Bäcklund transformations for fourth-order Painlevé-type equations

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Abstract

In this paper we study new forms of Bäcklund transformations for fourth-order ordinary differential equations of Painlevé-type. We present an algorithm which allows the construction of Bäcklund transformations between a given equation and a new fourth-order equation. The precise form of the new equations are also determined.

Keywords: *Painlevé equations, Painlevé-type equations, Bäcklund transformations.*

MSC2000 classification scheme numbers: *34M55, 37K35*

1 Introduction

One century ago Painlevé and his school studied the second-order ordinary differential equations (ODEs) of the form $v'' = F(z, v, v')$, where F is rational in v', v and locally analytic in z. They found 50 equations with the Painlevé property, that is, equations whose solutions have no movable critical points. Among all of them they found only six equations that define new functions, which are called now Painlevé transcendents and denoted by $PI - PVI$ [1].

Fourth-order ODEs with the Painlevé property were considered by Chazy [2], Bureau [3], Exton [4], Martynove [5], Harada and Oishi [6], Cosgrove [7], Muğan and Jrad [8], and Kudryashov [9]. Amongst the equations found in [2]–[9], the equations defining new transcendental functions need to be studied. One of the important properties that need to be studied is the existence of Bäcklund transformations (BTs), that is, transformations relating a particular fourth-order Painlevé-type equation either to itself (with possibly different values of the parameters appearing as coefficients), or to another fourth-order equation of Painlevé-type.

Fokas and Ablowitz [10] developed an algorithmic method to investigate the transformation properties of Painlevé equations. They used a transformation of the form

$$u = \frac{v' + av^2 + bv + c}{dv^2 + ev + f}, \tag{1}$$

where a, b, c, d, e, f are functions of z only. The inverse of the transformation (1) is either linear equation in v, and this yields second-order first-degree Painlevé-type equation for u, or quadratic equation in v and this yields second-order second-degree Painlevé-type equation for u.

In [11, 12] the same transformation was used to obtain second-order second-degree ODE for PI-PVI.

A transformation of the form

$$u = \frac{(v')^2 + (\Sigma_{i=0}^{2} a_i v^i)v' + \Sigma_{i=0}^{4} b_i v^i}{(\Sigma_{i=0}^{2} c_i v^i)v' + \Sigma_{i=0}^{4} d_i v^i}, \tag{2}$$

where $a_i,\ b_i,\ c_i,\ d_i$ are all functions of z only, was used in [13] to find second-order second-degree ODEs related to PI-PVI and was used in [14] to find second-order fourth-degree ODEs related to PI-PIV. An alternative approach to find BTs for Painlevé equation appears in [15, 16, 17].

For fourth-order Painlevé-type equations the study of BTs is more rich. They admit BTs of the following form

$$u = \frac{v^{(m)} - G(z, v, v', \ldots, v^{(m-1)})}{H(z, v, v', \ldots, v^{(m-1)})}, \tag{3}$$

where the functions G and H guarantee that the equation in u also of Painlevé-type and $m = 1,\ 2$, or 3. Differentiating (3) $(4-m)$-times and using the given equation to replace $v^{(4)}$ and the relation (3) and its derivatives to replace $v^{(n)},\ m \leq n \leq 3$, yields a polynomial equation in $v,\ v', \ldots,\ v^{(m-1)},\ u,\ u', \ldots,\ u^{(4-m)}$ with coefficients functions of z. Elimination of v between this last equation and equation (3) then yields the fourth-order equation for u. This elimination process can be simplified by insisting on special choices of the BT (3) such that the polynomial equation in $v,\ v', \ldots,\ v^{(m-1)},\ u,\ u', \ldots,\ u^{(4-m)}$ reduces to a polynomial equation in $v,\ v', \ldots,\ v^{(p)},\ u,\ u', \ldots,\ u^{(4-m)}$ for some

$p < m-1$; in particular we might ask that it reduces to a polynomial equation in $v,\ u,\ u',\ldots,\ u^{(4-m)}$.

In [18] the case $m = 1$ was considered with transformations of the form (1). The method was applied to a fourth-order ODE believed to define a new transcendental function. The same method has also been applied in [19] to the fourth-order analogue of PI and in [20] to the generalized fourth-order analogue of PII.

