Joint Conditional Quantiles Estimation For Strictly Stationary Process

Raid Salha
Department of Mathematics,
The Islamic University of Gaza,
e-mail: rbsalha@iugaza.edu

Abstract

In this paper, the kernel estimation of the conditional quantiles for a strictly stationary stochastic process satisfying the strong mixing condition, which was proposed by Abberger (1997) is studied. Under some mild conditions, the joint asymptotic normality of the kernel estimation of several conditional quantiles estimated at the same conditional point and the joint asymptotic normality of the kernel estimation of the same conditional quantile estimated at different conditional points are established. The performance of the estimations is tested by an application for a real life data .

Keywords: Kernel estimation, conditional quantile, joint asymptotic normality, mixing conditions.

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1 Introduction

In nonparametric estimation of the conditional distribution function $F(y|x)$ of Y given $X = x$, most investigation are concerned with the regression function, $m(x) = \int_{-\infty}^{\infty} y f(y|x)dy$, the conditional mean of Y given value x of a predictor X. However, new insights about the underling structures can be gained by considering other aspects of the conditional distribution function.

The conditional quantiles $q_\alpha(x), 0 < \alpha < 1$, of Y given $X = x$, coupled with the distribution function $F(y|x)$, are important in testing hypothesis and in constructing confidence intervals.

The asymptotic properties of nonparametric estimation of conditional quantiles using kernel or nearest neighbor methods has studied by several authors. For example, for iid data Samanta (1989) studied the strong consistency and asymptotic normality for a kernel - type nonparametric estimator of $q_\alpha(x)$. Similar results as in Samanta (1989) were obtained by Roussas (1969, 1991) for Markovian processes and for mixing data were obtained by Abberger (1997), Berlinet (2000) and Cai (2002).

Recently, some new methods of estimating conditional quantiles have been proposed. The first, an approach using a check function is presented by Fan et al. (1994). An alternative procedure is first to estimate the conditional distribution function using the double kernel local linear technique, and then to invert the conditional quantile, which was proposed by Yu and Jones (1998). However, some estimators are producing conditional distribution functions that are not constrained either to lie between zero and one or to be monotone increasing, Hall, et al. (1999) proposed the Reweighted Nadaraya - Watson (RNW) estimator to overcome these difficulties.

Our aim in this paper is to study the joint asymptotic properties of the kernel estimation of the conditional quantiles under an α-mixing condition. This mixing condition ensures asymptotically vanishing memory of the strictly stationary process. The α-mixing condition is satisfied if there exists a sequence of nonnegative numbers called mixing coefficients (α_k) such that for any set A in $\sigma\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ and set B in $\sigma\{(X_{n+k}, Y_{n+k}), \ldots\}$, $|P(A \bigcap B) - P(A)P(B)| \leq \alpha_k$ and $\lim_{k \longrightarrow \infty} \alpha_k = 0$.

The α-mixing condition is fulfilled for many stochastic processes, including many time series models and it is weaker than many other mixing conditions. For more details, see Roussas and Ioannides (1987) and De Gooiger and Zerom (2003).

Now, the kernel estimations of the conditional distribution function and the conditional quantile function from Abberger (1997) are presented.

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be two-dimensional random variables from a strictly stationary process distributed as the bivariate random variable (X, Y) with joint density function $f(x, y)$ and marginal density function $g(x)$ of X. Let $F(y|x)$ be the conditional distribution of Y given $X = x$. The conditional $\alpha-$ quantile $q_\alpha(x)$ is defined as

$$q_\alpha(x) = \inf\{y \in \mathbf{R} | F(y|x) \geq \alpha\}, \quad 0 < \alpha < 1. \tag{1}$$

The estimator of the conditional distribution function is defined as follows

$$F_n(y|x) = \frac{\sum_{i=1}^{n} K(\frac{x-X_i}{h_n}) I_{(Y_i \leq y)}}{\sum_{i=1}^{n} K(\frac{x-X_i}{h_n})}, \tag{2}$$

where K is a kernel function, $\{h_n\}$ is a sequence of positive numbers converges to zero, and I denotes the indicator function.

