Double Sequences and Double Series

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Abstract

This research considers two traditional important questions, which are interesting, at least to most mathematicians. The first question arises in the theory of double sequences of complex numbers, which concerns the relationship, if any, between the following three limits of a **double sequence** $s : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{C}$:

1. $\lim_{n,m\to\infty} s(n, m)$,
2. $\lim_{n\to\infty}(\lim_{m\to\infty} s(n, m))$,
3. $\lim_{m\to\infty}(\lim_{n\to\infty} s(n, m))$.

In particular, we'll address the question of when can we interchange the order of the limit for a double sequence $\{s(n, m)\}$; that is, when the limit (2) above equals the limit (3) above. The answer to this question is found in Theorem 2.13.

The second question arises in the theory of double series of complex numbers, which concerns the relationship, if any, between the following series:

4. $\sum_{n,m=1}^{\infty} z(n, m)$,
5. $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} z(n, m))$,
6. $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} z(n, m))$.

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In particular, we'll address the question of when can we interchange the order of summation in a **doubly indexed infinite series**; that is, when the series (5) above equals the series (6) above. The answers to this question are found in Theorems 7.5, 8.6, and 9.5.

The topics of the above-mentioned two questions have not received enough attention within the mathematical community, so that there has been scattered answers in the literature (see [1-7]). Up to this moment, one can't find a single textbook or a research paper that gives a full account to such topics. In this technical article, we'll, among other things, attempt to give such an expository account which will summarize facts from the basic theory of double sequences and double series and gives detailed proofs of them. Many of the results collected are well known and can be found in the supplied references.

1 Introduction

The theory of double sequences and double series is an extension of the single or ordinary sequences and series. To each double sequence $s : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{C}$, there corresponds three important limits; namely:

1. $\lim_{n,m\to\infty} s(n,m)$,
2. $\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m))$,
3. $\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m))$.

The important question that is usually considered in this regard is the question of when can we interchange the order of the limit for a double sequence $(s(n,m))$; that is, when the limit (2) above equals the limit (3) above. In the literature [1, 3, 5, 7], there has been several answers to this question. We combined these answers and came up the following result (see Theorem 2.13):

Let $\lim_{n,m\to\infty} s(n,m) = a$. *Then the iterated limits* $\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m))$ *and* $\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m))$ *exist and both are equal to* a *if and only if*

(i) $\lim_{m\to\infty} s(n,m)$ *exists for each* $n \in \mathbf{N}$, *and*

(ii) $\lim_{n\to\infty} s(n,m)$ *exists for each* $m \in \mathbf{N}$.

Apostol [1, Theorem 8.39] proved a partial converse of the above result by using **uniform convergence**; see Theorem 2.15. In the course of this research we'll develop a theory for double sequences that is parallel to the theory of of single sequences. More precisely, we'll **contribute** with the following study of double sequences and their limits:

- Double and iterated limits.
- Monotone double limits.
- Cauchy Criterion for double limits.
- Limit theorems for double sequences.
- Uniform convergence and double limits.
- Subsequences of double sequences and their convergence.
- Divergence theorem for double sequences and its applications.

The theory of **double series** is intimately related to the theory of **double sequences**. To each double sequence $z : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{C}$, there corresponds three important sums; namely:

1. $\sum_{n,m=1}^{\infty} z(n,m)$,
2. $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} z(n,m))$,
3. $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} z(n,m))$.

The most important question that is usually asked in this regard is the question of when can we interchange the order of summation in a **doubly indexed infinite series**; that is, when the series (2) above equals the series (3) above. In the literature [3, 5, 7], one possible answer to this question is perhaps the following result (see Theorem 7.5), which is a generalization of its counterpart for double sequences that is stated above:

If $\{z(n,m)\}$ *is a double sequence of complex numbers satisfying*

a. $\sum_{n,m=1}^{\infty} z(n,m)$ *is convergent, with sum* s*,*

b. for every $m \in N$*, the series* $\sum_{n=1}^{\infty} z(n,m)$ *is convergent, and* *c. for every $n \in N$, the series $\sum_{m=1}^{\infty} z(n,m)$ is convergent,*

then the iterated series

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} z(n,m)) = \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} z(n,m)) = s.$$

Another possible answer to the above-posed question which is based on known results found in [1, 7] is summarized in Theorem 9.5, which is analogous with the well-known Fubini's Theorem from measure theory. In the course of establishing these results, we'll **contribute** with the following study of the basic properties of double series and their associated iterated series:

- Double series, iterated series and their convergence.
- Double series of nonnegative terms and some convergence tests.
- Sufficient conditions for equality of iterated series.

Throughout this article, the symbols **R**, **C**, **Z** and **N** denote, respectively, the set of all real numbers, all complex numbers, all integers, and all natural numbers. The notation := means "equals by definition".

2 Double Sequences and Their Limits

In this section, we introduce double sequences of complex numbers and we shall give the definition of their convergence, divergence and oscillation. Then we study the relationship between double and iterated limits of double sequences.

2.1 Definition. A *double sequence* of complex numbers is a function $s : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{C}$. We shall use the notation $(s(n,m))$ or simply $(s_{n,m})$. We say that a double sequence $(s(n,m))$ *converges* to $a \in \mathbf{C}$ and we write $\lim_{n,m\to\infty} s(n,m) = a$, if the following condition is satisfied: For every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbf{N}$ such that

$$|s(n,m) - a| < \epsilon \qquad \forall\, n,m \geq N.$$

The number a is called the *double limit* of the double sequence $(s(n,m))$. If no such a exists, we say that the sequence $(s(n,m))$ *diverges.*

2.2 Definition. Let $(s(n,m))$ be a double sequence of real numbers.

(i) We say that $(s(n,m))$ *tends to* ∞, and we write $\lim_{n,m\to\infty} s(n,m) = \infty$, if for every $\alpha \in \mathbf{R}$, there exists $K = K(\alpha) \in \mathbf{N}$ such that if $n, m \geq K$, then $s(n,m) > \alpha$.

(ii) We say that $(s(n,m))$ *tends to* $-\infty$, and we write $\lim_{n,m\to\infty} s(n,m) = -\infty$, if for every $\beta \in \mathbf{R}$, there exists $K = K(\beta) \in \mathbf{N}$ such that if $n, m \geq K$, then $s(n,m) < \beta$.

We say that $(s(n,m))$ is *properly divergent* in case we have $\lim_{n,m\to\infty} s(n,m) = \infty$ or $\lim_{n,m\to\infty} s(n,m) = -\infty$. In case $(s(n,m))$ does not converge to $a \in \mathbf{R}$ and also it does not diverge properly, then we say that $(s(n,m))$ *oscillates finitely or infinitely* according as $(s(n,m))$ is also bounded or not. For example, the sequence $((-1)^{n+m})$ oscillates finitely, while the sequence $((-1)^{n+m}(n+m))$ oscillates infinitely.

2.3 Example.

(a) For the double sequence $s(n,m) = \frac{1}{n+m}$, we have

$$\lim_{n,m\to\infty} s(n,m) = 0.$$

To see this, given $\epsilon > 0$, choose $N \in \mathbf{N}$ such that $N > \frac{2}{\epsilon}$. Then $\forall\, n, m \geq N$, we have $\frac{1}{n}, \frac{1}{m} \leq \frac{1}{N}$, which implies that

$$| \; s(n,m) - 0 \; |=| \; \frac{1}{n+m} \; |< \frac{1}{n} + \frac{1}{m} < \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \epsilon.$$

(b) The double sequence $s(n,m) = \frac{n}{n+m}$ is divergent. Indeed, for all sufficiently large $n, m \in \mathbf{N}$ with $n = m$, we have $s(n,m) = \frac{1}{2}$, whereas for all sufficiently large $n, m \in \mathbf{N}$ with $n = 2m$, we have $s(n,m) = \frac{2}{3}$. It follows that $s(n,m)$ does not converge to a for any $a \in \mathbf{R}$ as $n, m \to \infty$.

(c) The double sequence $s(n,m) = n + m$ is properly divergent to ∞. Indeed, given $\alpha \in \mathbf{R}$, there exists $K \in \mathbf{N}$ such that $K > \alpha$. Then $n, m \geq K \quad \Rightarrow \quad n + m > \alpha$.

(d) The double sequence $s(n,m) = 1 - n - m$ is properly divergent to $-\infty$. Indeed, given $\beta \in \mathbf{R}$, there exists $K \in \mathbf{N}$ such that $K > -\frac{\beta}{2} + \frac{1}{2}$. Then $n, m \geq K \quad \Rightarrow \quad -n, -m < \frac{\beta}{2} - \frac{1}{2} \quad \Rightarrow \quad 1 - n - m < \beta$.

