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σ - Quasi Centralizers and Inner Derivations in a Closed Ideal of a Complex Banach Algebra.

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Abstract: *In this paper we show that, for a closed ideal J of a unital complex Banach algebra A and for a σ -quasi centralizer element a of J in A we have (i) under certain conditions if b is an element in the center of J and $\pi : J \to BL(X)$ is an irreducible representation of J on the Banach space X, then π(ba) is a scalar operator. (ii) If $\sigma_A(a)$ has empty interior and D_a^J is the restriction of the inner derivation of a to J then $(D_a^J)^3 = 0$.*

1. Introduction.

This paper is a continuation of our study in [1], [2], and [3] of quasi centralizers and inner derivations in a closed ideal of a complex Banach algebra, where here we focus our study in the case of σ-quasi centralizers. As in [2], and [3], in this paper we see that some results of Rennison in [7] remain true whenever the σ-quasi centrality conditions with respect to all the elements in the algebra given by Rennison is replaced by the same σ-quasi centrality conditions with respect to all the elements in a closed ideal.

Throughout this paper all linear spaces and algebras are assumed to be defined over ¢ , the field of complex numbers. **Let A be any complex normed algebra. Then we denote the center of A by Z(A) = { a∈A : ax = xa for all x ∈ A }, and the centralizer of a subset B of A by C(B) = { a∈A : ax = xa for all x∈B }. For a∈A, the spectrum in A of a will be denoted by $\sigma_A(a)$ and is defined by $\sigma_A(a)$ = { λ∈ ¢ : $(\lambda - a)^{-1}$ does not exist }. The resolvent set, its complement in ¢ , will be denoted by $\rho_A(a)$.**

In [7] Rennison defined the set of all σ-quasi central elements in a complex Banach algebra A by $Q_\sigma(A) = \bigcup_{k\geq 1} Q_\sigma(k, A)$, where $Q_\sigma(k, A)$ = { a∈A: $\| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all x∈A and all $\lambda \in \rho_A(a)$ }.

Similarly, in [1], for a subset B of a complex normed algebra A, we defined the σ-quasi centralizer of B is $QC_\sigma(B) = \bigcup_{k\geq 1} QC_\sigma(k,B)$, where $QC_\sigma(k, B) = \{ a\in A: \| x(\lambda - a)\| \leq k \| (\lambda - a)x \|$ for all $x\in B$ and all $\lambda\in \rho_A(a) \}$.

1.1 Theorem. [5, Theorem 7 p128]
Let π be an irreducible representation of a Banach algebra A on a normed space X such that $\pi(a) \in BL(X)$ and $a \in A$. Then π is continuous.

1.2 Lemma. [4, Lemma 5, p3]
Let A be a Banach algebra. If $x \in A$ and μ is a boundary point of $\sigma_A(a)$, then $(\lambda - a)^{-1}$ is unbounded as $\lambda \to \mu$ through $\rho_A(a)$.

1.3 Proposition. " Schur's lemma " [5, proposition 6 p121]
Let A be an algebra and X be an irreducible left A-module. Then $\{ T \in BL(X) : aTx = T(ax)$ for all $a \in A$ and all $x \in X \}$ is a division subalgebra of BL(X) containing the identity.

1.4 Theorem. [1, Theorem 2.1]
If A is a complex normed algebra and $D \subseteq B \subseteq A$, then for $k \geq 1$,

i) $C(B) \subseteq QC(k, B) = QC\sigma(k, B) \cap QC\rho(k, B)$.

(ii) $Q(k, A) = QC(k, A) \subseteq QC(k, B) \subseteq QC(k, D)$.

(iii) $Q\sigma(k, A) = QC\sigma(k, A) \subseteq QC\sigma(k, B) \subseteq QC\sigma(k, D)$.

(iv) $Q\rho(k, A) = QC\rho(k, A) \subseteq QC\rho(k, B) \subseteq QC\rho(k, D)$.

