

# $\tau_\alpha$ ON ARTINIAN $T_o$ - ALEXANDROFF SPACES.

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**ملخص:** في هذا البحث قمنا بدراسة الفضاء التوبولوجي  $\tau_\alpha$  المنبثق عن الفضاء التوبولوجي أرتينيان  $T_o$ -الكساندروف  $(X, \tau(\leq))$  والمكون من كل المجموعات  $\alpha$ -المفتوحة. أثبتنا أن  $\tau_\alpha$  هي أيضا أرتينيان  $T_o$ -الكساندروف. قمنا بوصف الترتيب  $\leq_\alpha$  المنبثق عن  $\tau_\alpha$ . أثبتنا باستخدام الترتيب الجديد بعض الخصائص المشتركة بين  $\tau$  و  $\tau_\alpha$  مثل:

1. المجموعة **D** تكون كثيفة في  $\tau$  إذا وفقط إذا كانت كثيفة في  $\tau_\alpha$ .
  2.  $\tau$  يحقق خاصية التقطع الشديد إذا وفقط إذا كان  $\tau_\alpha$  يحققها.
  3.  $\tau$  يحقق خاصية الترابط الفائق إذا وفقط إذا كان  $\tau_\alpha$  يحققها.
  4. المجموعة **A** تكون شبه-مفتوحة في  $\tau$  إذا وفقط إذا كانت شبه-مفتوحة في  $\tau_\alpha$ .
- التنانج الأساسية لهذا البحث وضعت في القسم الثالث منه.

**ABSTRACT.** In this paper, we study the topology  $\tau_\alpha$  of all  $\alpha$ -open sets on Artinian  $T_o$ -Alexandroff space  $(X, \tau(\leq))$ . We show that  $\tau_\alpha$  is Artinian  $T_o$ -Alexandroff space. Then we describe the induced partial order  $\leq_\alpha$ . We use the partial order  $\leq_\alpha$  to get some results and some common properties relating the original topology  $\tau(\leq)$  and  $\tau_\alpha$  such as:

- (1)  $D$  is dense in  $\tau$  if and only if  $D$  is dense in  $\tau_\alpha$ .
- (2)  $(X, \tau(\leq))$  is extremally disconnected if and only if  $(X, \tau_\alpha)$  is extremally disconnected.
- (3)  $(X, \tau(\leq))$  is hyperconnected if and only if  $(X, \tau_\alpha)$  is hyperconnected.
- (4)  $SO(X, \tau(\leq)) = SO(X, \tau_\alpha)$ , and many more.

Section three contains the main results of this paper.

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## 1. INTRODUCTION

Alexandroff spaces were first introduced by P. Alexandroff in 1937 in [25]. In the eighties, the interest in Alexandroff spaces was a consequence of the very important role of finite spaces in digital topology and the fact that Alexandroff spaces have all the properties of finite spaces relevant for such theory. An Alexandroff space  $(X, \tau)$  is a topological space in which an arbitrary intersection of open sets is open. Every Alexandroff space  $(X, \tau)$  is a smallest neighborhood space, where the minimal neighborhoods are defined as the intersection of all open sets containing  $x \in X$ . For every  $T_0$ -minimal neighborhood space  $(X, \tau)$ , there is a corresponding poset  $(X, \leq)$ , where a minimal neighborhood containing  $x$  is the set of all  $y \in X$  with  $x \leq y$ . The partial order is called (Alexandroff) specialization order and denoted by  $\leq_\tau$ . Using the specialization order,  $a \leq_\tau b$  if and only if  $a \in \overline{\{b\}}$ . A  $T_0$ -Alexandroff space is completely determined by its specialization order. We denote a  $T_0$ -Alexandroff space together with its specialization order by the pair  $(X, \tau(\leq))$  where the corresponding poset is  $(X, \leq)$ .

All finite spaces are Alexandroff spaces.  $T_0$ -finite spaces were studied in [1]. Locally finite spaces are defined by the requirement that every element  $x \in X$  belongs to a finite open set and a finite closed set. Locally finite spaces clearly include all finite spaces. In [10] we have studied wider class of topological spaces called Artinian  $T_0$ -Alexandroff spaces; those are  $T_0$ -Alexandroff spaces in which the corresponding poset satisfies the ascending chain condition (*ACC*). The corresponding posets of locally finite  $T_0$ -spaces satisfy both the ascending chain condition and the descending chain condition (*DCC*). In [11] we have proved among other things that for any subset  $A \subseteq X$ ,  $A$  is preopen if and only if  $A$  is  $\alpha$ -open in any Artinian  $T_0$ -Alexandroff space.

