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A GENERALIZATION OF D'ALEMBERT, JENSEN AND QUADRATIC FUNCTIONAL EQUATIONS ON SEMIGROUPS

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تصميم المعادلات الدالية دي أليمبرت وجينسن والمعادلة التربيعية على أشباه الزمرّ

ملخص: في هذا البحث تم إيجاد الحل العام للمعادلات الدالية التاليــة 1. تعمــيم معادلـــة دي أليمبرت 2. تعميمات لمعادلة جينسن 3. تعميم المعادلة التربيعية، مع العلم أن هذه المعـــادلات على أشباه زمر.

Abstract: *In this paper we shall find the general solution* $g : S_1\times S_2\times\ldots\times S_n\rightarrow F$ *of a generalization of d'Alembert equation* $g(x_1 + y_1 , x_2 + y_2 , \ldots, x_n + y_n) +g(x_1 + \sigma_1 y_1 , x_2 + \sigma_2 y_2 , \ldots, x_n + \sigma_n y_n) = 2g(x_1, x_2, \ldots, x_n) g(y_1, y_2,\ldots, y_n)$ $((x_1 , x_2,\ldots, x_n), (y_1 , y_2 ,\ldots, y_n) \in S_1\times S_2\times\ldots\times S_n)$,
the general solution $h : S_1\times S_2\times\ldots\times S_n\times T_1\times T_2\times\ldots\times T_m\rightarrow G$ *of generalizations of Jensen equation*
$h(x_1 + y_1 , x_2 + y_2 ,\ldots, x_n + y_n, t_1, t_2,.., t_m)+ h(x_1 + \sigma_1 y_1 , x_2 + \sigma_2 y_2 ,\ldots, x_n + \sigma_n y_n , t_1, t_2,.., t_m) = 2 h(x_1, x_2,\ldots, x_n, t_1, t_2,.., t_m)$ $((x_1 , x_2,\ldots, x_n, t_1, t_2,\ldots, t_m), (y_1 ,y_2,\ldots,y_n, t_1, t_2,.., t_m) \in S_1\times S_2\times\ldots\times S_n\times T_1\times T_2\times\ldots\times T_m)$,
and $h((x_1 + y_1, t_1), (x_2 + y_2, t_2),\ldots,(x_n + y_n, t_n)) + h((x_1 + \sigma_1 y_1, t_1), (x_2 + \sigma_2 y_2 , t_2), \ldots, (x_n + \sigma_n y_n, t_n)) = 2 h((x_1 , t_1), (x_2 , t_2), \ldots,(x_n, t_n))$ $((x_1 , x_2,\ldots, x_n), (y_1 ,y_2,\ldots,y_n), (t_1 ,t_2,\ldots, t_n) \in S_1\times S_2\times\ldots\times S_n$,
and the general solution $k : S_1\times S_2\times\ldots\times S_n\rightarrow H$ *of a generalization of the quadratic equation*
$k(x_1 + y_1 , x_2 + y_2 , \ldots, x_n + y_n) + k(x_1 + \sigma_1 y_1 , x_2 + \sigma_2 y_2 , \ldots, x_n + \sigma_n y_n) =2k(x_1, x_2, \ldots, x_n)+ 2k(y_1, y_2,\ldots, y_n)$ $((x_1 , x_2,\ldots, x_n), (y_1 , y_2 ,\ldots, y_n) \in S_1\times S_2\times\ldots\times S_n)$,
where $S_1,S_2,\ldots,S_n,T_1, T_2,\ldots,$ *and* T_m *are commutative semigroups ,F is a quadratically closed field of characteristic different from 2, G is a 2-cancelative abelian group, H is an abelian group uniquely divisible by 2 and for each* $i = 1, 2,\ldots n$, σ_i *is an endomorphism of* S_i *with* $\sigma_i (\sigma_i x_i) = x_i$.

Key Words: Functional Equation, Semigroup.

1. Introduction

Throughout this paper, (S, +), (S_1, +), (S_2, +), …,(S_n, +), (T_1, +), (T_2, +), … and (T_m, +) will denote semigroups (a semigroup is a set with an associative binary operation), F a quadratically closed field of characteristic different from 2, G a 2-cancelative abelian group, H an abelian group uniquely divisible by 2 and for each $i = 1, 2,\ldots n$, σ_i will denote an endomorphism of S_i with $\sigma_i(\sigma_i x_i) = x_i$.

A function h defined on $S \times S$ is called symmetric if $h(x, y) = h(y, x)$ for all $x, y \in S$, and it is called biaditive if $h(x + y, z) = h(x, z) + h(y, z)$ and $h(x, y + z) = h(x, y) + h(x, z)$ for all $x, y, z \in S$. A function $\chi : S_1 \times S_2 \times \ldots \times S_n \rightarrow F$ is called multiplicative if it satisfies $\chi(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = \chi(x, x_2, \ldots, x_n)\, \chi(y_1, y_2, \ldots, y_n)$ for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in S_1 \times S_2 \times \ldots \times S_n$.

In [2], Sinopoulos solve three functional equations, namely
the d'Alembert functional equation $g(x + y) + g(x + \sigma y) = 2g(x)\, g(y)$, $x, y \in S$,
the Jensen functional equation $g(x + y) + g(x + \sigma y) = 2g(x)$, $x, y \in S$,
and the quadratic functional equation $g(x + y) + g(x + \sigma y) = 2g(x) + 2g(y)$ $x, y \in S$.
It would be noted here that in the case where S is an abelian group and F = G = H = the complex field, the equations were solved by Stetkaer in [3].

In this paper we solved general equations of the above three functional equations, where our proofs are similar to that in [2], accordingly the above three equations become easy corollaries of our results in this paper.

Specifically, in this paper we shall find the general solutions of the following functional equations

$g(x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n) + g(x_1 + \sigma_1 y_1, x_2 + \sigma_2 y_2, \ldots\ldots, x_n + \sigma_n y_n) = 2g(x_1, x_2, \ldots, x_n)\, g(y_1, y_2, \ldots, y_n)$ ($(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in S_1 \times S_2 \times \ldots \times S_n$),

$h(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n, t_1, t_2, .., t_m) + h(x_1 + \sigma_1 y_1, x_2 + \sigma_2 y_2, \ldots, x_n + \sigma_n y_n, t_1, t_2, .., t_m) = 2\, h(x_1, x_2, \ldots, x_n, t_1, t_2, .., t_m)$ ($(x_1, x_2, \ldots, x_n, t_1, t_2, .., t_m), (y_1, y_2, \ldots, y_n, t_1, t_2, .., t_m) \in S_1 \times S_2 \times \ldots \times S_n \times T_1 \times T_2 \times \ldots \times T_m$,

$h((x_1 + y_1, t_1), (x_2 + y_2, t_2), \ldots, (x_n + y_n, t_n)) + h((x_1 + \sigma_1 y_1, t_1), (x_2 + \sigma_2 y_2, t_2), \ldots, (x_n + \sigma_n y_n, t_n)) = 2\, h((x_1, t_1), (x_2, t_2), \ldots, (x_n, t_n))$ ($(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n), (t_1, t_2, \ldots, t_n) \in S_1 \times S_2 \times \ldots \times S_n$, and

$k(x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n) + k(x_1 + \sigma_1 y_1, x_2 + \sigma_2 y_2, \ldots\ldots, x_n + \sigma_n y_n) = 2k(x_1, x_2, \ldots, x_n) + 2\, k(y_1, y_2, \ldots, y_n)$ ($(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in S_1 \times S_2 \times \ldots \times S_n$).