2. The Results.

If A is a complex normed algebra and $a\in A$, then the inner derivation corresponding to a is denoted by D_a, which is a bounded linear operator on A defined by $D_a x = ax - xa$. For J an ideal of A, we will use the symbol D_a^J, to denote the restriction of D_a to J.

In this section we generalize some results related to σ-quasi centrality in a complex Banach algebra that was obtained by Rennison in [7], where we see that some of these results remain true whenever the σ-quasi centrality conditions with respect to all the elements in the algebra given by Rennison is replaced by the same σ-quasi centrality conditions with respect to all the elements in a closed ideal. Hence some of Rennison's results becomes corollaries of our results. However, our proofs of these results are similar to that proofs of Rennison. We begin by the following Lemma:

2.1 Lemma. *Let J be an ideal of a unital complex normed algebra A. If $a \in QC_{\sigma}(k, J)$, then for all $x \in J$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \rho_A(a)$ we have,*

$$\| (\lambda_1 - a)^{-1} (\lambda_2 - a)^{-1} \cdots (\lambda_n - a)^{-1} (D_a^J)^n x \| \leq (k+1)^n \| x \|.$$

Proof: *By induction on n.*

The case in which $n = 1$ can be seen as follow:

Since $D_a^J x = x(\lambda - a) - (\lambda - a)x$, then $(\lambda - a)^{-1} D_a^J x = (\lambda - a)^{-1} x(\lambda - a) - x$ for all $x \in J$ and all $\lambda \in \rho_A(a)$. However, $(\lambda - a)^{-1} x$ belongs to the ideal J and $a \in QC_{\sigma}(k, J)$, then

$$\begin{aligned} \| (\lambda - a)^{-1} D_a^J x \| &\leq \| (\lambda - a)^{-1} x(\lambda - a) \| + \| x \| \\ &\leq k \| (\lambda - a)(\lambda - a)^{-1} x \| + \| x \| \\ &= (k+1) \| x \| \quad \ldots\ldots\ldots\ldots (1) \end{aligned}$$

Now, suppose that the result is true for $n = m \geq 1$, $x \in J$ and $\lambda_1, \lambda_2, \ldots, \lambda_{m+1} \in \rho_A(a)$. Since both of a and $(\lambda - a)$ commutes with $(\lambda_i - a)^{-1}$ for $i = 1, 2, \ldots, m$ and both $(D_a^J)^m x$ and $[(\lambda_1 - a)^{-1} (\lambda_2 - a)^{-1} \cdots (\lambda_m - a)^{-1} (D_a^J)^m x]$ belong to the ideal J, then one can use (1) and the hypothesis of induction to see that

$$\begin{aligned} &\| (\lambda_1 - a)^{-1} (\lambda_2 - a)^{-1} \cdots (\lambda_{m+1} - a)^{-1} (D_a^J)^{m+1} x \| \\ &= \| (\lambda_{m+1} - a)^{-1} (\lambda_{m+1} - a)(\lambda_1 - a)^{-1} (\lambda_2 - a)^{-1} \cdots (\lambda_{m+1} - a)^{-1} D_a^J (D_a^J)^m x \| \\ &= \| (\lambda_{m+1} - a)^{-1} (\lambda_1 - a)^{-1} (\lambda_2 - a)^{-1} \cdots (\lambda_m - a)^{-1} [a (D_a^J)^m x - (D_a^J)^m xa] \| \\ &= \| (\lambda_{m+1} - a)^{-1} [a(\lambda_1 - a)^{-1} (\lambda_2 - a)^{-1} \cdots (\lambda_m - a)^{-1} (D_a^J)^m x - (\lambda_1 - a)^{-1} (\lambda_2 - a)^{-1} \cdots (\lambda_m - a)^{-1} (D_a^J)^m xa] \| \\ &= \| (\lambda_{m+1} - a)^{-1} D_a^J [(\lambda_1 - a)^{-1} (\lambda_2 - a)^{-1} \cdots (\lambda_m - a)^{-1} (D_a^J)^m x] \| \\ &\leq (k+1) \| (\lambda_1 - a)^{-1} (\lambda_2 - a)^{-1} \cdots (\lambda_m - a)^{-1} (D_a^J)^m x \| \quad (\text{by } (1)) \end{aligned}$$

