm{st}(x)$ (see Figure 1).

Theorem 3.5. [19] The collection $\{\mu(x) : x \in {}^{\sigma}\mathbb{R}\}$ is a partition of $\mathcal{F}({}^*\mathbb{R})$.

Proof. Technically, to be a partition of $\mathcal{F}({}^*\mathbb{R})$, we have to show that $\mu(x) \cap \mu(y) \neq \emptyset$ implies $\mu(x) = \mu(y)$ and that $\bigcup\{\mu(x) : x \in {}^{\sigma}\mathbb{R}\} = \mathcal{F}({}^*\mathbb{R})$. For the first part, assume that there exists some $a \in \mu(x) \cap \mu(y)$. Then $a = \epsilon + x$, $a = \lambda + y$ where $\epsilon, \lambda \in \mathcal{I}({}^*\mathbb{R})$. But $\epsilon + x = \lambda + y$ implies that $\epsilon - \lambda = y - x$. This is only possible if $\epsilon - \lambda = 0$ since $y - x \in {}^{\sigma}\mathbb{R}$. Thus $x = y$ and so $\mu(x) = \mu(y)$.

For the second part, let $a \in \bigcup\{\mu(x) : x \in {}^{\sigma}\mathbb{R}\}$. Then $a = \epsilon + x$ for some $x \in {}^{\sigma}\mathbb{R}$. Then $|a| = |\epsilon + x| \leq |\epsilon| + |x| < |x| + 1$. Hence $a \in \mathcal{F}({}^*\mathbb{R})$. Consequently, $\bigcup\{\mu(x) : x \in {}^{\sigma}\mathbb{R}\} \subseteq \mathcal{F}({}^*\mathbb{R})$.

Now assume that $a \in \mathcal{F}({}^*\mathbb{R})$. Then there is some ${}^*x \in {}^{\sigma}\mathbb{R}^+$ such that $a < {}^*x$. So, consider the set $S := \{y : {}^*y < a\}$. This set is nonempty since $-x \in S$. Also since $a < {}^*x$, S is a set of real numbers that is bounded above, so it has a least upper bound z. Assume that $|z - a| \notin \mathcal{I}({}^*\mathbb{R})$. Then there is some $w \in \mathbb{R}$ such that $|{}^*z - a| > {}^*w$. Suppose that ${}^*z < a$, then $a - {}^*z > {}^*w$ implies that ${}^*z + {}^*w = {}^*(z + w) < a$, which in turn implies $z + w \in S$ and z is not a least upper bound of S. So, we must have $a < {}^*z$. This implies that ${}^*z - a > {}^*w$, and therefore $a < {}^*(z - w) < {}^*z$. But, $z - w$ is an upper bound for the set S. This contradicts the fact that z is the least upper bound of S. Hence, ${}^*z - a = \epsilon$ for some $\epsilon \in \mathcal{I}({}^*\mathbb{R})$, which implies that $a \in \mu(z)$. Therefore $\mathcal{F}({}^*\mathbb{R}) \subseteq \bigcup\{\mu(x) : x \in {}^{\sigma}\mathbb{R}\}$. □

Theorem 3.6 (Transfer Principle [7])**.** Let ϕ be any first order statement. Then ϕ holds in $\mathbb{R}$ if and only if ${}^*\phi$ holds in ${}^*\mathbb{R}$.

A first order statement ϕ in $\mathbb{R}$ (or ${}^*\phi$ in ${}^*\mathbb{R}$) is one referring to elements (fixed or variables) of $\mathbb{R}$ (respectively, ${}^*\mathbb{R}$), that uses the usual logical connectives and ($\wedge$), or ($\vee$), implies ($\Rightarrow$) and not ($\neg$). Quantification may be done over elements but not over relations or functions; i.e., $\forall x$, $\exists y$ are allowed, but $\forall f$, $\exists \mathcal{R}$ are not.

Example 3.7. The density of the rationals in the reals can be written as

$$\forall x \forall y (x < y \to \exists z (z \in \mathbb{Q} \wedge (x < z < y))),$$

an expression meaning, "between every two reals there is a rational". From the transfer principle we can therefore immediately conclude that the statement is true in ${}^*\mathbb{R}$; i.e., that the hyperrationals are dense in the hyperreals.

4. Topology on $\mathbb{R}$:

When we wish to examine the continuity, or otherwise, of a function f at a point a we find it necessary to consider that function's behavior at all points sufficiently near to a, if we are applying the standard criterion. In the case of the nonstandard criterion we would be concerned with all points infinitely close to a. What connects the two approaches is the fundamental idea of a neighborhood of a point.