2. A generalization of d'Alembert Equation

In this section we find the general solution of a generalized d'Alembert functional equation where we write the general solution $g : S_1 \times$

$S_2 \times \ldots \times S_n \to F$ in terms of multiplicative functions, i. e. in terms of solutions of the functional equation
$f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = f(x_1, x_2, \ldots \ldots, x_n) f(y_1, y_2, \ldots, y_n)$

Theorem 2.1

Let $(S_1,+)$, $(S_2,+)$, ……… $(S_n,+)$ be commutative semigroups and F be a quadratically closed field of characteristic different from 2. If for $i = 1, 2, \ldots\ldots n$, σ_i is an endomorphism on S_i with $\sigma_i(\sigma_i x_i) = x_i$ for all $x_i \in S_i$, then the general solution of
$g(x_1 + y_1\ x_2 + y_2\ \ldots\ldots, x_n + y_n) + g(x_1 + \sigma_1 y_1, x_2 + \sigma_2 y_2, \ldots\ldots, x_n + \sigma_n y_n) = 2g(x_1, x_2, \ldots, x_n) g(y_1, y_2, \ldots, y_n)$ *for all* $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ *in* $S_1 \times S_2 \times \ldots \times S_n$ ………………………………………………(1)
is given by $g(x_1, x_2, \ldots \ldots, x_n) = \frac{1}{2}[\chi(x_1, x_2, \ldots\ldots x_n) + \chi(\sigma_1 x_1, \sigma_2 x_2, \ldots\ldots \sigma_n x_n)]$ *for all* $(x_1, x_2, \ldots x_n) \in S_1 \times S_2 \times \ldots \times S_n$ ………………………………… (2),
where $\chi : S_1 \times S_2 \times \ldots \times S_n \to F$ *is an arbitrary multiplicative function.*

Proof:

First we use the property that χ is multiplicative and $\sigma_i(\sigma_i(y_i)) = y_i$ to check that (2) satisfies equation (1). For all $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ in $S_1 \times S_2 \times \ldots \times S_n$ we have,
$2g(x_1, x_2, \ldots, x_n) g(y_1, y_2, \ldots, y_n) = 2\{\frac{1}{2}[\chi(x_1, x_2, \ldots, x_n) + \chi(\sigma_1 x, \sigma_2 x_2, \ldots, \sigma_n x_n)]\} \{\frac{1}{2}[\chi(y_1, y_2, \ldots, y_n) + \chi(\sigma_1 y_1, \sigma_2 y_2, \ldots, \sigma_n y_n)]\} = \frac{1}{2}[\chi(x_1, x_2, \ldots, x_n)\chi(y_1, y_2, \ldots, y_n) + \chi(x_1, x_2, \ldots x_n)\chi(\sigma_1 y_1, \sigma_2 y_2, \ldots, \sigma_n y_n) + \chi(\sigma_1 x, \sigma_2 x_2, \ldots, \sigma_n x_n)\chi(y_1, y_2, \ldots, y_n) + \chi(\sigma_1 x, \sigma_2 x_2, \ldots \sigma_n x_n)\chi(\sigma_1 y_1, \sigma_2 y_2, \ldots, \sigma_n y_n)] = \frac{1}{2}[\chi(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) + \chi(x_1 + \sigma_1 y_1, x_2 + \sigma_2 y_2, \ldots, x_n + \sigma_n y_n) + \chi(\sigma_1 x_1 + y_1, \sigma_2 x_2 + y_2, \ldots, \sigma_n x_n + y_n) + \chi(\sigma x_1 + \sigma_1 y_1, \sigma_2 x_2 + \sigma_2 y_2, \ldots, \sigma_n x_n + \sigma_n y_n)] = g(x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n) + g(x_1 + \sigma_1 y_1, x_2 + \sigma_2 y_2, \ldots, x_n + \sigma_n y_n)$. Hence (2) is a solution of (1).

Conversly, assume that g is a solution of (1). We show that g must have the form (2). Throught the rest of the proof let $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ and $(t_1, t_2, \ldots, t_n)$ be any elements in $S_1 \times S_2 \times \ldots \times S_n$.

For all $i = 1, 2, \ldots\ldots, n$, replace in (1) y_i by $\sigma_i y_i$ and use the property that $\sigma_i(\sigma_i(y_i)) = y_i$ to get, $2g(x_1, x_2, \ldots, x_n) g(\sigma_1 y_1, \sigma_2 y_2, \ldots, \sigma_n y_n) =$
$g(x_1 + \sigma_1 y_1, x_2 + \sigma_2 y_2, \ldots, x_n + \sigma_n y_n) + g(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.
Then by (1) and using the assumption that the characteristic of F is not 2 to get,
$g(y_1, y_2, \ldots, y_n) = g(\sigma_1 y_1, \sigma_2 y_2, \ldots, \sigma_n y_n)$ ……………………………….. (3)
Again by using $\sigma_i(\sigma_i(y_i)) = y_i$ and (3) we get
$g(x_1 + \sigma_1 y_1, x_2 + \sigma_2 y_2, \ldots, x_n + \sigma_n y_n) = g(\sigma_1 x_1 + y_1, \sigma_2 x_2 + y_2, \ldots, \sigma_n x_n + y_n)$...(4).

We have two cases.

Case 1: $g(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) = g(x_1+\sigma_1 y_1, x_2+\sigma_2 y_2, \ldots, x_n+\sigma_n y_n)$.

Then by substituting in (1) and dividing by 2 we get,

$g(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) = g(x_1, x_2, \ldots, x_n)\, g(y_1, y_2, \ldots, y_n)$. Hence g is a multiplicative function. By (3) one can have $g(x_1, x_2, \ldots, x_n) = ½ [g(x_1, x_2, \ldots, x_n) + g(\sigma_1 x_1, \sigma_2 x_2, \ldots, \sigma_n x_n)]$. Hence we can write g in the form of (2).

Case 2: $g(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) \neq g(x_1+\sigma_1 y_1, x_2+\sigma_2 y_2, \ldots, x_n+\sigma_n y_n)$.

Then there exist $(u_1, u_2, \ldots, u_n)$, $(v_1, v_2, \ldots, v_n)$ in $S_1\times S_2\times\ldots\times S_n$ such that

$g(u_1+v_1, u_2+v_2, \ldots, u_n+v_n) - g(u_1+\sigma_1 v_1, u_2+\sigma_2 v_2, \ldots, u_n+\sigma_n v_n) \neq 0$ (5)

Define a function $f : S_1\times S_2\times\ldots\times S_n \to F$ by $f(x_1, x_2, \ldots, x_n) = g(x_1+v_1, x_2+v_2, \ldots, x_n+v_n) - g(x_1+\sigma_1 v_1, x_2+\sigma_2 v_2, \ldots, x_n+\sigma_n v_n)$. Then $f(u_1, u_2, \ldots, u_n) \neq 0$ and by using (3) and (4) one can show that $f(x_1, x_2, \ldots, x_n) = -f(\sigma_1 x, \sigma_2 x_2, \ldots \sigma_n x_n)$ …….(6)

Now, by (3) $f(x_1+\sigma_1 y_1, x_2+\sigma_2 y_2, \ldots, x_n+\sigma_n y_n) = g(x_1+\sigma_1 y_1+v_1, x_2+\sigma_2 y_2+v_2, \ldots, x_n+\sigma_n y_n+v_n) - g(x_1+\sigma_1 y_1+\sigma_1 v_1, x_2+\sigma_2 y_2+\sigma_2 v_2, \ldots, x_n+\sigma_n y_n+\sigma_n v_n)$

$= g(\sigma_1(x_1+\sigma_1 y_1+v_1), \sigma_2(x_2+\sigma_2 y_2+v_2), \ldots, \sigma_n(x_n+\sigma_n y_n+v_n)) - g(\sigma_1(x_1+\sigma_1 y_1+\sigma_1 v_1), \sigma_2(x_2+\sigma_2 y_2+\sigma_2 v_2), \ldots, \sigma_n(x_n+\sigma_n y_n+\sigma_n v_n)) = -f(\sigma_1 x_1+y_1, \sigma_2 x_2+y_2, \ldots, \sigma_n x_n+y_n)$.