The case $m = 3$ was considered in [21] where transformation of the form

$$u = \frac{v''' - G(z, v, v', v'')}{H(z, v, v', v'')} \tag{4}$$

was used to seek BT for fourth-order ODE. In [21] H and G were assumed to be linear in v''.

The aim of this paper is to obtain BTs (3) for fourth-order Painlevé equations with $m = 2$ and $m = 3$. That is we will consider the transformations of the forms

$$u = \frac{v'' - G(z, v, v')}{H(z, v, v')} \tag{5}$$

and (4), where G and H guarantee that the equation in u also has the Painlevé property. We will consider transformations of the form (4) with H and G being quadratic in v''.

In section (2) we will apply this algorithm to four fourth-order Painlevé-type equation using $m = 3$. In section (3) we will apply it to three fourth-order Painlevé-type equation using $m = 2$. In the last section we will give the conclusions.

2 Bäcklund transformations with $m = 3$

In this section we will consider BT for fourth-order ODEs

$$v^{(4)} = F(z, v, v', v'', v''') \tag{6}$$

of the form

$$u = \frac{v''' - G(z, v, v', v'')}{H(z, v, v', v'')}, \tag{7}$$

where $u(z)$ is a solution of another fourth-order Painlevé-type equation. We take G and H in the forms

$$G = a_0(v'')^2 + b_0 v'' + c_0, \qquad H = a_1(v'')^2 + b_1 v'' + c_1, \tag{8}$$

where $a_i, b_i, c_i, \quad i = 0, 1$ are rational functions of v and v' with coefficients functions of z only. We can rewrite the BT (7) as

$$v''' = A(v'')^2 + Bv'' + C \tag{9}$$

where $A = a_1 u + a_0, \quad B = b_1 u + b_0, \quad C = c_1 u + c_0$.

Differentiating equation (9) and using equation (6) to replace $v^{(4)}$ and equation (9) to replace v''', we get an equation of the form

$$\psi_3 (v'')^3 + \psi_2 (v'')^2 + \psi_1 v'' + \psi_0 = 0, \tag{10}$$

where $\psi_j, \quad j = 0, 1, 2, 3$ are polynomials in $v, \ u, \ v', \ u'$ with coefficients functions of z. Eliminating v between (10) and (9), one finds a fourth-order ODE for u.

In this article we consider the following simplification of this elimination procedure: we choose $A, \ B$, and C (that is $a_i, \ b_i$ and c_i) so that $\psi_j, \ j = 1, 2, 3$ are identically zero and equation (10) reduces to a second or first degree polynomial equation in v

$$\psi_0(v) = 0. \tag{11}$$

Elimination v between (11) and (9), in order to find the equation for u, is thus made much easier.

In this article we assume that A has the form $\dfrac{A_0}{v'}$, where A_0 is a function of $z, \ v$, and u. We now consider some examples.

2.1 Example (2.1)

As a first example we consider the equation

$$v^{(4)} = 20vv'' + 10(v')^2 - 40v^3 + \alpha v + z. \tag{12}$$

Equation (12) is a member of Painlevé-I hierarchy [22] and it was given also in [7] where it has been denoted by F-V. The solution of equation (12) is believed to define a new transcendent.

Applying the method discussed above we find that the coefficients ψ_j , $j = 0, 1, 2, 3$ in equation (10) are given by

$$\begin{aligned}
\psi_3 &= A_0(2A_0 - 1), \\
\psi_2 &= v'\Big[v'\frac{\partial B}{\partial v'} + 3A_0 B + \frac{\partial A_0}{\partial z} + v'\frac{\partial A_0}{\partial v}\Big], \\
\psi_1 &= v'\Big[v'\frac{\partial C}{\partial v'} + 2A_0 C + (v')^2\frac{\partial B}{\partial v} + \frac{\partial B}{\partial z}v' + (B^2 - 20v)v'\Big], \\
\psi_0 &= (v')^2\Big[BC + \frac{\partial C}{\partial v}v' + \frac{\partial C}{\partial z} - 10(v')^2 + 40v^3 - \alpha v - z\Big].
\end{aligned} \tag{13}$$