In [24], it was proved that the class of all  $\alpha$ -open sets  $\tau_\alpha$  is a topology on  $X$ . Here we prove that the topological space  $(X, \tau_\alpha)$  of Artinian  $T_0$ -Alexandroff space is an Artinian  $T_0$ -Alexandroff space. We describe its specialization order - which is denoted by  $\leq_\alpha$  - and we use the new order to conclude some results such as:

- (1) A subset  $A$  is  $\alpha$ -open in  $(X, \tau(\leq))$  if it is an up set with respect to the partial order  $\leq_\alpha$ .

- (2)  $A$  is dense with respect to  $\tau_\alpha$  if and only if it is dense with respect to  $\tau(\leq)$ .
- (3)  $(X, \tau(\leq))$  is hyperconnected if and only if  $(X, \tau_\alpha)$  is hyperconnected.
- (4) The class of all semi-open sets in  $(X, \tau(\leq))$  is the same as the class of all semi-open sets in  $(X, \tau_\alpha)$ .
- (5) The set of maximal elements in  $(X, \tau(\leq))$  is the same as the set of maximal elements in  $(X, \tau_\alpha)$ . Moreover the maximal elements greater than or equal any element  $x$  in  $(X, \tau(\leq))$  is the same set of maximal elements greater than or equal any element  $x$  in  $(X, \tau_\alpha)$ .
- (6) If  $A$  is semi-open, then  $D$  is dense in  $A$  in  $\tau$  if and only if  $D$  is dense in  $A$  in  $\tau_\alpha$ .

Our method of proofs in this paper depends basically on the corresponding poset rather than the topology itself, which proves to be an easier approach. This technique which was used in [10] and [11] introduces new formulations of some topological concepts.

In a  $T_o$ -Alexandroff space  $(X, \tau(\leq))$ , the subset  $A$  of  $X$  is open if and only if it is up set in the corresponding poset  $(X, \leq)$ , and it is closed set if and only if it is down set with respect to the corresponding poset.

## 2. PRELIMINARIES AND DEFINITIONS

**Definitions 2.1.** A subset  $A$  of a space  $(X, \tau)$  is

- (1) a *semi-open set* [23] if  $A \subseteq \overline{A^\circ}$ , and a *semi-closed set* [28] if  $A^\circ$  is semi-open. Thus  $A$  is semi-closed if and only if  $\overline{A}^\circ \subseteq A$ . If  $A$  is both semi-open and semi-closed then  $A$  is called *semi-regular* [7],
- (2) a *preopen set* [3] if  $A \subseteq \overline{A}^\circ$ , and a *preclosed set* [21] if  $A^\circ$  is preopen. Thus  $A$  is preclosed if and only if  $\overline{A^\circ} \subseteq A$ ,
- (3) an  $\alpha$ -open set [24] if  $A \subseteq \overline{A^\circ}^\circ$ , and an  $\alpha$ -closed set [15] if  $A^\circ$  is  $\alpha$ -open. Thus  $A$  is  $\alpha$ -closed if and only if  $\overline{A}^\circ \subseteq A$ .

The family of all semi-open (resp. preopen,  $\alpha$ -open) is denoted by  $SO(X)$  (resp.  $PO(X), \tau_\alpha$ ). In general,  $SO(X)$  and  $PO(X)$  need not be topologies on  $X$ . A set  $A$  is preopen [19] if and only if  $A = U \cap D$  where  $U$  is an open set and  $D$  is a dense set. In [15], it has been shown that a set is  $\alpha$ -open if and only if it is semi-open and preopen. If  $A \subseteq X$ ,

then  $pInt(A)$  (resp.  $sInt(A)$ ) is the largest preopen set (resp. semi-open set) inside  $A$ .  $pCl(A)$  (resp.  $sCl(A)$ ) is the smallest preclosed set (resp. semi-closed set) contains  $A$ .

**Definitions 2.2.** The space  $(X, \tau)$  is called

- (1) *extremally disconnected* if the closure of every open set is open,
- (2) *submaximal* [13] if each dense subset is open,
- (3) *hyperconnected* if every open subset of  $X$  is dense
- (4) *nodec* [18] if all nowhere dense sets are closed.

Recall that a poset  $(X, \leq)$  satisfies *the ascending chain condition (ACC)* if each increasing chain is finally constant. It satisfies *the descending chain condition (DCC)* if each decreasing chain is finally constant. The  $T_o$ -Alexandroff space  $(X, \tau(\leq))$  is *Artinian* [11] if the corresponding poset  $(X, \leq)$  satisfies *ACC*, and it is *Noetherian* [11] if the corresponding poset satisfies *DCC*.

