ON PRIMARY COMPACTLY PACKED BEZOUT MODULES

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ABSTRACT. Many results are proved concerning primary compactly packed modules and primary radical submodules. We also generalize some results that were proved on S-closed subset of modules and prime submodules to S-closed subset of modules and primary submodules. Furthermore, we find the conditions on an R-module M that make the following important result true, that is for a multiplication Bezout module M, M is primary compactly packed if and only if every primary submodule of M is primary compactly packed.

1. INTRODUCTION

Let M be a unitary R-module. A proper submodule N of M is primary if $rm \in N$, for $r \in R$ and $m \in M$ implies that either $m \in N$, or $r^nM \subseteq N$ for some positive integer n. In [2] we introduce the concepts of primary radical of submodules and primary compactly packed modules, which is a generalization to the concept of compactly packed modules that was introduced in [5].

Let N be a submodule of an R-module M, if there exist primary submodules that contain N, then the intersection of all primary submodules containing N is called the primary radical of N and denotes by $prad(N)$. A submodule N is called a primary radical submodule if $prad(N) = N$. A proper submodule N of M is primary compactly packed (*pcp*) if for each family $\{P_\alpha\}_{\alpha\in\lambda}$ of primary submodules of M with $N \subseteq \bigcup_{\alpha\in\lambda} P_\alpha, N \subseteq P_\beta$ for some $\beta \in \lambda$. A module M is called *pcp* if every submodule is *pcp*.

In section two of this paper we study the direct sum of primary submodules. We prove that the direct sum of a finite number of primary submodules is not necessarily primary, however the direct sum of finite number of submodules that include exactly

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one primary submodule is primary. We also prove that if $N = N_1 \oplus N_2$ is a proper submodule of an R-modules $M = M_1 \oplus M_2$, then $prad(N) \supseteq prad(N_1) \oplus Prad(N_2)$.

In section three we give a characterization of primary radical of submodules in case where $R = M$.

The concept of multiplicatively closed subsets of rings is an important concept that is usually used in commutative algebra to prove many important problems in this field. For this reason Chin Pi Lu in 1997, see [4], gave a generalization of this concept and introduce the concept of S-closed subsets of modules. She also explore a various properties of S-closed subsets, particulary, saturated S-closed subsets of modules.
The most importent result she obtained is that she found a condition under which a submodule of a finitely generated R-module M maximal with respect to exclusion of an S-closed subset to be a prime submodule of M. In 2002, see [5] it was proved that the hypothesis of finitely generated R-module is not needed.

In section four we generalize various results that was proved in [4] and [5] and we obtain a condition under which a submodule of any R-module M, not necessarily finitely generated, maximal with respect to exclusion of an S-closed subset to be a primary submodule of M. We also find the conditions on an R-module M that make the following important result true, that is M is *pcp* if and only if every primary submodule of M is *pcp*.

Throughout this paper, all rings are assumed to be commutative rings with unity and all modules will be unitary.

2. Direct Sum Of Primary radical of submodules

Proposition 2.1. *Let M_1 and M_2 be unitary R-modules over a ring R and let $M = M_1 \oplus M_2$. $N \subseteq M_1 \oplus M_2$ is primary submodule of M if and only if either N has the form*
(1) $N = Q \oplus M_2$, Q is primary submodule of M_1 or
(2) $N = M_1 \oplus Q$, Q is primary submodule of M_2.

Proof. We will prove (1) and the proof of (2) will be similar.
$(\longrightarrow)$ Let $N = Q \oplus M_2$, be primary submodule of M. We need to show that Q is primary submodule of M_1. Let $rq \in Q$, $q \notin Q$. We will show that there exists a positive integer n such that $r^n M_1 \subseteq Q$.
Since $q \notin Q$, then $(q, 0) \notin Q \oplus M_2$ while $r(q, 0) \in Q \oplus M_2$. Since $Q \oplus M_2$ is primary submodule of M, there exists a positive integer n such that $r^n(M_1 \oplus M_2) \subseteq Q \oplus M_2$. Hence $r^n M_1 \oplus r^n M_2 \subseteq Q \oplus M_2$. Therefore $r^n M_1 \subseteq Q$. So Q is primary submodule of M_1.
$(\longleftarrow)$ Suppose that Q is primary submodule of M_1. Let $r(q, k) \in Q \oplus M_2$ with $r \in R$ and $(q, k) \in M - (Q \oplus M_2)$, then $rq \in Q$. Since Q is primary submodule of M_1 and $q \notin Q, r^n M_1 \subseteq Q$ for some $n \in \mathbb{Z}^+$. But $r^n M_2 \subseteq M_2$. Thus $r^n M \subseteq Q \oplus M_2$. Therefore $Q \oplus M_2$ is primary submodule of M. □