Hence $f(x_1+\sigma_1 y_1, x_2+\sigma_2 y_2, \ldots, x_n+\sigma_n y_n) = -f(\sigma_1 x_1+y_1, \sigma_2 x_2+y_2, \ldots, \sigma_n x_n+y_n)$ ….(7)

Use the definition of f and the equation (1) to get

$f(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) + f(x_1+\sigma_1 y_1, x_2+\sigma_2 y_2, \ldots, x_n+\sigma_n y_n) = g(x_1+y_1+v_1, x_2+y_2+v_2, \ldots, x_n+y_n+v_n) - g(x_1+y_1+\sigma_1 v_1, x_2+y_2+\sigma_2 v_2, \ldots, x_n+y_n+\sigma_n v_n) + g(x_1+\sigma_1 y_1+v_1, x_2+\sigma_2 y_2+v_2, \ldots, x_n+\sigma_n y_n+v_n) - g(x_1+\sigma_1 y_1+\sigma_1 v_1, x_2+\sigma_2 y_2+\sigma_2 v_2, \ldots, x_n+\sigma_n y_n+\sigma_n v_n) = [g(x_1+v_1+y_1, x_2+v_2+y_2, \ldots, x_n+v_n+y_n) + g(x_1+v_1+\sigma_1 y_1, x_2+v_2+\sigma_2 y_2, \ldots, x_n+v_n+\sigma_n y_n)] - [g(x_1+\sigma_1 v_1+y_1, x_2+\sigma_2 v_2+y_2, \ldots, x_n+\sigma_n v_n+y_n) + g(x_1+\sigma_1 v_1+\sigma_1 y_1, x_2+\sigma_2 v_2+\sigma_2 y_2, \ldots, x_n+\sigma_n v_n+\sigma_n y_n)] = 2\, g(x_1+v_1, x_2+v_2, \ldots, x_n+v_n)\, g(y_1, y_2, \ldots, y_n) - 2\, g(x_1+\sigma_1 v_1, x_2+\sigma_2 v_2, \ldots, x_n+\sigma_n v_n)\, g(y_1, y_2, \ldots, y_n) = 2\, f(x_1, x_2, \ldots, x_n)\, g(y_1, y_2, \ldots, y_n)$. Hence

$f(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) + f(x_1+\sigma_1 y_1, x_2+\sigma_2 y_2, \ldots, x_n+\sigma_n y_n) = 2f(x_1, x_2, \ldots, x_n)\, g(y_1, y_2, \ldots, y_n)$.

Interchange $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in the above equation to get

$f(y_1+x_1, y_2+x_2, \ldots, y_n+x_n) + f(y_1+\sigma_1 x_1, y_2+\sigma_2 x_2, \ldots, y_n+\sigma_n x_n) = 2\, f(y_1, y_2, \ldots, y_n)\, g(x_1, x_2, \ldots, x_n)$.

Adding the last two equations and use (7) then divide the resulting equation by 2 to get,
$f(x_1+y_1, x_2+y_2,..,x_n+y_n) = f(x_1, x_2,.., x_n)\, g(y_1, y_2,.., y_n) + f(y_1, y_2,.., y_n)\, g(x_1, x_2,.., x_n)$ ….(8)
By using (8) we have, $f((x_1+t_1)+y_1, (x_2+t_2)+y_2,..,(x_n+t_n)+y_n) = f(x_1, x_2,.., x_n)\, g(t_1, t_2,.., t_n)\, g(y_1, y_2,.., y_n)+f(t_1, t_2,.., t_n)g(x_1, x_2,.., x_n)g(y_1, y_2,.., y_n)+f(y_1, y_2,.., y_n)g(x_1+t_1, x_2+t_2,., x_n+t_n)$, and, $f(x_1+(t_1+y_1), x_2+(t_2+y_2),\ldots, x_n+(t_n+y_n)) = f(x_1, x_2,.., x_n)\, g(t_1+y_1, t_2+y_2,.., t_n+y_n) + f(t_1, t_2,.., t_n)\, g(y_1, y_2,.., y_n)\, g(x_1, x_2,.., x_n) + f(y_1, y_2,.., y_n)\, g(t_1, t_2,.., t_n)\, g(x_1, x_2,.., x_n)$.

Subtract the last two equations then simplify to get $[g(x_1+t_1, x_2+t_2,., x_n+t_n) - g(x_1, x_2,.., x_n)\, g(t_1, t_2,.., t_n)]\, f(y_1, y_2,.., y_n) = [g(t_1+y_1, t_2+y_2,.., t_n+y_n) - g(t_1, t_2,.., t_n)\, g(y_1, y_2,.., y_n)]\, f(x_1, x_2,.., x_n)$. Let $(y_1, y_2,.., y_n) = (u_1, u_2, \ldots, u_n)$ in the above equation we have
$[g(x_1+t_1, x_2+t_2,., x_n+t_n) - g(x_1, x_2,.., x_n)\, g(t_1, t_2,.., t_n)]\, f(u_1, u_2,\ldots, u_n) = [g(t_1+u_1, t_2+u_2,.., t_n+u_n) - g(t_1, t_2,.., t_n)\, g(u_1, u_2,\ldots, u_n)]\, f(x_1, x_2,.., x_n)$.

Since $f(u_1, u_2,\ldots, u_n)$ is a nonzero element in the field F, then
$g(x_1+t_1, x_2+t_2,., x_n+t_n) - g(x_1, x_2,.., x_n)\, g(t_1, t_2,.., t_n) = h(t_1, t_2,.., t_n)\, f(x_1, x_2,.., x_n)$ ……....(9)
where $h(t_1, t_2,.., t_n) = [f(u_1, u_2,\ldots,u_n)]^{-1}[g(t_1+u_1, t_2+u_2,.., t_n+u_n) - g(t_1, t_2,.., t_n)g(u_1, u_2,..,u_n)]$.
Then by (9) $h(t_1, t_2,.., t_n)\, f(x_1, x_2,.., x_n) = h(x_1, x_2,.., x_n)\, f(t_1, t_2,.., t_n)$. Set $(x_1, x_2,.., x_n) = (u_1, u_2,\ldots,u_n)$, then $h(t_1, t_2,.., t_n)\, f(u_1, u_2,\ldots,u_n) = h(u_1, u_2,\ldots,u_n)\, f(t_1, t_2,.., t_n)$ and so $h(t_1, t_2,.., t_n)=[f(u_1, u_2,\ldots,u_n)]^{-1}\, h(u_1, u_2,\ldots,u_n)\, f(t_1, t_2,.., t_n)$.

Since F is quadratically closed, then there exists an $\alpha\in F$ such that $\alpha^2 = [f(u_1, u_2,\ldots,u_n)]^{-1}\, h(u_1, u_2,\ldots,u_n)$, so $h(t_1, t_2,.., t_n) = \alpha^2\, f(t_1, t_2,.., t_n)$.
In (9), replace $(t_1, t_2,.., t_n)$ by $(y_1, y_2,.., y_n)$ and use the above equation to get,
$g(x_1+y_1, x_2+y_2,., x_n+y_n) =g(x_1, x_2,.., x_n)g(y_1, y_2,..,y_n) + \alpha^2 f(y_1, y_2,..,y_n)f(x_1, x_2,.., x_n)$ …(10).

Multiplying (8) by α we get $\alpha\, f(x_1+y_1, x_2+y_2,..,x_n+y_n) = \alpha\, f(x_1, x_2,.., x_n)\, g(y_1, y_2,.., y_n) + \alpha\, f(y_1, y_2,.., y_n)\, g(x_1, x_2,.., x_n)$ …………………(11).
Adding (11) to (10) and simplifying to get,
$(g + \alpha f)(x_1+y_1, x_2+y_2,..,x_n+y_n) = (g + \alpha f)(x_1, x_2,.., x_n)\, (g + \alpha f)(y_1, y_2,..,y_n)$
Subtracting (11) from (10) and simplifying to get,
$(g - \alpha f)(x_1+y_1, x_2+y_2,..,x_n+y_n) = (g - \alpha f)(x_1, x_2,.., x_n)\, (g - \alpha f)(y_1, y_2,..,y_n)$.
Hence we can choose the functions $\chi_1 = g + \alpha f$ and $\chi_2 = g - \alpha f$ which are multiplicative functions. Moreover, $g(x_1, x_2,.., x_n) = \frac{1}{2}[\chi_1(x_1, x_2, \ldots, x_n) + \chi_2(x_1, x_2, \ldots, x_n)]$ as desired.