In order to make $\psi_3 = 0$ identically, we have to choose $A_0 = \dfrac{1}{2}$ or $A_0 = 0$. If $A_0 = 0$, then we do not have BT in the type considered here. Thus the only choice is $A_0 = \dfrac{1}{2}$, that is $A = \dfrac{1}{2v'}$. Then $\psi_2 = 0$ identically if $B = B_0(v')^{-3/2}$, where B_0 is an arbitrary function of z and v. In order to preserve the Painlevé property we must take $B_0 = 0$. Setting $\psi_1 = 0$, we find that $C = 10vv' + \dfrac{C_0}{v'}$, where C_0 is an arbitrary function of z, v. The equation $\psi_0 = 0$ now reads

$$\frac{\partial C_0}{\partial z} + v'\Big[\frac{\partial C_0}{\partial v} + 40v^3 - \alpha v - z\Big] = 0. \tag{14}$$

In order that equation (14) reduces to a polynomial equation in v we have to choose $\dfrac{\partial C_0}{\partial v} + 40v^3 - \alpha v - z = 0$ and hence $C_0 = -10v^4 + \frac{1}{2}\alpha v^2 + zv + C_{00}$, where C_{00} is an arbitrary function of z. Without loss of generality we may set $C_{00} = -u$. Then equation (14) gives

$$v = u' \tag{15}$$

and the transformation (9) becomes

$$v''' = \frac{1}{2v'}(v'')^2 + 10vv' - \frac{1}{v'}(10v^4 - \frac{\alpha}{2}v^2 - zv + u). \tag{16}$$

Substituting v from equation (15) into equation (16) yields the following fourth-order Painlevé-type equation for u

$$u^{(4)} = \frac{1}{2u''}(u''')^2 + 10u'u'' - \frac{1}{u''}[10(u')^4 - \frac{\alpha}{2}(u')^2 - zu' + u]. \tag{17}$$

Thus we obtain the BT (15-16) between equation (12) and equation (17).

2.2 Example (2.2)

As a second example we consider the equation

$$v^{(4)} = 18vv'' + 9(v')^2 - 24v^3 + \alpha v^2 + \frac{1}{9}\alpha^2 v + z. \tag{18}$$

The solution of equation (18) is a fourth-order Painlevé transcendent denoted by F-VI in [7].

Proceeding as in Example (2.1), we find that the coefficients $\psi_j,\ j = 1, 2, 3$ in equation (10) are identically zeros and the equation $\psi_0 = 0$ is a polynomial equation in v if $A = \frac{1}{2v'}$, $B = 0$ and $C = 9vv' + \frac{1}{v'}(-6v^4 + \frac{\alpha}{3}v^3 + \frac{1}{18}\alpha^2 v^2 + zv - u)$. Then equation $\psi_0(v) = 0$ reads

$$v = u' \tag{19}$$

and the transformation (9) has the form

$$v''' = \frac{1}{2v'}(v'')^2 + 9vv' - \frac{1}{v'}(6v^4 - \frac{\alpha}{3}v^3 - \frac{\alpha^2}{18}v^2 - zv + u). \tag{20}$$

Substituting v from equation (19) into equation (20) we get the following fourth-order Painlevé-type equation for u

$$u^{(4)} = \frac{1}{2u''}(u''')^2 + 9u'u'' - \frac{1}{u''}[6(u')^4 - \frac{\alpha}{3}(u')^3 - \frac{\alpha^2}{18}(u')^2 - zu' + u]. \tag{21}$$

Thus we obtain the BT transformation (19-20) between equation (18) and equation (21).

2.3 Example (2.3)

Our third example is the equation

$$v^{(4)} = 10v^2 v'' + 10v(v')^2 - 6v^5 - \beta(v'' - 2v^3) + zv + \alpha. \tag{22}$$

This equation was known to Martynov [5] and is a member of the so-called Painlevé-II hierarchy found by Ablowitz and Segur [23]. Some authors have been using the notation ${}_4P_2$ to denote this equation; see [24] for example, while Cosgrove [7] uses the notation F-XVIII to denote this equation. The solution of equation (22) define fourth-order Painlevé transcendent.