2.4 Theorem *(Uniqueness of Double Limits). A double sequence of complex numbers can have at most one limit.*

Proof: Suppose that a, a' are both limits of $(s(n,m))$. Then given $\epsilon > 0$, there exist natural numbers N_1, N_2 such that

$$n, m \geq N_1 \quad \Rightarrow \quad \mid s(n,m) - a \mid < \frac{\epsilon}{2} \tag{1}$$

and such that

$$n, m \geq N_2 \quad \Rightarrow \quad \mid s(n,m) - a' \mid < \frac{\epsilon}{2}. \tag{2}$$

Let $N := \max\{N_1, N_2\}$. Then for all $n, m \geq N$, implications (1) and (2) yield

$$\begin{aligned} 0 \leq |a - a'| &= |a - s(n,m) + s(n,m) - a'| \\ &\leq |s(n,m) - a| + |s(n,m) - a'| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

It follows that $a - a' = 0$, and hence the limit is unique. ∎

2.5 Definition. A double sequence $(s(n,m))$ is called *bounded* if there exists a real number $M > 0$ such that $|s(n,m)| \leq M \quad \forall\, n, m \in \mathbf{N}$.

2.6 Theorem. *A convergent double sequence of complex numbers is bounded.*

Proof: Suppose $s(n,m) \to a$ and let $\epsilon = 1$. Then there exists $N \in \mathbf{N}$ such that

$$n, m \geq N \quad \Rightarrow \quad |s(n,m) - a| < 1.$$

This and the triangle inequality yield that $|s(n,m)| < 1 + |a| \;\forall\, n, m \geq N$. Let

$$M := \max\{|s(1,1)|, |s(1,2)|, |s(2,1)|, \ldots, |s(N-1, N-1)|, |a| + 1\}.$$

Clearly, $|s(n,m)| \leq M \quad \forall\, n, m \in \mathbf{N}$. ∎

2.7 Definition. For a double sequence $(s(n,m))$, the limits

$$\lim_{n \to \infty} (\lim_{m \to \infty} s(n,m)), \qquad \text{and} \qquad \lim_{m \to \infty} (\lim_{n \to \infty} S(n,m))$$

are called *iterated limits.*

Outrightly there is no reason to suppose the equality of the above two iterated limits whenever they exist, as the following example shows.

2.8 Example. Consider the sequence $s(n,m) = \frac{n}{m+n}$ of Example 2.3(b). Then for every $m \in \mathbf{N}$, $\lim_{n \to \infty} s(n,m) = 1$ and hence

$$\lim_{m \to \infty} (\lim_{n \to \infty} s(n,m)) = 1.$$

While for every $n \in \mathbf{N}$, $\lim_{m\to\infty} s(n,m) = 0$ and hence

$$\lim_{n\to\infty}(\lim_{m\to\infty} S(n,m)) = 0.$$

Note that the double limit of this sequence does not exist, as has been shown in Example 2.3(b).

In the theory of double sequences, one of the most interesting questions is the following: For a convergent double sequence, is it always the case that the iterated limits exist? The answer to this question is no, as the following example shows.

2.9 Example. Consider the sequence $s(n,m) = (-1)^{n+m}(\frac{1}{n} + \frac{1}{m})$. Clearly, $\lim_{n,m\to\infty} s(n,m) = 0$. In fact, given $\epsilon > 0$, choose $N \in \mathbf{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Then we have

$$n, m \geq N \qquad \Rightarrow \qquad |(-1)^{n+m}(\frac{1}{n} + \frac{1}{m})| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \epsilon.$$

But $\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m))$ does not exist, since $\lim_{m\to\infty} s(n,m)$ does not exist, and also $\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m))$ does not exist, since $\lim_{n\to\infty} s(n,m)$ does not exist.

It should be noted that, in general, the existence and the values of the iterated and double limits of a double sequence $(s(n,m))$ depend on its form. While one of these limits exists, the other may or may not exist and even if these exist, their values may differ. Besides Examples 2.8 and 2.9, the following examples shed some light on these cases.

2.10 Example.

(a) For the sequence $s(n,m) = \frac{1}{n} + \frac{1}{m}$, note first that the double limit $\lim_{n,m\to\infty} s(n,m) = 0$. Indeed, given $\epsilon > 0$, $\exists\, N \in \mathbf{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Then,

$$n, m \geq N \qquad \Rightarrow \qquad |\frac{1}{n} + \frac{1}{m}| = \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \epsilon.$$

Moreover, since $\lim_{n\to\infty} s(n,m) = \frac{1}{m}$ and $\lim_{m\to\infty} s(n,m) = \frac{1}{n}$, it follows that the iterated limits also exist and

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = \lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = 0.$$

(b) Consider the sequence $s(n,m) = (-1)^m(\frac{1}{n} + \frac{1}{m})$. Clearly, by an argument similar to that given in part (a), we have $\lim_{n,m\to\infty} s(n,m) = 0$. Also, the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = 0$, since $\lim_{n\to\infty} s(n,m)) = \frac{(-1)^m}{m}$. But the other iterated limit $\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m))$ does not exist, since $\lim_{m\to\infty} s(n,m)$ does not exist.

(c) For the sequence $s(n,m) = (-1)^{n+m}$, it is clear that neither the double limit nor the iterated limits exist.

The next result gives a necessary and sufficient condition for the existence of an iterated limit of a convergent double sequence.

2.11 Theorem. *Let* $\lim_{n,m\to\infty} s(n,m) = a$. *Then* $\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = a$ *if and only if* $\lim_{n\to\infty} s(n,m)$ *exists for each* $m \in \mathbf{N}$.

Proof: The necessity is obvious. As for sufficiency, assume $\lim_{n\to\infty} s(n,m) = c_m$ for each $m \in \mathbf{N}$. We need to show that $c_m \to a$ as $m \to \infty$. Let $\epsilon > 0$ be given. Since $s(n,m) \to a$ as $n, m \to \infty$, there exists $N_1 \in \mathbf{N}$ such that

$$n, m \geq N_1 \qquad \Rightarrow \qquad |s(n,m) - a| < \frac{\epsilon}{2},$$

and since for each $m \in \mathbf{N}$, $s(n,m) \to c_m$ as $n \to \infty$, there exists $N_2 \in \mathbf{N}$ such that

$$n \geq N_2 \qquad \Rightarrow \qquad |s(n,m) - c_m| < \frac{\epsilon}{2}.$$

Now choose $n \geq \max\{N_1, N_2\}$. Then $\forall\, m \geq N_1$, we have

$$\begin{aligned} |c_m - a| &= |c_m - s(n,m) + s(n,m) - a| \\ &\leq |c_m - s(n,m)| + |s(n,m) - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, $c_m \to a$ as $m \to \infty$. ■

It should be noted that a theorem similar to Theorem 2.11 holds with an interchange of the n and m symbols. More precisely, we have

2.12 Theorem. *Let* $\lim_{n,m\to\infty} s(n,m) = a$. *Then* $\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = a$ *if and only if* $\lim_{m\to\infty} s(n,m)$ *exists for each* $n \in \mathbf{N}$.

Combining Theorems 2.11 and 2.12, we obtain the following result.

2.13 Theorem. *Let* $\lim_{n,m\to\infty} s(n,m) = a$. *Then the iterated limits*

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) \quad \textit{and} \quad \lim_{m\to\infty}(\lim_{n\to\infty} s(n,m))$$

exist and both are equal to a if and only if *(i)* $\lim_{m\to\infty} s(n,m)$ *exists for each* $n \in \mathbf{N}$*, and*

(ii) $\lim_{n\to\infty} s(n,m)$ *exists for each* $m \in \mathbf{N}$*.*

The following example shows that the converse of Theorem 2.11 or Theorem 2.12 is not true.

2.14 Example. Consider the sequence $s(n,m) = \frac{nm}{n^2+m^2}$. Clearly, for each $m \in \mathbf{N}$, $\lim_{n\to\infty} s(n,m) = 0$ and hence $\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = 0$. But $s(n,m) = \frac{1}{2}$ when $n = m$ and $s(n,m) = \frac{2}{5}$ when $n = 2m$, and hence it follows that the double limit $\lim_{n,m\to\infty} s(n,m)$ cannot exist in this case.

The next result can be viewed as a partial converse of Theorem 2.11. But, first, recall that a sequence of functions $\{f_n\}$ is said to *converge uniformly* to a function f on a set X if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ (depends only on ϵ) such that

$$n \geq N \qquad \Rightarrow \qquad |f_n(x) - f(x)| < \epsilon \qquad \forall\, x \in X.$$

2.15 Theorem. *If* $(s(n,m))$ *is a double sequence such that*

(i) the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = a$*, and*

(ii) the limit $\lim_{n\to\infty} s(n,m)$ *exists uniformly in* $m \in \mathbf{N}$*,*

then the double limit $\lim_{n,m\to\infty} s(n,m) = a$*.*

Proof: For each $n \in \mathbf{N}$, define a function f_n on $\mathbf{N}$ by

$$f_n(m) := s(n,m) \qquad \forall\, m \in \mathbf{N}.$$

Then, by hypothesis (ii), $f_n \to f$ uniformly on $\mathbf{N}$, where $f(m) := \lim_{n\to\infty} s(n,m)$. So given $\epsilon > 0$, there exists $N_1 \in \mathbf{N}$ such that

$$n \geq N_1 \qquad \Rightarrow \qquad |s(n,m) - f(m)| < \frac{\epsilon}{2} \qquad \forall\, m \in \mathbf{N}.$$

Since, by hypothesis (ii), $\lim_{m\to\infty} f(m) = \lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = a$, then for the same ϵ, there exists $N_2 \in \mathbf{N}$ such that

$$m \geq N_2 \qquad \Rightarrow \qquad |f(m) - a| < \frac{\epsilon}{2}.$$

Now, letting $N := \max\{N_1, N_2\}$, we have

$$\begin{aligned} n,m \geq N \quad \Rightarrow \quad |s(n,m) - a| &\leq |s(n,m) - f(m)| + |f(m) - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which means that $\lim_{n,m\to\infty} s(n,m) = a$. ∎

It should be noted that the hypothesis that $\lim_{n\to\infty} s(n,m)$ exists *uniformly* in $m \in \mathbf{N}$ in the theorem above cannot be weakened to $\lim_{n\to\infty} s(n,m)$ exists for every $m \in \mathbf{N}$. Indeed, reconsider the double sequence $s(n,m) = \frac{nm}{n^2+m^2}$ of Example 2.14. It can be easily seen that

(1) $(s(n,m))$ is bounded, since $|s(n,m)| \le 1 \; \forall\, n,m \in \mathbf{N}$.