Proposition 2.2. *Let $M_1, M_2, ..., M_n$ be unitary R-modules over a ring R and let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. Q_i is primary submodule of M_i, for some $i \in \{1, 2, ..., n\}$, if and only if ($Q_i \bigoplus_{j=1, j\neq i}^{n} M_j$) is primary submodule of M.*

Proof. We will take the case if $i = 1$, and similarly one can prove the result if i is any number in the set $\{2, ..., n\}$. that is we will prove that Q_1 is primary submodule

of M_1 if and only if $Q_1 \oplus M_2 \oplus ... \oplus M_n$ is primary submodule of M.
($\longrightarrow$) Suppose that Q_1 is primary submodule of M_1, and let
$r(q, k_2, ..., k_n) \in Q_1 \oplus M_2 \oplus ... \oplus M_n$ with $r \in R$ and
$(q, k_2, ..., k_n) \notin Q_1 \oplus M_2 \oplus ... \oplus M_n$. Then $rq \in Q_1$ and $q \notin Q_1$.
Since Q_1 is primary submodule of M_1, there exists a positive integer s such that $r^s M_1 \subseteq Q_1$. Thus $r^s M \subseteq Q_1 \oplus M_2 \oplus ... \oplus M_n$. Hence $Q_1 \oplus M_2 \oplus ... \oplus M_n$ is primary submodule of M.
($\longleftarrow$) Suppose that $Q_1 \oplus M_2 \oplus ... \oplus M_n$ is primary submodule of M. Let $rq \in Q_1$ with $q \notin Q_1$. Then $(q, 0, ..., 0) \notin Q_1 \oplus M_2 \oplus ... \oplus M_n$ while
$r(q, 0, ..., 0) \in Q_1 \oplus M_2 \oplus ... \oplus M_n$. Since $Q_1 \oplus M_2 \oplus ... \oplus M_n$ is primary submodule of M, there exists a positive integer s such that $r^s(M_1 \oplus M_2 \oplus ... \oplus M_n) \subseteq Q_1 \oplus M_2 \oplus ... \oplus M_n$. Hence $r^s M_1 \oplus r^s M_2 \oplus ... \oplus r^s M_n \subseteq Q_1 \oplus M_2 \oplus ... \oplus M_n$. Therefore $r^s M_1 \subseteq Q$. So Q_1 is primary submodule of M_1.

□

Now we note that the direct sum of primary submodules is not necessarily primary as in the following remark.

Remark 2.3. The direct sum of primary submodules is not necessarily primary. Note that if we take $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$, then $N = (4) \oplus (3)$ is not primary submodule of M. Because $3(4, 1) \in N$, but neither $(4, 1) \in N$ nor $3^n M \subseteq N$ for any positive integer n. However (4) and (3) are primary submodules of $\mathbb{Z}$.

Lemma 2.4. *Let M_1 and M_2 be R-modules and let $M = M_1 \oplus M_2$. If N is a proper submodule of M_1, then $m \in prad(N)$ if and only if $(m, 0) \in prad(N \oplus (0))$.*