We proceed as in example (2.1). Setting $\psi_j = 0,\ j = 1, 2, 3$ we find that $A_0 = \frac{1}{2}$, $B = 0$ and $C = \frac{1}{2}(10v^2 - \beta)v' + \frac{C_0}{v'}$. Then equation $\psi_0 = 0$ reads

$$\frac{\partial C_0}{\partial z} + v'\Big[\frac{\partial C_0}{\partial v} + 6v^5 - 2\beta v^3 - zv - \alpha\Big] = 0. \tag{23}$$

In order that equation (23) reduces to a polynomial equation in v we have to choose $\frac{\partial C_0}{\partial v} + 6v^5 - 2\beta v^3 - zv - \alpha = 0$ and hence $C_0 = -v^6 + \frac{1}{2}\beta v^4 + \frac{1}{2}zv^2 + \alpha v + C_{00}$, where C_{00} is an arbitrary function of z. Without loss of generality we may set $C_{00} = -u$ and as a result $C = (5v^2 - \frac{\beta}{2})v' + \frac{1}{v'}(-v^6 + \frac{\beta}{2}v^4 + \frac{z}{2}v^2 + \alpha v - u)$. Then equation (23) gives

$$v^2 = 2u' \tag{24}$$

and the transformation (9) has the form

$$v''' = \frac{1}{2v'}(v'')^2 + (5v^2 - \frac{\beta}{2})v' + \frac{1}{v'}(-v^6 + \frac{\beta}{2}v^4 + \frac{z}{2}v^2 + \alpha v - u). \tag{25}$$

Elimination of v between equation (24) and equation (25) gives the following fourth-order second-degree Painlevé-type equation for u

$$\Big[u''u^{(4)} - \frac{3(u'')^2}{2u'}(u''' - \frac{(u'')^2}{2u'}) - \frac{1}{2}(u''' - \frac{(u'')^2}{2u'})^2 - (u'')^2(10u' - \frac{\beta}{2}) + 16(u')^4 - 4\beta(u')^3 - 2u'(zu' - u)\Big]^2 = 8(u')^3\alpha^2. \tag{26}$$

Thus we obtain the BT (24-25) between equation (22) and equation (26). We now use this result to derive a Lie-point symmetry for equation (22).

2.3.1 Lie-point symmetry for equation (22)

We now use the BT obtained above to derive Lie-point symmetry for (22). Here we make use of the fact that equation (26) is invariant under the transformation $\alpha \to -\alpha$.

We begin by noting that the BTs (24-25) defines a mapping between solutions v of (22) and solutions u of (26). Changing the sign of α in this BT yields an alternative BT consisting of the two relations

$$v^2 = 2u' \tag{27}$$

and

$$v''' = \frac{1}{2v'}(v'')^2 + (5v^2 - \frac{\beta}{2})v' + \frac{1}{v'}(-v^6 + \frac{\beta}{2}v^4 + \frac{z}{2}v^2 - \alpha v - u). \tag{28}$$

between solutions v of (22) for parameter $-\alpha$ and solutions u of (26). Thus given a solution u of (26) we can obtain two solutions v and $\bar{v}$ of (22)

$$v^2 = 2u', \tag{29}$$

and

$$\bar{v}^2 = 2u', \tag{30}$$

for parameters α and $\bar{\alpha} = -\alpha$ respectively. Eliminating u' between (29) and (30), we obtain the Lie-point symmetry $\bar{v} = -v,\ \bar{\alpha} = -\alpha$ for equation (22).

2.4 Example (2.4)

As a last example we consider the equation

$$v^{(4)} = -5v'v'' + 5v^2v'' + 5v(v')^2 - v^5 + zv - \frac{1}{2}\alpha, \tag{31}$$

which has recently been proposed as defining new transcendent [7].

Proceeding as in the previous examples we obtain the BT

$$\begin{aligned} &v^2 = u', \\ &v''' = \frac{1}{2v'}(v'')^2 - \frac{5}{3}(v')^2 + \frac{5}{2}v^2v' - \frac{1}{2v'}(\frac{1}{3}v^6 - z\ v^2 + \alpha v - u). \end{aligned} \tag{32}$$

between equation (31) and the fourth-order Painlevé-type equation

$$\begin{aligned} u'\Big[u''u^{(4)} - \frac{3(u'')^2}{2u'}\Big(u''' - \frac{(u'')^2}{2u'}\Big) - \frac{5}{2}u'(u'')^2 \\ - \frac{1}{2}\Big(u''' - \frac{(u'')^2}{2u'}\Big)^2 + 2u'\Big(\frac{1}{3}(u')^3 - zu' + u\Big)\Big]^2 = \Big[2\alpha(u')^2 + \frac{5}{6}(u'')^3\Big]^2. \end{aligned} \tag{33}$$