(2) For each $m \in \mathbf{N}$, $\lim_{n\to\infty} s(n,m) = 0$.

(3) $\lim_{n\to\infty} s(n,m) \neq 0$ uniformly in $m \in \mathbf{N}$. Indeed, if for each $n \in \mathbf{N}$ we let

$$f_n(m) := s(n,m) = \frac{nm}{n^2+m^2}, \quad m \in \mathbf{N},$$

then we obtain

$$\| f_n - 0 \|_u := \sup\{|f_n(m) - 0| : m \in \mathbf{N}\} = \mathbf{\frac{1}{2}} \quad \forall\, \mathbf{n} \in \mathbf{N},$$

which implies that $\lim_{n\to\infty} \| f_n - 0 \|_u \neq 0$.

(4) $\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = 0$.

(5) $\lim_{n,m\to\infty} s(n,m)$ does not exist, as has been shown in Example 2.14.

We conclude this section with the following remark: In Theorem 2.11, the assumption that the limit $\lim_{n\to\infty} s(n,m)$ exists for each $m \in \mathbf{N}$ does not follow from the assumption that the double limit $\lim_{n,m\to\infty} s(n,m)$ exists, as the following example shows.

2.16 Example. Consider the double sequence $s(n,m) = \frac{(-1)^n}{m}$. Note, first, that

$$\lim_{n,m\to\infty} s(n,m) = \lim_{n,m\to\infty} \frac{(-1)^n}{m} = 0.$$

Indeed, given $\epsilon > 0$, choose $N \in \mathbf{N}$ such that $\frac{1}{N} < \epsilon$. Then for all $n,m \ge N$, we have $|\frac{(-1)^n}{m} - 0| = \frac{1}{m} \le \frac{1}{N} < \epsilon$.

On the other hand, $\lim_{n\to\infty} \frac{(-1)^n}{m}$ does not exist for each fixed $m \in \mathbf{N}$, since $\lim_{n\to\infty} (-1)^n$ does not exist.

3 Cauchy Double Sequences

We present in this section the important Cauchy Criterion for convergence of double sequences.

3.1 Definition. A double sequence $(s(n,m))$ of complex numbers is called a *Cauchy sequence* if and only if for every $\epsilon > 0$, there exists a natural number $N = N(\epsilon)$ such that

$$|s(p,q) - s(n,m)| < \epsilon \qquad \forall\, p \geq n \geq N \quad \text{and} \quad q \geq m \geq N.$$

3.2 Theorem*(Cauchy Convergence Criterion for Double Sequences). A double sequence $(s(n,m))$ of complex numbers converges if and only if it is a Cauchy sequence.*

Proof: $(\Rightarrow)$: Assume that $s(n,m) \to a \quad \text{as} \quad n,m \to \infty$. Then given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $|s(n,m) - a| < \frac{\epsilon}{2} \quad \forall\, n,m \geq N$. Hence, $\forall\, p \geq n \geq N$ and $\forall\, q \geq m \geq N$, we have

$$\begin{aligned} |s(p,q) - s(n,m)| &= |s(p,q) - a + a - s(n,m)| \\ &\leq |s(p,q) - a| + |s(n,m) - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon; \end{aligned}$$

that is, $(s(n,m))$ is a Cauchy sequence.

$(\Leftarrow)$: Assume that $(s(n,m))$ is a Cauchy sequence, and let $\epsilon > 0$ be given. Taking $m = n$ and writing $s(n,n) = b_n$, we see that there exists $K \in \mathbf{N}$ such that

$$|b_p - b_n| < \epsilon \qquad \forall\, p \geq n \geq K.$$

Therefore, by Cauchy's Criterion for single sequences, the sequence (b_n) converges, say to $a \in \mathbf{C}$. Hence, there exists $N_1 \in \mathbf{N}$ such that

$$|b_n - a| < \frac{\epsilon}{2} \qquad \forall\, n \geq N_1. \tag{3}$$

Since $(s(n,m))$ is a Cauchy sequence, there exists $N_2 \in \mathbf{N}$ such that

$$|s(p,q) - b_n| < \frac{\epsilon}{2} \qquad \forall\, p,q \geq n \geq N_2. \tag{4}$$

Let $N := \max\{N_1, N_2\}$ and choose $n \geq N$. Then, by (3) and (4), we have

$$\begin{aligned} |s(p,q) - a| &\leq |s(p,q) - b_n| + |b_n - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \forall\, p,q \geq N. \end{aligned}$$

Hence, $(s(n,m))$ converges (to a). ■

4 Monotone Double Sequences

In this section, we define increasing and decreasing double sequences of real numbers and we prove a monotone convergence theorems for such sequences that are parallel to their counterparts for single sequences.

4.1 Definition. Let $(s(n,m))$ be a double sequence of real numbers.

(i) If $s(n,m) \leq s(j,k) \quad \forall\ (n,m) \leq (j,k)$ in $\mathbf{N} \times \mathbf{N}$, we say the sequence is *increasing.*

(ii) If $s(n,m) \geq s(j,k) \quad \forall\ (n,m) \leq (j,k)$ in $\mathbf{N} \times \mathbf{N}$, we say the sequence is *decreasing.*

(ii) If $(s(n,m))$ is either increasing or decreasing, then we say it is *monotone.*

4.2 Theorem*(Monotone Convergence Theorem). A monotone double sequence of real numbers is convergent if and only if it is bounded. Further:*

(a) If $(s(n,m))$ *is increasing and bounded above, then*

$$\begin{aligned}\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) &= \lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = \lim_{n,m\to\infty} s(n,m) \\ &= \sup\{s(n,m) : n,m \in \mathbf{N}\}.\end{aligned}$$

(b) If $(s(n,m))$ *is decreasing and bounded below, then*

$$\begin{aligned}\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) &= \lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = \lim_{n,m\to\infty} s(n,m) \\ &= \inf\{s(n,m) : n,m \in \mathbf{N}\}.\end{aligned}$$

Proof: It was seen in Theorem 2.6 that a convergent sequence must be bounded.

Conversely, let $(s(n,m))$ be a bounded monotone sequence. Then $(s(n,m))$ is increasing or decreasing.

(a) We first treat the case that $(s(n,m))$ is increasing and bounded above. By the supremum principle of real numbers, the supremum $a^* := \sup\{s(n,m) : n,m \in \mathbf{N}\}$ exists. We shall show that the double and iterated limits of $(s(n,m))$ exist and are equal to a^*. If $\epsilon > 0$ is given, then $a^* - \epsilon$ is not an upper bound for the set $\{s(n,m) : n,m \in \mathbf{N}\}$; hence there exists natural numbers $K(\epsilon)$ and $J(\epsilon)$ such that $a^* - \epsilon < s(K,J)$. But since $(s(n,m))$ is increasing, it follows that

$$a^* - \epsilon < s(K,J) \leq s(n,m) \leq a^* < a^* + \epsilon \qquad \forall\ (n,m) \geq (K,J),$$

and hence

$$|s(n,m) - a^*| < \epsilon \qquad \forall\ (n,m) \geq (K,J).$$

Since $\epsilon > 0$ was arbitrary, it follows that $(s(n,m))$ converges to a^*.

Next, to show that

$$\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = \lim_{n,m\to\infty} s(n,m) = a^*,$$

note that since $(s(n,m))$ is bounded above, then, for each fixed $m \in \mathbf{N}$, the single sequence $\{s(n,m) : n \in \mathbf{N}\}$ is bounded above and increasing, so, by Monotone Convergence Theorem 3.3.2 of [2] for single sequences, we have

$$\lim_{n\to\infty} s(n,m)) = \sup\{s(n,m) : n \in \mathbf{N}\} =: l_m \qquad \forall\ m \in \mathbf{N}.$$

Hence, by Theorem 2.11, the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m))$ exists and

$$\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = \lim_{n,m\to\infty} s(n,m) = a^*.$$

Similarly, it can be shown that

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = \lim_{n,m\to\infty} s(n,m) = a^*.$$

(b) If $(s(n,m))$ is decreasing and bounded below, then the sequence $(-s(n,m))$ is increasing and bounded above. Hence, by part (a), we obtain

$$\begin{aligned}\lim_{m\to\infty}(\lim_{n\to\infty} -s(n,m)) &= \lim_{n\to\infty}(\lim_{m\to\infty} -s(n,m)) = \lim_{n,m\to\infty} -s(n,m)\\ &= \sup\{-s(n,m) : n,m \in \mathbf{N}\}\\ &= -\inf\{s(n,m) : n,m \in \mathbf{N}\}.\end{aligned}$$

Therefore it follows that

$$\begin{aligned}\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) &= \lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = \lim_{n,m\to\infty} s(n,m)\\ &= \inf\{s(n,m) : n,m \in \mathbf{N}\}. \qquad \blacksquare\end{aligned}$$

5 Theorems of Limits

In this section, we prove some results which enable us to evaluate the double and iterated limits of a double sequence.