Using Equation (2) the estimator for the conditional $\alpha-$quantile in Equation (1) is given by

$$q_{\alpha,n}(x) = \inf\{y \in \mathbf{R} | F_n(y|x) \geq \alpha\}, \quad 0 < \alpha < 1. \tag{3}$$

Let $f(y|x) = \dfrac{\partial F(y|x)}{\partial y} = \dfrac{f(x,y)}{g(x)}$ be the conditional density function of Y given $X = x$. The estimator of $f(y|x)$ is defined as

$$f_n(y|x) = \frac{f_n(x,y)}{g_n(x)},$$

where

$$f_n(x,y) = \frac{1}{nh_n^2} \sum_{i=1}^{n} K(\frac{x-X_i}{h_n}) K(\frac{y-Y_i}{h_n}),$$

and

$$g_n(x) = \frac{1}{nh_n} \sum_{i=1}^{n} K(\frac{x-X_i}{h_n}).$$

In this paper, the results of Abberger (1997) are considered and generalized in two different ways. Firstly, The joint asymptotic normality of the kernel estimation of several conditional quantiles estimated at the same conditional point for strictly stationary processes is shown. Secondly, in the same manner the asymptotic normality of the kernel estimation of the same conditional quantile estimated at different conditional points is established.

The paper is organized as follows. In the next section, the assumptions that allow us to derive the main results are stated. The main results, Theorems 1 and 2 are stated in Section 3. In Section 4, a rule for the bandwidth selection is presented. In Section 5, the performance of the conditional quantile estimation is tested by an application to a real life data. The technical arguments and proofs are collected in the appendix.

2 Assumptions

In this section, some regularity conditions that will be used in proving the asymptotic results in the paper are gathered together for easy reference.

Assumption (A1) The process $\{(X_i, Y_i)\}_{i=1}^n \subseteq \mathbf{R}\times\mathbf{R}$ is an α-mixing process with the mixing coefficient satisfies

$$\lim_{n\longrightarrow\infty} \frac{1}{nh_n^2} \sum_{i=1}^{n-1} \alpha_i = 0.$$

Assumption (A2) For fixed y and x, $0 < F(y|x) < 1$, $F^{(2,0)}(y|x) = \dfrac{\partial^2 F(y|x)}{\partial x^2}$ exists in a neighborhood of x.

Assumption (A3) The kernel function K is a probability density function satisfying the following:

i. K has a compact support;

ii. K is symmetric probability density function;

iii. K is Lipschitz continuous.

Assumption (A4) The bandwidth $\{h_n\}$ satisfies the following:

i. $\lim_{n\longrightarrow\infty} h_n = 0$;

ii. $\lim_{n\longrightarrow\infty} nh_n = \infty$;

iii. $\lim_{n\longrightarrow\infty} nh_n^5 = 0$.

Assumption (A5) let $p = p(n), q = q(n)$ be positive numbers with $p + q \leq n$ for all sufficiently large n and tending to infinity, and let k be the integral part of $n/(p+q)$, where the following holds,

i. $\lim_{n\to\infty} \dfrac{qk}{nh_n} = 0$;

ii. $\lim_{n\to\infty} \dfrac{p^2}{nh_n} = 0$;

iii. $\lim_{n\to\infty} \dfrac{n}{p}\,\alpha(q) = 0$.

3 Main Results

In this section, the two main theorems in the paper are stated. The technical arguments and proofs are collected in the Appendix.

Theorem 1. Suppose that $g(x) > 0$ and $f(q_{\alpha_i}(x)|x) > 0,\ i = 1, 2,\ 0 < \alpha_1 < \alpha_2 < 1$, then under the Assumptions A1-A5, the following holds

$$\sqrt{nh_n}[(q_{\alpha_1,n}(x) - q_{\alpha_1}(x), q_{\alpha_2,n}(x) - q_{\alpha_2}(x))^T] \xrightarrow{\text{d}} N(\mathbf{0}, \mathbf{\Gamma}),$$

where $\mathbf{0} = (0, 0)^T$ and the covariance matrix $\mathbf{\Gamma}$ is defined as

$$\mathbf{\Gamma} = \frac{\int_{-\infty}^{\infty} K^2(u)\, du}{g(x)} \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{bmatrix},$$

where

$$\gamma_{ij} = \frac{\alpha_i(1 - \alpha_j)}{f(q_{\alpha_i}(x)|x) f(q_{\alpha_j}(x)|x)},\ 1 \le i \le j \le 2.$$