$\leq$ (k + 1)(k + 1) ^{m} $\|x\|$ = (k + 1) $^{m+1}$ $\|x\|$. Therefore, the result is true for n = m + 1. Hence the induction is completed and the proof is complete □

Hence by using the above Lemma, we get the following Corollary that appears as Lemma 3.1 in [7].

2.2 Corollary. [7, Lemma 3.1]
Let A be a unital complex normed algebra. If $a \in Q_\sigma(k, A)$, then for all $x \in A$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \rho_A(a)$ we have,
$\| (\lambda_1 - a)^{-1} (\lambda_2 - a)^{-1} \ldots\ldots (\lambda_n - a)^{-1} (D_a)^n x \| \leq (k+1)^n \|x\|$.

Proof:
Use Theorem 1.4 (iii) and Lemma 2.1 with J = A to get the result □

2.3 Proposition.
Let J be a closed ideal of a unital complex Banach algebra A, $a \in QC_\sigma(k, J)$, and $b \in Z(J)$ *such that ab = ba. If* $\pi : J \to BL(X)$ *is an irreducible representation of J on the Banach space X, then* $Y = \{ \xi \in X : b(\lambda - a)^{-1} \xi$ *is bounded for all* $\lambda \in \rho_A(a) \}$ *is a submodule containing* $D_a^J x\, \xi$ for all $x \in J$ and all $\xi \in X$.

Proof:
Regard X as an irreducible left –J module with the module multiplication given by $x . \xi = \pi(x)\xi$ for $x \in J$ and $\xi \in X$. Since J is a closed ideal of a unital Banach algebra A, then by Theorem 1.1, π is continuous. So that there exists $C \geq 0$ such that $\| x\xi \| \leq C \|x\| \|\xi\|$ for all $x \in J$ and all $\xi \in X$. However, $a \in QC_\sigma(k, J)$, then by Lemma 2.1

$\| b(\lambda - a)^{-1} D_a^J x\, \xi \| \leq C \|b\| \| (\lambda - a)^{-1} D_a^J x \| \|\xi\|$
$\leq C (k + 1) \|x\| \|\xi\| \|b\|$ for all $x \in J$, $\xi \in X$

and $\lambda \in \rho_A(a)$. Hence $D_a^J x\, \xi \in Y$, and if $\xi, \gamma \in Y$ and $\mu \in ¢$, then $b(\lambda - a)^{-1} [\mu\xi + \gamma] = \pi(b(\lambda - a)^{-1})[\mu\xi + \gamma] = \mu\,\pi(b(\lambda - a)^{-1})\xi + \pi(b(\lambda - a)^{-1})\gamma = \mu\, b(\lambda - a)^{-1}\xi + b(\lambda - a)^{-1}\gamma$ which is bounded for all $\lambda \in \rho_A(a)$. Therefore, Y is a subspace of X.

Now, let $y \in J$ and $\xi \in Y$ be arbitrary. By using the continuity of π and the assumptions that $a \in QC_\sigma(k, J)$, and $b \in Z(J)$ such that $ab = ba$ we get that there is $M k \geq 0$ such that $\| b(\lambda - a)^{-1} y \xi \| = \| (\lambda - a)^{-1} y (\lambda - a) b (\lambda - a)^{-1} \xi \|$
$\leq M \| (\lambda - a)^{-1} y (\lambda - a) \| \| b(\lambda - a)^{-1}\xi \| \leq kM \| y \| \| b(\lambda - a)^{-1}\xi \|$.
However, $b(\lambda - a)^{-1} \xi$ is bounded for all $\lambda \in \rho_A(a)$, therefore, $b(\lambda - a)^{-1} y \xi$ is bounded. Hence $y\xi \in Y$. Therefore, Y is a submodule □