If  $(X, \tau(\leq))$  is an Artinian  $T_o$ -Alexandroff space, we define  $M$  to be the set of all maximal elements of  $X$ .  $M$  is the set of all isolated points in  $X$  which is not empty. For the point  $x \in X$ , we define  $\hat{x} = \uparrow x \cap M$ , where  $\uparrow x = \{y : x \leq y\}$ . If  $A$  is a subset of an Artinian  $T_o$ -Alexandroff space, then we define  $M(A)$  to be the set of all maximal elements of  $A$  under the induced order.

Dually, if  $X$  is a Noetherian  $T_o$ -Alexandroff space, we define  $m$  to be the set of all minimal elements of  $X$  which is not empty. we define  $\check{x} = \downarrow x \cap m$ . If  $A$  is a subset of a Noetherian  $T_o$ -Alexandroff space, then we define  $m(A)$  to be the set of all minimal elements of  $A$  under the induced order. If  $X$  is both Artinian and Neotherian  $T_o$ -Alexandroff space then it may happen that  $M \cap m$  is not empty.

For the results 2.3–2.15, see [11].

**Theorem 2.3.** *Let  $(X, \tau(\leq))$  be an Artinian  $T_o$ - Alexandroff space. Then*

- (1)  $A^\circ = \emptyset \Leftrightarrow A \cap M = \emptyset$ .
- (2)  $\overline{A} = \cup \{\downarrow x : x \in M(A)\} = \downarrow M(A)$ .
- (3)  $A' = \cup \{(\downarrow x) \setminus \{x\} : x \in M(A)\} = (\downarrow M(A)) \setminus M(A)$ .
- (4) *the subset  $A$  is dense if and only if  $M \subseteq A$ .*
- (5) *the subset  $A$  is nowhere dense if and only if  $M \cap A = \emptyset$ .*

- (6) if  $|M| = 1$ , then any subset is either dense or nowhere dense.
- (7) open sets and closed sets are convex (but not conversely, i.e. convex sets need not be open sets or closed sets).

**Proposition 2.4.** Let  $(X, \tau(\leq))$  be an Artinian  $T_o$ - Alexandroff space. If  $A$  is a preopen set then each maximal element in  $A$  belongs to  $M$  (i.e.,  $M(A) \subseteq M$ ).

**Theorem 2.5.** Let  $(X, \tau(\leq))$  be an Artinian  $T_o$ - Alexandroff space, the set  $A$  is preclosed if and only if  $\downarrow x \subseteq A$  for all  $x \in A \cap M$ .

**Corollary 2.6.** Let  $(X, \tau(\leq))$  be an Artinian  $T_o$ - Alexandroff space. Then

- (a) the set  $A$  is preopen if and only if  $\downarrow x \cap A = \emptyset$  for all  $x \in A^c \cap M$ . Equivalently,  $A$  is preopen if and only if  $\hat{x} \subseteq A$  for all  $x \in A$ .
- (b) if  $X$  contains a top element  $\top$ , then a nonempty subset  $A$  is preopen if and only if  $\top \in A$  if and only if  $A$  is dense.

**Theorem 2.7.** Let  $(X, \tau(\leq))$  be an Artinian  $T_o$ - Alexandroff space. A set  $A$  is semi-open if and only if  $M(A) \subseteq M$ .

Recall that a space  $(X, \tau)$  is called *resolvable* [8] if and only if  $X = D \cup D^c$  where both  $D$  and  $D^c$  are dense. A subset  $A$  is *resolvable* if the subspace  $(A, \tau|_A)$  is resolvable. A space  $(X, \tau)$  is *irresolvable* if it is not resolvable. It is *strongly irresolvable* [17] if no nonempty open set is resolvable, and it is *hereditarily irresolvable* [8] if no nonempty subset is resolvable.

**Corollary 2.8.** Let  $(X, \tau(\leq))$  be an Artinian  $T_o$ - Alexandroff space. Then

- (1) a subset  $A$  is semi-closed if and only if  $\forall x \notin A$ , there exists  $y \in M \setminus A$  such that  $x \leq y$ . Equivalently,  $A$  is semi-closed if and only if when  $x \notin A$ ,  $\hat{x} \not\subseteq A$ .
- (2)  $PO(X) \subseteq SO(X)$ , that is; if  $A$  is preopen then it is semi-open.
- (3)  $PO(X) = \tau_\alpha$ , that is; a set  $A$  is preopen if and only if it is  $\alpha$ -open.
- (4)  $X$  contains an open, dense and hereditarily irresolvable subspace.
- (5)  $X$  is strongly irresolvable.
- (7) for each dense subset  $D$  of  $X$ ,  $D^\circ$  is dense.