Theorem 2. Suppose that $g(x) > 0$ and $f(q_{\alpha,n}(x_i)|x_i) > 0,\ i = 1, 2,\ 0 < \alpha < 1$, then under the Assumptions A1-A5, the following holds

$$\sqrt{nh_n}[(q_{\alpha,n}(x_1) - q_\alpha(x_1), q_{\alpha,n}(x_2) - q_\alpha(x_2))^T] \xrightarrow{\text{d}} N(\mathbf{0}, \mathbf{D}),$$

where $\mathbf{0} = (0, 0)^T$ and $\mathbf{D}$ is a diagonal covariance matrix with the (i, i)th element

$$d_{ii} = \frac{\alpha(1 - \alpha)}{g(x_i) f^2(q_\alpha(x_i)|x_i)} \int_{-\infty}^{\infty} K^2(u)\, du,\ i = 1, 2.$$

4 Bandwidth Selection

The selection of the appropriate bandwidth is an important and basic problem in all kernel smoothing techniques. It is critical to the performance of the nonparametric kernel estimation. When the bandwidth is very small, the estimate will be very close to the origin data, and therefore will be very wiggly. On the other hand, if the bandwidth is very large, the estimate will be very smooth lying close to the mean of all the data.

Gannoun, et al. (2003) developed the following rule for the optimal bandwidth selection for the conditional quantile function,

$$h_{\alpha,n} = h_{mean} \left(\frac{\alpha(1 - \alpha)}{\phi(\Phi^{-1}(\alpha))^2} \right)^{\frac{1}{5}}, \tag{4}$$

where ϕ and Φ are respectively the standard normal density and distribution function, and h_{mean} is the bandwidth for the kernel estimation of the conditional mean function and it is given by

$$h_{mean} = \left(\frac{R_1(K)v(x)(1+\rho)}{nR_2(K)^2\{r''(x)\}^2 g(x)} \right)^{\frac{1}{5}}, \qquad (5)$$

where $R_1(K) = \int_{-\infty}^{\infty} K^2(u)\, du,\ R_2 = \int_{-\infty}^{\infty} u^2 K(u)\, du,\ r(x)$ and $v(x)$ are, respectively the conditional mean and conditional variance and ρ is the one - step autocorrelation coefficient.

5 Application

In this section, the performance of the conditional quantile estimator is tested via an application to a real life data. The time series of Cisco data from Tsay (2000) is considered. A time series plot of the data is shown in Figure 1. The data consists of 2275 observation. We have used the first 2264 observation to construct 90% prediction intervals for the last 12 observation. The Gaussian kernel is used and the bandwidth is selected as described in Section 4.

The true values and the corresponding prediction intervals are listed in Table 1, and a plot of the tail of the time series together with the prediction intervals is shown in Figure 2.

Table 1 shows that all the true values are contained in the corresponding prediction intervals. The average length of the prediction intervals equals 6.51 which is %31.64 of the range of the data. This application indicates that the performance of the conditional estimator in constructing prediction intervals is reasonably good.

Table 1: %90 prediction intervals for the Cisco data

i	True value	Pred. Interval	i	True value	Pred. Interval
2264	-2.128	(-2.780, 3.665)	2270	1.872	(-2.863, 3.665)
2265	2.065	(-3.021, 3.864)	2271	0.715	(-2.955, 3.673)
2266	1.835	(-2.980, 3.756)	2272	0	(-2.662, 3.665)
2267	3.512	(-2.655, 3.665)	2273	0.887	(-2.869, 3.660)
2268	0.723	(-2.664, 3.665)	2274	0.059	(-2.919, 3.660)
2269	-1.452	(-2.664, 3.665)	2275	0.879	(-2.820, 3.660)

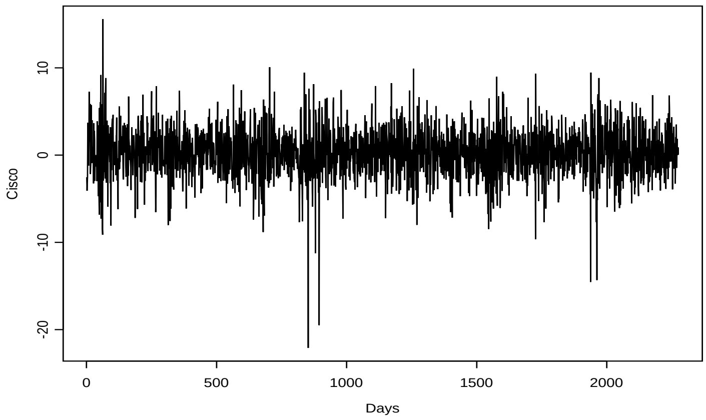

Figure 1: Time series plot of Cisco data.