Finally, By (3) and (6) one can note that, $\chi_1(\sigma_1 x_1, \sigma_2 x_2,\ldots, \sigma_n x_n) = g(\sigma_1 x, \sigma_2 x_2, \ldots \sigma_n x_n) + \alpha f(\sigma_1 x, \sigma_2 x_2, \ldots \sigma_n x_n) = g(x_1, x_2, \ldots, x_n) - \alpha f(x_1, x_2, \ldots, x_n) = \chi_2(x_1, x_2, \ldots, x_n)$.
Therefore, any g satisfies (1) must be of the form (2). Hence the general solution of (1) is given by (2) ■

Now, Sinopoulos's result becomes a corollary of our theorem, where it is the case of n =1 in our theorem.

Corollary 2.2 [2, Theorem 1]

Let (S,+) be a commutative semigroup and F is a quadratically closed field of characteristic different from 2. If σ is an endomorphism on S with $\sigma(\sigma x) = x$ for all $x\in S$, then the general solution of $g(x+y) + g(x+\sigma y) = 2g(x)\, g(y)$ for all x, y in S is given by $g(x) = ½[\chi(x)+ \chi(\sigma x)]$ for all $x\in S$, where $\chi: S\to F$ is an arbitrary multiplicative function.

3. Generalizations of Jensen Equation

This contains three theorems that discuss general solutions of three types of generalized Jensen functional equation from which we can get two corollaries appear as corollary and theorem2 in [2].

Theorem 3.1

Let $(S_1,+)$, $(S_2,+)$, ……, $(S_n,+)$, $(T_1,+)$, $(T_2,+)$, ……, and $(T_m,+)$ be commutative semigroups and let G be a 2-cancelative abelian group. Suppose that for each $i = 1, 2,\ldots\ldots n$, σ_i is an endomorphism on S_i with $\sigma_i(\sigma_i x_i) = x_i$ for all $x_i\in S_i$. Then the general solution $h: S_1\times S_2\times\ldots\times S_n\times T_1\times T_2\times\ldots\times T_m\to G$ of the functional equation
$h(x_1+ y_1, x_2+ y_2,\ldots, x_n+ y_n, t_1, t_2,.., t_m)+ h(x_1+\sigma_1 y_1, x_2+\sigma_2 y_2,\ldots, x_n+\sigma_n y_n, t_1, t_2,.., t_m)=2\, h(x_1, x_2,\ldots, x_n, t_1, t_2,.., t_m)$ for all $(x_1, x_2,\ldots, x_n, t_1, t_2,.., t_m),(y_1, y_2,\ldots,y_n, t_1, t_2,.., t_m)$ belong to $S_1\times S_2\times\ldots\times S_n\times T_1\times T_2\times\ldots\times T_m$………(12) is given by $h(x_1, x_2,\ldots, x_n, t_1,t_2,..,t_m)= \varphi(x_1, x_2, \ldots, x_n, t_1, t_2,.., t_m) + a(t_1, t_2,.., t_m)$ for all $(x_1, x_2,\ldots, x_n, t_1, t_2,.., t_m)\in S_1\times S_2\times\ldots\times S_n\times T_1\times T_2\times\ldots\times T_m$….. (13), where $a: T_1\times T_2\times\ldots\times T_m\to G$ and $\varphi: S_1\times S_2\times\ldots\times S_n\times T_1\times T_2\times\ldots\times T_m\to G$ are arbitrary functions with $\varphi(x_1+y_1, x_2+y_2,\ldots, x_n+y_n, t_1, t_2,.., t_m) = \varphi(x_1, x_2,\ldots, x_n, t_1, t_2,.., t_m) + \varphi(y_1, y_2,\ldots, y_n, t_1, t_2,.., t_m)$ and $\varphi(x_1, x_2, \ldots, x_n, t_1, t_2,.., t_m) = -\varphi(\sigma_1 x_1, \sigma_2 x_2, \ldots, \sigma_n x_n, t_1, t_2,.., t_m)$ for all $(x_1, x_2,\ldots, x_n, t_1, t_2,.., t_m)$, $(y_1, y_2,\ldots,y_n, t_1, t_2,.., t_m)\in S_1\times S_2\times\ldots\times S_n\times T_1\times T_2\times\ldots\times T_m$.

Proof:

In the proof we let $(x_1, x_2,\dots, x_n, t_1, t_2,.., t_m)$, $(y_1, y_2,\dots, y_n, t_1, t_2,.., t_m)$ and $(z_1, z_2,\dots, z_n, t_1, t_2,.., t_m)$ to be any elements in $S_1\times S_2\times\dots\times S_n \times T_1\times T_2 \times\dots\times T_m$.

First we show that (13) is a solution of (12). We use (13) and the given properties of φ to get, $h(x_1 + y_1 , x_2 + y_2 ,\dots, x_n + y_n, t_1, t_2,.., t_m) + h(x_1 + \sigma_1y_1, x_2 + \sigma_2y_2,\dots, x_n + \sigma_ny_n, t_1, t_2,.., t_m) = \varphi(x_1 + y_1 , x_2 + y_2 ,\dots, x_n + y_n, t_1, t_2,.., t_m) + a(t_1, t_2,.., t_m) + \varphi(x_1 + \sigma_1y_1 , x_2 + \sigma_2y_2,\dots, x_n + \sigma_ny_n , t_1, t_2,.., t_m) + a(t_1, t_2,.., t_m) = \varphi(x_1 , x_2,\dots, x_n, t_1, t_2,.., t_m) + \varphi (y_1, y_2,\dots, y_n, t_1, t_2,.., t_m) + a(t_1, t_2,.., t_m) + \varphi(x_1 , x_2, \dots, x_n, t_1, t_2,.., t_m) + \varphi(\sigma_1y_1, \sigma_2y_2 ,\dots, \sigma_ny_n, t_1, t_2,.., t_m) + a(t_1, t_2,.., t_m) = 2 [\varphi(x_1, x_2,\dots, x_n, t_1, t_2,.., t_m) + a(t_1, t_2,.., t_m)] = 2 h(x_1, x_2,\dots, x_n, t_1, t_2,.., t_m)$. Hence (13) is a solution of (12).

Conversly, we assume that h is a solution of (12) and we shall show that h has the form (13). In (12), for each $i = 1, 2,\dots n$, replace y_i by $y_i + \sigma_iy_i$ and use the property $\sigma_i(\sigma_i (y_i)) = y_i$ to get, $h(x_1 + y_1+ \sigma_1y_1, x_2 + y_2 + \sigma_2y_2,\dots, x_n + y_n+ \sigma_ny_n, t_1, t_2,.., t_m) + h(x_1 + \sigma_1y_1+ y_1, x_2 + \sigma_2y_2 + y_2,\dots, x_n + \sigma_ny_n+ y_n, t_1, t_2,.., t_m) = 2 h(x_1, x_2, \dots, x_n, t_1, t_2,.., t_m)$. Since the semi-groups are commutative and G is 2-cancelative,then $h(x_1 + y_1+ \sigma_1y_1, x_2 + y_2 + \sigma_2y_2,\dots, x_n + y_n+ \sigma_ny_n, t_1, t_2,.., t_m) = h(x_1, x_2, \dots, x_n, t_1, t_2,.., t_m)$ ………………. (14).

Again in (12), for each $i = 1, 2,\dots n$, replace x_i by $x_i + z_i$ to get $h(x_1 + z_1+ y_1, x_2 + z_2+ y_2,\dots, x_n + z_n+ y_n, t_1, t_2,.., t_m) + h(x_1 + z_1+ \sigma_1y_1, x_2 + z_2+ \sigma_2y_2,\dots, x_n + z_n + \sigma_ny_n, t_1, t_2,.., t_m) = 2 h(x_1 + z_1, x_2 + z_2,\dots, x_n + z_n, t_1, t_2,.., t_m)$, interchange y_i by z_i in the last equation to get $h(x_1 + y_1+ z_1, x_2 + y_2+ z_2,\dots, x_n + y_n + z_n, t_1, t_2,.., t_m) + h(x_1 + y_1+ \sigma_1z_1, x_2 + y_2+ \sigma_2z_2,\dots, x_n + y_n + \sigma_nz_n, t_1, t_2,.., t_m) = 2 h(x_1 + y_1, x_2 + y_2,\dots, x_n + y_n, t_1, t_2,.., t_m)$.