- (8)  $(X, \tau_\alpha)$  is submaximal.
- (9) for  $A \subseteq X$  where  $A^\circ = \emptyset$ ,  $A$  is nowhere dense.

**Theorem 2.9.** [13] *Let  $(X, \tau)$  be a topological space. Then the following are equivalent:*

- (1)  $X$  is submaximal.
- (2) Every preopen set is open.

The following theorem characterizes submaximality condition in a  $T_o$ -Alexandroff space.

**Theorem 2.10.** *Let  $(X, \tau(\leq))$  be a  $T_o$ -Alexandroff space. Then the following are equivalent:*

- (i)  $X$  is submaximal.
- (ii) Each element of  $X$  is either maximal or minimal.
- (iii)  $X$  is nodec.

**Theorem 2.11.** *In Artinian  $T_o$ -Alexandroff spaces, the following are equivalent:*

- (i)  $(X, \tau)$  is extremally disconnected.
- (ii)  $PO(X) = SO(X)$ .
- (iii) For all  $x \in X$ ,  $|\hat{x}| = 1$ .

**Corollary 2.12.** *If  $(X, \tau(\leq))$  is a submaximal, extremally disconnected  $T_o$ -Alexandroff space, then  $SO(X)$  is equal to the original topology.*

**Corollary 2.13.** *If  $X$  has a maximum element  $\top$ , then  $PO(X) = SO(X)$ , and so  $X$  is extremally disconnected.*

**Theorem 2.14.** *Let  $(X, \tau(\leq))$  be a  $T_o$ -Alexandroff space.*

- (1) *If  $X$  is a chain, then  $X$  is hyperconnected.*
- (2) *If  $X$  contains a maximum element  $\top$ , then  $X$  is hyperconnected.*
- (3) *If  $X$  satisfies the ACC, then  $X$  is hyperconnected if and only if  $X$  contains a top element  $\top$ .*
- (4) *It may happen that a hyperconnected  $T_o$ -Alexandroff space is not a chain and does not contain a maximum element.*

**Theorem 2.15.** *Let  $(X, \tau(\leq))$  be an Artinian  $T_o$ -Alexandroff space. If  $A$  is a subset of  $X$ , then*

- (a)  $pInt(A) = \{x \in A : \hat{x} \subseteq A\}$ .
- (b)  $sInt(A) = \{x \in A : \hat{x} \cap A \neq \emptyset\}$ .
- (c)  $pCl(A) = A \cup \{\downarrow x : x \in A \cap M\}$ .
- (d)  $sCl(A) = A \cup \{x : \hat{x} \subseteq A\}$ .

### 3. THE MAIN RESULT: $\tau_\alpha$ - Topology

In this section, we still work with Artinian  $T_o$ -Alexandroff spaces.

We know by Corollary 2.8 part (3) that  $\tau_\alpha = PO(X)$  in an Artinian  $T_o$ -Alexandroff space. So we may study the class of preopen sets rather than the class of  $\alpha$ -open sets.

**Lemma 3.1.** *If  $(X, \tau(\leq))$  is an Artinian  $T_o$ -Alexandroff space, then the space  $(X, \tau_\alpha)$  is an Alexandroff space (necessarily  $T_o$ ).*

*Proof.* Let  $\{U_\alpha\}_{\alpha \in \Delta}$  be a collection of preclosed subsets of  $X$ , and let  $x \in M \cap (\bigcup_{\alpha \in \Delta} U_\alpha)$ . So  $x \in M \cap U_{\alpha_o}$  for some  $\alpha_o \in \Delta$  and hence  $\downarrow x \subseteq U_{\alpha_o} \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ . Therefore by Theorem 2.5  $\bigcup_{\alpha \in \Delta} U_\alpha$  is preclosed.  $\square$

By this Lemma, we see that  $(X, \tau_\alpha)$  is a  $T_o$ -Alexandroff space. We denote its specialization order by  $\leq_\alpha$ . By Corollary 2.8 part (8),  $(X, \tau_\alpha)$  is a submaximal  $T_o$ -Alexandroff space, and by Theorem 2.10 each element of  $X$  is either maximal or minimal with respect to the partial order  $\leq_\alpha$ .

The following proposition describes the partial order  $\leq_\alpha$  on  $X$ .

**Proposition 3.2.** *Let  $(X, \tau(\leq))$  be an Artinian  $T_o$ -Alexandroff space. If  $x \in X$  then  $x \leq_\alpha y$  only for each element  $y \in \{x\} \cup \hat{x}$ .*

*Proof.* Let  $y \in \hat{x}$ . By Theorem 2.15 part (c),  $pCl\{y\} = \downarrow y$ , so we get that  $x \in pCl\{y\} = Cl_\alpha\{y\} = \downarrow y$  and hence  $x \leq_\alpha y$ .