Proof. ($\longrightarrow$) Suppose that $m \in prad(N)$. Let P be a primary submodule of M such that $(N \oplus (0)) \subseteq P$. Let $\bar{P} = \{x \in M_1 : (x, 0) \in P\}$. We want to show that either $\bar{P} = M_1$ or $\bar{P}$ is a primary submodule of M_1 with the property that $N \subseteq \bar{P}$. Suppose that $\bar{P} \neq M_1$. Let $rs \in \bar{P}$ with $r \in R$ and $s \in M_1 - \bar{P}$. Then $(rs, 0) \in P$. Thus $r(s, 0) \in P$. Since P is a primary submodule of M_1 and $s \notin \bar{P}$, so $(s, 0) \notin P$, then $r^n M \subseteq P$ for some $n \in \mathbb{Z}^+$. Therefore $(r^n m_1, 0) \in P$ for every $m_1 \in M_1$. Hence $r^n M_1 \subseteq \bar{P}$. Thus $\bar{P}$ is a primary submodule of M_1. It is clear that $N \subseteq \bar{P}$ because N is a proper submodule of M_1 and $N \oplus (0) \subseteq P$. Now since $m \in prad(N)$, then $m \in \bar{P}$. Therefore $(m, 0) \in P$ and it follows that $(m, 0) \in prad(N \oplus (0))$.
($\longleftarrow$) Suppose that $(m, 0) \in prad(N \oplus (0))$. Let Q be a primary submodule of M_1 such that $N \subseteq Q$. Then by (1) of Proposition 2.1, $Q \oplus M_2$ is a primary submodule of M with $N \oplus (0) \subseteq Q \oplus M_2$. Hence $(m, 0) \in Q \oplus M_2$. So that $m \in Q$. Therefore $m \in prad(N)$. □

In a similar way and by using (2) of Proposition 2.1 we can prove the following Lemma.

Lemma 2.5. *Let M_1 and M_2 be R-modules and let $M = M_1 \oplus M_2$. If N is a proper submodule of M_2, then $m \in prad(N)$ if and only if $(0, m) \in prad((0) \oplus N)$.*

Theorem 2.6. *Let M_1 and M_2 be R-modules. Let $M = M_1 \oplus M_2$. If $N = N_1 \oplus N_2$ is a proper submodule of M, then $prad(N) \supseteq prad(N_1) \oplus Prad(N_2)$.*

Proof. Let $(m_1, m_2) \in prad(N_1) \oplus prad(N_2)$, then $m_1 \in prad(N_1)$ and
$m_2 \in prad(N_2)$. Thus, by Lemmas 2.4 and 2.5 , $(m_1, 0) \in prad(N_1 \oplus (0))$
and $(0, m_2) \in prad((0) \oplus N_2)$. Since $N_1 \oplus (0) \subseteq N$ and $(0) \oplus N_2 \subseteq N$,

$prad(N_1 \oplus (0)) \subseteq prad(N)$ and $prad((0) \oplus N_2) \subseteq prad(N)$, see [2].
Thus $(m_1, 0) \in prad(N)$ and $(0, m_2) \in prad(N)$. Hence $(m_1, m_2) \in prad(N)$.
Therefore $prad(N_1) \oplus prad(N_2) \subseteq prad(N)$.

□

3. Characterization of primary radical of submodules when $R = M$

Note first that we take M to be an R-module of the commutative ring with unity R.

Definition 3.1. Let N be a submodule of an R-module. Recall that $(N : M) = \{r | r \in R : rM \subseteq N\}$.

Definition 3.2. Let N be a proper submodule of any R-module M. Let P be a primary ideal of R. Then we shall denote by $K(N, P)$ the following subset of M:
$K(N, P) = \{m \in M : cm \in PM + N$, where $c^n \in R - P$ for every $n \in \mathbb{Z}^+\}$.

Remark 3.3. It is clear that $K(N, P)$ is a submodule of M and $PM + N \subseteq K(N, P)$.

Lemma 3.4. *$K(N, P) = M$ or $K(N, P)$ is a primary submodule of M.*

Proof. Suppose $K(N, P) \neq M$. Note that $K(N, P) \neq \phi$, by Remark 3.2 . Let $r \in R, m \in M$ such that $rm \in K(N, P)$. We will prove that either $m \in K(N, P)$ or $r^s M \subseteq K(N, P)$ for some $s \in \mathbb{Z}^+$. Since $rm \in K(N, P)$, there exists c with $c^n \notin P$ for every $n \in \mathbb{Z}^+$ such that $crm \in PM + N$. If $r^s \in P$ for some $s \in \mathbb{Z}^+$, then by the definition of $K(N, P)$, $r^s M \subseteq K(N, P)$. Suppose that $r^n \notin P$ for every $n \in \mathbb{Z}^+$, then $(cr)^n \notin P$ for every $n \in \mathbb{Z}^+$ because P is a primary ideal of R. Hence $m \in K(N, P)$. Therefore $K(N, P)$ is a primary submodule of M. □

Let $M = R$ be an M-module. Let N be an ideal of R, it is clear that N is a submodule of M. Denote $prad_M(N)$ the primary radical of submodule N when $R = M$. Then we can describe $prad_M(N)$ as in the following Theorem.

