Topological Properties in Isotonic Spaces

Eissa D. Habil and **Khalid A. Elzenati** *
Department of Mathematics, Islamic University of Gaza
P.O. Box 108, Gaza, Palestine
E-mail: habil@iugaza.edu.ps

Abstract

An isotonic space (X, cl) is a set X with isotonic operator $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ which satisfies $cl(\emptyset) = \emptyset$ and $cl(A) \subseteq cl(B)$ whenever $A \subseteq B \subseteq X$. Many properties which hold in topological spaces hold in isotonic spaces as well.

We explore the topological concepts of lower separation axioms, higher separation axioms and connectedness for isotonic spaces, and we establish that they are topological properties in isotonic spaces.

Keywords: generalized closure spaces, isotonic spaces, neighborhood spaces, topological properties, lower separation axioms, higher separation axioms, connectedness, Z-connectedness, strong connectedness.

AMS Subject classification: 54A05

*This research is supported by a grant from the Deanery of Scientific Research, Islamic University of Gaza

1 Introduction

Closure spaces and (more generally) isotonic spaces have already been studied by Hausdorff [11], Day [2], Hammer [10, 9], Gnilka [4, 5, 6], Stadler [12, 13], and Habil and Elzenati [7]. In [12, 13], Stadler studied lower separation axioms and higher separation axioms in isotonic spaces. In [7], the notions of connectedness, Z-conectedness and strong connectedness in isotonic spaces have been studied. In this paper, we explore all these notions and axioms, and we further show that they form topological properties in isotonic spaces.

Let X be a set, $\mathscr{P}(X)$ its power set and $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ be an arbitrary set-valued set-function, called a *closure function*. We call $cl(A)$, $A \subseteq X$, the *closure* of A, and we call the pair (X, cl) a *generalized closure space*. Consider the following axioms of the closure function for all $A, B, A_\lambda \in \mathscr{P}(X), \lambda \in \Lambda$:

K0) $cl(\emptyset) = \emptyset$.

Kl) $A \subseteq B$ implies $cl(A) \subseteq cl(B)$ (isotonic).

K2) $A \subseteq cl(A)$ (expanding).

K3) $cl(A \cup B) \subseteq cl(A) \cup cl(B)$ (sub-additive).

K4) $cl(cl(A)) = cl(A)$ (idempotent).

K5) $\bigcup_{\lambda\in\Lambda} cl(A_\lambda) = cl(\bigcup_{\lambda\in\Lambda}(A_\lambda))$ (additive).

The dual of a closure function is the *interior function* $int : \mathscr{P}(X) \to \mathscr{P}(X)$ which is defined by

$$int(A) := X \setminus cl(X \setminus A). \tag{1}$$

Given the interior function $int : \mathscr{P}(X) \to \mathscr{P}(X)$, the closure function is recovered by

$$cl(A) := X \setminus (int(X \setminus A)) \tag{2}$$

for all $A \in \mathscr{P}(X)$. A set $A \in \mathscr{P}(X)$ is *closed* in the generalized closure space (X, cl) if $cl(A)$=A holds. It is *open* if its complement $X \setminus A$ is closed or equivalently $A = int(A)$.

It should be noted that the open and closed sets will not play a central role in our discussion. From now on, the word *space* will mean a generalized closure space.

Definition 1.1. [12] Let *cl* and *int* be a closure and its dual interior function on X. Then the *neighborhood function* $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ assigns to each $x \in X$ the collection

$$\mathscr{N}(x) := \{ N \in \mathscr{P}(X) \mid x \in int(N)\} \tag{3}$$

of its neighborhoods. A set V is a *neighborhood* of A, in symbols $V \in \mathscr{N}(A)$, if $V \in \mathscr{N}(x)\ \forall x \in A$.

The proof of the next lemma follows immediately from the definitions.

Lemma 1.1. [12, 13] *For any space* (X, cl)*,* $V \in \mathscr{N}(A)$ *if and only if* $A \subseteq int(V)$*.*

The next theorem illustrates the intimate relationship between closures of sets and neighborhoods of points.

Theorem 1.1. [12, 13] *Let* $\mathscr{N}$ *be the neighborhood function defined in equ.(3). Then* $x \in cl(A)$ *if and only if* $X \setminus A \notin \mathscr{N}(x)$*.*

It should be noted that there are equivalent properties for (Ki), $i = 0, 1, ..., 5$, which can be expressed in terms of interior or neighborhood functions (see[12, 13, 8]).