If  $x \leq_\alpha z$  and  $x \neq z$ , then  $z \in M$  and  $x \in pCl(z) = \downarrow z$  (to see this, if  $z \notin M$  then there is  $w \in M$  such that  $z \leq w$ , so  $z \in pCl(w) = \downarrow w$  which implies that  $x <_\alpha z <_\alpha w$  contradicting the fact  $\tau_\alpha$  is submaximal). Therefore  $x \leq z$  and hence  $z \in \hat{x}$ .  $\square$

We conclude from this theorem that  $x$  is a maximal element of  $X$  with respect to the order  $\leq$  if and only if it is maximal element of  $X$  with respect to the order  $\leq_\alpha$ , and since  $(X, \tau_\alpha)$  is a submaximal  $T_o$ -Alexandroff space, the graph of the poset  $(X, \leq_\alpha)$  consists of two rows, the row of

maximal elements and the row of minimal elements where the order is described in the above theorem. It is worth mentioning that these two rows may have an intersection, ie., one element can be maximal and minimal at the same time. By convention we will line up these elements in the top row in our examples.

**Notation.** For the notations with respect to the order  $\leq_\alpha$ , we will use the following notations

$$\begin{aligned} M_\alpha & \text{ the set of all maximal elements,} \\ M_\alpha(A) & \text{ the set of all maximal elements of } A, \\ m_\alpha & \text{ the set of all minimal elements,} \\ m_\alpha(A) & \text{ the set of all minimal elements of } A, \\ \uparrow_\alpha x & := \{y \in X : x \leq_\alpha y\}, \\ \downarrow_\alpha x & := \{y \in X : x \geq_\alpha y\}, \\ \uparrow_\alpha A & := \{y \in X : \exists x \in A, x \leq_\alpha y\}, \\ \downarrow_\alpha A & := \{y \in X : \exists x \in A, x \geq_\alpha y\}, \\ \hat{x}_\alpha & := \uparrow_\alpha x \cap M_\alpha, \\ \check{x}_\alpha & := \downarrow_\alpha x \cap m_\alpha. \end{aligned}$$

For a subset  $A$  of  $X$ , we can use the description of the partial order  $\leq_\alpha$  to find  $pCl(A) = Cl_\alpha(A)$  which is the smallest down set with respect to  $\leq_\alpha$  contains  $A$ , and  $pInt(A) = int_\alpha(A)$  which is the largest up set with respect to  $\leq_\alpha$  inside  $A$ . This gives another proof of parts (a) and (c) of Theorem 2.15. To see this, note that

$$pCl(A) = Cl_\alpha(A) = \downarrow_\alpha M_\alpha(A) = A \cup \{\downarrow x : x \in A \cap M\},$$

and

$$pInt(A) = int_\alpha(A) = \{x \in A : \uparrow_\alpha x \subseteq A\} = \{x \in A : \hat{x} \subseteq A\}.$$

One more time, we can use the description of the partial order  $\leq_\alpha$  rather than Theorem 2.5 and Corollary 2.6 part (1) to determine whether a subset  $A$  of  $X$  is preopen, preclosed or neither. If  $A$  is an up set (resp. a down set) with respect to  $\leq_\alpha$  then  $A$  is preopen (resp. preclosed). The following example illustrates this fact.



**Example 3.3.** Let  $X = \{a, b, c, d, r, e\}$  be a poset with the order as in figure 1 below:

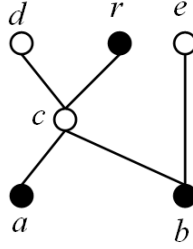


FIGURE 1

Let  $A = \{a, b, r\}$ , since  $\downarrow r \not\subseteq A$  and  $\downarrow d \not\subseteq A^c$ , by Theorem 2.5 and Corollary 2.6 it is not preclosed and not preopen.

The induced order  $\leq_\alpha$  on  $X$  is given in figure 2 below:

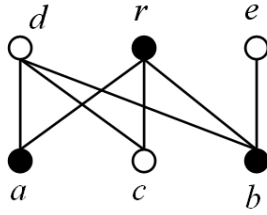


FIGURE 2

Clearly,  $A$  is not an up set and not a down set with respect to the order  $\leq_\alpha$ . Moreover,

$$pCl(A) = \downarrow_\alpha r = \{a, b, c, r\} \text{ and } pInt(A) = \uparrow_\alpha r = \{r\}.$$

**Theorem 3.4.** For a space  $(X, \tau(\leq))$ ,

- (i)  $M_\alpha = M$ ,
- (ii)  $M \cap m = M_\alpha \cap m_\alpha$ ,
- (iii)  $m_\alpha = (X \setminus M) \cup (m \cap M)$ ,

- (iv)  $\uparrow_\alpha x = \{x\} \cup \hat{x}$ ,
- (v) if  $x \notin M$ ,  $\downarrow_\alpha x = \{x\}$ , and if  $x \in M$ ,  $\downarrow_\alpha x = \downarrow x$
- (vi)  $\hat{x}_\alpha = \hat{x}$ .