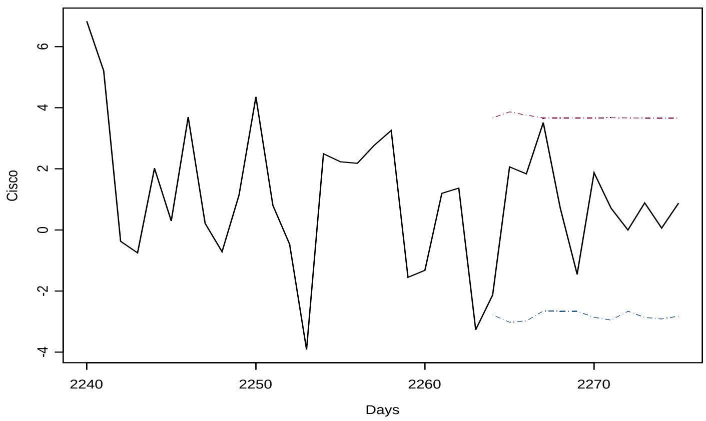

Figure 2: Time series plot of the tail of Cisco data together with the prediction intervals for the last 12 observation .

Appendix

Here, just the details of the proof of Theorem 1 will be given. The proof of Theorem 2 will be omitted, since it is similar to that of Theorem 1.

Now, the following notations will be considered.
Let

$$S_{nr} = F_n(y_r|x) - EF_n(y_r|x), \quad r = 1, 2,$$

$$W_{ni}(x) = \frac{K(\frac{x-X_i}{h_n})}{\sum_{j=1}^{n} K(\frac{x-X_i}{h_n})}, \quad i = 1, 2, \ldots, n,$$

and

$$Z_{ri} = W_{ni}(x)I_{(Y_i \leq y_r)} - EW_{ni}(x)I_{(Y_i \leq y_r)}, \; i = 1, 2, \ldots, n, \; r = 1, 2.$$

Therefore,

$$S_{nr} = \sum_{i=1}^{n} Z_{ri} = (F_n(y_1|x) - EF_n(y_1|x), F_n(y_2|x) - EF_n(y_2|x))^T.$$

Set that

$$S_n = \sum_{r=1}^{2} c_r S_{nr},$$

where c_r are constants, $r = 1, 2$.

The proof of Theorem will be established through the following five lemmas.

Lemma 1. Under the Assumptions A1- A4, and as $n \longrightarrow \infty$, the following holds

(i) $\left|\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} W_{ni}(x)W_{nj}(x)Cov(I_{(Y_i \leq y_1)}, I_{(Y_j \leq y_2)})\right| \longrightarrow 0.$

(ii) $f_n(y|x) \xrightarrow{p} f(y|x).$

(iii) $q_{\alpha,n}(x) \xrightarrow{p} q_\alpha(x).$

Proof.
(i) Suppose that $K(u) \leq M < \infty$ for all $u \in \mathbf{R}$. Following the same lines as in Abberger (1997), we have

$$\left|\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} W_{ni}(x)W_{ni}(x)Cov(I_{(Y_i \leq y_1)}, I_{(Y_j \leq y_2)})\right| = \left|\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{K(U_i)K(U_j)}{[\sum_{i=1}^{n} K(U_i)]^2} Cov(I_{(Y_i \leq y_1)}, I_{(Y_j \leq y_2)})\right|$$

$$\leq \frac{2}{n^2h_n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} K(U_i)K(U_j)\left|Cov(I_{(Y_i\leq y_1)}, I_{(Y_j\leq y_2)})\right|$$
$$\leq \frac{2}{n^2h_n^2}M^2\sum_{i=1}^{n-1}(n-i)\left|Cov(I_{(Y_i\leq y_1)}, I_{(Y_j\leq y_2)})\right|$$
$$\leq \frac{8}{nh_n^2}M^2\sum_{i=1}^{n-1}\alpha_i \longrightarrow 0,$$

by Assumption (A1).

