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The Asymptotic Distribution of the Estimated Conditional Mode at a Finite Number of Distinct Points Under Dependence Conditions

Raid Salha[1] and Dimitris Ioannides[2]
[1]Department of Mathematics, Islamic University of Gaza
e-mail: rbsalha@mail.iugaza.edu
[2]Department of Economics University of Macedonia, Greece
e-mail:dimioan@uom.gr

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Abstract

The problem of estimating the mode of a conditional pdf based on a sample of i.i.d. random variables $(X_1, Y_1), \ldots, (X_n, Y_n)$, with joint pdf $f(x, y)$, has been considered by Samanta and Thavaneswaram [7]. They have shown under some regularity conditions that, the estimate of the conditional mode, obtained by maximizing a kernel estimate of the conditional density is strongly consistent and asymptotically normally distributed.

The result is applied to the problem of estimating the conditional mode evaluated at a finite number of distinct continuity points of f, based on a sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ of a strictly stationary process satisfying a weak dependence condition.

Keywords: *Kernel estimation, conditional distribution, conditional mode, multivariate distribution, strong mixing, stochastic process, large sample theory.*

ملخص: : درس [7] Samanta and Thavaneswaram مشكلة تقدير منوال دالة الكثافة الإحتمالية المشروطة المعتمدة على عينة من متغيرات عشوائية $(X_1,Y_1),....,(X_n,Y_n)$ مستقلة ولها دالة كثافة إحتمالية مشتركة $f(x,y)$. تحت شروط معينة أثبتوا أن تقدير kernel للمنوال الشرطى ذو تناسق قوى و يتقـارب لتوزيـع طبيعى.

في هذا البحث نطبق هذه النتيجة على مشكلة تقدير المنوال المشروط مقدراً عند عدد محدود من نقاط اتصال مختلفة للدالة f معتمدة على عينة $(X_1,Y_1),....,(X_n,Y_n)$ من عملية متوقفة قطعاً تحقق شرط تبعية ضعيف.

1 Introduction

Assume that $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ a sample of i.i.d. random variables with joint pdf $f(x, y)$. The marginal pdf of X_1 is $g(x) = \int_{-\infty}^{\infty} f(x, y)dy$, and the conditional pdf of Y_1 given $X_1 = x$ is $f(y|x) = \frac{f(x,y)}{g(x)}$. Assume that for each x, $f(y|x)$ is uniformly continuous in y and it follows that $f(y|x)$ possesses a mode $M(x)$, assume that $M(x)$ is unique, defined by

$$f(M(x)|x) = \max_{-\infty<y<\infty} f(y|x).$$

Let K be a Borel function that satisfies some properties mention in assumption A(2), and $\{h_n\}$ be a sequence of positive numbers converging to zero. Consider the following estimates of $f(x, y)$, $g(x)$, and $f(y|x)$:

$$f_n(x, y) = (nh_n^2)^{-1} \sum_{i=1}^{n} K(\frac{x - X_i}{h_n})K(\frac{y - Y_i}{h_n}),$$

$$g_n(x) = (nh_n)^{-1} \sum_{i=1}^{n} K(\frac{x - X_i}{h_n}), \quad f_n(y|x) = \frac{f_n(x, y)}{g_n(x)}.$$

For every sample sequence and for each x, $f_n(y|x)$ is a continuous function of y and tends to zero as y tends to $\pm\infty$. In sequentially, there is a random variable $M_n(x)$ such that

$$f_n(M_n(x)|x) = \max_{-\infty<y<\infty} f_n(y|x). \tag{1}$$

Samanta and Thavaneswaran [7] considered $M_n(x)$ as an estimate of $M(x)$ and established conditions under which the estimate is strongly consistent and asymptotically normally distributed.

In this paper, we study the joint asymptotic normality of the estimated conditional mode, proposed by Samanta and Thavaneswaram [7], evaluated at a finite number of distinct points, when the sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ is taken from a strictly stationary process satisfying a weak dependence condition. We assume that the stochastic process (X_t, Y_t), $t \in \mathbf{Z}$ satisfies the strong mixing ($\alpha-$mixing) condition, see Rousses and Ioanides [5].

