The Islamic University Journal (Series of Natural Studies and Engineering)
Vol.18, No.1, pp 17 - 23, **2010**, ISSN 1726-6807, http://www.iugaza.edu.ps/ara/research/

Primary Ideals and Primary Modules over Noncommutative Rings

Arwa Eid Ashour

Assistant Professor, Department of Mathematics,
Islamic University of Gaza,
P.O. Box 108, Gaza Strip, Palestine,
Tel: +970-8-2060420,
E-mail: arashour@iugaza.edu.ps

Abstract: The concepts of prime ideals and prime modules were introduced over noncommutative rings. In this article we generalize these concepts and introduce the concepts of primary ideals and primary modules over noncommutative rings. Also we give equivalent definitions of primary ideals over noncommutative rings. Finally, we study some properties of primary submodules over noncommutative rings.

Key Words: Primary ideals over noncommutative rings, radical annihilator of an R-module, prime modules over noncommutative rings, primary modules over noncommutative rings, primary submodules over noncommutative rings.

المثاليات والمقاسات الإبتدائية المعرفة على الحلقات غير الإبدالية

ملخص: من المعروف أن المثاليات الإبتدائية والمقاسات الإبتدائية قد تم تعريفها علـــى الحلقـــات الإبدالية.في هذا البحث تم تعريف المثاليات الإبتدائية والمقاسات الإبتدائية علـــى الحلقـــات غيـــر الإبدالية، وقد تم إثبات أن هذا التعريف هو تعميم لتعريف المثاليات الإبتدائية والمقاسات الإبتدائيـــة على الحلقات الإبدالية، كذلك تم إثبات أنه تعميم لتعريف المثاليات الأولية والمقاسات الأولية على الحلقات الغير الإبدالية في نهاية بحثنا قمنا بدراسة بعض خواص المقاسات الجزئية الإبتدائية على الحلقات الغير الإبدالية.

"2000 Mathematics Subject Classification": Primary 16D25, Secondary 16D80, 16N60.

1. Introduction

Among the most fundamental objects of study in commutative ring theory are the prime and the primary ideals. Prime and primary ideals are also quite useful in the study of noncommutative rings. Prime ideals were defined over noncommutative rings as early as 1928 by W. Krull [1], as follows:

An ideal P in an arbitrary ring R is said to be a prime ideal if $P \neq R$ and, for ideals A and B of R satisfying $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

He also defined prime rings as follows: Let R be an arbitrary ring. If the zero ideal of R is prime, we call R a prime ring.

In Section 2 of this paper, we generalize the concept of prime ideals in noncommutative rings to the concept of primary ideals in noncommutative rings as follows:

An ideal P in an arbitrary ring R is said to be a primary ideal if $P \neq R$ and, for ideals A and B of R satisfying $AB \subseteq P$ implies that either $A \subseteq P$ or for some positive integer *n*, $B^n = \{b^n \in R \mid b \in B\} \subseteq P$. We also define the primary ring. Thus we say that an arbitrary ring R is primary if the zero ideal of R is primary.

We show that our definition of primary ideal in an arbitrary ring is a generalization of the definition of primary ideal in a commutative ring. We also give an equivalent definition of primary ideals over noncommutative rings. Thus we show that an ideal P is primary in an arbitrary ring R if and only if the quotient ring R/P is a primary ring.

In the beginning of Section 3 we define the radical annihilator of an *R*-module *M*, as follows : for an *R*-module *M*, the radical annihilator of *M* is denoted by rann(*M*) and is defined by $rann(M) = \{r \mid r \in R$ and $r^n \in ann(M)$ for some positive integer $n\}$. Then we introduce the concept of primary modules. Thus we say that an *R*-module N is primary if $N \neq 0$ and $rann(N) = rann(\bar{N})$ for every nonzero submodule $\bar{N} \subseteq N$. We also show that this definition of a primary module is a generalization of the definition of a prime module - which are modules all of whose nonzero submodules have the same annihilator - in an arbitrary ring, as it is known in the commutative rings.

Finally in Section 4, we study some properties of primary submodules and show that if f is an isomorphism from an R-module B to an R-module D, then a proper submodule C of D is primary if and only if $f^{-1}(C)$ is a primary submodule of B.

Throughout this paper, we will work exclusively with left modules and all rings are assumed to be rings with identity.

2. Primary Ideals over Noncommutative Rings

We first recall the following definitions [1],[2]:

Definition 2.1 An ideal P in an arbitrary ring R is said to be a prime ideal if $P \neq R$ and, for ideals A and B of R satisfying $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

Definition 2.2 Let R be an arbitrary ring. If the zero ideal of R is prime, we call R a prime ring.

Now we introduce the following definitions.