2.4 Theorem.
Let J be a closed ideal of a unital complex Banach algebra A. If $a \in QC_\sigma(k, J)$, $b \in Z(J)$, *ab = ba and* $\pi : J \to BL(X)$ *is an irreducible representation of J on the Banach space X such that ,* $\sigma_{BL(X)}(\pi ba) \cap \partial\sigma_A(a) \neq \varphi$ and $(\lambda - \pi ba)^{-1} = \pi(b)(\lambda - \pi a)^{-1}$ *for all* $\lambda \in \rho_A(a)$*, then* $\pi(ba)$ *is a scalar operator.*

Proof:

Since $\sigma_{BL(X)}(\pi ba) \cap \partial\sigma_A(a) \neq \varphi$, then by using Lemma 1.2 we have that $(\lambda - \pi ba)^{-1}$ is unbounded in $BL(X)$ for all $\lambda \in \rho_A(a)$ and so is $\pi(b)(\lambda - \pi a)^{-1}$. Let Y be as in proposition 2.3. To show that $Y \neq X$, suppose on the contrary that $Y = X$, then for any $\lambda \in \rho_A(a)$, $b(\lambda - a)^{-1} \xi$ is bounded for every $\xi \in X$, and so $\pi(b)(\lambda - \pi a)^{-1}\xi$ is bounded. By uniform boundedness Theorem $\pi(b)(\lambda - \pi a)^{-1}$ is bounded in $BL(X)$ for all $\lambda \in \rho_A(a)$ which is a contradiction. Therefore $Y \neq X$. However, π is irreducible, then the only submodule of X is $\{0\}$ and X itself, but by Proposition 2.3 Y is a submodule of X, hence $Y = \{0\}$.
Again by Proposition 2.3, we have $D_a^J x \xi \in Y$ for all $x \in J$ and $\xi \in X$, but $bx \in J$, then $D_a^J bx \xi \in Y$. Hence $0 = D_a^J bx \xi = (abx - bxa) \xi = \pi(abx - bxa) \xi = \pi(bax - xba) \xi$. In other words, $\pi ba \, \pi x = \pi x \, \pi ba$ for all $x \in J$. Therefore, $\pi ba \in Z(\pi(J))$. Finally, by using Schur's Lemma (Proposition 1.3) we get πba is a scalar operator □

Hence by using the above Theorem, we get the following Corollary that appears as Theorem 3.3 in [7].

2.5 Corollary. [7, Theorem 3.3]

Suppose that a is σ-quasi central *of a unital complex Banach algebra A and that* $\pi : A \rightarrow BL(X)$ is *an irreducible representation of A on the Banach space X such that,* $\sigma_{BL(X)}(\pi a) \cap \partial\sigma_A(a) \neq \varphi$. *Then* $\pi(a)$ *is a scalar operator.*

Proof:

Use Theorem 1.4 (iii) and Theorem 2.3 with J = A and b = e to get the result □

2.6 Theorem.

Let J be a closed ideal of a unital complex Banach algebra A and $a \in QC_\sigma(J)$. *If* $\sigma_A(a)$ has empty interior then $(D_a^J)^3 = 0$.

Proof:

For any fixed $x \in J$, define $f : \rho_A(a) \rightarrow J$ by $f(\lambda) = (\lambda - a)^{-1} (D_a^J)^3 x$. Then for all $\lambda \in \rho_A(a)$, (by Lemma 2.1) we have, $\| f(\lambda) \| = \| (\lambda - a)^{-1} (D_a^J)^3 x) \| \leq (k+1) \| (D_a^J)^2 x \|$. Hence f is bounded on $\rho_A(a)$.