2 Isotonic Spaces

Definition 2.1. [12, 13] An *isotonic space* is a pair (X, cl), where X is a set and $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ satisfies the axioms $(K0)$ and $(K1)$. An isotonic space (X, cl) that satisfies $(K2)$ is called a *neighborhood space*. A *closure space* is a neighborhood space that satisfies $(K4)$. A *topological space* is a closure space that satisfies $(K3)$.

Lemma 2.1. [9, Lemma10] *The following conditions are equivalent for an arbitrary closure function* $cl : \mathscr{P}(X) \to \mathscr{P}(X)$*:*

(K1) $A \subseteq B \subseteq X$ *implies* $cl(A) \subseteq cl(B)$*.*

(K1^I) $cl(A) \cup cl(B) \subseteq cl(A \cup B)$ *for all* $A, B \in \mathscr{P}(X)$*.*

(K1^{II}) $cl(A \cap B) \subseteq cl(A) \cap cl(B)$ *for all* $A, B \in \mathscr{P}(X)$*.*

It is easy to derive equivalent conditions for the associated interior function by repeated application of $int(A) = X \setminus cl(X \setminus A)$, as the following lemma shows.

Lemma 2.2. [12] The following conditions are equivalent for an arbitrary interior function: $int : \mathscr{P}(X) \to \mathscr{P}(X)$:

(K1^{III}) $A \subseteq B \subseteq X$ implies $int(A) \subseteq int(B)$.

(K1^{IV}) $int(A) \cup int(B) \subseteq int(A \cup B)$ for all $A, B \in \mathscr{P}(X)$.

(K1^{V}) $int(A \cap B) \subseteq int(A) \cap int(B)$ for all $A, B \in \mathscr{P}(X)$.

An isotonic space can be described by means of interior and neighborhood functions, as the following two lemmas show. Their proofs follow immediately from the definitions and, therefore, are omitted.

Lemma 2.3. *A space (X, cl) is isotonic if and only if the interior function $int : \mathscr{P}(X) \to \mathscr{P}(X)$ satisfies:*

I0) $int(X) = X$*;*

I1) $int(A) \subseteq int(B) \quad \forall A \subseteq B \subseteq X$.

Lemma 2.4. *A space (X, cl) is isotonic if and only if the neighborhood function $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ satisfies:*

N0) $X \in \mathscr{N}(x) \quad \forall x \in X$*;*

N1) $N \in \mathscr{N}(x), N \subseteq N_1$ *implies* $N_1 \in \mathscr{N}(x)$.

The next theorem shows that (N1)(or equivalently (K1)) is a necessary and sufficient condition for defining the closure function in terms of neighborhoods.

Theorem 2.1. [2, Theorem 3.1, Corollary 3.2] *Let (X, cl) be a space and $c(A) := \{x \in X \mid \forall N \in \mathscr{N}(x) : A \cap N \neq \emptyset\}$ for all $A \subseteq X$. Then*

(i) $c(A) \subseteq cl(A)$.

(ii) $c : \mathscr{P}(X) \to \mathscr{P}(X)$ *is isotonic (i.e., satisfies (K1)).*

(iii) $c(A) = cl(A)$ *if and only if cl is isotonic.*

3 Continuous Functions

The purpose of this section is to define continuous functions on a space (X, cl) with arbitrary closure function, and establish their elementary properties.

Definition 3.1. [12, 13] Let (X, cl_X) and (Y, cl_Y) be two spaces. A function $f : X \to Y$ is *continuous* if $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ $\forall B \in \mathscr{P}(Y)$.

Proposition 3.1. [13] *Let (X, cl_X) and (Y, cl_Y) be two spaces and $f : X \to Y$. Then the following statements are equivalent:*

(1) *f is continuous.*

(2) $f^{-1}(int_Y(B)) \subseteq int_X(f^{-1}(B))$ $\quad \forall B \in \mathscr{P}(Y)$.

(3) $(\forall x \in X)$ *[$B \in \mathscr{N}_Y(f(x))$ implies $f^{-1}(B) \in \mathscr{N}_X(x)$ $\forall B \in \mathscr{P}(Y)$].*

Definition 3.2. [12] Let (X, cl_X) and (Y, cl_Y) be two spaces. We say that $f : X \to Y$ is *continuous at $x \in X$* if $\forall B \in \mathscr{P}(Y)$ and $B \in \mathscr{N}_Y(f(x))$, we have $f^{-1}(B) \in \mathscr{N}_X(x)$.