Add the last two equations and simplify we obtain, $2 h(x_1 + z_1+ y_1, x_2 + z_2+ y_2,\dots, x_n + z_n+ y_n, t_1, t_2,.., t_m) + h(x_1 + z_1+ \sigma_1y_1, x_2 + z_2+ \sigma_2y_2,\dots, x_n + z_n + \sigma_ny_n, t_1, t_2,.., t_m) + h(x_1 + y_1+ \sigma_1z_1, x_2 + y_2+ \sigma_2z_2,\dots, x_n + y_n + \sigma_nz_n, t_1, t_2,.., t_m) = 2 h(x_1 + z_1, x_2 + z_2,\dots, x_n + z_n, t_1, t_2,.., t_m) + 2 h(x_1 + y_1, x_2 + y_2,\dots, x_n + y_n, t_1, t_2,.., t_m)$. By using (12) and $\sigma_i (\sigma_i (y_i)) = y_i$ we have, $h(x_1 + z_1+ \sigma_1y_1, x_2 + z_2+ \sigma_2y_2,\dots, x_n + z_n + \sigma_ny_n, t_1, t_2,.., t_m) + h(x_1 + y_1+ \sigma_1z_1, x_2 + y_2+ \sigma_2z_2,\dots, x_n + y_n + \sigma_nz_n, t_1, t_2,.., t_m) = h(x_1 + (z_1+ \sigma_1y_1), x_2 + (z_2+ \sigma_2y_2),\dots, x_n + (z_n + \sigma_ny_n), t_1, t_2,.., t_m) + h(x_1 + \sigma_1(z_1+ \sigma_1y_1), x_2 + \sigma_2(z_2+ \sigma_2y_2),\dots, x_n + \sigma_n(z_n + \sigma_ny_n), t_1, t_2,.., t_m) = 2 h(x_1, x_2,\dots, x_n, t_1, t_2,.., t_m)$. But G is 2-cancelative then, $h(x_1 + z_1+ y_1, x_2 + z_2+ y_2,\dots, x_n + z_n+ y_n, t_1, t_2,.., t_m) + h(x_1, x_2,\dots, x_n, t_1, t_2,.., t_m) = h(x_1 + z_1, x_2 + z_2,\dots, x_n+z_n, t_1, t_2,.., t_m) + h(x_1 + y_1, x_2 + y_2,\dots, x_n + y_n, t_1, t_2,.., t_m)$. Setting $z_i = \sigma_i x_i$ and obtain, $h(x_1 + \sigma_1x_1 + y_1, x_2 + \sigma_2x_2 + y_2,\dots, x_n + \sigma_nx_n + y_n, t_1, t_2,.., t_m) + h(x_1, x_2,\dots, x_n, t_1, t_2,.., t_m) = h(x_1 + \sigma_1x_1, x_2 + \sigma_2x_2,\dots, x_n+ \sigma_nx_n, t_1, t_2,.., t_m) + h(x_1 + y_1, x_2 + y_2,\dots, x_n + y_n, t_1, t_2,.., t_m)$.

By interchanging x_i and y_i in (14) and using the last equation we obtain that, $h(y_1, y_2,\dots,y_n, t_1, t_2,.., t_m) + h(x_1, x_2,\dots, x_n, t_1, t_2,.., t_m) = h(x_1 + \sigma_1x_1, x_2 + \sigma_2x_2,\dots, x_n + \sigma_nx_n, t_1, t_2,.., t_m) + h(x_1 + y_1, x_2 + y_2,\dots, x_n + y_n, t_1, t_2,.., t_m)$ ………… (15). Interchanging x_i and y_i, $h(x_1, x_2,\dots, x_n, t_1, t_2,.., t_m) + h(y_1, y_2,\dots,y_n, t_1, t_2,.., t_m) = h(y_1+ \sigma_1y_1, y_2+ \sigma_2y_2,\dots, y_n + \sigma_ny_n, t_1, t_2,.., t_m) + h(y_1 + x_1, y_2 + x_2,\dots, y_n + x_n, t_1, t_2,.., t_m)$.

Then by the last two equations we have, $h(x_1 + \sigma_1x_1, x_2 + \sigma_2x_2,\dots, x_n+ \sigma_nx_n, t_1, t_2,.., t_m) = h(y_1+ \sigma_1y_1, y_2+ \sigma_2y_2,\dots, y_n + \sigma_ny_n, t_1, t_2,.., t_m)$. Thus $h(x_1 + \sigma_1x_1, x_2 + \sigma_2x_2,\dots, x_n+ \sigma_nx_n, t_1, t_2,.., t_m)$ is constant in the first n variables; that is it is a function of $(t_1, t_2,.., t_m)$, say $a(t_1, t_2,.., t_m)$. So that (15) gives $h(x_1 + y_1, x_2 + y_2,\dots, x_n + y_n, t_1, t_2,.., t_m) - a(t_1, t_2,.., t_m) = [h(x_1, x_2,\dots, x_n, t_1, t_2,.., t_m) - a(t_1, t_2,.., t_m)] + [h(y_1, y_2,\dots,y_n, t_1, t_2,.., t_m) - a(t_1, t_2,.., t_m)]$.

Finally, by taking $\varphi(x_1 , x_2,\dots, x_n, t_1, t_2,.., t_m) = h(x_1, x_2,\dots, x_n, t_1, t_2,.., t_m) - a(t_1, t_2,.., t_m)$ we have h in the form (13). Moreover, $\varphi(x_1 + y_1 , x_2 + y_2 ,\dots, x_n + y_n, t_1, t_2,.., t_m) = h(x_1 + y_1 , x_2 + y_2 ,\dots, x_n + y_n, t_1, t_2,.., t_m) - a(t_1, t_2,.., t_m) = \varphi(x_1 , x_2,\dots, x_n, t_1, t_2,.., t_m) + \varphi (y_1, y_2,\dots, y_n, t_1, t_2,.., t_m)$ and $0 = h(x_1 + \sigma_1x_1, x_2 + \sigma_2x_2,\dots, x_n+ \sigma_nx_n, t_1, t_2,.., t_m) - a(t_1, t_2,.., t_m) = \varphi (x_1 + \sigma_1x_1, x_2 + \sigma_2x_2,\dots, x_n+ \sigma_nx_n, t_1, t_2,.., t_m) = \varphi(x_1 , x_2,\dots, x_n, t_1, t_2,.., t_m) + \varphi (\sigma_1x_1, \sigma_2x_2,\dots, \sigma_nx_n, t_1, t_2,.., t_m)$ which implies that $\varphi(x_1, x_2,\dots, x_n, t_1, t_2,.., t_m) = -\varphi(\sigma_1x_1, \sigma_2x_2,\dots, \sigma_nx_n, t_1, t_2,.., t_m)$.
Therefore, the general solution of (12) is given by (13) ■

Similarly one can prove the following theorem.