*Proof.* (i) If  $x \in M$  then  $\hat{x} = \{x\}$  which implies that  $x \leq_\alpha x$  only, and so  $x \in M_\alpha$ .

If  $z \in M_\alpha$  then  $\hat{z} = \{z\}$ . Therefore  $x \in M$ .

(ii) If  $x \in M \cap m$  then  $x \in M = M_\alpha$ . Since  $x \in m$ , there is no element  $y \in X$  with  $x \in \hat{y}$ . Therefore, there is no  $y \in X$  with  $y \leq_\alpha x$ . This implies that  $x \in m_\alpha$ .

For the reverse inclusion, if  $z \in M_\alpha \cap m_\alpha$ , then  $z \in M_\alpha = M$ . If  $z \notin m$ , then there is  $y \in X$  such that  $y < z$ , so  $z \in \hat{y}$ , and hence  $y \leq_\alpha z$ . This implies that  $z \notin m_\alpha$  contradicting  $z \in M_\alpha \cap m_\alpha$ . Therefore,  $z \in M \cap M_\alpha$ .

(iii) Since  $(X, \tau_\alpha)$  is submaximal, each element of  $X$  is either maximal or minimal, so  $X = M_\alpha \cup m_\alpha$ . Hence, by parts (i) and (ii) we get that

$$\begin{aligned} m_\alpha &= (X \setminus M_\alpha) \cup (M_\alpha \cap m_\alpha) \\ &= (X \setminus M) \cup (M \cap m). \end{aligned}$$

(iv) If  $z \in \uparrow_\alpha x$  then  $x \leq_\alpha z$ , so either  $z = x$  or  $z \in \hat{x}$ . Therefore  $z \in \{x\} \cup \hat{x}$ .

If  $y \in \{x\} \cup \hat{x}$  then by Proposition 3.2,  $x \leq_\alpha y$ , so  $y \in \uparrow_\alpha x$ .

(v) If  $x \notin M$  then for each element  $y \in X$ ,  $x \notin \hat{y}$ , which implies that there is no element  $y \in X$  such that  $y \leq_\alpha x$ , so  $x \in m_\alpha$ . Therefore  $\downarrow_\alpha x = \{x\}$ .

Let  $x \in M$ . If  $z \in \downarrow_\alpha x$  then  $x \in \hat{z}$  and hence  $z \leq_\alpha x$  therefore  $z \in \downarrow_\alpha x$ .

For the other inclusion, let  $w \in \downarrow_\alpha x$ , so  $w \leq_\alpha x$ , and hence  $x \in \hat{w}$ . Therefore  $w \in \downarrow x$ .

(vi) If  $y \in \hat{x}_\alpha$  then  $y \in M_\alpha$  and  $x \leq_\alpha y$ . Hence,  $y \in \hat{x}$ .

If  $y \in \hat{x}$  then  $y \in M = M_\alpha$  and  $x \leq_\alpha y$ . So we get that  $y \in \uparrow_\alpha x \cap M_\alpha$ , hence  $y \in \hat{x}_\alpha$ .  $\square$

The following two theorems follow now as corollaries.

**Theorem 3.5.** *A subset  $D$  is dense with respect to the topology  $\tau(\leq)$  if and only if it is dense with respect to the topology  $\tau_\alpha$ .*

*Proof.* Since  $M_\alpha = M$  so  $M \subseteq A$  if and only if  $M_\alpha \subseteq A$ .  $\square$

If  $(X, \tau(\leq))$  is submaximal, then it is clear that the two orders  $\leq, \leq_\alpha$  are the same and  $\tau(\leq) = \tau_\alpha$ , and we get Theorem 2.9 of  $T_o$ -Alexandroff spaces.

Janković [6] showed that in any topological spaces,  $(\tau_\alpha)_\alpha = \tau_\alpha$ . In Artinian  $T_o$ -Alexandroff spaces, this fact is obvious here, since  $(X, \tau_\alpha)$  is submaximal.