Corollary 3.1. [12] *Let (X, cl_X) and (Y, cl_Y) be two spaces. Then $f : X \to Y$ is continuous if and only if it is continuous at every $x \in X$.*

Definition 3.3. [12] Let (X, cl_X) and (Y, cl_Y) be two spaces. Then $f : X \to Y$ is *closure-preserving* if for all $A \in \mathscr{P}(X)$, $f(cl_X(A)) \subseteq cl_Y(f(A))$.

Theorem 3.1. [12] *Let (X, cl) and (Y, cl) be isotonic spaces. Then the following properties are equivalent:*

(i) *$f : X \to Y$ is continuous.*

(ii) *$f : X \to Y$ is closure-preserving.*

(iii) *$f(A) \subseteq B$ implies $f(cl_X(A)) \subseteq cl_Y(B)$ for all $A \in \mathscr{P}(X)$ and $B \in \mathscr{P}(Y)$.*

Lemma 3.1. [7] *Let X, Y, Z be spaces and suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous functions. Then $g \circ f : X \to Z$ is continuous.*

Definition 3.4. If X and Y are two spaces, a function f from X to Y is a *homeomorphism* iff f is a continuous bijection, and f^{-1} is also continuous. In this case, we say X and Y are *homeomorphic*.

Definition 3.5. A *topological property* is a property which, if possessed by a space X, then it is possessed by all spaces homeomorphic to X.

Theorem 3.2. *If (X, cl) and (Y, cl) are isotonic spaces and $f : (X, cl) \to (Y, cl)$ is a bijection, then f is a homeomorphism iff $f(cl_X(A)) = cl_Y(f(A))$ for each $A \subseteq X$.*

Proof. It is immediate from Theorem 3.1 and Definition 3.4. $\square$

Remark 3.1. It should be noted that in Theorem 3.2 we can replace $f(cl_X(A)) = cl_Y(f(A))$ by $f(int_X(A)) = int_Y(f(A))$, and hence if $U \in \mathscr{N}(x)$ then $f(U) \in \mathscr{N}(f(x))$ for all $x \in X$.

4 Lower Separation Axioms

Definition 4.1. [12] A space (X, cl) is a $T_0 - space$ if and only if $\forall\, x, y \in X, x \neq y$, $\exists\, N_x \in \mathscr{N}(x)$ such that $y \notin N_x$ or $\exists\, N_y \in \mathscr{N}(y)$ such that $x \notin N_y$.

An equivalent definition of a T_0-space can be given by using closure functions, as the following result shows.

Proposition 4.1. [12] *An isotonic space (X, cl) is a T_0-space if and only if whenever x and y are distinct points in X, we have $x \notin cl(\{y\})$ or $y \notin cl(\{x\})$.*

Definition 4.2. A space (X, cl) is a $T_1 - space$ if $\forall\, x, y \in X,\ x \neq y,\ \exists\, N' \in \mathscr{N}(x)$ and $N'' \in \mathscr{N}(y)$ such that $x \notin N'',\ y \notin N'$.

Proposition 4.2. [12] *An isotonic space (X, cl) is a $T_1 - space$ if and only if $cl(\{x\}) \subseteq \{x\}\ \forall\, x \in X$.*

Definition 4.3. A space (X, cl) is a $T_2 - space$ if and only if $\forall x, y \in X$, $x \neq y,\ \exists\, N' \in \mathscr{N}(x)$ and $N'' \in \mathscr{N}(y)$ such that $N' \cap N'' = \emptyset$.

The next lemma is a generalization of ([3, Lemma VII-1.2])

Lemma 4.1. *An isotonic space (X, cl) is a $T_2 - space$ if and only if for any two distinct points $x, y \in X$, there is $U \in \mathscr{N}(x)$ such that $y \notin cl(U)$.*

Proof. Let (X, cl) be an isotonic space and $x \neq y$ in X. Then (X, cl) is T_2 if and only if there are $U \in \mathscr{N}(x)$ and $V \in \mathscr{N}(y)$ such that $U \cap V = \emptyset$ if and only if there are $U \in \mathscr{N}(x)$ and $V \in \mathscr{N}(y)$ such that $V \subseteq X \setminus U$ if and only if there are $U \in \mathscr{N}(x)$ and $V \in \mathscr{N}(y)$ such that $y \in int(V) \subseteq int(X \setminus U)$ if and only if there exists $U \in \mathscr{N}(x)$ such that $X \setminus U \in \mathscr{N}(y)$ if and only if (by Theorem 1.1) there exists $U \in \mathscr{N}(x)$ such that $y \notin cl(U)$. $\square$

Definition 4.4. A space (X, cl) is a $T_{2\frac{1}{2}} - space$ if and only if $\forall x, y \in X, x \neq y$, $\exists N' \in \mathscr{N}(x)$ and $N'' \in \mathscr{N}(y)$ such that $cl(N') \cap cl(N'') = \emptyset$.