Theorem 3.2

Let $(S_1,+),(S_2,+),\dots\dots,(S_n,+)$ be commutative semigroups and let G be a 2-cancelative abelian group. Suppose that for each $i = 1, 2,\dots\dots n$, σ_i is an endomorphism on S_i with $\sigma_i (\sigma_i x_i) = x_i$ for all $x_i\in S_i$. Then the general solution $h: S_1\times S_2 \times\dots\times S_n \to G$ of the functional equation $h((x_1 + y_1 , t_1), (x_2 + y_2 , t_2),\dots\dots,(x_n + y_n, t_n)) + h((x_1 + \sigma_1 y_1 , t_1), (x_2 + \sigma_2 y_2 , t_2) ,\dots\dots, (x_n + \sigma_n y_n, t_n)) = 2 h((x_1 , t_1), (x_2 , t_2), \dots,(x_n, t_n))$ for all $(x_1 , x_2,\dots, x_n)$, $(y_1 ,y_2,\dots,y_n,)$, $(t_1 ,t_2,\dots, t_n)$ belong to $S_1\times S_2 \times\dots\times S_n$ is given by $h((x_1 , t_1), (x_2 , t_2),\dots\dots,(x_n, t_n)) = B((x_1 , t_1), (x_2 , t_2),\dots\dots,(x_n, t_n)) + a(t_1, t_2,.., t_n)$ for all $(x_1, x_2,\dots, x_n)$, $(t_1, t_2,.., t_n)\in S_1\times S_2\times\dots\times S_n$, where $a : S_1\times S_2 \times\dots\times S_n \to G$ and $B : (S_1\times S_1) \times(S_2 \times S_2) \times\dots\times (S_n\times S_n) \to G$ are arbitrary functions with the properties $B((x_1 + y_1 , t_1), (x_2 + y_2 , t_2), \dots, (x_n + y_n, t_n))= B((x_1 , t_1), (x_2 , t_2),\dots,(x_n, t_n))+ B((y_1 , t_1), (y_2 , t_2), \dots,(y_n, t_n))$

and $B((x_1, t_1), (x_2, t_2), \ldots, (x_n, t_n)) = -B((\sigma_1 x_1, t_1), (\sigma_2 x_2, t_2), \ldots, (\sigma_n x_n, t_n))$ for all $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$, $(t_1, t_2, \ldots, t_n)$ belong to $S_1 \times S_2 \times \ldots \times S_n$.

Proof:

Left to the reader ■

Now the corollary in [2] follows directly from theorem 3.2.

Corollary 3.3 [2, Corollary]

Let S be a commutative semigroup and let G be a 2-cancelative abelian group. Suppose that σ *is an endomorphism on S such that* $\sigma(\sigma(x)) = x$ *for all* $x \in S$. *Then the general solution* $h: S\times S \to G$ *of the functional equation*

$h(x+y, t) + h(x+\sigma y, t) = 2h(x, t)$ *for all* $x, y, t \in S$ *is given by* $h(x, t) = \varphi(x, t) + a(t)$ *for all* $x, t \in S$, *where* $a: S \to G$ *is an arbitrary function and* $\varphi: S\times S \to G$ *is an arbitrary function additive in the first variable with* $\varphi(x, t) = -\varphi(\sigma x, t)$ *for all* $x, t \in S$.

Similarly as in the proof of Theorem 3.1 one can prove the following theorem.

Theorem 3.4

Let $(S_1,+), (S_2,+), \ldots\ldots, (S_n,+)$ *be commutative semi-groups and let G be a 2-cancelative abelian group. Suppose that for each* $i = 1, 2, \ldots\ldots n$, σ_i *is an endomorphism on* S_i *with* $\sigma_i(\sigma_i(x_i)) = x_i$ *for all* $x_i \in S_i$. *Then the general solution* $g: S_1\times S_2\times\ldots\times S_n \to G$ *of the functional equation* $g(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) + g(x_1+\sigma_1 y_1, x_2+\sigma_2 y_2, \ldots, x_n+\sigma_n y_n) = 2g(x_1, x_2, \ldots, x_n)$ *for all* $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in S_1\times S_2\times\ldots\times S_n$ *is given by* $g(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + a$ *for all* $(x_1, x_2, \ldots, x_n) \in S_1\times S_2\times\ldots\times S_n$, *where* $a \in G$ *is an arbitrary constant and* $\varphi: S_1\times S_2\times\ldots\times S_n \to G$ *is an arbitrary function with*

$\varphi(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) = \varphi(x_1, x_2, \ldots, x_n) + \varphi(y_1, y_2, \ldots, y_n)$ *and* $\varphi(x_1, x_2, \ldots, x_n) = -\varphi(\sigma_1 x_1, \sigma_2 x_2, \ldots, \sigma_n x_n)$ *for all* $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in S_1\times S_2\times\ldots\times S_n$.

Proof:

Left to the reader ■

Hence Theorem 2 in [2] becomes an easy corollary of Theorem 3.4 above.

Corollary 3.5 [2, Theorem 2]

Let (S,+) be a commutative semigroup and let G be a 2-cancelative abelian group. Suppose that σ *is an endomorphism on S with* $\sigma(\sigma x) = x$ *for all* $x \in S$*. Then the general solution* $g: S \to G$ *of the functional equation* $g(x + y) + g(x + \sigma y) = 2g(x)$ *for all* $x,y \in S$ *is given by* $g(x) = \varphi(x) + a$ *for all* $x \in S$*, where* $a \in G$ *is an arbitrary constant and* $\varphi : S \to G$ *is an arbitrary additive function with* $\varphi(x) = -\varphi(\sigma x)$ *for all* $x \in S$*.*

4. A generalization of quadratic Equation

In this last section of this paper we find the general solution of a generalized quadratic functional equation from which we can get theorem 3 in [2] as a corollary of our result.

Theorem 4.1

Let $(S_1,+), (S_2,+), \ldots (S_n,+)$ *be additive semigroups and let H be an abelian group, uniquely divisible by 2. Suppose that for* $i = 1, 2, \ldots n$, σ_i *is an endomorphism of* S_i *with* $\sigma_i(\sigma_i x_i) = x_i$ *for all* $x_i \in S_i$*. then the general solution* $k: S_1\times S_2 \times \ldots \times S_n \to H$ *of the functional equation*

$k(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) + k(x_1 + \sigma_1 y_1, x_2 + \sigma_2 y_2, \ldots, x_n + \sigma_n y_n) = 2k(x_1, x_2, \ldots, x_n) + 2k(y_1, y_2, \ldots, y_n)$ *for all* $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)$ *in* $S_1\times S_2 \times \ldots \times S_n$ ………….(16)

is given by $k(x_1, x_2, \ldots, x_n) = B((x_1, x_1),(x_2, x_2), \ldots,(x_n, x_n)) + \varphi(x_1, x_2, \ldots, x_n)$ *for all* $(x_1, x_2, \ldots x_n) \in S_1\times S_2\times \ldots\times S_n$ ……………………….(17),

where $B : (S_1\times S_1) \times (S_2\times S_2) \times \ldots \times (S_n\times S_n) \to H$ *and* $\varphi : S_1\times S_2\times \ldots \times S_n \to H$ *are arbitrary functions with the properties:* $B((x_1, t_1),(x_2, t_2),\ldots,(x_n, t_n)) = B((t_1, x_1),(t_2, x_2),\ldots,(t_n, x_n))$, $B((x_1 + y_1, t_1),(x_2 + y_2, t_2),\ldots,(x_n + y_n, t_n)) = B((x_1, t_1),(x_2, t_2),\ldots,(x_n, t_n)) + B((y_1, t_1),(y_2, t_2),\ldots,(y_n, t_n))$, $B((x_1, t_1+ y_1),(x_2, t_2+ y_2),\ldots,(x_n, t_n+ y_n)) = B((x_1, t_1),(x_2, t_2),\ldots,(x_n, t_n)) + B((x_1, y_1),(x_2, y_2),\ldots,(x_n, y_n))$, $B((\sigma_1 x_1, y_1),(\sigma_2 x_2, y_2),\ldots,(\sigma_n x_n, y_n)) = -B((x_1, y_1),(x_2, y_2),\ldots,(x_n, y_n))$, $\varphi(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = \varphi(x_1, x_2,\ldots, x_n) + \varphi(y_1, y_2,\ldots, y_n)$ *and* $\varphi(x_1, x_2, \ldots, x_n) = \varphi(\sigma_1 x_1, \sigma_2 x_2, \ldots, \sigma_n x_n)$ *for all* $(x_1, x_2,\ldots, x_n), (y_1, y_2,\ldots, y_n)$ *and* $(t_1, t_2,\ldots,t_n)$ *in* $S_1\times S_2 \times \ldots \times S_n$*.*

Proof:

In the proof we let $(x_1, x_2,\ldots, x_n)$, $(y_1, y_2,\ldots, y_n)$ and $(t_1, t_2,.., t_n)$ to be any elements in $S_1\times S_2\times\ldots\times S_n$.