Theorem 3.5. *For any submodule N of M,*
$prad_M(N) = \bigcap\{K(N, P) : P$ *is a primary ideal of* R $\}$.

Proof. Let $m \in B = \bigcap\{K(N, P) : P$ be a primary ideal of R $\}$. Let L be a primary submodule of M containing N. Since $m \in B$, and $(L : M)$ is primary ideal of R, see [2], $m \in K(N, (L : M))$. Thus $m \in M$ and $\exists c$ with $c^n \in R - (L : M)$ for every $n \in \mathbb{Z}^+$ such that $cm \in (L : M)M + N$. Therefore $cm = hs + n$, where $h \in (L : M), s \in M$ and $n \in N$. Thus $hs \in L$ and $n \in L$ because $hM \subseteq L$ and $N \subseteq L$. Hence $cm \in L$. Since $c^n M \nsubseteq L$ for every $n \in \mathbb{Z}^+$, and L is a primary submodule of M, $m \in L$. Thus $B \subseteq prad(N)$.
Now let $m \in prad_M(N)$, and let $K(N, P) \in B$. Then by the previous lemma either $K(N, P) = M$ or $K(N, P)$ is a primary submodule of M. If $K(N, P) = M$, then it is trivial that $m \in K(N, P)$. So let $K(N, P) \neq M$. Since $K(N, P)$ is a primary submodule of M with $N \subseteq K(N, P), m \in K(N, P)$. Thus $prad_M(N) \subseteq B$. □

4. S-Closed Subset of Modules and Primary Submodules

First we recall the following Definition, see [3].

Definition 4.1. Let R be a ring. A non empty subset S of R is said to be multiplicatively closed subset of R if $1 \in S, 0 \notin S$ and $ab \in S$ for all $a, b \in S$.

Definitions 4.2. Let S be a multiplicatively closed subset of a ring R and M be an R-module.
1) A non-empty subset S^* of M is said to be S-closed if $se \in S^*$ $\forall s \in S$ and $e \in S^*$.
2) An S-closed subset S^* is said to be saturated if the following condition is satisfied:

Whenever $ae \in S^*$ for $a \in R$ and $e \in M$, then $a \in S$ and $e \in S^*$.

Definition 4.3. Let N be a submodule of an R-module M.

$(N : M)^{\frac{1}{n}} = \{r | r \in R : r^n M \subseteq N\}$

Proposition 4.4. *If N is a primary submodule of an R-module M, then $(N : M)^{\frac{1}{n}}$ is a primary ideal of R for every $n \in \mathbb{Z}^+$.*

Proof. Let $rs \in (N : M)^{\frac{1}{n}}$, then $(rs)^n M \subseteq N$. Thus $(rs)^n \in (N : M)$, and hence $r^n s^n \in (N : M)$. Since N is a primary submodule of M, by [2] $(N : M)$ is a primary ideal of R. Thus either $(r^n)^k \in (N : M)$ for some $k \in \mathbb{Z}^+$ or $s^n \in (N : M)$. If $(r^n)^k \in (N : M)$, then $r^k \in (N : M)^{\frac{1}{n}}$, this implies that $(N : M)^{\frac{1}{n}}$ is a primary ideal of R. If $s^n \in (N : M)$, then $s \in (N : M)^{\frac{1}{n}}$ and hence $(N : M)^{\frac{1}{n}}$ is a primary ideal of R. $\square$

The following Example is due to Chin Pi Lu, see [4].

Example 4.5. *Let $\{P_i\}_{i\in I}$ be a collection of prime submodules of M with $p_i = (P_i : M)$ $\forall i \in I$. Then $S^* = M - \bigcup_{i\in I} P_i$ is a saturated S-closed subset of M, where $S = R - \bigcup_{i\in I} p_i$.*

Now consider the following example.