Theorem 4.1. *The lower separation axioms T_0, T_1, T_2 and $T_{2\frac{1}{2}}$ are topological properties in any isotonic space.*

Proof. Let (X, cl_X) and (Y, cl_Y) be isotonic spaces and $f : (X, cl_X) \to (Y, cl_Y)$ be a homeomorphism.

(T_0) If X is a $T_0 - space$, then for distinct points x and y in Y, we have $f^{-1}(x) \neq f^{-1}(y)$ in X, hence $f^{-1}(y) \notin cl_X(\{f^{-1}(x)\})$ or $f^{-1}(x) \notin cl_X(\{f^{-1}(y)\})$. Now, by Theorem 3.2, $f(f^{-1}(y)) = y \notin f(cl_X(\{f^{-1}(x)\})) = cl_Y(f(\{f^{-1}(x)\})) = cl_Y(\{x\})$, or $f(\{f^{-1}(x)\}) = x \notin f(cl_X(\{f^{-1}(y)\})) = cl_Y(f(\{f^{-1}(y)\})) = cl_Y(\{y\})$. Therefore by Proposition 4.1, Y is a T_0-space.

(T_1) If X is T_1 and $y \in Y$, then $f^{-1}(y) \in X$ and, by Proposition 4.2, $cl_X(\{f^{-1}(y)\}) \subseteq \{f^{-1}(y)\}$. Hence, by Theorem 3.2,

$$cl_Y(\{y\}) = cl_Y(f(\{f^{-1}(y)\})) = f(cl_X(\{f^{-1}(y)\})) \subseteq f(\{f^{-1}(y)\}) = \{y\},$$

and therefore by Proposition 4.2, Y is a T_1-space.

(T_2) If X is T_2 and $x, y \in Y$, $x \neq y$, we have $f^{-1}(x) \neq f^{-1}(y)$ in X. Hence there are $N' \in \mathscr{N}(f^{-1}(x))$ and $N'' \in \mathscr{N}(f^{-1}(y))$ such that $N' \cap N'' = \emptyset$. Since $f(N') \in \mathscr{N}(x)$ and $f(N'') \in \mathscr{N}(y)$ and $f(N') \cap f(N'') = \emptyset$, it follows that Y is a $T_2 - space$.

($T_{2\frac{1}{2}}$) Suppose that X is $T_{2\frac{1}{2}}$, $x, y \in Y, x \neq y$. Then there are $N' \in \mathscr{N}(f^{-1}(x))$ and $N'' \in \mathscr{N}(f^{-1}(y))$ such that $cl(N') \cap cl(N'') = \emptyset$. Now $f(N') \in \mathscr{N}(x)$, $f(N'') \in \mathscr{N}(y)$ and, by Theorem 3.2, $f(cl(N') \cap cl(N'')) = cl(f(N')) \cap cl(f(N'')) = \emptyset$. Therefore Y is a $T_{2\frac{1}{2}} - space$. □

5 Regular and Completely Regular Spaces

Definition 5.1. The function $v : X \to [0, 1]$ -where X is any space with a closure function and $[0, 1]$ is the closed unit interval in $\mathbb{R}$ with the usual topology- is called a *Urysohn function separating* A and B in X if v is continuous, $v(A) \subseteq \{0\}$ and $v(B) \subseteq \{1\}$. Two subsets A and B of X are *Urysohn separated* if there is a Urysohn function $v : X \to [0, 1]$ separating A and B. In this case, we write $A \parallel_v B$ or simply $A \parallel B$.

Definition 5.2. Let X be a space and $A, B \subseteq X$. We say that A is *completely within* B, and we write $A \Subset B$, if there is a continuous function $v : X \to [0,1]$ such that $v(A) \subseteq \{0\}$ and $v(X \setminus B) \subseteq \{1\}$.