**Theorem 3.6.** *Let  $(X, \tau(\leq))$  be an Artinian  $T_o$ -Alexandroff space. Then*

- (a)  *$(X, \tau(\leq))$  is extremally disconnected if and only if  $(X, \tau_\alpha)$  is extremally disconnected.*
- (b)  *$(X, \tau(\leq))$  is hyperconnected if and only if  $(X, \tau_\alpha)$  is hyperconnected.*
- (c) *For a subset  $A$  of  $X$ ,  $sInt(A) = sInt_\alpha(A)$  and  $sCl(A) = sCl_\alpha(A)$ .*

*Proof.* The proofs of parts (a) and (c) follow directly from the fact that  $\hat{x} = \hat{x}_\alpha$  using Theorem 2.11 and Theorem 2.15 parts (b) and (d).

(b) By Theorem 2.14 part (3),  $(X, \tau(\leq))$  is hyperconnected if and only if  $|M| = 1$  if and only if  $|M_\alpha| = 1$  if and only if  $(X, \tau_\alpha)$  is hyperconnected.  $\square$

**Lemma 3.7.** *For each subset  $A$  of  $X$ ,  $M(A) \subseteq M_\alpha(A)$ .*

*Proof.* Suppose that  $x \notin M_\alpha(A)$ . Then there exists  $y \neq x$  in  $A$  such that  $x \leq_\alpha y$ . Since each element of  $X$  is either maximal or minimal, we get that  $y \in M_\alpha = M$  and we get that  $y \in M \cap A$ , with  $y > x$  which implies that  $x \notin M(A)$ .  $\square$

**Proposition 3.8.** *Let  $(X, \tau(\leq))$  be an Artinian  $T_o$ -Alexandroff space. If  $A$  is a semi-open set then  $M(A) = M_\alpha(A)$*

*Proof.* Let  $A$  be a semi-open set. Lemma 3.7 shows that  $M(A) \subseteq M_\alpha(A)$ . For the reverse inclusion, let  $x \in M_\alpha(A)$ , i.e.,  $x$  is maximal in  $A$  with respect to the order  $\leq_\alpha$  and two cases induced for  $x$ , one case is that  $x \in M_\alpha = M$  which implies that  $x \in M(A)$ . In the other case, if  $x \in m_\alpha$  and if  $x \notin M(A)$ , there exists  $y \in M(A)$ ,  $y \neq x$  such that  $x < y$ . Since  $A$  is a semi-open set, by Theorem 2.7  $y \in M = M_\alpha$  which implies that  $y \in A \cap M_\alpha$ , and hence  $x \notin M_\alpha(A)$  which is a contradiction.  $\square$

The converse of the proposition is not true, i.e., if  $M(A) = M_\alpha(A)$ , then  $A$  need not be semi-open. The Sierpinski space  $X = \{a, b\}$  with the topology  $\tau(\leq) = \{\emptyset, X, \{b\}\}$ , with the chain order  $a \leq b$  is a counter example, since  $A = \{a\}$  is not semi-open while  $M(A) = M_\alpha(A) = \{a\}$ .

The inclusion in Lemma 3.7 may be proper as the following example shows.

**Example 3.9.** Let  $X = \{a, b, c\}$  with the chain order  $a \leq b \leq c$ . So the new order  $\leq_\alpha$  is the relation  $a \leq_\alpha c, b \leq_\alpha c$  and  $a, b$  are incomparable, as figure 3 shows

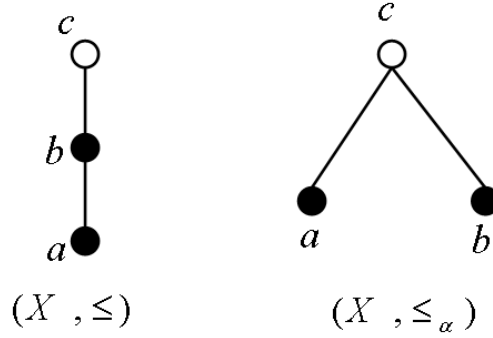


FIGURE 3

Let  $A = \{a, b\}$ , so  $M(A) = \{b\}$  and  $M_\alpha(A) = \{a, b\}$ ,  $M(A) \subsetneq M_\alpha(A)$  is proper inclusion. Note that  $A$  is not semi-open

**Theorem 3.10.** *Let  $(X, \tau(\leq))$  be an Artinian  $T_0$ -Alexandroff space, and let  $A$  be a subset of  $X$ . Then  $A$  is semi-open with respect to the space  $(X, \tau(\leq))$  if and only if it is semi-open with respect to the space  $(X, \tau_\alpha)$ , that is,  $SO(X, \tau(\leq)) = SO(X, \tau_\alpha)$ .*

*Proof.* We will use Theorem 2.7 in the proof of both directions.