2 Assumptions

(A1) $(X_1,Y_1),\ldots,(X_n,Y_n)$, is a sample of a strictly stationary process and strongly mixing with mixing coefficient $\alpha(n)$, and joint pdf $f(x,y)$, where the following are hold,

(i) the marginal pdf of X, $g(x)$ is uniformly continuous.

(ii) $f^{(i,j)}(x,y)=\dfrac{\partial^{i+j}f(x,y)}{\partial x^i\partial y^j}$ exist and are bounded for $1\leq i+j\leq 4$.

(A2) The kernel K is a Borel function and satisfies the following

(i) $K(u)$ tends to zero as u tends to $\pm\infty$.

(ii) $K(u)$ and its first two derivatives are functions of bounded variation.

(iii) $\lim\limits_{|u|\to\infty}|u^2K^{(i)}(u)|=0, \quad (i=0,1)$.

(iv) $\displaystyle\int_{-\infty}^{\infty}u^iK(u)du=1, \quad i=0(=0,\text{if } i=1,2)$.

(v) $\displaystyle\int_{-\infty}^{\infty}|u|^3K(u)du<\infty$.

(A3) h_n is a sequence of positive numbers tending to zero, and satisfies the following $\lim\limits_{n\to\infty}nh_n^8=\infty$ and $\lim\limits_{n\to\infty}nh_n^{10}=0$.

(A4) There exist a sequence of positive constants c_n, such that

(i) $c_n=o(h_n^{-4})$.

(ii) $\lim\limits_{n\to\infty}h_n^{-2(1-\gamma)}\sum\limits_{l=c_n}^{\infty}\alpha^{(1-\gamma)}(l)=0$, for some $\gamma\in(0,1)$.

(A5) For all $i\geq 2$, the random variables $Z_1=(X_1,Y_1)$ and $Z_i=(X_i,Y_i)$ have a joint pdf f_{Z_1,Z_i} with respect to lebesgue measure such that

$$|f_{Z_1,Z_i}(u,v,w,z)-f(u,v)f_(w,z)|<C,$$

for all $(u,v),(w,z)$ in $\mathbf{R}^2$, where C is a constant.

(A6) let $p = p(n), q = q(n)$ be positive numbers with $p + q \leq n$ for all sufficiently large n and tending to infinity, and let k be the integral part of $n/(p+q)$, where the following hold,

(i) $\lim_{n\to\infty} \frac{qk}{n} = 0$.

(ii) $\lim_{n\to\infty} \frac{p^2}{nh_n^2} = 0$.

(iii) $\lim_{n\to\infty} k\alpha(q) = 0$.

For example, $f(x,y) = \frac{15}{32}xy^2, 0 < y < x < 2$, $g(x) = \frac{5}{32}x^4$ satisfy assumption A(1), while assumption A(2) is satisfied if we choose

$$K(u) = \frac{15}{8}\left(1 - \frac{2}{3}u^2 + \frac{1}{15}u^4\right)(1/\sqrt{2\pi})e^{-u^2/2}, -\infty < u < \infty.$$

For more details see Samanta and Thavaneswaran [7].

For notational convenience, set

$$\frac{\partial^j f_n(x,y)}{\partial y^j} = f_n^{(0,j)}(x,y) = (nh_n^{j+2})^{-1}\sum_{i=1}^{n} K(\frac{x-X_i}{h_n})K^{(j)}(\frac{y-Y_i}{h_n}), \quad (j = 1,2),$$

$$K_i = h_n^{-3}K(\frac{x-X_i}{h_n})K^{(1)}(\frac{y-Y_i}{h_n}), K_{ri} = h_n^{-3}K(\frac{x_r-X_i}{h_n})K^{(1)}(\frac{y-Y_i}{h_n}),$$

$i = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, \ell.$

3 Main Results

Firstly we will prove the following two lemmas, since they will be helpful in proving our main theorem.

Lemma 1. Under assumption(A2)(ii),(iii), and (iv), the following hold

(i) $\lim_{n\to\infty} h_n^4 \mathrm{Var}(K_{ri}) = f(x,y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(K(u)K^1(v))^2 du dv, \quad f(x,y) \in C(f)$.

$$\text{(ii)} \lim_{n\to\infty} n^{-1}h_n^4 \sum_{i=1}^{n} \text{Cov}(K_{ri}, K_{si}) = 0, \quad (x_r, y), (x_s, y) \in C(f), x_r \neq x_s.$$

Proof.