By definition, we have $A \Subset B$ iff $A \parallel X \setminus B$. The question now is whether there is a relation between A and B whenever $A \Subset B$.

Lemma 5.1. *Let X be a space and $A, B \subseteq X$ such that $A \Subset B$. Then $A \subseteq B$.*

Proof. Let $A \Subset B$. Then there is a continuous function $v : X \to [0,1]$ such that $v(A) \subseteq \{0\}$ and $v(X \setminus B) \subseteq \{1\}$, which implies $A \subseteq v^{-1}(\{0\})$ and $X \setminus B \subseteq v^{-1}(\{1\})$. Now $v^{-1}(\{0\}) \subseteq X \setminus v^{-1}(\{1\}) \subseteq B$. Hence $A \subseteq B$. $\square$

Theorem 5.1. [13] *If (X, cl) is an isotonic space, then $A \Subset B$ implies $A \cup cl(A) \Subset int(B) \cap B$.*

Definition 5.3. An isotonic space (X, cl) is an *Urysohn space*, if any two distinct points $x, y \in X$ are Urysohn separated; that is, $\{x\} \parallel \{y\}$.

Lemma 5.2. [13] *Every isotonic, Urysohn space satisfies $(T_{2\frac{1}{2}})$.*

A separation condition stronger than T_2 is obtained by replacing one of the points by a closed set.

Definition 5.4. [12, 13] A space (X, cl) is *regular* if for all $x \in X$ and all nonempty $A \in \mathscr{P}(X)$ such that $x \notin cl(A)$, $\exists\ U \in \mathscr{N}(x)$ and $V \in \mathscr{N}(A)$ such that $U \cap V = \emptyset$. A space (X, cl) is a T_3-*space* if it is T_0 and *regular*.

Theorem 5.2. [12, 13] *In an isotonic space (X, cl), the following statements are equivalent:*

(i) *(X, cl) is a regular space.*

(ii) *For all $x \in X$ and all $N \in \mathscr{N}(x)$, $\exists\ U \in \mathscr{N}(x)$ such that $cl(U) \subseteq N$.*

Definition 5.5. A space (X, cl) is *completely regular* if for all $x \in X$ and all $N \in \mathscr{N}(x)$ there is $N' \in \mathscr{N}(x)$ such that $N' \Subset N$. A space (X, cl) is $T_{3\frac{1}{2}}$-space if it is T_1 and completely regular.

Lemma 5.3. *Every completely regular, isotonic space is regular, and hence $(T_{3\frac{1}{2}})$ implies (T_3).*

Proof. Let (X, cl) be a completely regular, isotonic space. Fix $x \in X$. Then for all $N \in \mathscr{N}(x)$ there is $U \in \mathscr{N}(x)$ such that $U \Subset N$. Now, by Theorem 5.1, we have $U \cup cl(U) \Subset int(N) \cap N$; so, by Lemma 5.1, $U \cup cl(U) \subseteq int(N) \cap N$, which implies $cl(U) \subseteq N$. Therefore, by Theorem 5.2, X is regular. □

Theorem 5.3. *Regularity, T_3, complete regularity and $T_{3\frac{1}{2}}$ are topological properties in any isotonic space.*

Proof. Let (X, cl_X) and (Y, cl_Y) be isotonic spaces and $f : (X, cl_X) \to (Y, cl_Y)$ be a homeomorphism.

(*Regularity*) Let X be a regular space, and let $y \in Y$, $A \subseteq Y$ be such that $y \notin cl_Y(A)$. Then $f^{-1}(y) \notin f^{-1}(cl_Y(A)) = cl_X(f^{-1}(A))$, and hence, by regularity of X, there are $U \in \mathscr{N}(f^{-1}(y))$ and $V \in \mathscr{N}(f^{-1}(A))$ such that $U \cap V = \emptyset$. Since $f(U) \in \mathscr{N}(y)$, $f(V) \in \mathscr{N}(A)$ and $f(U) \cap f(V) = \emptyset$, it follows that Y is a regular space.

(T_3) Immediate from (regularity) and Theorem 4.1.

(*Complete regularity*) Let X be a completely regular space, $y \in Y$ and $N \in \mathscr{N}_Y(y)$. Then $f^{-1}(N) \in \mathscr{N}_X(f^{-1}(y))$ which implies that there is $U \in \mathscr{N}_X(f^{-1}(y))$ such that $U \Subset f^{-1}(N)$. It is easy to show that $f(U) \in \mathscr{N}_Y(y)$ and $f(U) \Subset N$. Therefore Y is completely regular.