Example 4.6. *Let $\{Q_i\}_{i\in I}$ be a collection of primary submodules of M with $q_i = (Q_i : M)$ $\forall i \in I$. Then $S^* = M - \bigcup_{i\in I} Q_i$ is a saturated S-closed subset of M, where $S = R - \bigcup_{i\in I} q_i^{\frac{1}{n}}$, $n \in \mathbb{Z}^+$.*

Proof. To prove that S^* is a saturated S-closed subset of M, we must prove the following three identities:
(1) $S = R - \bigcup_{i\in I} q_i^{\frac{1}{n}}$, $n \in \mathbb{Z}^+$ is multiplicatively closed subset of R.
(2) $S^* = M - \bigcup_{i\in I} Q_i$ is S-closed subset of M.
(3) S^* is saturated.

Now we will prove the previous three identities.
(1) Since $\bigcup_{i\in I} q_i^{\frac{1}{n}}$, $n \in \mathbb{Z}^+$ is a proper ideal of R, $S \neq \phi$. It is clear that $1 \in S$ and $0 \notin S$. Now let $a, b \in S$ and suppose that $ab \notin S$. Then $ab \in q_j^{\frac{1}{m}}$, for some $m \in \mathbb{Z}^+$, for some $j \in I$. Thus $(ab)^m = a^m b^m \in q_j = (Q_j : M)$. Since Q_j is a primary submodule of M, by Proposition 4.4, $(Q_j : M)$ is a primary ideal of R. Thus either $(a^m)^k \in (Q_j : M)$, for some $k \in \mathbb{Z}^+$, or $b^m \in (Q_j : M)$.
If $(a^m)^k = a^{mk} \in (Q_j : M)$, for some $k \in \mathbb{Z}^+$, then $a \in (Q_j : M)^{\frac{1}{mk}} = q_j^{\frac{1}{mk}}$. Thus $a \notin S$, which is a contradiction to the assumption that $a \in S$.
If $b^m \in (Q_j : M)$, then $b \in (Q_j : M)^{\frac{1}{m}} = q_j^{\frac{1}{m}}$. Thus $b \notin S$, which is also contradiction to the assumption that $b \in S$.
Therefore $ab \in S$ for all $a, b \in S$. Hence S is a multiplicatively closed subset of R.
(2) Since $\bigcup_{i\in I} Q_i$ is a proper submodule of M, $S^* \neq \phi$. Now let $s \in S$ and $e \in S^*$, then $s \notin \bigcup_{i\in I} q_i^{\frac{1}{n}}$, for any $n \in \mathbb{Z}^+$. and $e \notin \bigcup_{i\in I} Q_i$. Thus $s^n \notin q_i = (Q_i : M)$, for

any $i \in I$ for any $n \in \mathbb{Z}^+$, and $e \notin Q_i$, for any $i \in I$. Since Q_i is a primary submodule of M for every $i \in I$, $se \notin Q_i$, for any $i \in I$. Therefore $se \in M - \bigcup_{i \in I} Q_i = S^*$. Therefore S^* is S-closed subset of M.
(3) Let $ae \in S^*$ for some $a \in R$ and $e \in M$, then $ae \notin Q_i$ for any $i \in I$. Since Q_i is a primary submodule of M for every $i \in I$, $a^n \notin (Q_i : M) = q_i$ for every $n \in \mathbb{Z}^+$ and $e \notin Q_i$ for every $i \in I$. Hence $a \in R - \bigcup_{i \in I} q_i^{\frac{1}{n}} = S$ and $e \in M - \bigcup_{i \in I} Q_i = S^*$. Thus S^* is saturated.
Therefore by (1), (2), and (3) S^* is a saturated S-closed subset of M. □

Theorem 4.7. *Let S be a multiplicatively closed set of a ring R and let S^* be an S-closed subset of the R-module M. Let N be a submodule of M which is maximal in $M - S^*$. If the ideal $(N : M)$ is maximal in $R - S$, then N is a primary submodule of M.*

Proof. Clearly $N \neq M$. Suppose N is not primary. Then there exists $r \notin (N : M)^{\frac{1}{n}}$ for every $n \in \mathbb{Z}^+$ and $m \in M - N$ with the property that $rm \in N$. Since N is maximal in $M - S^*$ and the ideal $(N : M)$ is maximal in $R - S$, the submodule $(Rm + N) \cap S^* \neq \phi$ and the ideal $(< r > +(N : M)) \cap S \neq \phi$. That is there exists $x \in (Rm + N) \cap S^*$ and $t \in S \cap (< r > +(N : M))$. Then $x = a_1 m + n$, $t = a_2 r + h$ for $a_1, a_2 \in R$, $n \in N$ and $h \in (N : M)$. Hence $tx \in N \cap S^*$ which is a contradiction. Thus N is a primary submodule of M. □