Definition 4.1 (Standard Neighborhood [20, 21])**.**
If a is any point in $\mathbb{R}$ and if $r \in \mathbb{R}^+$, then we denote by $B(a, r)$ the set of all real points x whose distance from a is less than r:

$$B(a, r) = \{x \in \mathbb{R} : |a - x| < r\}.$$

Any set $M \subseteq \mathbb{R}$ will be called a *neighborhood* of a point $a \in \mathbb{R}$ iff there exists some $r > 0$ such that

$$a \in B(a, r) \subseteq M.$$

Theorem 4.2 (Nonstandard Neighborhood [12])**.**
A set $M \subseteq \mathbb{R}$ is a neighborhood of a point $a \in \mathbb{R}$ iff every hyperreal x which is infinitely close to a necessarily belongs to the nonstandard extension ${}^*M$ of M; i.e., $M \subseteq \mathbb{R}$ is a neighborhood of $a \in \mathbb{R}$ iff:

$$\mu(a) \subseteq {}^*M.$$

Proof. First, let M be a neighborhood of a. Then there exists $r_0 \in \mathbb{R}^+$ such that $B(a, r_0) \subseteq M$. Let $x = [A] \in {}^*\mathbb{R}$ be such that $x \in \mu(a)$. Then for any real $r \in \mathbb{R}^+$ we have $|a - A_n| < r$ for almost all values of n; in particular this is true for r_0. It follows that $A_n \in B(a, r_0) \subseteq M$ for almost all values of n. So that $x \in {}^*M$.
On the other hand, let $\mu(a) \subseteq {}^*M$ and suppose, on the contrary, that M is not a neighborhood of a. Then for each $n \in \mathbb{N}$ we can find a point $A_n \in \mathbb{R}$ such that:

$$|a - A_n| < \frac{1}{n} \text{ and } A_n \in \mathbb{R} - M.$$

But this means that $x = [A]$ is a hyperreal which belongs to $\mu(a)$ but not to ${}^*M$, a contradiction. Hence M is a neighborhood of a. □

Using Theorem (4.2), we can define the open sets in $\mathbb{R}$ as follows.

Definition 4.3 (Nonstandard Open Set [1]).
A set $A \subseteq \mathbb{R}$ is *open* iff for every $x \in A$ we have $\mu(x) \subseteq {}^*A$. (In other words: for any $x \in A$ and $y \approx x$, we have $y \in {}^*A$).

Theorem 4.4. [19, 22] Let $A \subseteq \mathbb{R}$, $p \in \mathbb{R}$. Then:

(i) p is an *accumulation point* of A iff $\mu(p) \cap {}^*A \neq \emptyset$. (In other words: $\exists\, y \in {}^*A$ such that $p \approx y$ but $p \neq y$).

(ii) p is an *isolated point* of A iff $\mu(p) \cap {}^*A = \{p\}$.

Proof. (i) Let $p \in \mathbb{R}$ be an accumulation point for $A \subseteq \mathbb{R}$. Then,

$$\forall\, x((x \in \mathbb{R}^+) \Rightarrow \exists\, y((y \in A) \wedge |y - p| < x)).$$

Then, by transfer principle, we have

$$\forall\, x((x \in {}^*\mathbb{R}^+) \Rightarrow \exists\, y((y \in {}^*A) \wedge |y - p| < x)),$$

so that $y \in \mu(p) \cap {}^*A$.
Conversely, assume that $\mu(p) \cap {}^*A \neq \emptyset$. By Theorem (4.2), $\mu(p) \subseteq {}^*(-w + p, p + w)$ $\forall\, w \in \mathbb{R}^+$. Hence, letting $y \in \mu(p) \cap {}^*A$ and $w \in \mathbb{R}^+$, we have

$$\exists\, y((y \in {}^*A) \wedge |y - p| < w),$$

so that, by transfer principle, we have

$$\forall\, x((x \in \mathbb{R}^+) \Rightarrow \exists\, y((y \in A) \wedge |y - p| < x)).$$

Hence p is an accumulation point for A.
(ii) Suppose that p is an isolated point for A. Then there exists some $w \in \mathbb{R}^+$ such that

$$(-w + p, p + w) \cap A = \{p\}.$$

Hence, by transfer principle, we have ${}^*(-w + p, p + w) \cap {}^*A = {}^*\{p\} = \{p\}$, and since $p \in \mu(p) \subseteq {}^*(-w + p, p + w)$, we have $\mu(p) \cap {}^*A = \{p\}$.
Conversely, suppose that p is not an isolated point of A, there is a sequence $x_n \in A$, $x_n \neq p$ with $x_n \rightarrow p$. Hence, by transfer principle, for any $\Lambda \in {}^*\mathbb{N}$, there is some $p_\Lambda \in {}^*A$ such that $p_\Lambda \approx p$; i.e., $p_\Lambda \in \mu(p)$ and so $p_\Lambda \in \mu(p) \cap {}^*A$. Therefore $\mu(p) \cap {}^*A \neq \{p\}$. □

Definition 4.5 (Point of Closure [1]).
A point $x \in \mathbb{R}$ is said to be a *point of closure* of a set $F \subseteq \mathbb{R}$ iff there exists $y \in {}^*F$ such that $x \approx y$.