($T_{3\frac{1}{2}}$) Immediate from (complete regularity) and Theorem 4.1. □

6 Normal and Completely Normal Spaces

In the last section, we have discussed the separation of points, lower separation axioms, the separation of points from closed sets, regular and completely regular spaces. Carrying out this theme further, we may consider spaces in which disjoint closed sets can be separated by means of neighborhoods. Such spaces are called normal spaces.

Four kinds of normal spaces will be given in this section. Every one has a special name with a special definition; t-normal, quasi-normal, normal and Urysohn-normal spaces.

Definition 6.1. [13] *An isotonic space (X, cl) is*

(tN) *t-normal if any two non-empty disjoint closed sets* $A, B \subseteq X$ *are separated; that is, there are neighborhoods* $U \in \mathscr{N}(A)$ *and* $V \in \mathscr{N}(B)$ *such that* $U \cap V = \emptyset$*;*

(QN) *quasi-normal if, for all non-empty sets* $A, B \subseteq X$ *satisfying* $cl(A) \cap cl(B) = \emptyset$*, there are neighborhoods* $U \in \mathscr{N}(A)$ *and* $V \in \mathscr{N}(B)$ *such that* $U \cap V = \emptyset$*;*

(N) *normal if, for all non-empty sets* $A, B \subseteq X$ *satisfying* $cl(A) \cap cl(B) = \emptyset$*, there are neighborhoods* $U \in \mathscr{N}(cl(A))$ *and* $V \in \mathscr{N}(cl(B))$ *such that* $U \cap V = \emptyset$*;*

(UN) *Urysohn-normal if, for all non-empty sets* $A, B \subseteq X$ *satisfying* $cl(A) \cap cl(B) = \emptyset$*, there is a Urysohn function separating* A *and* B*,* $A \parallel B$*.*

Theorem 6.1. [13] *If* (X, cl) *is an isotonic space, then* $(UN) \Rightarrow (N) \Rightarrow (tN)$ *and* $(QN) \Rightarrow (tN)$*.*

Definition 6.2. [12] *An isotonic space* (X, cl) *is* (T_4) *if it satisfies* (T_1) *and* (QN)*.*

Definition 6.3. *An isotonic space* (X, cl) *is* (T_4U) *if it satisfies* (T_1) *and* (UN)*.*

Definition 6.4. In an isotonic space (X, cl), two subsets $A, B \subseteq X$ are called *semi-separated* if $cl(A) \cap B = A \cap cl(B) = \emptyset$.

Lemma 6.1. [13] *In an isotonic space* (X, cl)*, the following two conditions are equivalent for all* $A, B \subseteq X$*:*

(SS) A, B *are semi-separated.*

(SS') *There are* $U \in \mathscr{N}(A)$ *and* $V \in \mathscr{N}(B)$ *such that* $A \cap V = U \cap B = \emptyset$.

Lemma 6.2. [12] *Let* (X, cl_X) *and* (Y, cl_Y) *be spaces and let* $f : (X, cl_X) \to (Y, cl_Y)$ *be a continuous function. If* $A, B \subseteq Y$ *are semi-separated, then* $f^{-1}(A)$ *and* $f^{-1}(B)$ *are semi-separated in* X*.*

Definition 6.5. [13] *A space* (X, cl) *is*

(CN) *completely normal if any two semi-separated sets are separated;*

(T_5) *if it is* (T_1) *and completely normal;*

(CUN) *completely Urysohn-normal if each pair of semi-separated sets is Urysohn-separated;*

(T_5U) *if it is* (T_1) *and completely Urysohn-normal.*

Lemma 6.3. *In isotonic spaces,* (CUN) *implies* (CN).