(ii) Let $L(x)$ be an upper bound for $\dfrac{f(x,y)}{g(x)}$, $y \in \mathbf{R}$.

$$\begin{aligned}
\sup_{y\in\mathbf{R}} |f_n(y|x) - f(y|x)| &= \sup_{y\in\mathbf{R}}\left|\frac{f_n(x,y)}{g_n(x)} - \frac{f(x,y)}{g(x)}\right| \\
&= \sup_{y\in\mathbf{R}}\left|\frac{f_n(x,y)}{g_n(x)} - \frac{f(x,y)}{g_n(x)} + \frac{f(x,y)}{g_n(x)} - \frac{f(x,y)}{g(x)}\right| \\
&\leq \sup_{y\in\mathbf{R}}\left|\frac{f_n(x,y)}{g_n(x)} - \frac{f(x,y)}{g_n(x)}\right| + \sup_{y\in\mathbf{R}}\left|\frac{f(x,y)}{g(x)}\right|\left|\frac{g(x)}{g_n(x)} - 1\right| \\
&\leq \frac{\sup_{y\in\mathbf{R}} |f_n(x,y) - f(x,y)|}{g_n(x)} + L(x)\left|\frac{g(x)}{g_n(x)} - 1\right|.
\end{aligned}$$

Now, if $g(x) > 0$, then an application of Theorem 1 in Samanta and Thavaneswaran (1990) implies that $f_n(y|x) \xrightarrow{\text{p}} f(y|x)$.

(iii) Follows easily from Theorem 3 and Corrollary 2 in Abberger (1997).

Lemma 2. Under the Assumptions A1, A3, A4(i),(ii), and as $n \longrightarrow \infty$, the following is true

$$Cov(S_{n1}, S_{n2}) \longrightarrow \frac{F(y_1|x)[1 - F(y_2|x)]}{nh_n g(x)}\int_{-\infty}^{\infty} K^2(u)du.$$

Proof.

$$\begin{aligned}
Cov(S_{n1}, S_{n2}) &= Cov(F_n(y_1|x), F_n(y_2|x)) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} W_{ni}(x)W_{nj}(x)Cov(I_{(Y_i\leq y_1)}, I_{(Y_j\leq y_2)}) \\
&= \sum_{i=1}^{n}[W_{ni}(x)]^2 Cov(I_{(Y_i\leq y_1)}, I_{(Y_i\leq y_2)}) \\
&+ 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} W_{ni}(x)W_{nj}(x)Cov(I_{(Y_i\leq y_1)}, I_{(Y_j\leq y_2)})
\end{aligned}$$

$$= \quad A_1 + A_2.$$

$$A_1 = \frac{\sum_{i=1}^{n} K^2(\frac{x-X_i}{h_n}) Cov(I_{(Y_i \le y_1)}, I_{(Y_i \le y_2)})}{[\sum_{i=1}^{n} K^2(\frac{x-X_i}{h_n})]^2} \longrightarrow \frac{F(y_1|x)[1-F(y_2|x)]}{nh_n g(x)} \int_{-\infty}^{\infty} K^2(u)du,$$

by Theorem 1 in Abberger (1997).
By Lemma 1(i), $A_2 \longrightarrow 0$, as $n \longrightarrow \infty$.

Lemma 3. Under the Assumptions of Lemma 2, and as $n \longrightarrow \infty$, the following is true

$$Var(S_n) \longrightarrow \sigma^2,$$

where

$$\sigma^2 = \sum_{r=1}^{2} c_r^2 \frac{F(y_r|x)[1-F(y_r|x)]}{nh_n g(x)} \int_{-\infty}^{\infty} K^2(u)du + 2c_1c_2 \frac{F(y_1|x)[1-F(y_2|x)]}{nh_n g(x)} \int_{\infty}^{\infty} K^2(u)du.$$

Proof. The proof follows directly from Lemma 2.