3 Bäcklund transformations with $m = 2$

In this section we will seek BT transformations for fourth-order ODE (6) of the form

$$u = \frac{v'' - G(z, v, v')}{H(z, v, v')}, \tag{34}$$

where $u(z)$ is a solution of another fourth-order Painlevé-type equation. We assume that G and H have the following forms

$$\begin{aligned} G = a_0(v')^2 + b_0v' + c_0, \\ H = a_1(v')^2 + b_1v' + c_1, \end{aligned} \tag{35}$$

where $a_0,\ b_0,\ c_0,\ a_1,\ b_1$ and c_1 are functions of z and v. We can rewrite the BT (34) as

$$v'' = A(v')^2 + Bv' + C, \tag{36}$$

where

$$\begin{aligned} A = a_1u + a_0, \\ B = b_1u + b_0, \\ C = c_1u + c_0. \end{aligned} \tag{37}$$

Differentiating equation (36) twice and using equation (6) to replace $v^{(4)}$, equation (36) to replace v'' and its derivative to replace v''', we get an equation of the form

$$\phi_4(v')^4 + \phi_3(v')^3 + \phi_2(v')^2 + \phi_1v' + \phi_0 = 0, \tag{38}$$

where $\phi_j, j = 0,1,2,3,4$ are polynomials in v, u, u', u'' with coefficients functions of z. Elimination of v between (38) and (36) leads to fourth-order ODE in u.

Here, as in section 2, we consider the following simplification of this elimination procedure: we choose A, B, and C (that is a_i, b_i, and c_i) so that $\phi_j, j = 1,2,3,4$ are identically zero and equation (38) reduces to a second or first degree polynomial equation in v

$$\phi_0(v) = 0. \tag{39}$$

In this paper we assume that $A = \dfrac{A_0}{v}$ where A_0 is a function of z and u and we will illustrate our method by three examples.

3.1 Example (3.1)

As a first example we consider the equation

$$v^{(4)} = \frac{5v'}{v}v''' - 5\Big[\frac{(v')^2}{v^2} - \alpha v^2\Big]v'' - 5\alpha v(v')^2 - \alpha^2 v^5 + z\ v + 1. \tag{40}$$

Equation (40) is solvable in terms of the transcendental function defined by the equation (31) [21].

Using the method introduced above, we find that the coefficients ϕ_j in (38) are given by

$$\begin{aligned}
\phi_4 &= A_0(6A_0^2 - 17A_0 + 12),\\
\phi_3 &= -v\Big[v^2\frac{\partial^2 B}{\partial v^2} + 5(A_0-1)v\frac{\partial B}{\partial v} + (12A_0^2 - 21A_0 + 5)B + A_0'(6A_0 - 7)\Big],\\
\phi_2 &= v\Big[v^2\frac{\partial^2 C}{\partial v^2} + v(3A_0 - 5)\frac{\partial C}{\partial v} + (8A_0^2 - 15A_0 + 5)C + 2v^2\frac{\partial^2 B}{\partial v \partial z}\\
&\quad + 4v^2 B\frac{\partial B}{\partial v} - 5\alpha(A_0 - 1)v^3 + (7A_0 - 5)B^2 v + (A_0'' + 5A_0' B)v + (4A_0 - 5)v\frac{\partial B}{\partial z}\Big],\\
\phi_1 &= v^3\Big[\frac{\partial^2 B}{\partial z^2} + 2\frac{\partial^2 C}{\partial v \partial z} + 3B\frac{\partial B}{\partial z} + 3C\frac{\partial B}{\partial v} + B^3 - 5\alpha v^2 B\Big]\\
&\quad + v^2\Big[(2A_0 - 5)\frac{\partial C}{\partial z} + 4A_0' C + (8A_0 - 5)BC\Big],\\
\phi_0 &= v^3\Big[\frac{\partial^2 C}{\partial z^2} + B\frac{\partial C}{\partial z} + C\frac{\partial C}{\partial v} + 2C\frac{\partial B}{\partial z} + CB^2 - 5\alpha v^2 C + \alpha^2 v^5 - zv - 1\Big]\\
&\quad + 2C^2 A_0 v^2.
\end{aligned} \tag{41}$$