Proof. Assume that (X, cl) is a (CUN) isotonic space, and let $A, B \subseteq X$ be semi-separated sets. Then there is a continuous function $\nu : X \to [0,1]$ such that $\nu(A) \subseteq \{0\}$ and $\nu(B) \subseteq \{1\}$. Choose $x \in A$ and $\epsilon > 0$ such that $[0,\epsilon) \in \mathscr{N}(\nu(x))$. Then there is $N_x \in \mathscr{N}(x)$ such that $N_x = \nu^{-1}([0,\epsilon))$; so that $\nu(N_x) \subseteq [0,\epsilon)$. Define $U := \bigcup\{N_x \in \mathscr{N}(x) \mid x \in A\}$. Then, by (N1), $U \in \mathscr{N}(x)\, \forall\, x \in A$, and hence $U \in \mathscr{N}(A)$ and $\nu(U) \subseteq [0,\epsilon]$. A similar argument yields $V \in \mathscr{N}(B)$ such that $\nu(V) \subseteq [1-\epsilon, 1]$. Setting $\epsilon < \frac{1}{2}$ implies $U \cap V = \emptyset$. Therefore X is completely normal. □

Theorem 6.2. (tN)*,* (QN)*,* (N)*,* (UN)*,* (T_4)*,* (T_4U)*,* (CN)*,* (T_5)*,* (CUN) *and* (T_5U) *are topological properties in any isotonic space.*

Proof. Let (X, cl_X) and (Y, cl_Y) be isotonic spaces and $f : (X, cl_X) \to (Y, cl_Y)$ be a homeomorphism.

(tN) Let X be a $t-normal$ space, and let A, B be disjoint closed subsets of Y. Now it is easy to show that $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed subsets of X. Since X is $t-normal$, there exist $U \in \mathscr{N}_X(f^{-1}(A))$ and $V \in \mathscr{N}_X(f^{-1}(B))$ such that $U \cap V = \emptyset$. So by Proposition 3.1, $f(U) \in \mathscr{N}_Y(A)$ and $f(V) \in \mathscr{N}_Y(B)$. Since $f(U) \cap f(V) = \emptyset$, it follows that Y is a $t-normal$ space.

(QN) Let X be a QN-space, and let A, B be nonempty subsets of Y such that $cl_Y(A) \cap cl_Y(B) = \emptyset$. Then it is easy to see (using Theorem 3.2) that $cl_X(f^{-1}(A)) \cap cl_X(f^{-1}(B)) = \emptyset$. Since X is (QN), there exist $U \in \mathscr{N}_X(f^{-1}(A))$ and $V \in \mathscr{N}_X(f^{-1}(B))$ such that $U \cap V = \emptyset$. Since, by Remark 3.1, $f(U) \in \mathscr{N}_Y(A)$ and $f(V) \in \mathscr{N}_Y(B)$, and since $f(U) \cap f(V) = \emptyset$, it follows that Y is a QN-space.

(N) Let X be an N-space, and let A, B be nonempty subsets of Y such that $cl_Y(A) \cap cl_Y(B) = \emptyset$. Then, as noted above, $cl_X(f^{-1}(A)) \cap cl_X(f^{-1}(B)) = \emptyset$. Since X is an $N-$space, there exist $U \in \mathscr{N}_X(cl_X(f^{-1}(A)))$ and $V \in \mathscr{N}_X(cl_X(f^{-1}(B)))$ such that $U \cap V = \emptyset$. By Remark 3.1, we have $f(U) \in \mathscr{N}_Y(f(cl_X(f^{-1}(A))))$ and $f(V) \in \mathscr{N}_Y(f(cl_X(f^{-1}(B))))$. Since, by Theorem 3.2 (applied to the homeomorphism f^{-1}), $cl_Y(A) = f(cl_X(f^{-1}(A)))$

and $cl_Y(B) = f(cl_X(f^{-1}(B)))$, it follows that $f(U) \in \mathscr{N}_Y(cl_Y(A))$ and $f(V) \in \mathscr{N}_Y(cl_Y(B))$. Since $f(U) \cap f(V) = \emptyset$, it follows that Y is an $N-$space.

(UN) Let X be an $UN-$space, and let A, B be nonempty subsets of Y such that $cl_Y(A) \cap cl_Y(B) = \emptyset$. Then $cl_X(f^{-1}(A)) \cap cl_X(f^{-1}(B)) = \emptyset$. Since X is a $UN-$space, there exists a continuous function $\nu : X \to [0, 1]$ such that $\nu(f^{-1}(A)) \subseteq \{0\}$ and $\nu(f^{-1}(B)) \subseteq \{1\}$. Let $\mu := \nu \circ f^{-1}$. Then $\mu : Y \to [0, 1]$ is continuous, $\mu(A) = \nu(f^{-1}(A)) \subseteq \{0\}$ and $\mu(B) = \nu(f^{-1}(B)) \subseteq \{1\}$. Hence, $A \| B$, and therefore Y is a $UN-$space.