Definition 2.3 An ideal P in an arbitrary ring R is said to be a primary ideal if $P \neq R$ and, for ideals A and B of R satisfying $AB \subseteq P$ implies that either $A \subseteq P$ or for some positive integer n, $B^n = \{b^n \in R \mid b \in B\} \subseteq P$.

Definition 2.4 Let R be an arbitrary ring. If the zero ideal of R is primary, we call R a primary ring. That is if A and B are ideals such that $AB = 0$, then $A = 0$ or B is a nil ideal.

Remark 2.5 It is clear that every prime ideal in an arbitrary ring R is primary, but the converse is not true. Because if p is a prime integer, then the principal ideal (p^n) in Z, when n is a positive integer, (p^n) is a primary ideal, but it is not necessarily prime. For example the ideal (4) is primary ideal in Z, but it is not prime ideal in Z.

The following results show that our definition of primary ideals in an arbitrary ring is a generalization of the definition of primary ideal in a commutative ring.

Theorem 2.6 Let P be an ideal in an arbitrary ring R such that $P \neq R$. If for all $a, b \in R$; $ab \in P$ implies that ($a \in P$ or $b^n \in P$ for some positive integer n), (1) then P is primary. Conversely, if P is primary and R is commutative, then P satisfies condition (1).

Proof. Suppose that P is an ideal in an arbitrary ring R such that $P \neq R$ and that P satisfies condition (1). Let I and J be ideals such that $IJ \subseteq P$, $J^n \neq P$ and $J^n \not\subset P$ for every positive integer n, then there exists an element $j \in J$ such that $j^n \notin P$ for every positive integer n. Now for every $i \in I$, $ij \in IJ \subseteq P$, whence $i \in P$ or $j^n \in P$ for some positive integer n. Since $j^n \notin P$ for every positive integer n, we must have $i \in P$ for all $i \in I$; that is, $I \subseteq P$. Therefore P is primary ideal of R. Conversely, if P is any ideal and $ab \in P$, then the principal ideal (ab) is contained in P. If R is commutative, then $(a)(b) \subseteq (ab)$, whence $(a)(b) \subseteq P$. If P is primary, then either $(a) \subseteq P$ or $(b)^n \subseteq P$ for some positive integer n, whence $a \in P$ or $b^n \in P$ for some positive integer n. □

Remark 2.7 Commutivity is a necessary condition for the converse in the previous theorem.

Proof. Let R be the ring of 2×2 matrices over the set of real numbers, then $P = 0$ is a primary ideal of R. However $a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are elements in R with $ab \in P$ but neither $a \in P$ nor $b^n \in P$ for any positive integer n. □

Now we give another equivalent definition of primary ideals over noncommutative rings.

Proposition 2.8 P is a primary ideal over an arbitrary ring R if and only if R/P is a primary ring.

Proof. It is easy to check that if P is a primary ideal over an arbitrary ring R, then R/P is a primary ring. Now suppose that R/P is a primary ring. Let $\prod : R \to R/P$ be the canonical epimorphism. If I and J are ideals of R such that $IJ \subseteq P$, then $\prod(I)$, $\prod(J)$ are ideals of R/P such that $\prod(I)\prod(J) = \prod(IJ) = 0$. Since R/P is primary ring, either $\prod(I) = 0$ or $\prod^n(J) = 0$ for some positive integer n; that is,

$I \subseteq P$ or $J^n \subseteq P$ for some positive integer n. Therefore, P is a primary ideal of R. □

3. Primary Modules over Noncommutative Rings

We first recall the following definitions [3], [4]:

Definition 3.1 Let M be an R-module. The annihilator of M, denoted by $ann(M)$ and is defined by $ann(M) = \{r \mid r \in R \text{ and } rM = 0\}$.

Definition 3.2 An R-module N is prime if $N \neq 0$ and $ann(N) = ann(\bar{N})$ for every nonzero submodule $\bar{N} \subseteq N$.

Definition 3.3 Let M be an R-module. The radical annihilator of M is denoted by $rann(M)$ and is defined by $rann(M) = \{r \mid r \in R$ and $r^n \in ann(M)$ for some positive integer $n\}$.

Remark 3.4 It is clear that $ann(M)$ is a subset of $rann(M)$ for any R-module M, but the converse is not necessarily true. Consider the following example:

Let $R = Z$ and $M = Z_4$. Then $ann(Z_4) = 4Z$, but $rann(Z_4) = 2Z$.

Now we define the primary modules over noncommutative rings as follows:

Definition 3.5 An R-module N is primary if $N \neq 0$ and $rann(N) = rann(\bar{N})$ for every nonzero submodule $\bar{N} \subseteq N$.

Remark 3.6 Every prime module is primary, but the converse is not true.