First we show that (17) is a solution of (16). We use (17) and the given properties of B and φ to get, $k(x_1 + y_1, x_2 + y_2,\ldots, x_n + y_n) + k(x_1 + \sigma_1 y_1, x_2 + \sigma_2 y_2,\ldots, x_n + \sigma_n y_n) = B((x_1 + y_1, x_1 + y_1), (x_2 + y_2, x_2 + y_2), \ldots,(x_n$

$+ y_n, x_n + y_n)) + \varphi(x_1 + y_1, x_2 + y_2,\ldots, x_n + y_n) + B((x_1 + \sigma_1 y_1, x_1 + \sigma_1 y_1), (x_2 + \sigma_2 y_2, x_2 + \sigma_2 y_2),\ldots, (x_n + \sigma_n y_n, x_n + \sigma_n y_n) + \varphi(x_1 + \sigma_1 y_1, x_2 + \sigma_2 y_2,\ldots, x_n + \sigma_n y_n) = B((x_1, x_1), (x_2, x_2), \ldots,(x_n, x_n)) + B((x_1, y_1), (x_2, y_2),\ldots,(x_n, y_n)) + B((y_1, x_1), (y_2, x_2), \ldots,(y_n, x_n)) + B((y_1, y_1), (y_2, y_2), \ldots,(y_n, y_n)) + \varphi(x_1, x_2,\ldots, x_n) + \varphi(y_1, y_2,\ldots, y_n) + B((x_1, x_1), (x_2, x_2),\ldots, (x_n, x_n)) + B((x_1, \sigma_1 y_1), (x_2, \sigma_2 y_2),\ldots, (x_n, \sigma_n y_n)) + B((\sigma_1 y_1, x_1), (\sigma_2 y_2, x_2),\ldots, (\sigma_n y_n, x_n)) + B((\sigma_1 y_1, \sigma_1 y_1), (\sigma_2 y_2, \sigma_2 y_2),\ldots, (\sigma_n y_n, \sigma_n y_n)) + \varphi(x_1, x_2,\ldots, x_n) + \varphi(\sigma_1 y_1, \sigma_2 y_2,\ldots, \sigma_n y_n) = 2 B((x_1, x_1), (x_2, x_2), \ldots,(x_n, x_n)) + 2 B((x_1, y_1), (x_2, y_2),\ldots,(x_n, y_n)) + B((y_1, y_1), (y_2, y_2), \ldots,(y_n, y_n)) + 2 B((\sigma_1 y_1, x_1), (\sigma_2 y_2, x_2),\ldots, (\sigma_n y_n, x_n)) + ((\sigma_1 y_1, \sigma_1 y_1), (\sigma_2 y_2, \sigma_2 y_2),\ldots, (\sigma_n y_n, \sigma_n y_n)) + 2 \varphi(x_1, x_2,\ldots, x_n) + 2 \varphi(y_1, y_2,\ldots, y_n) = 2 [B((x_1, x_1), (x_2, x_2), \ldots,(x_n, x_n)) + \varphi(x_1, x_2,\ldots, x_n)] + B((y_1, y_1), (y_2, y_2),\ldots,(y_n, y_n)) - B((y_1, \sigma_1 y_1), (y_2, \sigma_2 y_2),\ldots, (y_n, \sigma_n y_n)) + 2 \varphi(y_1, y_2,\ldots, y_n)$

$= 2k(x_1, x_2, \ldots, x_n) + 2 [B((y_1, y_1), (y_2, y_2),\ldots,(y_n, y_n)) + \varphi(y_1, y_2,\ldots, y_n)]$

$= 2 k(x_1, x_2, \ldots, x_n) + 2 k(y_1, y_2,\ldots, y_n)$. Hence (17) is a solution of (16).

Conversly, we assume that k is a solution of (16) and we shall show that h has the form (17) with the properties in the theorem. In (16), for each i = 1, 2,…n, replace y_i by $\sigma_i y_i$ and use the property $\sigma_i(\sigma_i(y_i)) = y_i$ to get, $k(x_1 + \sigma_1 y_1 , x_2 + \sigma_2 y_2 ,\ldots, x_n + \sigma_n y_n) + k(x_1 + y_1, x_2 + y_2,\ldots, x_n + y_n) = 2k(x_1, x_2,\ldots, x_n) + 2k(\sigma_1 y_1, \sigma_2 y_2,\ldots, \sigma_n y_n)$. Hence by (16), $2k(x_1, x_2,\ldots, x_n) + 2k(\sigma_1 y_1, \sigma_2 y_2,\ldots, \sigma_n y_n) = 2k(x_1, x_2, \ldots, x_n) + 2k(y_1, y_2,\ldots, y_n)$.

However G is an abelian group that uniquely divisible by 2, then we have,

$k(\sigma_1 y_1, \sigma_2 y_2,\ldots, \sigma_n y_n) = k(y_1, y_2,\ldots, y_n)$ ……………………………..…..(18).

Also in (16), for each i = 1, 2,…n, we replace first x_i by $x_i + t_i$ and then by $x_i + \sigma_i t_i$, to get $k(x_1 + t_1+ y_1, x_2 + t_2 + y_2,\ldots, x_n + t_n + y_n) + k(x_1 + t_1+ \sigma_1 y_1, x_2 + t_2 + \sigma_2 y_2,\ldots, x_n + t_n + \sigma_n y_n) = 2k(x_1+ t_1, x_2+ t_2, \ldots, x_n+ t_n) + 2k(y_1, y_2,\ldots, y_n)$,

$k(x_1 + \sigma_1 t_1+ y_1, x_2 + \sigma_2 t_2 + y_2,\ldots, x_n + \sigma_n t_n + y_n) + k(x_1 + \sigma_1 t_1+ \sigma_1 y_1, x_2 + \sigma_2 t_2 + \sigma_2 y_2,\ldots, x_n + \sigma_n t_n + \sigma_n y_n) = 2k(x_1+ \sigma_1 t_1, x_2+ \sigma_2 t_2, \ldots, x_n+ \sigma_n t_n) + 2k(y_1, y_2,\ldots, y_n)$.

By subtraction we obtain $k(x_1 + t_1+ y_1, x_2 + t_2 + y_2,\ldots, x_n + t_n + y_n) + k(x_1 + t_1+ \sigma_1 y_1, x_2 + t_2 + \sigma_2 y_2,\ldots, x_n + t_n + \sigma_n y_n) - k(x_1 + \sigma_1 t_1+ y_1, x_2 + \sigma_2 t_2 + y_2,\ldots, x_n + \sigma_n t_n + y_n) - k(x_1 + \sigma_1 t_1+ \sigma_1 y_1, x_2 + \sigma_2 t_2 + \sigma_2 y_2,\ldots, x_n + \sigma_n t_n + \sigma_n y_n) = 2 [k(x_1+ t_1, x_2+ t_2, \ldots, x_n+ t_n) -k(x_1+ \sigma_1 t_1, x_2+ \sigma_2 t_2, \ldots, x_n+ \sigma_n t_n)]$.