(T_4) Immediate from (QN) and Theorem 4.1.

(T_4U) Immediate from (UN) and Theorem 4.1.

(CN) Let X be *completely normal* space, and let $A, B \subset Y$ be semi-separated. Since f is continuous, by Lemma 6.2, $f^{-1}(A)$ and $f^{-1}(B)$ are semi-separated in X. So, as X is (CN), $f^{-1}(A)$ and $f^{-1}(B)$ are separated in X. Hence, there exist $U \in \mathscr{N}_X(f^{-1}(A)))$, $V \in \mathscr{N}_X(f^{-1}(B))$, such that $U \cap V = \emptyset$. Now by continuity of f^{-1} and Remark 3.1, we have $f(U) \in \mathscr{N}_Y(A)$, $f(V) \in \mathscr{N}_Y(B)$, and $f(U) \cap f(V) = \emptyset$. Thus A, B are separated, and therefore Y is (CN).

(T_5) Immediate from (CN) and Theorem 4.1.

(CUN) Let X be an $CUN-$space, and let $A, B \subseteq Y$ be semi-separated. Then, by Lemma 6.2, $f^{-1}(A)$ and $f^{-1}(B)$ are semi-separated in X. So, as X is (CUN), $f^{-1}(A)$ and $f^{-1}(B)$ are Urysohn-separated in X. Hence, there exists a continuous function $\nu : X \to [0, 1]$ such that $\nu(f^{-1}(A)) \subseteq \{0\}$ and $\nu(f^{-1}(B)) \subseteq \{1\}$. Let $\mu := \nu \circ f^{-1}$. Then $\mu : Y \to [0, 1]$ is continuous, $\mu(A) = \nu(f^{-1}(A)) \subseteq \{0\}$ and $\mu(B) = \nu(f^{-1}(B)) \subseteq \{1\}$. Thus, $A \| B$, and therefore Y is (CUN).

(T_5U) Immediate from (CUN) and Theorem 4.1. □

7 Connectedness in Isotonic Spaces

Definition 7.1. [12] A set $Y \in \mathscr{P}(X)$ is *connected* in a space (X, cl) if it is not a disjoint union of a nontrivial semi-separated pair of sets $A, Y \setminus A$, $A \neq \emptyset, Y$. We say that a space (X, cl) is *connected* if X is connected in (X, cl).

Definition 7.2. Let (X, cl) be a space and $x \in X$. The *component* $C(x)$ of x in X is the union of all connected subsets of X containing x.

Definition 7.3. A space (X, cl) is *totally disconnected* if for each $x \in X$ the component $C(x) = \{x\}$.

Theorem 7.1. [7] *If $f : (X, cl_X) \rightarrow (Y, cl_Y)$ is a continuous function between isotonic spaces, and X is connected, then $f(X)$ is connected in Y.*

Definition 7.4. [1] Let (Z, cl) be an isotonic space with more than one element. An isotonic space (X, cl) is called *Z-connected* if and only if any continuous function $f : X \rightarrow Z$ is constant.

Lemma 7.1. [7] *Any continuous image of a Z-connected isotonic space is Z-connected.*

Definition 7.5. A space (X, cl) is *strongly connected* if there is no countable collection of pairwise semi-separated sets $\{A_i\}$ such that $X = \bigcup A_i$.

Lemma 7.2. [7] *A continuous image of a strongly connected isotonic space is strongly connected.*

Theorem 7.2. *Connectedness, total disconnectedness, Z-connectedness, and strong connectedness are topological properties in any isotonic space.*

Proof.

($Connectedness$) Immediate from Theorem 7.1.

($Total\ disconnectedness$) Let (X, cl_X) be a totally disconnected isotonic space, (Y, cl_Y) be an isotonic space, and $f : (X, cl_X) \rightarrow (Y, cl_Y)$ be a homeomorphism. Assume that (Y, cl_Y) is not totally disconnected, so for some $y \in Y$, $\exists\ A \subseteq Y$ a connected set, such that $\{y\} \subseteq A$. Now $f^{-1}(A)$ is connected in X, and $f^{-1}(\{y\}) \subseteq f^{-1}(A)$. Since X is connected, we have $f^{-1}(\{y\}) = f^{-1}(A)$. Therefore Y is totally disconnected.

($Z - connectedness$) Immediate from Lemma 7.1.

($Strong\ connectedness$) Immediate from Lemma 7.2. □

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