5.1 Theorem. *If* $(s(n,m))$ *can be written as* $s(n,m) = a_n a_m$ *such that the limits* $\lim_{n\to\infty} a_n = l_1$ *and* $\lim_{m\to\infty} a_m = l_2$*, then*

$$\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = \lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = \lim_{n,m\to\infty} s(n,m) = l_1 l_2.$$

Proof: By hypothesis, we have

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = \lim_{n\to\infty}(\lim_{m\to\infty} a_n a_m) = \lim_{n\to\infty} a_n \lim_{m\to\infty} a_m = l_1 l_2$$

and

$$\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = \lim_{m\to\infty}(\lim_{n\to\infty} a_m a_n) = \lim_{m\to\infty} a_m \lim_{n\to\infty} a_n = l_2 l_1.$$

Hence

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = \lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = l_1 l_2.$$

Next, to show that $\lim_{n,m\to\infty} s(n,m) = l_1 l_2$, let Let $\epsilon > 0$ be given. Since (a_n) is bounded (being convergent), there exists $K \in \mathbf{N}$ such that

$$|a_n| \le K \qquad \forall\, n \in \mathbf{N},$$

and since $a_n \to l_1$ and $a_m \to l_2$, there exists a natural number $N = N(\epsilon)$ such that

$$|a_n - l_1| < \frac{\epsilon}{2b} \qquad \text{and} \qquad |a_m - l_2| < \frac{\epsilon}{2b} \qquad \forall\, n, m \ge N,$$

where $b := \max\{K, |l_1|\}$. Hence it follows that

$$\begin{aligned} n,m \ge N \quad \Rightarrow \quad |s(n,m) - l_1 l_2| &\le |s(n,m) - a_n l_2| + |a_n l_2 - l_1 l_2| \\ &= |a_n||a_m - l_2| + |l_2||a_n - l_1| \\ &< K\frac{\epsilon}{2b} + |l_2|\frac{\epsilon}{2b} \\ &\le 2b\frac{\epsilon}{2b} = \epsilon. \end{aligned}$$

Therefore it follows that $\lim_{n,m\to\infty} s(n,m) = l_1 l_2$. ■

5.2 Example. Consider the double sequence $s(n,m) = \frac{1}{nm}$, $n, m \in \mathbf{N}$. We claim that

$$\lim_{n,m\to\infty} s(n,m) = \lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = \lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = 0.$$

Indeed, write $s(n,m) = a_n a_m = (\frac{1}{n})(\frac{1}{m}) \quad \forall n, m \in \mathbf{N}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \lim_{m\to\infty} a_m = \lim_{m\to\infty} \frac{1}{m} = 0.$$

It follows from Theorem 4.1 that

$$\lim_{n,m\to\infty}\frac{1}{nm}=\lim_{n\to\infty}(\lim_{m\to\infty}\frac{1}{nm})=\lim_{m\to\infty}(\lim_{n\to\infty}\frac{1}{nm})=(\lim_{n\to\infty}\frac{1}{n})(\lim_{m\to\infty}\frac{1}{m})=0.$$

5.3 Theorem. *If $(s(n,m))$ can be written as $s(n,m)=a_n+a_m$ such that the limits $\lim_{n\to\infty} a_n=l_1$ and $\lim_{m\to\infty} a_m=l_2$, then*

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m))=\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m))=\lim_{n,m\to\infty} s(n,m)=l_1+l_2.$$

Proof: By hypothesis, we have

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m))=\lim_{n\to\infty}(\lim_{m\to\infty}(a_n+a_m))=\lim_{n\to\infty} a_n+\lim_{m\to\infty} a_m=l_1+l_2,$$

and

$$\lim_{m\to\infty}(\lim_{n\to\infty} s(n,m))=\lim_{m\to\infty}(\lim_{n\to\infty}(a_m+a_n))=\lim_{n\to\infty} a_n+\lim_{m\to\infty} a_m=l_1+l_2.$$

Next, to show that $\lim_{n,m\to\infty} s(n,m)=l_1+l_2$, let $\epsilon>0$ be given. By hypothesis, there exists a natural number $N=N(\epsilon)$ such that

$$|a_n-l_1|<\frac{\epsilon}{2} \qquad \text{and} \qquad |a_m-l_2|<\frac{\epsilon}{2} \qquad \forall\, n,m\geq N.$$

Hence, we have

$$\begin{aligned} n,m\geq N \quad \Rightarrow \quad |s(n,m)-(l_1+l_2)| &= |a_n+a_m-l_1-l_2| \\ &\leq |a_n-l_1|+|a_m-l_2| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon. \end{aligned}$$

Therefore, it follows that $\lim_{n,m\to\infty} s(n,m)=l_1+l_2$. ■

5.4 Example. Consider the double sequence $s(n,m)=\frac{1}{n}+\frac{1}{m}$, $n,m\in\mathbf{N}$. We showed in Example 2.10 that

$$\lim_{n,m\to\infty} s(n,m)=\lim_{n,m\to\infty}(\frac{1}{n}+\frac{1}{m})=0.$$

Now we can prove this result, by using Theorem 4.3, as follows:

$$\lim_{n,m\to\infty}(\frac{1}{n}+\frac{1}{m})=\lim_{n\to\infty}\frac{1}{n}+\lim_{m\to\infty}\frac{1}{m}=0.$$

Moreover, by Theorem 5.3, we have

$$\lim_{n\to\infty}(\lim_{m\to\infty}(\frac{1}{n}+\frac{1}{m})=\lim_{m\to\infty}(\lim_{n\to\infty}(\frac{1}{n}+\frac{1}{m})=\lim_{n\to\infty}\frac{1}{n}+\lim_{m\to\infty}\frac{1}{m}=0.$$

5.5 Theorem*(The Sandwich Theorem). Suppose that* $(x(n,m))$, $(s(n,m))$, *and* $(y(n,m))$ *are double sequences of real numbers such that*

$$x(n,m) \leq s(n,m) \leq y(n,m) \qquad \forall\ n,m \in \mathbf{N},$$

and that $\lim_{n,m\to\infty} x(n,m) = \lim_{n,m\to\infty} y(n,m)$. *Then* $(s(n,m))$ *is convergent and*

$$\lim_{n,m\to\infty} x(n,m) = \lim_{n,m\to\infty} s(n,m) = \lim_{n,m\to\infty} y(n,m).$$

Proof: Let $a := \lim_{n,m\to\infty} x(n,m) = \lim_{n,m\to\infty} y(n,m)$. Then given $\epsilon > 0$, there exists a natural number N such that

$$n,m \geq N \quad \Rightarrow \quad |x(n,m) - a| < \epsilon \quad \text{and} \quad |y(n,m) - a| < \epsilon.$$

Since the hypothesis implies that

$$x(n,m) - a \leq s(n,m) - a \leq y(n,m) - a \qquad \forall\ n,m \in \mathbf{N},$$

it follows that

$$-\epsilon < s(n,m) - a < \epsilon \qquad \forall\ n,m \in \mathbf{N}.$$

Since $\epsilon > 0$ was arbitrary, this implies that $\lim_{n,m\to\infty} s(n,m) = a$. ■

6 Double Subsequences

In this section, we study double subsequences and we prove some results about their convergence and its relation to the convergence of the original double sequence.

6.1 Definition. Let $(s(n,m))$ be a double sequence of complex numbers and let $(k_1, r_1) < (k_2, r_2) < \ldots < (k_n, r_n) < \ldots$ be a strictly increasing sequences of pairs of natural numbers. Then the sequence $(s(k_n, r_m))$ is called a *subsequence* of $(s(n,m))$.

Double subsequences of convergent double sequences also converge to the same limit, as the following result shows.