Now, let p, q and k be as in Assumption A5 and divide the set $\{1, 2, \ldots, n\}$ into p large blocks and q small blocks and set

$$y_{nrm} = \sum_{i=k_m}^{k_m+p-1} Z_{ri}, \qquad k_m = (m-1)(p+q)+1,$$

$$y'_{nrm} = \sum_{j=l_m}^{l_m+q-1} Z_{rj}, \qquad l_m = (m-1)(p+q)+p+1,$$

$$y'_{nrk} = \sum_{l=k(p+q)+1}^{n} Z_{rl}, \qquad m = 1, 2, \ldots, k.$$

Also, set

$$S'_{nr} = \sum_{m=1}^{k} y_{nrm}, \qquad S''_{nr} = \sum_{m=1}^{k} y'_{nrm}, \qquad S'''_{nr} = y'_{nrk},$$

and

$$S'_n = \sum_{r=1}^{2} c_r S'_{nr}, \qquad S''_n = \sum_{r=1}^{2} c_r S''_{nr}, \qquad S'''_n = \sum_{r=1}^{2} c_r y'_{nrk}.$$

This implies that,

$$S_{nr} = \sum_{i=1}^{n} Z_{ri} = S'_{nr} + S''_{nr} + S'''_{n}, \qquad (6)$$

and

$$S_{n} = \sum_{r=1}^{2} c_r S_{nr} = S'_{n} + S''_{n} + S'''_{n}. \qquad (7)$$

Lemma 4. Under the Assumptions A1, A3, A4(i), (ii) and A5, and as $n \longrightarrow \infty$, the following is true

(i) $E(S''_n)^2 \longrightarrow 0$,

(ii) $E(S'''_n)^2 \longrightarrow 0$.

Proof. The proof of part (i) will be given and the proof of part (ii) will be similar. Using the definitions of the notations S''_n and S''_{nr} the following two equations are obtained.

$$E(S''_n)^2 = E(\sum_{r=1}^{2} c_r S''_{nr})^2 = \sum_{r=1}^{2} c_r^2 E(S''_{nr})^2 + 2c_1c_2 Cov(S''_{n1}, S''_{n2}). \qquad (8)$$

$$E(S''_{nr})^2 = \sum_{m=1}^{k} Var(y'_{nrm}) + 2 \sum_{1\le i<j\le k} Cov(y'_{nri}, y'_{nrj}). \qquad (9)$$

Now, let C denotes a positive constant.

$$\begin{aligned}
\sum_{m=1}^{k} Var(y'_{nrm}) &= \sum_{m=1}^{k} Var(\sum_{i=l_m}^{l_m+q-1} Z_{ri}) \\
&= \sum_{m=1}^{k}\{ \sum_{i=l_m}^{l_m+q-1} Var(W_{ni}(x) I_{(Y_i \le y_r)}) \\
&+ 2 \sum_{l_m \le i<j<l_m+q-1} Cov(W_{ni}(x) I_{(Y_i\le y_r)}, W_{nj}(x) I_{(Y_j \le y_r)})\} \\
&\le \frac{Cqk}{nh_n} \longrightarrow 0, \qquad (10)
\end{aligned}$$

as $n \longrightarrow \infty$ by Assumption A5.
Also,

$$2 \sum_{1\le i<j\le k} Cov(y'_{nri}, y'_{nrj}) = 2 \sum_{1\le i<j\le k} Cov(\sum_{v=l_i}^{l_i+q-1} Z_{rv}, \sum_{l=l_j}^{l_j+q-1} Z_{rl})$$

$$
\begin{aligned}
&= 2\sum_{1\le i<j\le k}\sum_{v=l_i}^{l_i+q-1}\sum_{l=l_j}^{l_j+q-1} Cov(Z_{rv}, Z_{rl}) \\
&= 2\sum_{1\le i<j\le k}\sum_{v=l_i}^{l_i+q-1}\sum_{l=l_j}^{l_j+q-1} Cov(W_{nv}(x)I_{(Y_v\le y_r)}, W_{nl}(x)I_{(Y_l\le y_r)}) \\
&\le C\sum_{i=1}^{n-p}\sum_{j=i+1}^{n} |Cov(W_{ni}(x)I_{(Y_i\le y_1)}, W_{nj}(x)I_{(Y_j\le y_2)})| \\
&\le C\sum_{1\le i<j\le n} |Cov(W_{ni}(x)I_{(Y_i\le y_1)}, W_{nj}(x)I_{(Y_j\le y_2)})| \longrightarrow 0, \quad (11)
\end{aligned}
$$

as $n \longrightarrow \infty$ by Lemma 1.
Now, a combination of Equations (9), (10) and (11) implies that

$$E(S''_{nr})^2 \longrightarrow 0, \text{ as } n \longrightarrow \infty \qquad (12)$$