Definition 4.6 (Nonstandard Closed Set [1]).
A set $F \subseteq \mathbb{R}$ is *closed* iff $y \in {}^*F$ and $y \approx x \in \mathbb{R}$ always implies that $x \in F$. (In other words: $\mu(x) \cap {}^*F \neq \emptyset$ implies $x \in F$ for each $x \in \mathbb{R}$).

Theorem 4.7. $F \subseteq \mathbb{R}$ is closed iff its complement $G = \mathbb{R} - F$ is open.

Proof. Suppose that G is open and let y be any hyperreal in ${}^*F$. If $x := \text{st}(y) \in G$, then, since G is open, we must have

$$\mu(x) \subseteq {}^*G = {}^*(\mathbb{R} - F) = {}^*\mathbb{R} - {}^*F.$$

In particular $y \in {}^*\mathbb{R} - {}^*F$, which is a contradiction. Hence x must belong to $\mathbb{R} - G = F$ and so, by Definition (4.6), F is closed.

Conversely, suppose that F is closed and let x be any point of G. If there exists $y \approx x$ such that $y \in {}^*F = {}^*(\mathbb{R} - G)$, then, since F is closed, we must have $x \in F = \mathbb{R} - G$, which is a contradiction. Hence,

$$\mu(x) \subseteq {}^*\mathbb{R} - {}^*F = {}^*(\mathbb{R} - F) = {}^*G,$$

and so G is open. □

Theorem 4.8 (Boundedness [19])**.**
A nonempty set $A \subseteq \mathbb{R}$ is bounded iff ${}^*A \subseteq \mathcal{F}({}^*\mathbb{R})$.

Proof. Suppose that A is bounded. Then there is some positive real number x such that, for each $y \in A$, $|y| \leq x$. By transfer principle, for any $a \in {}^*A$ we have $|a| \leq {}^*x$. Consequently, ${}^*A \subseteq \mathcal{F}({}^*\mathbb{R})$.
Conversely, if A is not bounded, then for any $n \in \mathbb{N}$ there is some $x_n \in A$ such that $|x_n| > n$. Hence, by transfer principle, for any $\Lambda \in {}^*\mathbb{N}$ there is some $p_\Lambda \in {}^*A$ such that $|p_\Lambda| > \Lambda$. Choose $\Lambda \in \mathbb{N}_\infty = {}^*\mathbb{N} - \mathbb{N}$. Then $p_\Lambda \notin \mathcal{F}({}^*\mathbb{R})$. Hence ${}^*A \nsubseteq \mathcal{F}({}^*\mathbb{R})$. □

Theorem 4.9 (Continuity [1])**.**
A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$

iff ${}^*f(x) \approx {}^*f(a)$ whenever $x \in {}^*\mathbb{R}$ and $x \approx a$.

Proof. Suppose that f is continuous at $a \in \mathbb{R}$ and let x be a hyperreal such that $x \approx a$. We have to prove that $|{}^*f(x) - {}^*f(a)| < \epsilon$ for each $\epsilon \in \mathbb{R}^+$. For any such ϵ choose $\delta \in \mathbb{R}^+$ such that

$$|y - a| < \delta \Rightarrow |f(y) - f(a)| < \epsilon \,\forall\, y \in$$

$\mathbb{R}$. Then, by transfer principle, we have

$$|y - a| < \delta \Rightarrow |{}^*f(y) - {}^*f(a)| < \epsilon \,\forall\, y \in {}^*\mathbb{R}.$$