Theorem 4.8. *Let $M = Rm$ be a cyclic R-module over a ring R. Let S^* be an S-closed subset of M relative to a multiplicatively closed subset S of R, and N a submodule of M maximal in $M - S^*$. If S^* is saturated, then the ideal $(N : M)$ is maximal in $R - S$ and therefore N is a primary in M.*

Proof. Assume that $(N : M)$ is not maximal in $R - S$. Then there must exist an ideal J in $R - S$ such that $(N : M) \subsetneq J$. Hence $N = (N : M)M \subsetneq JM$. So that there exists $rm \in S^*$ for some $r \in J$ by the maximality of N in $M - S^*$. Since S^* is saturated, $r \in S$ which contradicts the fact that $J \cap S = \phi$. Thus $(N : M)$ is maximal in $R - S$ and consequently by Theorem 4.7 N is primary. □

The following Proposition is a combination of results proved in [4] and [5].

Proposition 4.9. *Let M be an R-module and let S be a multiplicatively closed set of a ring R, S^* be a saturated S-closed subset of M, and I be an ideal of R such that $I \cap S = \phi$. If one of the following conditions holds:*

i) R is a Bezout ring.

ii) M is a Bezout module.

iii) M is a cyclic module.

Then $IM \cap S^* = \phi$.

Proof. Suppose that $IM \cap S^* \neq \phi$. Thus there exists $x \in IM \cap S^*$. Then $x = \sum_{i=1}^{n} r_i m_i$ where $r_i \in I$ and $m_i \in M$ for each $i \in \{1, 2, ..., n\}$. Now:

i) Let $J = (r_1, r_2, ..., r_n)$, then J is a finitely generated ideal of R. Since R is Bezout ring, there exists $r \in J$ such that $J = (r)$. That is $x = rm$ where $r \in I$ and $m \in M$ and also $x = rm \in S^*$. Since S^* is a saturated S-closed subset of M, then $r \in S$ which contradicts the fact that $I \cap S = \phi$.

ii) Let $N = Rm_1 + Rm_2 + ... + Rm_n$. It is clear that N is a finitely generated submodule of M. Since M is Bezout module, there exists $m \in N$ such that $N = Rm$. Thus $x = rm$ where $r \in I$ and $m \in M$ and also $x = rm \in S^*$. Now again Since S^* is saturated S-closed subset of M, then $r \in S$ which contradicts the fact that $I \cap S = \phi$.

iii) Since M is a cyclic module, $M = Rm$ for some $m \in M$. Thus $x = rm$ where $r \in I$ and $m \in M$. As before this yields to a contradiction. □

Theorem 4.10. *Let M be a multiplication R-module and let S^* be a saturated S-closed subset of M relative to the multiplicatively closed set S of R. Suppose that $IM \subseteq M - S^*$ for each ideal I contained in $R - S$. If N is a submodule of M maximal in $M - S^*$, then N is a primary submodule of M.*

Proof. By Theorem 4.7 we only have to show that $(N : M)$ is maximal in $R - S$. Assume that $(N : M)$ is not maximal in $R - S$. Then there exists an ideal J in $R - S$ such that $(N : M) \subsetneq J$. Hence $JM \subseteq M - S^*$. But $N = (N : M)M \subseteq JM$, so that
$JM = N$ because N is maximal in $M - S^*$, that is, $J \subseteq (N : M)$ and by this contradiction we conclude that $(N : M)$ is maximal in $R - S$. □

From Proposition 4.9 and Theorems 4.8 and 4.10, we have the following Corollary.

Corollary 4.11. *Let M be a multiplication R-module and let S^* be a saturated S-closed subset of M relative to the multiplicatively closed set S of R. If N is a submodule of M maximal in $M - S^*$ and one of the following conditions is satisfied*

i) R is a Bezout ring

ii) M is a Bezout module

iii) M is a cyclic module

Then N is a primary submodule of M.

The following Proposition can be proved by Zorn's Lemma as follow.