6.2 Theorem. *If a double sequence* $(s(n,m))$ *of complex numbers converges to a complex number* a*, then any subsequence of* $(s(n,m))$ *also converges to* a*.*

Proof: Let $(s(k_n, r_m))$ be a subsequence of $(s(n,m))$ and let $\epsilon > 0$ be given. Then there exists $N = N(\epsilon) \in \mathbf{N}$ such that

$$p, q \geq N \qquad \Rightarrow \qquad |s(p,q) - a| < \epsilon.$$

Since $k_1 \leq k_2 \leq \ldots \leq k_n \leq \ldots$ and $r_1 \leq r_2 \leq \ldots \leq r_m \leq \ldots$, we have $k_n \geq n,\ r_m \geq m\ \ \forall\, n, m \in \mathbf{N}$. Hence, it follows that

$$n, m \geq N \quad \Rightarrow \quad k_n, r_m \geq N \quad \Rightarrow \quad |s(k_n, r_m) - a| < \epsilon,$$

and therefore $\lim_{n,m\to\infty} s(k_n, r_m) = a$. ∎

6.3 Theorem. *If the iterated limits of a double sequence $(s(n,m))$ exist and satisfy*

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = \lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = a,$$

then the iterated limits for any subsequence $(s(p_n, q_m))$ exist and satisfy

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(p_n, q_m)) = \lim_{m\to\infty}(\lim_{n\to\infty} s(p_n, q_m)) = a.$$

Proof: Claim 1: If, for every $n \in \mathbf{N}$, $\lim_{m\to\infty} s(n,m) =: f(n)$ exists, then

$$\lim_{m\to\infty} s(p_n, q_m) = f(p_n) \qquad \forall\, n \in \mathbf{N}.$$

Indeed, the hypothesis that $\lim_{m\to\infty} s(n,m) =: f(n)$ exists for all $n \in \mathbf{N}$ implies that $\lim_{m\to\infty} s(p_n, m) =: f(p_n)$ exists for all $n \in \mathbf{N}$. Since $(s(p_n, q_m))_{m=1}^{\infty}$ is a subsequence of of the single sequence $(s(p_n, m))_{m=1}^{\infty}$, it follows that $\lim_{m\to\infty} s(p_n, q_m) = f(p_n)$ for all $n \in \mathbf{N}$.

Claim 2: If $\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = a$, then $\lim_{n\to\infty}(\lim_{m\to\infty} s(p_n, q_m)) = a$. Indeed, the hypothesis implies that $\lim_{m\to\infty} s(n,m) =: f(n)$ exists for each $n \in \mathbf{N}$ and $\lim_{n\to\infty} f(n) = a$. Hence, by Claim 1,

$$\lim_{m\to\infty} s(p_n, q_m) = f(p_n) \qquad \forall\, n \in \mathbf{N}. \tag{5}$$

Since $(f(p_n))_{n=1}^{\infty}$ is a subsequence of the sequence $(f(n))_{n=1}^{\infty}$ and $\lim_{n\to\infty} f(n) = a$, it follows that $\lim_{n\to\infty} f(p_n) = a$. This and (5) yield that

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(p_n, q_m)) = a,$$

which proves the claim.

By interchanging the roles of n and m in Claims 1 and 2 we get that

$$\text{if} \quad \lim_{m\to\infty}(\lim_{n\to\infty} s(n,m)) = a, \quad \text{then} \quad \lim_{m\to\infty}(\lim_{n\to\infty} s(p_n,q_m)) = a.$$

Now this and Claim 2 prove the theorem. ■

While not every double sequence of real numbers is monotone, we will now show that every double sequence has a monotone subsequence.

6.4 Theorem. *Every double sequence of real numbers has a monotone subsequence.*

Proof: To serve this proof we will say that the term $s(p,q)$ is a *peak* if

$$s(p,q) \geq s(n,m) \qquad \forall\ (n,m) \in \mathbf{N}\times\mathbf{N} \text{ with } (p,q) \leq (n,m).$$

That is, $s(p,q)$ is never exceeded by any term that follow it. We will consider two cases, depending on whether $(s(n,m))$ has infinitely many, or finitely many, peaks.

Case 1: $(s(n,m))$ has infinitely many peaks. In this case, we order the peaks by increasing subscripts. So we have the peaks $s(p_1,q_1), s(p_2,q_2), \ldots, s(p_k,q_k), \ldots$, where $(p_1,q_1) < (p_2,q_2) < \ldots < (p_k,q_k) < \ldots$. Since each of the terms is a peak, we have

$$s(p_1,q_1) \geq s(p_2,q_2) \geq \ldots > s(p_k,q_k) \geq \ldots.$$

Hence the subsequence $(s(p_k,q_k))$ of peaks is a decreasing subsequence of $(s(n,m))$.

Case 2: $(s(n,m))$ has a finite number (possibly zero) of peaks. Let these peaks be $s(p_1,q_1), s(p_2,q_2), \ldots, s(p_j,q_j)$. Let $k_1 := p_j + 1$ and $r_1 := q_j + 1$. Then (k_1,r_1) is the first index beyond the last peak. Since $s(k_1,r_1)$ is not a peak, there exists $(k_2,r_2) > (k_1,r_1)$ such that $s(k_1,r_1) < s(k_2,r_2)$. Since $s(k_2,r_2)$ is not a peak, there exists $(k_3,r_3) > (k_2,r_2)$ such that $s(k_2,r_2) < s(k_3,r_3)$. Continuing in this process, we obtain an increasing subsequence $(s(k_n,r_m))$ of $(s(n,m))$. ■

As a consequence of Theorem 6.4 and the Monotone Convergence Theorem 4.2, we obtain a Bolzano-Weierstrass Theorem for double sequences.

6.5 Theorem*(Bolzano-Weierstrass Theorem). A bounded double sequence of real numbers has a convergent monotone subsequence.*

Proof: Let $(s(n,m))$ be a bounded double sequence of real numbers. By Theorem 6.4, it has a monotone subsequence $(s(k_n,r_m))$. Since this subsequence is also bounded, it follows from the Monotone Convergence Theorem 4.2 that the subsequence is convergent. ■

6.6 Corollary. *If $(s(n,m))$ is a bounded double sequence of real numbers, then there exists a convergent subsequence $(s(k_n, r_m))$ such that the iterated limits*

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(k_n, r_m)) \quad \textit{and} \quad \lim_{m\to\infty}(\lim_{n\to\infty} s(k_n, r_m))$$

exist, and are both equal to the double limit $\lim_{n,m\to\infty} s(k_n, r_m)$.

Proof: Since $(s(n,m))$ is a bounded double sequence of real numbers, it follows from Theorem 6.5 that it has a monotone subsequence $(s(k_n, r_m))$ such that $\lim_{n,m\to\infty} s(k_n, r_m)$ exists. Since this subsequence is also bounded, it follows from the Monotone Convergence Theorem 4.2 that

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(k_n, r_m)) = \lim_{m\to\infty}(\lim_{n\to\infty} s(k_n, r_m)) = \lim_{n,m\to\infty} s(k_n, r_m). \qquad \blacksquare$$

Subsequences of double sequences can be utilized to provide a test for the divergence of a double sequence.

6.7 Theorem*(Divergence Criterion). Let $(s(n,m))$ be a double sequence of complex numbers. Then the following statements are equivalent:*

(i) The sequence $(s(n,m))$ does not converge to $a \in \mathbf{C}$.

(ii) There exists an $\epsilon_0 > 0$ such that for any $k \in \mathbf{N}$, there exist $n_k, m_k \in \mathbf{N}$ such that $n_k, m_k \geq k$ and $|s(n_k, m_k) - a| \geq \epsilon_0$.

(iii) There exists an $\epsilon_0 > 0$ and a subsequence $(s(n_k, m_k))$ of $(s(n,m))$ such that $|s(n_k, m_k) - a| \geq \epsilon_0$ for all $k \in \mathbf{N}$.

Proof: (i) $\Rightarrow$ (ii): If $(s(n,m))$ does not converge to a, then there exists $\epsilon_0 > 0$ such that $\forall\, k \in \mathbf{N}$, the statement $(n, m \geq k \quad \Rightarrow \quad |s(n,m) - a| < \epsilon_0)$ is false; that is, $\forall\, k \in \mathbf{N}$, there are natural numbers $n_k, m_k \geq k$ such that $|s(n_k, m_k) - a| \geq \epsilon_0$.

(ii) $\Rightarrow$ (iii): Let ϵ_0 be as in (ii) and let $n_1, m_1 \in \mathbf{N}$ be such that $n_1, m_1 \geq 1$ and $|s(n_1, m_1) - a| \geq \epsilon_0$. Now let $n_2, m_2 \in \mathbf{N}$ be such that $n_2 \geq n_1 + 1$, $m_2 \geq m_1 + 1$ and $|s(n_2, m_2) - a| \geq \epsilon_0$. Let $n_3, m_3 \in \mathbf{N}$ be such that $n_3 \geq n_2 + 1$, $m_3 \geq m_2 + 1$ and $|s(n_3, m_3) - a| \geq \epsilon_0$. Continuing in this process, we obtain a strictly increasing sequence $\{(n_k, m_k)\}$ of ordered pairs in $\mathbf{N}\times\mathbf{N}$, and hence a subsequence $(s(n_k, m_k))$ of $(s(n,m))$ such that $|s(n_k, m_k) - a| \geq \epsilon_0$.

(iii) $\Rightarrow$ (i): Suppose there exists an $\epsilon_0 > 0$ and a subsequence $(s(n_k, m_k))$ of $(s(n,m))$ such that $|s(n_k, m_k) - a| \geq \epsilon_0$ for all $k \in \mathbf{N}$. Then $(s(n,m))$ cannot converge to a. For, if $s(n,m) \to a$, then, by Theorem 6.2, the subsequence $(s(n_k,m_k))$ would converge to a, which is impossible in view of our supposition. ■

As an application of the Divergence Criterion, we obtain the following interesting result.