(i)Easily from the definition of K_{ri}, we have

$$\begin{aligned} h_n^4 \text{Var}(K_{ri}) &= h_n^{-2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (K(\frac{u}{h_n})K^1(\frac{v}{h_n}))^2 f(x-u, y-v)dudv \\ &- h_n^2 (h_n^{-2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\frac{u}{h_n})K^1(\frac{v}{h_n}) f(x-u, y-v)dudv)^2. \end{aligned}$$

An application of Lemma 1 in Samanta and Thavaneswaram [7], completes the proof of (i).

(ii) Suppose that without the loss of generality that $x_s > x_r$.
Let $\delta = x_s - x_r$, and $\delta_n = \frac{\delta}{h_n}$.

$$\begin{aligned} h_n^4 E(K_{ri}K_{si}) &= h_n^{-2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\frac{x_r-u}{h_n})K(\frac{x_s-u}{h_n}) \left(K^{(1)}(\frac{y-v}{h_n})\right)^2 f(u,v)dudv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(u)K(\delta_n + u) \left(K^{(1)}(v)\right)^2 f(x_r - h_n u, y - h_n v)dudv \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} K(u)K(\delta_n + u) g(x_r - h_n u)du\right] \\ &\times \left(K^{(1)}(v)\right)^2 f(y - h_n v | x_r - h_n u)dv. \end{aligned} \quad (2)$$

Next,

$$\begin{aligned} \int_{-\infty}^{\infty} K(u)K(\delta_n + u)g(x_r - h_n u)du &= \int_{|u|<\frac{\delta_n}{2}} K(u)K(\delta_n + u)g(x_r - h_n u)du \\ &+ \int_{|u|\geq\frac{\delta_n}{2}} K(u)K(\delta_n + u)g(x_r - h_n u)du \\ &\leq \sup_{|u|<\frac{\delta_n}{2}} K(\delta_n + u) \int_{-\infty}^{\infty} K(z)g(x_1 - h_n z)dz \\ &+ \sup_{|u|\geq\frac{\delta_n}{2}} K(u) \int_{-\infty}^{\infty} K_1(\delta_n + z)g(x_r - h_n z)dz \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{|u|\geq \frac{\delta_n}{2}} K(u)\cdot O(1) + \sup_{|u|\geq \frac{\delta_n}{2}} K(u)\cdot O(1) \\
&= 2 \sup_{|u|\geq \frac{\delta_n}{2}} K(u)\cdot O(1) \\
&\leq \frac{4}{\delta_n} \sup_{|u|\geq \frac{\delta_n}{2}} |uK(u)|\cdot O(1) \\
&= \frac{4h_n}{\delta} \sup_{|u|\geq \frac{\delta_n}{2}} |uK(u)|\cdot O(1) \\
&= O(h_n). \qquad (3)
\end{aligned}
$$

From (2) and (3) we have that

$$\lim_{n\to\infty} h_n^4 E(K_{ri}K_{si}) = 0. \qquad (4)$$

It is easy to see by another application of Lemma 1 in Samanta and Thavaneswaram [7] that

$$\lim_{n\to\infty} h_n^4 E(K_{ri})E(K_{si}) = 0. \qquad (5)$$

Hence a combination of (4) and (5) gives that $\lim_{n\to\infty} h_n^4 \mathrm{Cov}(K_{ri}, K_{si}) = 0$, which completes the proof of (ii). □

Lemma 2. Under the assumptions (A1) through (A5), and for $(x_r, y), (x_s, y) \in C(f)$, and $r \neq s$, the following holds,

$$\lim_{n\to\infty} \frac{h_n^4}{n} \sum_{1\leq i<j\leq n} \mathrm{Cov}(K_{ri}, K_{sj}) = 0.$$

Proof.