Now the aim is to choose A_0, B and C so that $\phi_j,\ j = 1,2,3,4$ are identically zero and the equation $\phi_0 = 0$ is reduced to linear equation in v. In order to make $\phi_4 = 0$, we should choose $A_0 = 0$, $A_0 = \dfrac{4}{3}$ or $A_0 = \dfrac{3}{2}$. It turns out that the first two choices for A_0 are not compatible with $\phi_j = 0,\ j = 1, 2, 3$ so we have only the choice $A_0 = \dfrac{3}{2}$.

Now setting $\phi_3 = 0$ implies $2v^2 \dfrac{\partial^2 B}{\partial v^2} + 5v \dfrac{\partial B}{\partial v} + B = 0$ and hence $B = B_1 v^{-1} + B_2 v^{-1/2}$, where B_1 and B_2 are functions of z and u. In order to preserve the Painlevé property we should take $B_2 = 0$.

The condition $\phi_2 = 0$ now gives $2v^2 \dfrac{\partial^2 C}{\partial v^2} - v \dfrac{\partial C}{\partial v} + C = 5\alpha v^3 - 3B_1^2 v^{-1} + 2B_1'$. Thus $C = \dfrac{\alpha}{2} v^3 + C_1 v + 2B_1' - B_1^2 v^{-1} + C_2 v^{1/2}$, where C_1 and C_2 are functions of z and u. Again to preserve the Painlevé property we should take $C_2 = 0$ and without loss of generality we may take $C_1 = u$. Substituting these values of A_0, B and C into the condition $\phi_1 = 0$, we obtain $B_1 = 0$.

As a result equation $\phi_0(v) = 0$ has the form

$$(u'' + 4u^2 - z)v - 1 = 0 \tag{42}$$

and the transformation (36) becomes

$$v'' = \frac{3}{2v}(v')^2 + \frac{\alpha}{2} v^3 + uv. \tag{43}$$

Elimination of v between (42) and (43) leads to the following fourth-order Painlevé-type equation for u

$$\begin{aligned} &(u'' + 4u^2 - z)(u^{(4)} + 8(u')^2 + 8uu'') \\ &\quad - \tfrac{1}{2}(u''' + 8uu' - 1)^2 + u(u'' + 4u^2 - z)^2 + \frac{\alpha}{2} = 0. \end{aligned} \tag{44}$$

Thus we obtain the BT (42-43) between equation (40) and equation (44).

The substitution $u = -\frac{3}{2} y$ transforms equation (44) into the equation

$$\begin{aligned} &2(y'' - 6y^2 + \tfrac{2}{3}z)(y^{(4)} - 12yy'' - 12(y')^2) \\ &\quad - (y''' - 12yy' + \tfrac{2}{3})^2 - 3y(y'' - 6y^2 + \tfrac{2}{3}z)^2 + \frac{4\alpha}{9} = 0, \end{aligned} \tag{45}$$

which is the first integral of Cosgrove's Fif-I equation [7, 25] with $\lambda \neq 0$. Therefore we have rederived the known relation between Cosgrove's Fif-I equation and equation (40) [7].

3.2 Example (3.2)

As a second example of the application of our method we consider the equation

$$\begin{aligned} v^{(4)} = \frac{5v'v'''}{v} &+ \frac{15}{4v}(v'')^2 - \Big[\frac{65}{4v^2} v'^2 - \frac{5}{4}\alpha v^2\Big] v'' \\ &+ \frac{135}{16v^3}(v')^4 + \frac{5}{8}\alpha v (v')^2 - \frac{1}{16}\alpha^2 v^5 + zv - 2. \end{aligned} \tag{46}$$

Equation (46) is solvable in terms of the transcendental function defined by equation (31) [21].