Proposition 4.12. *Let S be a multiplicatively closed set of a ring R and let S^* be an S-closed subset of the R-module M. Let N be a submodule of M such that $N \subseteq M - S^*$, then there exists a submodule L containing N and maximal in $M - S^*$.*

Proof. Let $T = \{K : K$ submodule of M and $N \subseteq K$ with the property that $K \cap S^* = \phi\}$. It is clear that $N \in T$. Thus $T \neq \phi$. Let $\{N_i\}_{i \in I}$ be a sequence of elements on T. It is clear that $\bigcup_{i \in I} N_i$ is a submodule of M that includes N and $(\bigcup_{i \in I} N_i) \cap S^* = \bigcup_{i \in I}(N_i \cap S^*) = \phi$. Thus by Zorn's Lemma T contains a maximal element L. Thus there exists a submodule L containing N and maximal in $M - S^*$. □

Proposition 4.13. *Let S be a multiplicatively closed set of a ring R and let S^* be an S-closed subset of the R-module M. If the S-closed subset S^* is saturated, then $M - S^* \neq \phi$ and it is a union of submodules each of them is maximal in $M - S^*$.*

Proof. Since S^* is a saturated S-closed subset of M, then $0 \in M - S^*$. Thus $M - S^* \neq \phi$. And also for all $x \in M - S^*, Rx \subseteq M - S^*$. From the previous Proposition, there exists a submodule L_i containing Rx and maximal in $M - S^*$. Thus $M - S^* = \bigcup L_i$, where L_i is a submodule maximal in $M - S^*$ for every i. □

From the previous Proposition it is clear that we can always have a submodule L maximal in $M - S^*$. Thus by Theorem 4.7 if $(L : M)$ is maximal in $R - S$, then L is a primary submodule of M.

Theorem 4.14. *Let R be a ring and let S be a multiplicatively closed set of R. Let M be a multiplication R-module and S^* be a non-empty subset of M. Suppose that $T^* \cap IM = \phi$ for each saturated S-closed subset T^* of M and for each ideal I in $R - S$. Then S^* is a saturated S-closed subset of M if and only if the set $M - S^*$ is a union of primary submodules Q_i ($i \in I$) of M and the set $R - S$ is a union of $(Q_i : M)^{\frac{1}{n}}$, $i \in I$ and $n \in \mathbb{Z}^+$.*

Proof. We have seen in Example 4.6 that the sufficiency holds for any submodule. To prove the necessity suppose S^* is saturated S-closed subset of M, then
$IM \cap S^* = \phi$, for each I in $R - S$. Since S^* is saturated, then $0 \in M - S^*$, that is $M - S^* \neq \phi$. Let e be any element of $M - S^*$, then $Re \subseteq M - S^*$, and by Proposition 4.12 there exists a submodule Q containing Re and maximal in $M - S^*$. The submodule Q must be primary by Theorem 4.10 Hence $M - S^* = \bigcup_{i \in I} Q_i$ where Q_i is a primary submodule of M for every $i \in I$. Next put $S_0 = R - \bigcup_{i \in I} (Q_i : M)^{\frac{1}{n}}$, $n \in \mathbb{Z}^+$.
We shall show that $S = S_0$.
If $s \in S$ and $m \in S^*$, then $sm \in S^* = M - \bigcup_{i \in I} Q_i$. so that $sm \notin Q_i$ for every $i \in I$. Since each Q_i is primary and $m \notin Q_i$ for every $i \in I$, $s \notin (Q_i : M)^{\frac{1}{n}}$ for every $i \in I$, for every $n \in \mathbb{Z}^+$. Hence $s \in S_0$. Therefore $S \subseteq S_0$. On the other hand, if $\grave{s} \in S_0$, then $\grave{s} \notin (Q_i : M)^{\frac{1}{n}}$ for every $i \in I$, for every $n \in \mathbb{Z}^+$. And $\grave{s}\grave{m} \in S^*$ for all $\grave{m} \in S^*$ due to that each Q_i is a primary submodule of M. It follows that $\grave{s} \in S$ as S^* is saturated S-closed subset. We conclude that $S_0 \subseteq S$ and therefore $S = S_0$. □

By Proposition 4.9 and Theorem 4.14, we have the following Corollary.

Corollary 4.15. *Let M be a multiplication R-module, S be a multiplicatively closed set of R and S^* be a non-empty subset of M. If one of the following conditions holds:*

i) R is a Bezout ring.

ii) M is a Bezout module.

iii) M is a cyclic module.