Let S_1, and S_2 be defined by

$$S_1 = \{(i,j)|i,j \in \{1,2,\ldots,n\}, 1 < j-i < c_n\},$$

$$S_2 = \{(i,j)|i,j \in \{1,2,\ldots,n\}, c_n + 1 \leq j-i \leq n-1\}.$$

Then

$$\frac{h_n^4}{n}\sum_{1<i<j<n}|\mathrm{Cov}(K_{ri},K_{sj})| = \frac{h_n^4}{n}\sum_{S_1}|\mathrm{Cov}(K_{ri},K_{sj})| - \frac{h_n^4}{n}\sum_{S_2}|\mathrm{Cov}(K_{ri},K_{sj})|. \qquad (6)$$

First we consider the summed over S_1,

$$\begin{aligned}\frac{h_n^4}{n}\sum_{S_1}|\mathrm{Cov}(K_{ri},K_{sj})| &\le \frac{h_n^4}{n}[\sum_{S_1}\frac{1}{h_n^6}\int_{\mathbb{R}^4}K(\frac{x_r-u}{h_n})K^{(1)}(\frac{y-v}{h_n})K(\frac{x_s-w}{h_n})\\ &\times K^{(1)}(\frac{y-z}{h_n})|f_{Z_iZ_j}(u,v,w,z)-f(u,v)f(w,z)|dudvdwdz]\\ &= \frac{h_n^4}{n}[\sum_{S_1}\frac{1}{h_n^2}\int_{\mathbb{R}^4}K(u)K^{(1)}(v)K(w)K^{(1)}(z)\\ &\times|f_{Z_i,Z_j}(x_r-h_nu,y-h_nv,x_s-h_nw,y-h_nz)\\ &-f(x_r-h_nu,y-h_nv)f(x_s-h_nw,y-h_nz)|dudvdwdz]\\ &\le \frac{h_n^4}{n}Cnc_n\\ &= Cc_nh_n^4\to 0, \qquad (7)\end{aligned}$$

by $\Lambda(4)$(i) .

Next, consider the summed over S_2. For some $\gamma\in(0,1)$, we have

$$\begin{aligned}|\mathrm{Cov}(K_{ri},K_{sj})| &\le 10\alpha^{1-\gamma}(j-i)[E(K_{ri})^{2/\gamma}]^{\gamma/2}[E(K_{sj})^{2/\gamma}]^{\gamma/2}\\ &= 10\alpha^{1-\gamma}(j-i)[h_n^{-2\frac{(3-\gamma)}{\gamma}}q_i(x_r)]^{\gamma/2}[h_n^{-2\frac{(3-\gamma)}{\gamma}}q_j(x_s)]^{\gamma/2},\end{aligned}$$

where

$$\begin{aligned}q_i(x_r) &= h_n^{-2}\int_{-\infty}^{\infty}(k(u)K^1(v))^{2/\gamma}f(x_r-h_nu,y-h_nv)dudv\\ &\to f(x_r,y)\int_{-\infty}^{\infty}(K(u)K^1(v))^{2/\gamma}dudv = q(x_r).\end{aligned}$$

and

$$q_j(x_s) = \int_{-\infty}^{\infty} (K(u)K^1(v))^{2/\gamma} f(x_s - h_n u, y - h_n v) du dv$$
$$\to f(x_s, y) \int_{-\infty}^{\infty} (K(u)K^1(v))^{2/\gamma} du dv = q(x_s).$$

Thus,

$$\frac{h_n^4}{n} \sum_{S_2} |\mathrm{Cov}(k_{ri}, K_{si})| \le \frac{10}{n} h_n^4 \left[\sum_{l=c_n+1}^{n-1} \alpha^{1-\gamma}(l) \right] \left[\sum_{i=1}^{n-l} \frac{q_i^{\gamma}(x_r)}{h_n^{2(3-\gamma)}} \right]^{1/2} \left[\sum_{i=1}^{n-l} \frac{q_{l+i}^{\gamma}(x_s)}{h_n^{2(3-\gamma)}} \right]^{1/2}$$

$$\le 10 \left[h_n^{2(\gamma-1)} \sum_{l=c_n}^{\infty} \alpha^{1-\gamma}(l) \right] \left[n^{-1} \sum_{i=1}^{n} q_i^{\gamma}(x_r) \right]^{1/2} \left[n^{-1} \sum_{j=1}^{n} q_j^{\gamma}(x_s) \right]^{1/2} \to 0, \tag{8}$$

since

$$h_n^{-2(1-\gamma)} \sum_{l=c_n}^{\infty} \alpha^{1-\gamma}(l) \to 0,$$

by (A4)(ii) and,

$$\left[n^{-1} \sum_{i=1}^{n} q_i^{\gamma}(x_r) \right]^{1/2} \left[n^{-1} \sum_{j=1}^{n} q_j^{\gamma}(x_s) \right]^{1/2} \to [q^{\gamma}(x_r)]^{1/2} [q^{\gamma}(x_s)]^{1/2}.$$

Therefore a combination of (6), (7), and (8) completes the proof. ❑

Now, we will introduce and prove our main result in this paper.