6.8 Theorem. *Let $(s(n,m))$ be a bounded double sequence of complex numbers and let $a \in \mathbf{C}$ have the property that every convergent subsequence of $(s(n,m))$ converges to a. Then the sequence $(s(n,m))$ converges a.*

Proof: Assume, on the contrary, that the sequence $(s(n,m))$ does not converge to a. Then, by the Divergence Theorem 6.7, there exists an $\epsilon_0 > 0$ and a subsequence $(s(n_k,m_k))$ of $(s(n,m))$ such that

$$|s(n_k,m_k) - a| \geq \epsilon_0 \qquad \forall\ k \in \mathbf{N}. \tag{6}$$

Since the sequence $(s(n,m))$ is bounded, then so is the subsequence $(s(n_k,m_k))$. It follows from the Bolzano-Weierstrass Theorem 6.5 that $(s(n_k,m_k))$ has a convergent subsequence, say $(s(n_p,m_q))$. Hence, by hypothesis, $\lim_{p,q\to\infty} s(n_p,m_q) = a$. This means that there exists $N = N(\epsilon_0)$ such that

$$|s(n_p,m_q) - a| \geq \epsilon_0 \qquad \forall\ p,q \geq N. \tag{7}$$

Since every term of $(s(n_p,m_q))$ is also a term of $(s(n_k,m_k))$, we see that (6) gives a contradiction to (7). ■

7 Double Series

In this section, we introduce double series and we shall give the definition of their convergence and divergence. Then we study the relationship between double and iterated series, and we give a sufficient condition for equality of iterated series.

7.1 Definition. Let $z : \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{C}$ be a double sequence of complex numbers and let $(s(n,m))$ be the double sequence defined by the equation

$$s(n,m) := \sum_{i=1}^{n}(\sum_{j=1}^{m} z(i,j)).$$

The pair (z,s) is called a *double series* and is denoted by the symbol $\sum_{n,m=1}^{\infty} z(n,m)$ or, more briefly by $\sum z(n,m)$. Each number $z(n,m)$ is called a *term* of the double series and each $s(n,m)$ is called a *partial sum*. We say that the double series

$\sum_{n,m=1}^{\infty} z(n,m)$ is *convergent* to the sum s if $\lim_{n,m\to\infty} s(n,m) = s$. If no such limit exists, we say that the double series $\sum_{n,m=1}^{\infty} z(n,m)$ is *divergent.* The series

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} z(n,m)) \qquad \text{and} \qquad \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} z(n,m))$$

are called *iterated series.*

7.2 Theorem. *If the double series $\sum_{n,m=1}^{\infty} z(n,m)$ is convergent, then*

$$\lim_{n,m\to\infty} z(n,m) = 0.$$

Proof: Since the double series $\sum_{n,m=1}^{\infty} z(n,m)$ is convergent, say to s, then its sequence of partial sums $(s(n,m))$ converges to s. So given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that

$$|s(n,m) - s| < \frac{\epsilon}{4} \qquad \forall\, n,m \geq N.$$

It follows that for all $n,m \geq N$, we have

$$\begin{aligned} |z(n,m)| &= |s(n,m) + s(n-1,m-1) - s(n,m-1) - s(n-1,m)| \\ &\leq |s(n,m) - s| + |s(n-1,m-1) - s| + |s(n,m-1) - s| \\ &\quad +|s(n-1,m) - s| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

Therefore, $\lim_{n,m\to\infty} z(n,m) = 0$. ■

7.3 Theorem*(Cauchy Convergence Criterion for Double Series.) A double series $\sum_{n,m=1}^{\infty} z(n,m)$ of complex numbers converges if and only if its sequence of partial sums $(s(n,m))$ is Cauchy.*

Proof: It follows immediately from the Definition 7.1 and Cauchy Convergence Criterion for Double Sequences (Theorem 3.2). ■

The proof of the following theorem is straightforward, and therefore it is omitted.

7.4 Theorem. *If $\sum_{n,m=1}^{\infty} z(n,m)$ converges to s and $\sum_{n,m=1}^{\infty} w(n,m)$ converges to s', then*

(a) $\sum_{n,m=1}^{\infty} z(n,m) + w(n,m)$ converges to $s + s'$, and

(b) $\sum_{n,m=1}^{\infty} c\, z(n,m)$ converges to $c\, s$ for any $c \in \mathbf{C}$.

The next result gives necessary and sufficient conditions for the convergence and equality of the iterated series of a convergent double series. The proof of this result follows immediately from Theorem 2.13.

7.5 Theorem. *Suppose that the double series $\sum_{n,m=1}^{\infty} z(n,m)$ is convergent, with sum s. Then the iterated series $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} z(n,m))$ and $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} z(n,m))$ are both convergent with sum s if and only if*

(a) for every $m \in \mathbf{N}$, the series $\sum_{n=1}^{\infty} z(n,m)$ is convergent, and

(b) for every $n \in \mathbf{N}$, the series $\sum_{m=1}^{\infty} z(n,m)$ is convergent.

8 Double Series of Nonnegative Terms

In this section, we give some tests of convergence for double series of nonnegative terms.

8.1 Theorem. *A double series of nonnegative terms $\sum_{n,m=1}^{\infty} z(n,m)$ converges if and only if the set of partial sums $\{s(n,m) : n, m \in \mathbf{N}\}$ is bounded.*

Proof: Let $\sum_{n,m=1}^{\infty} z(n,m)$ be a double series with $z(n,m) \geq 0 \,\forall\, n,m \in \mathbf{N}$. If $\sum_{n,m=1}^{\infty} z(n,m)$ converges, then its double sequence $(s(n,m))$ of partial sums converges and, hence, bounded, by Theorem 2.6. It follows that the set $\{s(n,m) : n, m \in \mathbf{N}\}$ is bounded.

Suppose, conversely, that the set of partial sums $\{s(n,m) : n, m \in \mathbf{N}\}$ is bounded. Then the double sequence of partial sums $(s(n,m))$ is bounded. Since the terms $z(n,m)$ of the double series are nonnegative, it is clear that the sequence $(s(n,m))$ is increasing. It follows from the Monotone Convergence Theorem 4.2 that $(s(n,m))$ converges and, hence, $\sum_{n,m=1}^{\infty} z(n,m)$ converges. ∎

8.2 Corollary. *A double series of nonnegative terms $\sum_{n,m=1}^{\infty} z(n,m)$ either converges to a finite number s or else it diverges properly to ∞.*

Proof: Let $S := \{s(n,m) : n, m \in \mathbf{N}\}$ be the set of partial sums of the double series $\sum_{n,m=1}^{\infty} z(n,m)$. Then either the set S is bounded; that is, $\sup S = s \geq 0$, and hence, by Monotone Convergence Theorem 4.2 or by Theorem 8.1, the sequence of partial sums $(s(n,m))$ converges to s and hence $\sum z(n,m) = s$, or else the set S is unbounded, and in this case it is easy to show that $\lim_{n,m\to\infty} s(n,m) = \infty$, and hence the double series $\sum z(n,m)$ properly diverges to ∞. ∎

8.3 Example. The double series $\sum_{n,m=1}^{\infty} \frac{1}{2^n 3^m}$ is convergent. Indeed, for each $n, m \in \mathbf{N}$, the partial sum

$$s(n,m) \leq (\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n})(\frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^m}).$$

Since $\sum \frac{1}{2^n}$ and $\sum \frac{1}{3^m}$ are convergent, there exists $M > 0$ such that

$$(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n})(\frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^m}) \leq M \quad \forall\, n, m \in \mathbf{N}.$$

It follows that $0 \leq s(n,m) \leq M \; \forall n, m \in \mathbf{N}$. Thus the set $\{s(n,m) : n, m \in \mathbf{N}\}$ is bounded, and therefore, by Theorem 8.1, $\sum_{n,m=1}^{\infty} \frac{1}{2^n 3^m}$ converges.

8.4 Theorem*(Comparison Test). Suppose that*

$$0 \leq u(n,m) \leq v(n,m) \quad \textit{for every} \quad n, m \in \mathbf{N}.$$

(i). If $\sum_{n,m=1}^{\infty} v(n,m)$ is convergent, then $\sum_{n,m=1}^{\infty} u(n,m)$ is convergent.

(ii). If $\sum_{n,m=1}^{\infty} u(n,m)$ is divergent, then $\sum_{n,m=1}^{\infty} v(n,m)$ is divergent

Proof: (i). Suppose $\sum_{n,m=1}^{\infty} v(n,m)$ is convergent, and let $\epsilon > 0$ be given. If $(s'(n,m))$ denotes the sequence of partial sums of the series $\sum_{n,m=1}^{\infty} v(n,m)$, then there exists $N \in \mathbf{N}$ such that

$$|s'(p,q) - s'(n,m)| < \epsilon \quad \forall\, p \geq n \geq N \text{ and } q \geq m \geq N,$$

and so if $(s(n,m))$ denotes the sequence of partial sums of the series $\sum_{n,m=1}^{\infty} u(n,m)$, then we have

$$|s(p,q) - s(n,m)| \leq (|s'(p,q) - s'(n,m)|) < \epsilon \quad \forall\, p \geq n \geq N \text{ and } q \geq m \geq N.$$

Hence, $\sum_{n,m=1}^{\infty} u(n,m)$ converges.