( $\Rightarrow$ ) Suppose  $A$  is semi-open in  $(X, \tau(\leq))$ . Then  $M(A) \subseteq M$ . By Proposition 3.8 we have that:

$$M_\alpha(A) = M(A) \subseteq M = M_\alpha.$$

Hence  $A$  is semi-open in  $(X, \tau_\alpha)$ .

( $\Leftarrow$ ) Suppose  $A$  is semi-open in  $(X, \tau_\alpha)$ . By Lemma 3.7 we get that

$$M(A) \subseteq M_\alpha(A) \subseteq M_\alpha = M.$$

Hence  $A$  is semi-open in  $(X, \tau(\leq))$ .  $\square$

In fact, the above result is true in any topological space, i.e., for any topological space  $(X, \tau)$  we have that  $SO(X, \tau) = SO(X, \tau_\alpha)$  - a result that was proved by D.S. Janković in [6], however here we introduced a simpler proof that goes along the theme that has been used in this paper.

**Corollary 3.11.** *If  $A$  is semi-open, then*

- (1)  *$A$  is hyperconnected subspace in  $\tau$  if and only if  $A$  is hyperconnected subspace in  $\tau_\alpha$ .*
- (2)  *$D$  is dense in the subspace  $A$  in  $\tau$  if and only if  $D$  is dense in the subspace  $A$  in  $\tau_\alpha$ .*

*Proof.* (1) Since  $A$  is semi-open,  $M(A) = M_\alpha(A)$ .  $A$  is hyperconnected in  $\tau$  if and only if  $|M(A)| = 1$  if and only if  $|M_\alpha(A)| = 1$  if and only if  $A$  is hyperconnected in  $\tau_\alpha$ .

(2)  $D$  is dense in  $A$  in  $\tau$  if and only if  $M(A) \subseteq D$  if and only if  $M_\alpha(A) \subseteq D$  if and only if  $D$  is dense in  $A$  in  $\tau_\alpha$ .  $\square$

The condition that  $A$  is semi-open is necessary in the above corollary. To see this, back to Example 3.9 where  $X = \{a, b, c\}$  with the two partial orders  $\leq$  and  $\leq_\alpha$  shown in Figure 3. The subset  $A = \{a, b\}$  is not a semi-open set in  $\tau$  with  $|M(A)| = 1$  and  $|M_\alpha(A)| = 2$ , so  $A$  is hyperconnected in  $\tau$  while  $A$  is hyperdisconnected ( $=X$  is not hyperconnected) in  $\tau_\alpha$ . The subset  $D = \{b\}$  of  $A$  contains  $M(A)$ , so it is dense in  $A$  in  $\tau$  while it is not dense in  $A$  in  $\tau_\alpha$ , since  $M_\alpha(A) = \{a, b\}$  is not a subset of  $D$ . This fact implies that the class of  $\alpha$ -open sets in the subspace  $(A, \tau(\leq)|_A)$  need not be the same class  $\tau_\alpha|_A$ . When they are equal, then by Theorem 3.6  $A$  is hyperconnected in  $\tau$  iff  $A$  is hyperconnected in  $\tau_\alpha$ .

**Theorem 3.12.** *Let  $(X, \tau(\leq))$  be an Artinian  $T_o$ - Alexandroff space. Then  $A$  is clopen if and only if  $A$  is preclopen.*

*Proof.* In any topological space, if  $A$  is clopen then it is preclopen.

For the converse, suppose that  $A$  is preclopen, then  $A^c$  is preclopen. In the trivial case, when  $A = X$ , the result holds. Let  $x \in A$ . If  $y \in \downarrow x \cap A^c$  or  $y \in \uparrow x \cap A^c$  then  $\hat{x} \cap \hat{y} \neq \emptyset$ , so either  $\hat{x} \not\subseteq A$  or  $\hat{y} \not\subseteq A^c$ . By Corollary 2.6 part (a), either  $A$  or  $A^c$  is not preopen contradicting  $A$  and  $A^c$  are preclopen. So, both  $\uparrow x$  and  $\downarrow x$  are in  $A$  and hence  $A$  is clopen.  $\square$

Recall that a space  $(X, \tau)$  is *preconnected* if  $X$  cannot be represented as the disjoint union of two preopen subsets [29]).

**Theorem 3.13.**  *$(X, \tau(\leq))$  is connected if and only if  $(X, \tau_\alpha)$  is connected. Equivalently,  $(X, \tau(\leq))$  is connected if and only if it is preconnected.*

*Proof.*  $(X, \tau(\leq))$  is disconnected if and only if there exists a nontrivial clopen subset  $A$  of  $X$  if and only if  $A$  is preclopen if and only if  $(X, \tau_\alpha)$  is disconnected.  $\square$

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