Theorem 1. Suppose that $x_1, x_2, \ldots, x_t$ are distinct points, where $f(x_i, y) > 0$, and $(x_i, y) \in C(f)$, $(i = 1, 2, \ldots, t)$. Then under the assumptions (A1) through (A6),

$$(nh_n^4)^{\frac{1}{2}} \{M_n(x_1) - M(x_1), \ldots, M_n(x_t) - M(x_t)\}^T,$$

where T denotes the transpose, is asymptotically multivariate normal with mean vector zero and diagonal covariance matrix $C = [c_{ij}]$, with

$$c_{ii} = \frac{f(x_i, M(x_i))}{\{f^{(0,2)}(x_i, M(x_i)\}^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{K(u)K^{(1)}(v)\}^2 du dv.$$

Proof.

First, we will show that

$$(nh_n^4)^{\frac{1}{2}}\{f_n^{(0,1)}(x_1,y) - Ef^{(0,1)}(x_1,y),\ldots,f_n^{(0,1)}(x_t,y) - Ef^{(0,1)}(x_t,y)\}^T \tag{9}$$

is asymptotically multivariate normal with mean vector zero and diagonal covariance matrix $\Gamma = [\gamma_{ij}]$, with

$$\gamma_{ii} = f(x_i,y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{K(u)K^{(1)}(v)\}^2 dudv, \quad (i = 1,2,\ldots,t).$$

Follows the same lines as the proof of Theorem 3.1 in Roussas and Tran [4].

Let

$$\psi_{nr} = \psi_n(x_r) = (nh_n^4)^{\frac{1}{2}}(f_n^{(0,1)}(x_r,y) - Ef^{(0,1)}(x_r,y)),\ (r = 1,2,\ldots,t),$$

$$\psi_n = \psi_n(x_1,\ldots,x_t) = \sum_{r=1}^{t} c_r\psi_{nr},\ c_r \text{ is a constant.}$$

$$\mathrm{Var}(\psi_n) = \sum_{r=1}^{t} c_r^2\mathrm{Var}(\psi_{nr}) + 2\sum_{1\le r<s\le t} c_r c_s \mathrm{Cov}(\psi_{nr},\psi_{ns}). \tag{10}$$

$$\begin{aligned}\mathrm{Var}(\psi_{nr}) &= nh_n^4\mathrm{Var}f_n^{(0,1)}(x_r,y)\\ &\to f(x_r,y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{K(u)K^{(1)}(v)\}^2 dudv = \tau_r^2,\end{aligned} \tag{11}$$

by Lemma 2 in Samanta and Thavaneswaram [7].

Since

$$f^{(0,1)}(x_r,y) = \frac{1}{n}\sum_{i=1}^{n} K_{ri},\ (r = 1,2,\ldots,t),$$

then

$$\begin{aligned}\mathrm{Cov}(\psi_{nr},\psi_{ns}) &= nh_n^4\mathrm{Cov}(f_n^{(0,1)}(x_r,y), f_n^{(0,1)}(x_s,y))\\ &= n^{-1}h_n^4\sum_{i=1}^{n}\mathrm{Cov}(K_{ri},K_{si}) + n^{-1}h_n^4\sum_{i\neq j}\mathrm{Cov}(K_{ri},K_{sj})\end{aligned} \tag{12}$$

by Lemma 1(ii), and Lemma 2. Hence by a combination of (10), (11), and (12) we get that

$$\mathrm{Var}(\psi_n) = \sum_{r=1}^{t} c_r^2 \tau_r^2 = \sigma_{\mathbf{c}}^2,\ \mathbf{c} = (c_1, \ldots, c_t). \tag{13}$$