Now an application of Minkowski Inequality and Equation (12) implies that as $n \longrightarrow \infty$

$$[E(S''_n)^2]^{1/2} = [E(\sum_{r=1}^{2} c_r S''_{nr})^2]^{1/2} \le \sum_{r=1}^{2} |c_r|[ES''_{nr})^2]^{1/2} \longrightarrow 0. \qquad (13)$$

Lemma 5. Under the assumptions A1-A5, and as $n \longrightarrow \infty$,

$$(F_n(y_1|x) - EF_n(y_1|x), F_n(y_2|x) - EF_n(y_2|x))^T \xrightarrow{\text{d}} N(\mathbf{0}, \Sigma),$$

where $\mathbf{0} = (0,0)^T$ and Σ is the covariance matrix with the (i, j)th element σ_{ij}, $1 \le i \le j \le 2$ is given by

$$\sigma_{ij} = \frac{F(y_i|x)[1 - F(y_j|x)]}{nh_n g(x)} \int_{-\infty}^{\infty} K^2(u)du.$$

Proof.
Equation (7) implies that $S'_n = S_n - (S''_n + S'''_n)$.
Therefore,

$$E(S'_n)^2 = E(S_n)^2 + E(S''_n + S'''_n)^2 - 2E(S_n(S''_n + S'''_n)).$$

Now, by Lemma 4 and as $n \longrightarrow \infty$ we obtain

$$[E(S''_n + S'''_n)^2]^{1/2} \le [E(S''_n)^2]^{1/2} + [ES'''_n)^2]^{1/2} \longrightarrow 0,$$

and

$$E|S_n(S''_n + S'''_n)| \le (E(S^2_n)]^{1/2}[E(S''_n + S'''_n)^2]^{1/2} \longrightarrow 0.$$

This implies that

$$E(S'_n)^2 \longrightarrow E(S_n)^2.$$

Now,

$$\begin{aligned} Var(S'_n)^2 &= Var(\sum_{r=1}^{2} c_r S'_{nr}) = c_1^2 Var(S'_{n1}) + c_2^2 Var(S'_{n2}) \\ &+ 2Cov(S'_{n1}, S'_{n2}) \longrightarrow \sigma^2, \text{ as } n \longrightarrow \infty. \end{aligned}$$

Therefore

$$E(S'_{n1})^2 \longrightarrow \tau_1^2,\ c_1 \neq 0,\ c_2 = 0,$$

and

$$E(S'_{n2})^2 \longrightarrow \tau_2^2,\ c_2 \neq 0,\ c_1 = 0,$$

where

$$\tau_r^2 = \frac{F(y_r|x)[1 - F(y_r|x)]}{nh_n g(x)} \int_{-\infty}^{\infty} K^2(u)du.$$

Since

$$E(S'_{nr})^2 = \sum_{m=1}^{k} Var(y_{nrm}) + 2 \sum_{1 \le i < j \le k} Cov((y_{nri}, y_{nrj}),$$

this implies

$$\sum_{m=1}^{k} Var(y_{nrm}) \longrightarrow \tau_r^2, \quad r = 1, 2.$$

Now, let $Y_{nrm}, m = 1, 2, \ldots, k, r = 1, 2$ be independent random variables such that Y_{nrm} is distributed as $c_r y_{nrm}$ and $X_{nrm} = Y_{nrm}/S_n$, where $\sum_{k=1}^{2} \sum_{m=1}^{k} Var(Y_{nrm}) \longrightarrow \sigma^2$.
Following the same technique as in Roussas et al. (1992), we get that

$$\sum_{k=1}^{2} \sum_{m=1}^{k} X_{nrm} \xrightarrow{\text{d}} N(0, 1).$$

This means that

$$\sum_{k=1}^{2} \sum_{m=1}^{k} c_r y_{nrm} \xrightarrow{\text{d}} N(0, \sigma_c^2),$$

which implies $S'_n \xrightarrow{\text{d}} N(0, \sigma^2)$ and therefore, $S_n \xrightarrow{\text{d}} N(0, \sigma^2)$.
This implies that

$$\sum_{r=1}^{2} c_r(F_n(y_r|x) - EF_n(y_r|x)) \xrightarrow{\text{d}} N(0, \sigma^2).$$

Now, an application of the Cramér-Wold theorem completes the proof of the lemma.