Hence $[k(x_1 + t_1+ y_1, x_2 + t_2 + y_2,\ldots, x_n + t_n + y_n) - k(x_1 + \sigma_1 t_1+ y_1, x_2 + \sigma_2 t_2 + y_2,\ldots, x_n + \sigma_n t_n + y_n)] + [k(x_1 + t_1+ \sigma_1 y_1, x_2 + t_2 + \sigma_2 y_2,\ldots, x_n + t_n + \sigma_n y_n) - k(x_1 + \sigma_1 t_1+ \sigma_1 y_1, x_2 + \sigma_2 t_2 + \sigma_2 y_2,\ldots, x_n + \sigma_n t_n + \sigma_n y_n)] = 2 [k(x_1+ t_1, x_2+ t_2, \ldots, x_n+ t_n) -k(x_1+ \sigma_1 t_1, x_2+ \sigma_2 t_2, \ldots, x_n+ \sigma_n t_n)]$ which can be written as,

$h((x_1+y_1, t_1), (x_2+y_2, t_2),\ldots, (x_n + y_n), t_n)) + h((x_1+\sigma_1 y_1, t_1), (x_2 +\sigma_2 y_2, t_2),\ldots, (x_n+\sigma_n y_n), t_n)) = 2\, h((x_1, t_1), (x_2, t_2),\ldots, (x_n , t_n))$ …………………… (19), where $h((x_1, t_1), (x_2, t_2),\ldots, (x_n , t_n)) = \frac{1}{4} [k(x_1+ t_1, x_2+ t_2, \ldots, x_n+ t_n) - k(x_1+ \sigma_1 t_1, x_2+ \sigma_2 t_2, \ldots, x_n+ \sigma_n t_n)] = \frac{1}{2} [k(x_1+ t_1, x_2+ t_2,\ldots, x_n+ t_n) - k(x_1, x_2,\ldots, x_n) - k(t_1, t_2,\ldots, t_n)]$…….(20).

According to Theorem3.2 we have $h((x_1, t_1), (x_2, t_2), \ldots,(x_n, t_n)) = B((x_1, t_1), (x_2, t_2), \ldots, (x_n, t_n)) + a(t_1, t_2,.., t_n)$ where $a : S_1\times S_2 \times\ldots\times S_n \rightarrow H$ and
$B : (S_1\times S_1) \times(S_2 \times S_2) \times\ldots\times (S_n\times S_n) \rightarrow H$ are arbitrary functions with the properties $B((x_1 + y_1, t_1), (x_2 + y_2, t_2), \ldots, (x_n + y_n, t_n)) = B((x_1, t_1), (x_2, t_2),\ldots,(x_n, t_n)) + B((y_1, t_1), (y_2, t_2),\ldots,(y_n, t_n))$ and $B((x_1, t_1), (x_2, t_2),\ldots,(x_n, t_n)) = -B((\sigma_1 x_1, t_1), (\sigma_2 x_2, t_2),\ldots,(x_n, \sigma_n t_n))$. Hence by using these properties we have
$h((x_1, t_1), (x_2, t_2),\ldots,(x_n, t_n)) + h((\sigma_1 x_1, t_1), (\sigma_2 x_2, t_2),\ldots, (x_n, \sigma_n t_n)) = 2\, a(t_1, t_2,.., t_n)$.

From (20) one can easily see that $h((x_1, t_1), (x_2, t_2),\ldots,(x_n, t_n)) = h((t_1, x_1), (t_2, x_2), \ldots,(t_n, x_n))$. By using (18) and (20) we have $h((x_1, t_1), (x_2, t_2),\ldots,(x_n, t_n)) = - h((\sigma_1 x_1,t_1), (\sigma_2 x_2, t_2), \ldots, (x_n, \sigma_n t_n))$. So that $a(t_1, t_2,.., t_n) = 0$ which implies that $h((x_1, t_1), (x_2, t_2),\ldots,(x_n, t_n)) = B((x_1, t_1), (x_2, t_2),\ldots,(x_n, t_n))$.
Thus $B((x_1, t_1), (x_2, t_2), \ldots\ldots, (x_n, t_n)) = B((t_1, x_1), (t_2, x_2), \ldots\ldots, (t_n, x_n))$ and
$B((x_1, t_1+ y_1), (x_2, t_2+ y_2), \ldots, (x_n, t_n+ y_n)) = B((x_1, t_1), (x_2, t_2),\ldots,(x_n, t_n)) + B((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n))$.

For each $i = 1, 2,\ldots n$, in (16) replace y_i by x_i and in (20) replace t_i by x_i to get, $k(2x_1, 2x_2, \ldots, 2x_n) + k(x_1 + \sigma_1 x_1, x_2 + \sigma_2 x_2, \ldots, x_n + \sigma_n x_n) = 4k(x_1, x_2, \ldots, x_n)$, and $h((x_1, x_1), (x_2, x_2),\ldots, (x_n , x_n)) = \frac{1}{2} k(2x_1, 2x_2,\ldots, 2x_n) - k(x_1, x_2,\ldots, x_n)$. Together
$h((x_1, x_1), (x_2, x_2),\ldots, (x_n , x_n)) = k(x_1, x_2,\ldots, x_n) - \frac{1}{2} k(x_1 + \sigma_1 x_1, x_2 + \sigma_2 x_2, \ldots, x_n +\sigma_n x_n)$. Hence $k(x_1, x_2,\ldots,x_n) = B((x_1,x_1), (x_2, x_2),\ldots,(x_n , x_n)) +\frac{1}{2} k(x_1+\sigma_1 x_1, x_2+\sigma_2 x_2,\ldots,x_n +\sigma_n x_n)$.
To end we define $\varphi : S_1\times S_2\times\ldots\times S_n \rightarrow H$ by $\varphi(x_1 ,x_2, \ldots, x_n) = \frac{1}{2} k(x_1 + \sigma_1 x_1, x_2 + \sigma_2 x_2, \ldots, x_n +\sigma_n x_n)$.

Finally we show that $\varphi(x_1 + y_1, x_2 + y_2 ,\ldots, x_n + y_n) = \varphi(x_1 , x_2,\ldots, x_n) + \varphi(y_1, y_2,\ldots, y_n)$ and $\varphi(x_1 ,x_2,\ldots \ldots, x_n) = \varphi(\sigma_1 x_1, \sigma_2 x_2,\ldots\ldots, \sigma_n x_n)$.
For each $i = 1, 2,\ldots n$, replace x_i by $x_i + \sigma_i x_i$ and y_i by $y_i + \sigma_i y_i$ in (16) and use $\sigma_i (\sigma_i y_i) = y_i$ to obtain,

$k(x_1 + \sigma_1 x_1 + y_1 + \sigma_1 y_1, x_2 + \sigma_2 x_2 + y_2 + \sigma_2 y_2, \ldots, x_n + \sigma_n x_n + y_n + \sigma_n y_n) = k(x_1 + \sigma_1 x_1, x_2 + \sigma_2 x_2, \ldots, x_n + \sigma_n x_n) + k(y_1 + \sigma_1 y_1, y_2 + \sigma_2 y_2, \ldots, y_n + \sigma_n y_n)$.
Hence
$\varphi(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = \varphi(x_1, x_2, \ldots, x_n) + \varphi(y_1, y_2, \ldots, y_n)$. By using $\sigma_i(\sigma_i x_i) = x_i$ we have, $\varphi(\sigma_1 x_1, \sigma_2 x_2, \ldots\ldots, \sigma_n x_n) = \varphi(x_1, x_2, \ldots \ldots, x_n)$. Thus k has the requiered form.
Therefore, (17) is the general solution of (15).

Hence Theorem 3 in [2] follows directly from our Theorem 4.1 above.

Corollary 4.2 [2, Theorem 3]

Let (S,+) be an additive semigroup and let H be an abelian group, uniquely divisible by 2. Suppose that σ is an endomorphism of S with $\sigma(\sigma x) = x$ for all $x \in S$. Then the general solution $k: S \to H$ of the functional equation
$k(x + y) + k(x + \sigma y) = 2k(x_1) + 2k(y)$ *for all x, y in S is given by*
$k(x) = B(x, x) + \varphi(x)$ *for all $x \in S$, where $B : S \times S \to H$ is an arbitrary symmetric biadditive function with $B(\sigma x, y) = -B(x, y)$ and $\varphi : S \to H$ is an arbitrary additive function with $\varphi(x) = \varphi(\sigma x)$.*

References

1- J. Aczel, and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 1989.
2- P. Sinopoulos, Functional equations on semigroups, *Aequationes Math. 59 (2000) 255-261.*
3- H. Stetkaer, Functional equations on abelian groups with involution, *Aequationes Math. 54 (1997) 144-172.*