Let

$$z_{ri} = h_n^2 (K_{ri} - EK_{ri}).$$

$$s_{nr} = \sum_{i=1}^{n} z_{ri}, \qquad s_n = \sum_{i=1}^{n} c_r s_{nr},$$

so that

$$s_{nr} = n^{\frac{1}{2}} \psi_{nr}, \qquad s_n = n^{\frac{1}{2}} \psi_n.$$

Let p, q, and k be as in assumption (A6). We divide the set $(1, 2, \ldots, n)$ into p large blocks and q small blocks and set

$$y_{nrm} = \sum_{i=k_m}^{k_m+p-1} z_{ri}, \qquad k_m = (m-1)(p+q)+1, \tag{14}$$

$$y'_{nrm} = \sum_{j=l_m}^{l_m+q-1} z_{rj}, \qquad l_m = (m-1)(p+q)+p+1, \tag{15}$$

$$y'_{nrk} = \sum_{l=k(p+q)+1}^{n} z_{rl}, \qquad m = 1, 2, \ldots, k. \tag{16}$$

Also, set

$$s'_{nr} = \sum_{m=1}^{k} y_{nrm}, \qquad s''_{nr} = \sum_{m=1}^{k} y'_{nrm}, \qquad s'''_{nr} = y'_{nrk}, \tag{17}$$

and

$$s'_n = \sum_{r=1}^{t} c_r s'_{nr}, \qquad s''_n = \sum_{r=1}^{t} c_r s''_n, \qquad s'''_n = \sum_{r=1}^{t} c_r y'_{nrk}. \tag{18}$$

Then

$$\begin{aligned}\mathrm{Var}(z_{ri}) &= h_n^4 \mathrm{Var}(K_{ri}) \le h_n^4 E K_{ri}^2 \\ &= h_n^4 \left[\frac{1}{h_n^6} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(K(u) K^{(1)}(v) \right)^2 f(x_r - u h_n, y - v h_n) du dv \right] \\ &\to f(x_r, y) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(K(u) K^{(1)}(v) \right)^2 du dv,\end{aligned}$$

so that $\mathrm{Var}(z_{ri}) \le C, \quad (i = 1, 2, \ldots, n),$ and $(r = 1, 2, \ldots, t)$. Hence

$$\frac{1}{n} \sum_{m=1}^{k} \sum_{i=1}^{q} \mathrm{Var}(z_{ri}) \le \frac{C(qk)}{n}. \tag{19}$$

Furthermore,

$$\mathrm{Cov}(z_{ri}, z_{rj}) = h_n^4 \mathrm{Cov}(K_{ri}, K_{rj}).$$

Thus,

$$\begin{aligned}\frac{1}{n} \sum_{m=1}^{k} \sum_{l_m \le i < j \le l_m + q - 1} |\mathrm{Cov}(z_{ri}, z_{rj})| &= \frac{h_n^4}{n} \sum_{m=1}^{k} \sum_{l_m \le i < j \le l_m + q - 1} |\mathrm{Cov}(K_{ri}, K_{rj})| \\ &\le \frac{h_n^4}{n} \sum_{1 \le i < j \le n} |\mathrm{Cov}(K_{ri}, K_{rj})| \to 0. \quad (20)\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{n} \sum_{m=1}^{k} \mathrm{Var}(y'_{nrm}) &= \frac{1}{n} \sum_{m=1}^{k} \sum_{i=1}^{q} \mathrm{Var}(z_{ri}) + \frac{2}{n} \sum_{m=1}^{k} \sum_{l_m \le i < j \le l_m + q - 1} \mathrm{Cov}(z_{ri}, z_{rj}) \\ &\le C\left(\frac{qk}{n}\right) + \frac{h_n^4}{n} \sum_{1 \le i < j \le n} |\mathrm{Cov}(K_{ri}, K_{rj})| \to 0, \quad (21)\end{aligned}$$

by (A6)(i), and (20).

Next,

$$\mathrm{Cov}(y'_{nri}, y'_{nrj}) = \mathrm{Cov}\left(\sum_{v=l_i}^{l_i+q-1} z_{rv}, \sum_{l=l_j}^{l_j+q-1} z_{rl} \right) = \sum_{v=l_i}^{l_i+q-1} \sum_{l=l_j}^{l_j+q-1} \mathrm{Cov}(z_{rv}, z_{rl}).$$