Let $\lambda, \mu \in \rho_A(a)$. Then as in the proof of Theorem 2.1 in [2], $(\mu - \lambda)(\lambda - a)^{-1}(\mu - a)^{-1} = [(\mu - a) - (\lambda - a)](\lambda - a)^{-1}(\mu - a)^{-1} = (\lambda - a)^{-1} - (\mu - a)^{-1}$, and

$$\frac{df}{d\lambda}(\mu) = \lim_{\lambda \rightarrow \mu} \frac{f(\lambda) - f(\mu)}{\lambda - \mu}$$

$$= \lim_{\lambda \rightarrow \mu} \frac{[(\lambda - a)^{-1} - (\mu - a)^{-1}](D_a^J)^3 x}{\lambda - \mu}$$

$$= \lim_{\lambda \rightarrow \mu} \frac{(\mu - \lambda)(\lambda - a)^{-1} (\mu - a)^{-1} (D_a^J)^3 x)}{\lambda - \mu}$$

$= -(\mu - a)^{-2}(D_a^J)^3 x$. Hence f is analytic on $\rho_A(a)$.

For all $\lambda, \mu \in \rho_A(a)$, (by Lemma 2.1) we have,

$$\| f(\lambda) - f(\mu) \| = \| [(\lambda - a)^{-1} - (\mu - a)^{-1}] (D_a^J)^3 x \|$$
$$= \| (\mu - \lambda)(\lambda - a)^{-1}(\mu - a)^{-1}(D_a^J)^3 x \|$$
$$\leq (k+1)^2 \, | \lambda - \mu | \, \| (D_a^J)^3 x \|.$$

Hence f is uniformly Lipschitz on $\rho_A(a)$. Therefore, f is bounded analytic and uniformly Lipschitz on $\rho_A(a)$. However $\sigma_A(a)$ has

empty interior, then f has a unique extension to the whole complex plane.
For all distinct points λ, ξ, μ ∈ ρ_A (a), (by Lemma 2.1) we have,

$$\left\| \frac{f(\lambda) - f(\mu)}{\lambda - \mu} - \frac{f(\xi) - f(\mu)}{\xi - \mu} \right\| =$$

$$\left\| \frac{(\lambda - a)^{-1} - (\mu - a)^{-1}}{\lambda - \mu} - \frac{(\xi - a)^{-1} - (\mu - a)^{-1}}{\xi - \mu} (D_a^J)^3 x \right\| = \| [-(\lambda - a)^{-1}(\mu - a)^{-1} + (\xi - a)^{-1}(\mu - a)^{-1}] (D_a^J)^3 x \|$$

$$= \| (\mu - a)^{-1} [(\xi - a)^{-1} - (\lambda - a)^{-1}] (D_a^J)^3 x \|$$

$$= |\lambda - \xi| \, \| (\mu - a)^{-1} (\lambda - a)^{-1} (\xi - a)^{-1} (D_a^J)^3 x \| \leq (k+1)^3 |\lambda - \xi| \, \| x \|.$$

Since f has a unique extension to the whole complex plane, then the above inequality holds for all complex numbers λ, ξ, and μ. This implies that f extend to be differentiable at each complex number μ; that is this extension is entire hence f must be constant.

As in the proof Theorem 2.1 in [2], let λ∈ ¢ such that $|\lambda| > \|a\|$, then by [6, p. 398], $(\lambda - a)^{-1} = \sum_{n=0}^{\infty} a^n \lambda^{-n-1}$, and so $f(\lambda) = \sum_{n=0}^{\infty} a^n \lambda^{-n-1} (D_a^J)^3 x$. But f is constant, therefore, $(D_a^J)^3 x = 0$, which implies $(D_a^J)^3 = 0$ since x was arbitrary in J □

Hence by using the above Theorem, we get the following Corollary that appears as Theorem 3.7 (iii) in [7].

2.7 Corollary. [7, Theorem 3.7(iii) **]**
Suppose that a is σ-quasi central *of a unital complex Banach algebra A and that $\sigma_A(a)$* has empty interior then $D_a^3 = 0$.

Proof:
Use Theorem 1.4 (iii) and Theorem 2.6 with J = A to get the result□

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