We proceed as in the previous example. In order that $\phi_j = 0$, $j = 1, 2, 3, 4$ identically, we have to choose $A_0 = \frac{3}{2}$, $B = 0$ and $C = \frac{1}{2}\alpha v^3 + uv$. Then the equation $\phi_0(v) = 0$ gives

$$(u'' + \frac{1}{4}u^2 - z)v + 2 = 0 \tag{47}$$

and the transformation (36) takes the form

$$v'' = \frac{3}{2v}(v')^2 + \frac{1}{2}\alpha v^3 + uv. \tag{48}$$

Substituting v from (47) into (48) gives the following fourth -order Painlevé-type equation for u

$$\begin{aligned}&(u'' + \frac{1}{4}u^2 - z)(u^{(4)} + \frac{1}{2}(u')^2 + \frac{1}{2}uu'') - \frac{1}{2}(u''' + \frac{1}{2}uu' - 1)^2\\ &+u(u'' + \frac{1}{4}u^2 - z)^2 + 2\alpha = 0.\end{aligned} \tag{49}$$

The transformation $u = -12y$ transforms (49) into the equation

$$\begin{aligned}&2[y'' - 3y^2 + \tfrac{1}{12}z][y^{(4)} - 6yy'' - 6(y')^2]\\ &-[y''' - 6yy' + \tfrac{1}{12}]^2 + 24y[y'' - 3y^2 + \tfrac{1}{12}z]^2 + \frac{\alpha}{36} = 0\end{aligned} \tag{50}$$

which is the first integral of Cosgrove's Fif-II equation with $\lambda \neq 0$ [7, 25]. Therefore we have rederived the known relation between Cosgrove's Fif-II equation and equation (46) [7].

3.3 Example(3.3)

As our third example we consider the equation

$$\begin{aligned}v^{(4)} = &\frac{5v'}{v}v''' + \frac{5(v'')^2}{2v} - [\frac{25}{2v^2}(v')^2 - \frac{5}{2}\alpha v^2 + \beta]v''\\ &+\frac{45}{8v^3}(v')^4 - [\frac{5}{4}\alpha v^2 - \frac{3}{2}\beta]\frac{(v')^2}{v} - \frac{3}{8}\alpha^2 v^5 + \frac{1}{2}\beta\alpha v^3 + zv - 2\epsilon,\end{aligned} \tag{51}$$

where $\epsilon = \pm 1$. Equation (51) is solvable in terms of transcendental function define by equation (22) [21].

We proceed as in Section 3.1. In this case the equation $\phi_0 = 0$ reads

$$(u'' + \frac{3}{2}u^2 + \beta u - z)v + 2\epsilon = 0. \tag{52}$$

and the transformation (36) has the form

$$v'' = \frac{3}{2v}(v')^2 + \frac{1}{2}\alpha v^3 + uv. \tag{53}$$

Substituting v from equation (52) into equation (53), we obtain the following fourth-order Painlevé-type equation for u

$$\begin{aligned}&[u'' + \frac{3}{2}u^2 + \beta u - z][u^{(4)} + 3(u')^2 + (3u+\beta)u''] \\ &\quad = \frac{1}{2}[u''' + (3u+\beta)u' - 1]^2 - u[u'' + \frac{3}{2}u^2 + \beta u - z]^2 - 2\epsilon^2\alpha.\end{aligned} \tag{54}$$

Thus once again we see that our approach allows the derivation of the BT (52-53) between equation (51) and the new equation (54).

4 Conclusions

We have presented new types of BTs for fourth-order Painlevé-type equations. We have derived new BTs between fourth-order Painlevé-type equations whose solutions are transcendental functions and other fourth-order Painlevé-type equations of first and second degrees.

Many generalization of this work are possible. One generalization is to consider higher order Painlevé-type equations. Another generalization is to use other restrictions on the coefficients of equations (10) and (38).

References

[1] Ince, E. L., **Ordinary Differential Equations.** Dover, New york (1956).

[2] Chazy, J., 1911 - *Sur les équations différentielles du troisiéme ordre et d'ordre supérieur dont 1'intégrale générale a ses points critiques fixes*, Acta Mathematica, Vol. 34, p: 317-385.

[3] Bureau, F., 1964 - *Sur les équations différentielles du quatriéme ordre dont 1'intégrale générale est á points critiques fixes*, Comptes rendus de l'Académie des Sciences, Paris, Vol. 258, p: 38-40.

[4] Exton, H., 1971 - *On nonlinear ordinary differential equations with fixed critical points*, Rendiconti di Matematica, Vol. 4, p: 385-448.