Hence,

$$\frac{1}{n}\sum_{1\le i<j\le k}|\mathrm{Cov}(y'_{nri},y'_{nrj})|\le\frac{1}{n}\sum_{1\le i<j\le k}\sum_{v=l_i}^{l_i+q-1}\sum_{l=l_j}^{l_j+q-1}|\mathrm{Cov}(z_{rv},z_{rl})|$$

$$<Cn^{-1}\sum_{i=1}^{n-p}\sum_{j=i+p}^{n}|\mathrm{Cov}(z_{ri},z_{rj})|$$

$$=Cn^{-1}h_n^4\sum_{i=1}^{n-p}\sum_{j=i+p}^{n}|\mathrm{Cov}(K_{ri},K_{rj})|$$

$$\le Cn^{-1}h_n^4\sum_{1\le i<j\le n}|\mathrm{Cov}(K_{ri},K_{rj})|\to 0. \quad (22)$$

Therefore by (21) and (22) we obtain for each $r=1,2,\ldots,t$

$$\frac{1}{n}\mathrm{Var}(s''_{nr})-\frac{1}{n}\sum_{m=1}^{k}\mathrm{Var}(y'_{nrm})+\frac{2}{n}\sum_{1\le i<j\le k}\mathrm{Cov}(y'_{nri},y'_{nrj})\to 0.$$

By the Minkowski inequality

$$[\frac{1}{n}E(s''_{nr})^2]=[\frac{1}{n}E(\sum_{r=1}^{t}c_r s''_{nr})^2]^{\frac{1}{2}}\le\sum_{r=1}^{t}|c_r|[\frac{1}{n}E(s''_{nr})^2]^{\frac{1}{2}}$$

$$=\sum_{r=1}^{t}|c_r|[\frac{1}{n}\mathrm{Var}(s''_{nr})]^{\frac{1}{2}}\to 0.$$

This implies that

$$\frac{1}{n}E(s''_{nr})^2\to 0. \quad (23)$$

Using the same technique, we can prove that

$$\frac{1}{n}E(s'''_{nr})^2\to 0. \quad (24)$$

Since

$$\psi_n=n^{-\frac{1}{2}}s_n=n^{-\frac{1}{2}}(s'_n+s''_n+s'''_n),$$

we have

$$n^{-\frac{1}{2}}s'_n=\psi_n-n^{-\frac{1}{2}}(s''_n+s'''_n). \quad (25)$$

$$n^{-1}E(s'_n)^2 = E\psi_n^2 + n^{-1}E(s''_n + s'''_n)^2 - 2n^{-1}E(s_n(s''_n + s'''_n)), \qquad (26)$$

and

$$[n^{-1}E(s''_n + s'''_n)^2]^{\frac{1}{2}} \leq [n^{-1}E(s''_n)^2]^{\frac{1}{2}} + [n^{-1}E(s'''_n)^2]^{\frac{1}{2}} \to 0, \qquad (27)$$

and

$$\begin{aligned} n^{-1}E|s_n(s''_n + s'''_n)| &\leq [n^{-1}E(s_n^2)]^{\frac{1}{2}} \times [n^{-1}E(s''_n - s'''_n)]^{\frac{1}{2}} \\ &= [E\psi_n^2]^{\frac{1}{2}}[n^{-1}E(s''_n + s'''_n)]^{\frac{1}{2}} \to 0. \end{aligned} \qquad (28)$$

Thus a combination of (13), (26), (27) and (28) implies that

$$n^{-1}E(s'_n)^2 \to \sigma_c^2. \qquad (29)$$

Applying (29) with $c_r \neq 0$ and $c_s = 0, s \neq r$ we obtain

$$n^{-1}E(s'_{nr}) \to \tau_r^2,\ r = 1, 2, \ldots, t, \qquad (30)$$

$$E(s'_{nr})^2 = \sum_{m=1}^{k} \mathrm{Var}(y_{nrm}) + 2 \sum_{1 \leq i < j \leq k} \mathrm{Cov}(y_{nri}, y_{nrj}). \qquad (31)$$

Now,

$$\begin{aligned} n^{-1} \sum_{1 \leq i < j \leq k} \mathrm{Cov}(y_{nri}, y_{nrj}) &\leq Cn^{-1}h_n^4 \sum_{i=1}^{n-q} \sum_{j=i+q}^{n} |\mathrm{Cov}(K_{ri}, K_{rj})| \\ &\leq Cn^{-1}h_n^4 \sum_{1 < i < j \leq k} |\mathrm{Cov}(K_{ri}, K_{rj})| \to 0. \end{aligned} \qquad (32)$$