(ii). Suppose $\sum_{n,m=1}^{\infty} u(n,m)$ is divergent. Then, by Corollary 8.2, we have $\lim_{n,m\to\infty} s(n,m) = \infty$. Since, by hypothesis, $s(n,m) \leq s'(n,m) \quad \forall\, n, m \in \mathbf{N}$, it follows that $\lim_{n,m\to\infty} s'(n,m) = \infty$. Therefore, $\sum_{n,m=1}^{\infty} v(n,m)$ diverges. ■

8.5 Example. The double series $\sum_{n,m=1}^{\infty} \sin \frac{1}{2^n 3^m}$ is convergent. Indeed, this follows from the facts that $\sin \frac{1}{2^n 3^m} \leq \frac{1}{2^n 3^m} \quad \forall\, n, m \in \mathbf{N}$, $\sum_{n,m=1}^{\infty} \frac{1}{2^n 3^m}$ is convergent (by Example 8.3), and by applying the Comparison Test.

We now prove the following rearrangement theorem for later use.

8.6 Theorem. *Let $a(n,m) \in [0,\infty]$ for each $(n,m) \in \mathbf{N} \times \mathbf{N}$ and let ϕ be a one-to-one mapping of $\mathbf{N}$ onto $\mathbf{N} \times \mathbf{N}$. Then* *(i)* $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} a(n,m)) = \sum_{k=1}^{\infty} a(\phi(k))$, *and*

(ii) $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} a(n,m)) = \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a(n,m))$.

Proof: (i) First let α be any real number less that the right side of (i). Choose $k_0 \in \mathbf{N}$ such that

$$\sum_{k=1}^{k_0} a(\phi(k)) > \alpha.$$

Next choose $n_0, m_0 \in \mathbf{N}$ such that

$$\{\phi(k) : 1 \le k \le k_0\} \subseteq \{(n,m) : 1 \le n \le n_0,\ 1 \le m \le m_0\}.$$

Then we have

$$\alpha < \sum_{k=1}^{k_0} a(\phi(k)) \le \sum_{n=1}^{n_0}(\sum_{m=1}^{m_0} a(n,m)) \le \sum_{n=1}^{n_0}(\sum_{m=1}^{\infty} a(n,m)) \le \sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} a(n,m)).$$

Since α was arbitrary, it follows that

$$\sum_{k=1}^{\infty} a(\phi(k)) \le \sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} a(n,m)). \tag{8}$$

Now let β be any real number less than the left hand side of (i) and then choose $n_1 \in \mathbf{N}$ such that

$$\beta < \sum_{n=1}^{n_1}(\sum_{m=1}^{\infty} a(n,m)).$$

Using induction on n_1, we have

$$\sum_{n=1}^{n_1}(\sum_{m=1}^{\infty} a(n,m)) = \sum_{m=1}^{\infty}(\sum_{n=1}^{n_1} a(n,m));$$

hence, we can choose $m_1 \in \mathbf{N}$ such that

$$\beta < \sum_{m=1}^{m_1}(\sum_{n=1}^{n_1} a(n,m)). \tag{9}$$

Now choose $k_1 \in \mathbf{N}$ such that

$$\{(n,m) : 1 \le n \le n_1,\ 1 \le m \le m_1\} \subseteq \{\phi(k) : 1 \le k \le k_1\}.$$

Then we have

$$\sum_{n=1}^{n_1}(\sum_{m=1}^{m_1} a(n,m)) \le \sum_{k=1}^{k_1} a(\phi(k)) \le \sum_{k=1}^{\infty} a(\phi(k)). \tag{10}$$

Combining (9) and (10) yields

$$\beta < \sum_{k=1}^{\infty} a(\phi(k)). \tag{11}$$

Finally (11) and the arbitrariness of β prove the reverse of inequality (8) and so (i) obtains.

(ii) Note that equality (ii) will follow from equality (i) if we can show that

$$\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a(n,m)) = \sum_{k=1}^{\infty} a(\phi(k)). \tag{12}$$

To this end, note first that

$$\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a(n,m)) = \sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} a(m,n)); \tag{13}$$

so writing

$$b(n,m) := a(m,n), \tag{14}$$

and $\psi(k) = (n,m)$ if $\phi(k) = (m,n)$, we have

$$b(\psi(k)) = a(\phi(k)), \tag{15}$$

and ψ is a one-to-one mapping of $\mathbf{N}$ onto $\mathbf{N} \times \mathbf{N}$. Thus applying part (i) of this theorem to $b(n,m)$ and ψ, we have

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} b(n,m)) = \sum_{k=1}^{\infty} b(\psi(k)). \tag{16}$$

In view of (13), (14) and (15), (12) is just (16). ■

8.7 Remark. Theorem 8.6 can fail without the hypothesis that $a(n,m) \geq 0 \; \forall \, (n,m) \in \mathbf{N} \times \mathbf{N}$. For example, let

$$a(n,m) := \begin{cases} 1, & \text{if } n = m+1, \; m = 1,2,\ldots, \\ -1, & \text{if } n = m-1, \; m = 1,2,\ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} a(n,m)) = 1, \qquad \text{but } \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a(n,m)) = -1,$$

and, for any ϕ as in Theorem 8.6, the series

$$\sum_{k=1}^{\infty} a(\phi(k))$$

diverges, since its terms do not tend to 0.

Other sufficient conditions for equality of the iterated series

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} a(n,m)) \qquad \text{and} \qquad \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a(n,m))$$

are examined in the next section.

9 Absolute Convergence of Double Series

In this section, we study absolute convergence of a double series of complex numbers and we give a sufficient condition for equality of iterated series.

9.1 Definition. A double series $\sum_{n,m=1}^{\infty} z(n,m)$ of complex numbers is said to be *absolutely convergent* if the double series $\sum_{n,m=1}^{\infty} \mid z(n,m) \mid$ is convergent. The iterated series $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} z(n,m))$ is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} |\sum_{m=1}^{\infty} z(n,m)|$ is convergent.

9.2 Theorem. *Every absolutely convergent double series is convergent.*

Proof: Suppose that the double series $\sum_{n,m=1}^{\infty} z(n,m)$ converges absolutely. Then $\sum_{n,m=1}^{\infty} \mid z(n,m) \mid$ converges, and hence, by Cauchy Convergence Criterion 7.3, its sequence $(s'(n,m))$ of partial sums is Cauchy. So given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that

$$|s'(p,q) - s'(n,m)| < \epsilon \qquad \forall\ p \geq n \geq N \text{ and } q \geq m \geq N.$$

Letting $(s(n,m))$ denotes the sequence of partial sums of $\sum_{n,m=1}^{\infty} z(n,m)$, it is easy to see that

$$|s(p,q) - s(n,m)| \leq |s'(p,q) - s'(n,m)| < \epsilon \qquad \forall\ p \geq n \geq N \text{ and } q \geq m \geq N.$$

It follows then that the sequence $(s(n,m))$ is Cauchy and therefore, by Cauchy Convergence Criterion 7.3, the series $\sum_{n,m=1}^{\infty} z(n,m)$ converges. ∎

The following rearrangement theorem, which is analogous with the well-known Fubini's Theorem from measure theory, tells us that general rearrangements cannot alter sums in the presence of *absolute* convergence. It further gives a sufficient condition for equality of the iterated series of a double series of complex numbers.

9.3 Theorem. *Let* $z(n,m) \in \mathbf{C}$ *for each* $(n,m) \in \mathbf{N} \times \mathbf{N}$ *and let* ϕ *be a one-to-one mapping of* $\mathbf{N}$ *onto* $\mathbf{N} \times \mathbf{N}$. *If any of the three sums* *(i)* $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty}|z(n,m)|)$, $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty}|z(n,m)|)$, $\sum_{k=1}^{\infty}|z(\phi(k))|$ *is finite, then all of the series*

(ii) $\sum_{m=1}^{\infty} z(n,m) \qquad (n \in \mathbf{N})$,

(iii) $\sum_{n=1}^{\infty} z(n,m) \qquad (m \in \mathbf{N})$,

(iv) $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} z(n,m))$, $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} z(n,m))$, $\sum_{k=1}^{\infty} z(\phi(k))$

are absolutely convergent and the three series in (iv) all have the same sum.

Proof: By Theorem 8.6, the three series in (i) all have the same sum and so, by hypothesis, they are all finite. Since no term of a convergent series of nonnegative terms can equal ∞, it follows that all the series in (ii) and (iii) are absolutely convergent; hence convergent.

Write

$$b_n := \sum_{m=1}^{\infty} z(n,m) \qquad (n \in \mathbf{N}).$$

Since

$$|b_n| = \lim_{q\to\infty} |\sum_{m=1}^{q} z(n,m)| \le \lim_{q\to\infty} \sum_{m=1}^{q} |z(n,m)| = \sum_{m=1}^{\infty} |z(n,m)|$$

for all n, the Comparison Test 8.4 yields

$$\sum_{n=1}^{\infty} |\sum_{m=1}^{\infty} z(n,m)| = \sum_{n=1}^{\infty} |b_n| \le \sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} |z(n,m)|) < \infty,$$

and so the first series in (iv) is absolutely convergent. Similarly the second series in (iv) is absolutely convergent and, by Theorem 8.6 again, so is the third. Write

$$\sum_{k=1}^{\infty} z(\phi(k)) = s \in \mathbf{C}.$$

We shall next show that $\sum_{n=1}^{\infty} b_n = s$; i.e., the first and the third series in (iv) have the same sum. That the second and the third have same sum can be proved similarly.