Then S^* is a saturated S-closed subset of M if and only if the set $M - S^*$ is a union of of primary submodules Q_i $(i \in I)$ of M, and the set $R - S$ is a union of $(Q_i : M)^{\frac{1}{n}}$, $i \in I$, $n \in \mathbb{Z}^+$.

Theorem 4.16. *Let M be a cyclic module over a ring R, S a multiplicatively closed subset of R, and S^* a non empty subset of M. Then S^* is a saturated S-closed subset of M if and only if $S^* = M - \bigcup_{i \in I} Q_i$, where Q_i is a primary submodule of M for every $i \in I$ and $S = R - \bigcup_{i \in I} (Q_i : M)^{\frac{1}{n}}$, $n \in \mathbb{Z}^+$.*

Proof. We have seen in Example 4.6 that the sufficiency holds for any submodule. To prove the necessity, let e be any non-zero element of $M - S^*$. Then $Re \cap S^* = \phi$. Since S^* is saturated. By Proposition 4.12, there exists a submodule Q containing Re and maximal in $M - S^*$. The submodule Q must be primary by Theorem 4.8 Hence $M - S^* = \bigcup_{i \in I} Q_i$, a union of primary submodules Q_i, $i \in I$. Next put

$S_0 = R - (\bigcup_{i\in I}(Q_i : M)^{\frac{1}{n}})$, $n \in \mathbb{Z}^+$. As we show in the proof of Theorem 4.14 $S = S_0$ and the proof is complete. □

Proposition 4.17. *Let M be a multiplication R-module. If one of the following conditions holds:*

i) R is a Bezout ring.

ii) M is a Bezout module.

iii) M is a cyclic module.

Then M is pcp if and only if every primary submodule of M is pcp.

Proof. The necessity is trivial. To prove the sufficiency, suppose every primary submodule of M is *pcp*. Let N be a proper submodule of M with $N \subseteq \bigcup_{\alpha\in\lambda} Q_\alpha$ where Q_α is a primary submodule of M for each $\alpha \in \lambda$. We have two cases:
1) $\bigcup_{\alpha\in\lambda} Q_\alpha = M$. Since N is a proper submodule of a multiplication module then by [1], there exists a prime, hence primary submodule Q that contains N. By assumption Q is *pcp* and $N \subseteq Q \subseteq M = \bigcup_{\alpha\in\lambda} Q_\alpha$ therefore $Q \subseteq Q_\beta$ for some $\beta \in \lambda$ that is $N \subseteq Q_\beta$ for some $\beta \in \lambda$.
2) $\bigcup_{\alpha\in\lambda} Q_\alpha \subsetneq M$. Let $S^* = M - \bigcup_{\alpha\in\lambda} Q_\alpha$. Then S^* is a saturated S-closed subset of M where $S = R - \bigcup_{\alpha\in\lambda}(Q_\alpha : M)^{\frac{1}{n}}$, $n \in \mathbb{Z}^+$, by Example 4.6 Since $N \subseteq \bigcup_{\alpha\in\lambda} Q_\alpha$, then $N \subseteq M - S^*$ and by Proposition 4.12 there exists a submodule Q containing N and maximal in $M - S^*$. By Corollary 4.11 Q must be primary. But $Q \subseteq \bigcup_{\alpha\in\lambda} Q_\alpha$, then by the hypothesis, there exists $\beta \in \lambda$ such that $Q \subseteq Q_\beta$ and hence $N \subseteq Q_\beta$. Thus N is *pcp*, hence M is *pcp*. □

References

[1] Athab, E. A. :Prime and semiprime submodules, M.sc. Thesis, College of science, University of Baghdad, 1996.

[2] El-Atrash, M. S. , Ashour, A. E. : On Primary compactly packed modules, Islamic University Journal, ( )2005 Gaza, Palestine.

[3] Kaplansky,I. : Commutative rings, The university of Chicago press, 1974.

[4] Lu, C. Pi : Union of prime submodules, Houston J. Math. 23 (2)(1997), 203-213.

[5] Naoum, A. G., Al-Hashimi, B. A., Al-Ani, Z. A. : On compactly packed modules, At the second Islamic University Conference in Math., Gaza, Palestine, 27-28, August, 2002 .

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