[5] Martynov, I. P., 1973 - *Differential Equations with Stationary Critical Singularities*, Differential Equations, Vol. 9, p: 1368-1376.

[6] Harada, H., Oishi, S., 1985 - *A new approach to completely integrable partial differential equations by means of the singularity analysis*, J. Phys. Soc. Japan, Vol. 54, p: 51-56.

[7] Cosgrove, C. M., 2000 - *Higher-Order Painlevé Equations in the Polynomial Class 1, Bureau Symbol P2*, Stud. Appl. Math., Vol. 104, p: 1-65.

[8] Muğan, U. , Jrad, F.,1999 - *Painlevé test and the first Painlevé Hierarchy*, J. Phys. A, Vol. 32, p: 7933-7952.

[9] Kudryashov,N. A., 2002 - *Fouth-order analogies to the Painlevé equations*, J. Phys. A, Vol. 35, p:4617-4632.

[10] Fokas, A. S., Ablowitz, M. J., 1982 - *On a unified approach to transformations and elementary solutions of Painlevé equations*, J. Math. Phys., Vol. 23, p: 2033-2042.

[11] A. Sakka and U. Muğan, 1997-*Second-order second degree Painlevé equations related with Painlevé I, II, III equations*, J. Phys. A., Vol.30, p: 5159-5177.

[12] Sakka, A., Muğan, U., 1998 - *Second-order second-degree Painlevé equations related to Painlevé IV, V, VI equations*, J. Phys. A., Vol. 31, p: 2471-2490.

[13] Muğan, U., Sakka, A., 1999 - *Second-order second-degree Painlevé equations related with Painlevé I-VI equations and Fuchsian-type transformations*, J. Math. Phys., Vol. 40, p: 3569-3587.

[14] Sakka, A., 2001 - *Second-order fourth-degree Painlevé-type equations*, J. Phys. A, Vol. 34, p: 623-631.

[15] Gordoa, P. R., Joshi, N., Pickering, A., 1999 - *Mappings preserving locations of movable poles: a new extension of the truncation method to ordinary differential equations*, Nonlinearity, Vol. 12, p: 955-968.

[16] Gordoa, P. R., Joshi, N., Pickering, A., 2001 - *Truncation-type methods and Bäcklund transformations for ordinary differential equations: the third and fifth Painlevé equations*, Glasgow Math. J., Vol. 43A, p: 23-32.

[17] Gordoa, P. R., Joshi, N., Pickering, A., 2001 - *Mappings preserving locations of movable poles II: The third and fifth Painlevé equations*, Nonlinearity, Vol. 14, p: 567-582.

[18] Gordoa, P. R., Pickering, A., 2001 - *New integrable equations of fourth order and higher degree related to Cosgrove's equation*, J.Math.Phys., Vol. 42, p: 1697-1701.

[19] Gordoa, P. R. 2001 - *Bäcklund transformations for the second member of the first Painlevé hierarchy*, Phys. Lett. A., Vol. 287, p: 365-370.

[20] Gordoa, P. R., 2003 - *Obtaining Bäcklund transformations for higher order ordinary differential equations*, Proceedings of the X fall Workshop on Geometry and Physics (Publications de la Real Sociedad Matematica Espanola, Vol. 4 Madrid, p: 255-264.

[21] Gordoa, P. R., Muğan, U., Pickering, A., Sakka, A., 2004 - *Bäcklund transformations for higher order Painlevé equations*, Chaos, Solitons and Fractals, Vol. 22, p: 1103-1115.

[22] Kudryashov, N. A., 1997 - *The first and second Painlevé equations of higher order and some relations between them*, Phys. Lett. A, Vol. 224, p: 353-360.

[23] Ablowitz, M. J., Segur, H., 1977 - *Exact linearization of a Painlevé transcendent*, Phys.Rev.Lett., Vol. 38, p: 1103-1106.

[24] Yezhou, L., Yuzan, H., 2002 - *On analytic properties of higher analogs of the second Painlevé equations*, J. Math. Phys., Vol. 43, p: 1106-1115.

[25] Sakka, A. H., 2005 - *Bäcklund transformations for fifth-order Painlevé equations*, Z. Naturforsch A, Vol. 60, p: 681-686.