Taking $y = x$, since $x \approx a$, we have $|x - a| < \delta$ and so $|{}^*f(x) - {}^*f(a)| < \epsilon$.
Conversely, assume that ${}^*f(x) \approx {}^*f(a)$ whenever $x \approx a$, and let $\epsilon \in \mathbb{R}^+$ be given. Pick any positive infinitesimal $\delta \in \mathbb{R}^+$. Then $x \in {}^*\mathbb{R}$ and $|x - a| < \delta$ implies $x \approx a$; so, by assumption,
$\exists\, \delta \in {}^*\mathbb{R}$, $\delta \in \mathbb{R}^+$, $(|x - a| < \delta \Rightarrow |{}^*f(x) - {}^*f(a)| < \epsilon)$.
Then, by transfer principle, we have

$$|x - a| < \delta \Rightarrow (|f(x) - f(a)| < \epsilon \,\forall\, x \in \mathbb{R}).$$

Hence f is continuous at a. □

Theorem 4.10. [12] A function $f : \mathbb{R} \to \mathbb{R}$ is continuous (on $\mathbb{R}$) iff the inverse image $f^{-1}(A) = \{x \in \mathbb{R} : f(x) \in A\}$ of any open set A is itself always an open set.

Proof. Suppose that f is continuous and let A be an open set in $\mathbb{R}$ and $x \in f^{-1}(A)$. Then $f(x) \in A$. If $y = [B] \in {}^*\mathbb{R}$ is such that $y \approx x$, then, by the continuity of f,

$${}^*f(x) \approx {}^*f(y).$$

But, since A is open, this implies that ${}^*f(y) \in {}^*A$. This means that $f(B_n) \in A$ for almost all n and therefore that $B_n \in f^{-1}(A)$ for almost all n. Thus $y \approx x$ always implies that $y \in {}^*(f^{-1}(A))$, and so $f^{-1}(A)$ is an open set.
Conversely, suppose that the inverse image under f of every open set A is always itself an open set. Let $x \in \mathbb{R}$ and $y \in {}^*\mathbb{R}$ be such that $y \approx x$. If it is false that ${}^*f(x) \approx {}^*f(y)$, then for some $r \in \mathbb{R}^+$ we must have $|{}^*f(x) - {}^*f(y)| > r$. Thus, ${}^*f(y)$ /

$\in {}^*A$ where

$$A = (f(x) - r,\, f(x) + r).$$

It follows that $y \notin {}^*(f^{-1}(A))$. This contradicts the hypothesis that $f^{-1}(A)$ is open and yet we have

$$y \approx x \in f^{-1}(A).$$ □

Definition 4.11. [1] A set $A \subseteq \mathbb{R}$ is *compact* iff for each $b \in {}^*A$ there is some $p \in A$ such that $b \in \mu(p)$ (i.e., $b \approx p$) iff ${}^*A \subseteq \bigcup\{\mu(p) : p \in A\}$.
Theorem 4.12 (Heine−Borel [12])**.**
A nonempty $A \subseteq \mathbb{R}$ is compact iff it is closed and bounded.

Proof. Assume that A is compact. Using Theorem (3.5), we have $^*A \subseteq \bigcup\{\mu(p) : p \in A\} \subseteq \mathcal{F}(^*\mathbb{R})$. Then, by Theorem (4.8), A is bounded. Now let $\mu(q) \cap {}^*A \neq \emptyset$ for some $q \in \mathbb{R}$. Since $^*A \subseteq \bigcup\{\mu(p) : p \in A\}$, $\mu(q) \cap \mu(p) \neq \emptyset$ for some $p \in A$. Hence Theorem (3.5) implies that $q = p$. Thus $q \in A$. Hence, by Definition (4.6), A is closed.

Conversely, assume that A is closed and bounded. Since A is bounded, by Theorem (4.8), we have $^*A \subseteq \bigcup\{\mu(p) : p \in \mathbb{R}\}$ $\mathcal{F}(^*\mathbb{R})$. Also, $A \neq \mathbb{R}$. Since A is closed, Definition (4.6) implies that $\mu(q) \cap {}^*A = \emptyset$ for any $q \in \mathbb{R} - A$. Thus $^*A \subseteq \bigcup\{\mu(p) : p \in A\}$. Hence, by Definition (4.11), A is compact. □

Theorem 4.13 (Bolzano−Weierstrass [12]). If $A \subseteq \mathbb{R}$ is an infinite, compact subset of $\mathbb{R}$, then every infinite subset of A has an accumulation point in A.

Proof. If A has an infinite subset B, then we can choose a sequence C_n of distinct points of B which defines a hyperreal $y = [C]$. Then $y \in {}^*A$ (since $C_n \in B \subseteq A$ for all n), y is finite (since A is bounded) and $x = \mathrm{st}(y)$ exists and belongs to A (by compactness of A and Definition (4.11)). Finally, $x \approx y$ but $x \neq y$ (since the C_n are all distinct). It follows from Theorem (4.4) that $x \in A$ is an accumulation point of B. □

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