Proof. Let P be a prime R-module of a ring R. Let $N \neq 0$ be a submodule of P. Let $r \in rann(P)$, then $r^n \in ann(P) = ann(N)$ for some positive integer n. Thus $r \in rann(N)$. Hence $rann(P) \subseteq rann(N)$. Similarly, we can prove that $rann(N) \subseteq rann(P)$. Therefore $rann(N) = rann(P)$. Thus P is a primary R-module. Now note that Z_4 is a primary Z-module, however it is not a prime Z-module. □

4. Primary Submodules over Noncommutative Rings

We define the primary submodules over noncommutative rings as follows:

Definition 4.1 Let M be an R-module. Let N be a submodule of M. N is said to be a primary submodule of M if $N \neq 0$ and $rann(N) = rann(\bar{N})$ for every nonzero submodule $\bar{N} \subseteq N$.

Proposition 4.2 Let $f : B \to D$ be an R-module isomorphism and $C(\neq D)$ a submodule of D. Then C is a primary submodule of D if and only if $f^{-1}(C)$ is a primary submodule of B.

Proof. Suppose that C is a primary submodule of D, then $C \neq 0$. Thus $f^{-1}(C) \neq 0$. Let $\bar{N}$ be a nonzero submodule of $f^{-1}(C)$. Then $f(\bar{N})$ is a nonzero submodule of C. Since C is primary, then $rann(C) = rann(f(\bar{N}))$. Note that $r \in rann(f^{-1}(C))$ if and only if $r^n f^{-1}(C) = 0$ for some positive integer n if and only if $f^{-1}(r^n C) = 0$ for some positive integer n if and only if $r^n C = 0$ for some positive integer n if and only if $r \in rann(C) = rann(f(\bar{N}))$ if and only if $r^m f(\bar{N}) = 0$ for some positive integer m if and only if $f(r^m \bar{N}) = 0$ for some positive integer m if and only if $r^m \bar{N} = 0$ for some positive integer m if and only if $r \in rann(\bar{N})$. Therefore, $rann(f^{-1}(C)) = rann(\bar{N})$ and hence $f^{-1}(C)$ is a primary submodule of B.

Conversely , Let $f^{-1}(C)$ be a primary submodule of B, then $f^{-1}(C) \neq 0$. Thus $C \neq 0$. Let $\bar{C}$ be a nonzero submodule of C, then $f^{-1}(\bar{C})$ is a nonzero submodule of $f^{-1}(C)$. Since $f^{-1}(C)$ is a primary submodule of B, then
$rann(f^{-1}(\bar{C})) = rann(f^{-1}(C))$. Note that $s \in rann(C)$ if and only if $s^n C = 0$ for some positive integer n if and only if $f^{-1}(s^n C) = 0$ for some positive integer n if and only if $s^n f^{-1}(C) = 0$ for some positive integer n if and only if $s \in rann(f^{-1}(C)) = rann(f^{-1}(\bar{C}))$ if and only if $s^m f^{-1}(\bar{C}) = 0$ for some positive integer m if and only if $f^{-1}(s^m \bar{C}) = 0$ for some positive integer m if and only if $s^m \bar{C} = 0$ for some positive integer m if and only if $s \in rann(\bar{C})$. Thus $rann(C) = rann(\bar{C})$ and hence C is a primary submodule of D. □

Finally, we end this paper by the following result.

Proposition 4.3 If N is a primary R-module, then in the center of R $rann(N)$ is a prime ideal of the center of R.

Proof. Let $rs \in rann(N)$ with $r, s \in$ center of R and $s \notin rann(N)$, then $s^m N \neq 0$ for every positive integer m. But $(rs)^k N = 0$ for some positive integer k. Thus $r^k(s^k N) = 0$ for some positive integer k. Therefore $r \in rann(s^k N)$ for some positive integer k. Since $\bar{N} = s^k N \neq 0$ and N is a primary R-module, then $rann(\bar{N}) = rann(N)$. Thus $r \in rann(N)$. Hence $rann(N)$ is a prime ideal of R. □

References:

[1] Krull, W.: Primide Alketten in Allgemeinen Ring Bereichen Sitzungsberichte, Heidelberg. Akad. Wissenschaft, **7**, (1928) 3-14.

[2] Hungerford, T.W.: Algebra, Springer-Verlag, New York Inc., (1974).

[3] Annin, S.: Attached Primes over Noncommutative Rings, Journal of Pure and Applied Algebra, **212**, (2008) 510 – 521 .

[4] Lam, T. Y.: A First Course in Noncommutative Rings, Graduate Texts In Mathematics, Volume 131,Springer-Verlag Berlin-Heidelberg-New York, (1991).