From (29), (31), and (32), we obtain that

$$n^{-1} \sum_{m=1}^{k} \mathrm{Var}(y_{nrm}) \to \tau_r^2,\ r = 1, 2, \ldots, t. \qquad (33)$$

Therefore,

$$\sum_{r=1}^{t} n^{-1} \sum_{m=1}^{k} \mathrm{Var}(y_{nrm}) \to \sigma_c^2. \qquad (34)$$

Let Y_{nrm}, $m = 1, 2, \ldots, k$, $r = 1, 2, \ldots, t$ be independent random variables such that Y_{nrm} is distributed as $c_r n^{-\frac{1}{2}} y_{nrm}$, and $X_{nrm} = Y_{nrm}/s_n$, where

$$s_n^2 = \sum_{r=1}^{t} \sum_{m=1}^{k} \mathrm{Var}(Y_{nrm}) = \sum_{r=1}^{t} n^{-1} \sum_{m=1}^{k} \mathrm{Var}(c_r y_{nrm}) \to \sigma_{\mathbf{c}}^2.$$

Since $EX_{nrm} = 0$ and $\sum_{r=1}^{t} \sum_{m=1}^{k} \mathrm{Var}(X_{nrm}) = 1$, it follows that

$$\sum_{r=1}^{t} \sum_{m=1}^{k} X_{nrm} \to N(0, 1) \tag{35}$$

in distribution, if and only if, for every $\epsilon > 0$,

$$g(\epsilon) = \sum_{r=1}^{t} \sum_{m=1}^{k} EX_{nrm}^2 I_{(|X_{nrm}| \geq \epsilon)} \to 0. \tag{36}$$

Now,

$$\begin{aligned} g(\epsilon) &= \frac{1}{s_n^2} EY_{nrm}^2 I_{(|Y_{nrm}| \geq s_n \epsilon)} \\ &= \frac{c_r^2}{n s_n^2} Ey_{nrm}^2 I_{(|y_{nrm}| \geq n^{\frac{1}{2}} s_n \epsilon / c_r)} \\ &\leq \frac{c_r^2}{n s_n^2} \cdot \frac{p^2 C^2}{h_n^2} \cdot P(|y_{nrm}| \geq n^{\frac{1}{2}} s_n \epsilon / c_r) \\ &\leq \frac{c_r^2 C^2}{s_n^2} \cdot \frac{p^2}{n h_n^2} \cdot \frac{c_r^2 \,\mathrm{Var}(y_{nrm})}{n s_n^2 \epsilon^2}. \end{aligned}$$

This implies that

$$g(\epsilon) \leq \frac{c_r^2 C^2}{s_n^2 \epsilon^2} \cdot \frac{p^2}{n h_n^2} \cdot tk \frac{c_r^2 \, Var(y_{nrm})}{n s_n^2}, \tag{37}$$

which converges to zero, by assumption (A6)(ii). Thus, (36) holds. Now, by A(6)(iii) and (36) we obtain that

$$\sum_{r=1}^{t} n^{-\frac{1}{2}} \sum_{m=1}^{k} c_r y_{nrm} \to N(0, \sigma_c^2), \quad (38)$$

in distribution.
Since

$$n^{-\frac{1}{2}} s'_n = n^{-\frac{1}{2}} \sum_{r=1}^{t} c_r s'_{nr} = \sum_{r=1}^{t} n^{-\frac{1}{2}} \sum_{m=1}^{k} c_r y_{nrm},$$

we have that,

$$n^{-\frac{1}{2}} s'_n \to N(0, \sigma_c^2), \quad (39)$$

in distribution, and since

$$\psi_n = n^{-\frac{1}{2}} (s'_n + s''_n + s'''_n),$$

thus a combination of (23) and (24) and (39) implies that

$$\psi_n \to N(0, \sigma_c^2) \text{ in distribution.} \quad (40)$$

The convergence in (40) is the same as

$$(nh_n^4)^{\frac{1}{2}} \sum_{r=1}^{t} c_r (f_n^{(0,1)}(x, y) - E f_n^{(0,1)}(x, y)) \to N(0, \sigma_c^2), \quad (41)$$

in distribution. Therefore by the Cramér-Wold device (9) holds. The proof of Theorem 1 is completed by making use of Lemma 3 and Lemma 4 in Samanta and Thavaneswaram [7]. □

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