Proof of Theorem 1.
Now, by expanding $F_n(q_{\alpha_i,n}(x)|x)$ around $q_{\alpha_i}(x),\ i = 1, 2$, we get

$$F(q_{\alpha_i}(x)|x) = F_n(q_{\alpha_i,n}(x)|x) = F_n(q_{\alpha_i}(x)|x) + (q_{\alpha_i,n}(x_i) - q_{\alpha_i}(x)) f_n(q^*_{\alpha_i}(x)|x).$$

where $q^*_{\alpha_i}(x)$ is some random point between $q_{\alpha_i,n}(x)$ and $q_{\alpha_i}(x)$.
This implies

$$q_{\alpha_i,n}(x) - q_{\alpha_i}(x) = \frac{F(q_{\alpha_i}(x)|x) - F_n(q_{\alpha_i}(x)|x)}{f_n(q^*_{\alpha_i}(x)|x)}, \tag{14}$$

From Equation (14), we obtain that

$$\sqrt{nh_n}\begin{bmatrix} q_{\alpha_1,n}(x) - q_{\alpha_1}(x) \\ q_{\alpha_2,n}(x) - q_{\alpha_2}(x) \end{bmatrix} = \begin{bmatrix} \sqrt{nh_n}(\dfrac{F(q_{\alpha_1}(x)|x) - F_n(q_{\alpha_1}(x)|x)}{f_n(q^*_{\alpha_1}(x)|x)}) \\ \sqrt{nh_n}(\dfrac{F(q_{\alpha_2}(x)|x) - F_n(q_{\alpha_2}(x)|x)}{f_n(q^*_{\alpha_2}(x)|x)}) \end{bmatrix} = \mathbf{W}_n\mathbf{X}_n,$$

where

$$\mathbf{W}_n = \begin{bmatrix} \dfrac{1}{f_n(q^*_{\alpha_1}(x)|x))} & 0 \\ 0 & \dfrac{1}{f_n(q^*_{\alpha_2}(x)|x))} \end{bmatrix}, \mathbf{X}_n = \begin{bmatrix} \sqrt{nh_n}(F(q_{\alpha_1}(x)|x) - F_n(q_{\alpha_1}(x)|x)) \\ \sqrt{nh_n}(F(q_{\alpha_2}(x)|x) - F_n(q_{\alpha_2}(x)|x)) \end{bmatrix}.$$

From Corrollary 1 in Abberger (1997), the following holds

$$\sqrt{nh_n}(EF_n(y|x) - F(y|x)) = o(1). \tag{15}$$

Now, a combination of Lemma 5 and Equation (15) implies that

$$\mathbf{X}_n \xrightarrow{\text{d}} \mathbf{X} \sim N(\mathbf{0}, \mathbf{\Delta}), \tag{16}$$

where $\mathbf{0} = (0, 0)^T$ and $\mathbf{\Delta}$ is the covariance matrix with the (i, j)th element δ_{ij}, $1 \le i \le j \le 2$ is given by

$$\begin{aligned} \delta_{ij} &= \frac{F(q_{\alpha_i}(x)|x)[1 - F(q_{\alpha_j}(x)|x)]}{g(x)} \int_{-\infty}^{\infty} K(u)^2\, du \\ &= \begin{bmatrix} \alpha_1(1-\alpha_1) & \alpha_1(1-\alpha_2) \\ \alpha_1(1-\alpha_2) & \alpha_2(1-\alpha_2) \end{bmatrix} \frac{\int_{-\infty}^{\infty} K(u)^2\, du}{g(x)}. \end{aligned} \tag{17}$$

Let

$$\mathbf{W} = \begin{bmatrix} \dfrac{1}{f(q_{\alpha_1}(x)|x))} & 0 \\ 0 & \dfrac{1}{f(q_{\alpha_2}(x)|x))} \end{bmatrix}.$$

From Lemma 1, we have

$$f_n(q^*_{\alpha_i}(x)|x) \xrightarrow{\text{p}} f(q_{\alpha_i}(x)|x),\ i = 1, 2. \tag{18}$$

Now, Equation (18) implies that

$$tr\{(\mathbf{W}_n - \mathbf{W})^T(\mathbf{W}_n - \mathbf{W})\} \xrightarrow{\text{p}} 0.$$

Now, using Slutsky Theorem (see Theorem 3.4.3 in Paranb and Julio (1993, pp. 130)), we obtain

$$\mathbf{W}_n\mathbf{X}_n \xrightarrow{\text{d}} \mathbf{W}\mathbf{X},$$

which completes the proof of the Theorem.

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