Let $\epsilon > 0$ be given. Choose $k_0 \in \mathbf{N}$ such that

$$\sum_{k=k_0+1}^{\infty} |z(\phi(k))| < \frac{\epsilon}{3} \tag{17}$$

and

$$|s - \sum_{k=1}^{k_0} z(\phi(k))| < \frac{\epsilon}{3}. \tag{18}$$

Next choose $p_0, q_0 \in \mathbf{N}$ such that

$$\{\phi(k) : 1 \leq k \leq k_0\} \subseteq \{(n,m) : 1 \leq n \leq p_0,\ 1 \leq m \leq q_0\}.$$

Then, whenever $p \geq p_0$ and $q \geq q_0$, each term of the finite sum $\sum_{k=1}^{k_0} z(\phi(k))$ appears as a term $z(n,m)$ in the finite sum $\sum_{n=1}^{p}(\sum_{m=1}^{q} z(n,m))$, and so, subtracting those terms from the latter sum and using (17), we obtain

$$|\sum_{n=1}^{p}(\sum_{m=1}^{q} z(n,m)) - \sum_{k=1}^{k_0} z(\phi(k))| \leq \sum_{k=k_0+1}^{\infty} |z(\phi(k))| < \frac{\epsilon}{3}. \tag{19}$$

We claim that

$$p \geq p_0 \qquad \Rightarrow \qquad |s - \sum_{n=1}^{p} b_n| < \epsilon. \tag{20}$$

If (20) can be established, then $\sum_{n=1}^{\infty} b_n = s$, and the proof will be complete. To this end, fix $p \geq p_0$. Since

$$\lim_{q\to\infty} \sum_{m=1}^{q} z(n,m) = b_n \qquad \text{for each } n \in \mathbf{N},$$

it follows, by induction, that

$$\lim_{q\to\infty} \sum_{n=1}^{p}(\sum_{m=1}^{q} z(n,m)) = \sum_{n=1}^{p} b_n.$$

Thus we may choose some $q \geq q_0$ such that

$$|\sum_{n=1}^{p}(\sum_{m=1}^{q} z(n,m)) - \sum_{n=1}^{p} b_n| < \frac{\epsilon}{3}. \tag{21}$$

Combining (18), (19), and (21), we obtain

$$|s - \sum_{n=1}^{p} b_n| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Since $p \geq p_0$ was arbitrary, (20) has been established. ■

9.4 Remarks. (**a**) In the notation of the preceding theorem, the hypothesis that the series

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} |z(n,m)|) \qquad \text{or} \qquad \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} |z(n,m)|)$$

is finite cannot be weakened to the requirement that iterated series

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} z(n,m)) \qquad \text{or} \qquad \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} z(n,m))$$

is absolutely convergent. For example, the example of Remark 8.7 shows it can happen that all of the series in (ii) and (iii) and the first two series in (iv) of Theorem 9.3 are absolutely convergent, but that the first two of (iv); i.e., the iterated series, do not have the same sum.

(**b**) It can easily happen that

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} z(n,m)) = \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} z(n,m))$$

even though the hypothesis of Theorem 9.3 fails. As a simple example, let $z(n,m) = \frac{(-1)^{n+m}}{nm}$, $(n,m) \in \mathbf{N} \times \mathbf{N}$. It is easy to check that

$$\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{nm}) = (\log 2)^2 = \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} \frac{(-1)^{n+m}}{nm}).$$

9.5 Theorem. *Let $z(n,m) \in \mathbf{C}$ for each $(n,m) \in \mathbf{N} \times \mathbf{N}$ and let ϕ be a one-to-one mapping of $\mathbf{N}$ onto $\mathbf{N} \times \mathbf{N}$. Then*

(i) $\sum_{k=1}^{\infty} z(\phi(k))$ converges absolutely if and only if the series $\sum_{n,m=1}^{\infty} z(n,m)$ converges absolutely.

(ii) If $\sum_{n,m=1}^{\infty} z(n,m)$ converges absolutely to the sum s, then $\sum_{k=1}^{\infty} z(\phi(k)) = s$.

Proof: (i) Define, for each $k \in \mathbf{N}$, $T_k := |z(\phi(1))| + \ldots + |z(\phi(k))|$ and, for each $(p,q) \in \mathbf{N} \times \mathbf{N}$, define

$$S(p,q) := \sum_{n=1}^{p}\sum_{m=1}^{q} |z(n,m)|.$$

Then, for each $k \in \mathbf{N}$, there exists a pair $(p,q) \in \mathbf{N} \times \mathbf{N}$ such that $T_k \leq S(p,q)$ and conversely, for each pair $(p,q) \in \mathbf{N} \times \mathbf{N}$, there exists $r \in \mathbf{N}$ such that $S(p,q) \leq T_r$. It follows from these inequalities that the series $\sum_{k=1}^{\infty} |z(\phi(k))|$ has bounded set of partial sums if and only if the series $\sum_{n,m=1}^{\infty} |z(n,m)|$ has bounded set of partial sums. Hence, by Theorem 8.1, (i) holds.

(ii) Assume that $\sum_{n,m=1}^{\infty} z(n,m)$ converges absolutely to the sum s. Then, by part (i), $\sum_{k=1}^{\infty} z(\phi(k))$ converges absolutely, say to the sum s'. To prove (ii), we need to show that $s' = s$. To this end, let

$$T := \lim_{p,q\to\infty} S(p,q).$$

Given $\epsilon > 0$, choose $N \in \mathbf{N}$ so that

$$0 \leq T - S(p,q) < \frac{\epsilon}{2} \qquad \forall\, p,q > N. \tag{22}$$

Now write

$$t_k := \sum_{n=1}^{k} z(\phi(n)), \qquad s(p,q) := \sum_{n=1}^{p}\sum_{m=1}^{q} z(n,m).$$

Choose M so that t_M includes all terms of the form $z(n,m)$ with

$$1 \le n \le N+1, \qquad 1 \le m \le N+1.$$

Then $t_M - s(N+1, N+1)$ is a sum of terms with either $n > N$ or $m > N$. Therefore, if $n \ge M$, it follows from (22) that

$$|t_n - s(N+1,N+1)| \le T - S(N+1,N+1) < \frac{\epsilon}{2}, \tag{23}$$

and that

$$|s - s(N+1,N+1)| \le T - S(N+1,N+1) < \frac{\epsilon}{2}. \tag{24}$$

Thus (23) and (24) yield that

$$|t_n - s| < \epsilon \qquad \forall\, n \ge M.$$

Since $\epsilon > 0$ was arbitrary, this shows that $\lim_{n\to\infty} t_n = s$. But then $s' = \lim_{n\to\infty} t_n = s$, as desired. ■

Combining Theorems 9.3 and 9.5 together yields the following main result, which gives a sufficient condition for equality of the iterated series of a double series of complex numbers.

9.6 Theorem. *Let $z(n,m) \in \mathbf{C}$ for each $(n,m) \in \mathbf{N}\times\mathbf{N}$. If any of the three series*

$$\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} |z(n,m)|\Big), \quad \sum_{m=1}^{\infty}\Big(\sum_{n=1}^{\infty} |z(n,m)|\Big), \quad \sum_{n,m=1}^{\infty} |z(n,m)|$$

converges, then all of the series

(i) $\sum_{m=1}^{\infty} z(n,m) \qquad (n \in \mathbf{N})$,

(ii) $\sum_{n=1}^{\infty} z(n,m) \qquad (m \in \mathbf{N})$,

(iii) $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} z(n,m)), \quad \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} z(n,m)), \quad \sum_{n,m=1}^{\infty} z(n,m)$

are absolutely convergent and the three series in (iii) all have the same sum.

As an application of Theorem 9.6, we prove the following theorem concerning the multiplication of series.

9.7 Theorem. *Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be any two absolutely convergent series of complex numbers with sums* a *and* b*, respectively. Define a double sequence* $z : \mathbf{N} \times \mathbf{N} \to \mathbf{C}$ *by*

$$z(n,m) := a_n b_m, \qquad \textit{if } (n,m) \in \mathbf{N} \times \mathbf{N}.$$

Then the double series $\sum_{n,m=1}^{\infty} z(n,m)$ *converges absolutely and*

$$\sum_{n,m=1}^{\infty} z(n,m) = ab.$$

Proof: We have

$$\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} |z(n,m)|\Big) = \sum_{n=1}^{\infty}\Big(|a_n| \sum_{m=1}^{\infty} |b_m|\Big) = \Big(\sum_{m=1}^{\infty} |b_m|\Big)\Big(\sum_{n=1}^{\infty} |a_n|\Big) < \infty,$$

and so Theorem 9.6 applies and yields that the double series converges absolutely and $\sum_{n,m=1}^{\infty} z(n,m) = ab$. ■

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