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A NEW TECHNIQUE FOR ONE-DIMENSIONAL SCATTERING FROM DIRAC COMB

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طريقة جديدة لدراسة التشتت في بعد واحد من عدد من الحواجز الجهدية على شكل دالة ديراك

ملخص في هذا البحث نستخدم طريقة المصفوفات لدراسة نفاذ وانعكاس الموجات الكهرمغناطيسية من وسط متعدد الطبقات وكتابة معاملي الانعكاس والنفاذ باستخدام الدوال الأولية المتماثلة التي تستخدم في النظرية الرياضية لكثيرة الحدود ومن الممكن تطبيق هذا الأسلوب على التشتت في ميكانيكا الكم، ثم نحاول أن نطبق هذا المنهج لدراسة الانعكاس والنفاذ من نظام متكون من عدد N من الحواجز الجهدية التي لها شكل دالة ديراك.

ABSTRACT Using the well-known matrix formulation of the reflection and transmission of electromagnetic waves by a stratified planner structure, we show that the reflection and transmission coefficients of any number of isotropic media can be written by a simple general formula. This formula uses the so-called elementary symmetric functions that are extensively used in the mathematical theory of polynomials. The approach is then applied to the quantum scattering. We show that the reflection and transmission coefficients of any number of quantum wells or barriers can be written in the similar way. Finally, one-dimensional scattering from a series of delta-function barriers (a system that is called Dirac Comb) is studied. The computed numerical illustrations compared with the earlier results based on the transfer matrix and Chebychev polynomials reveal an excellent agreement.

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I- INTRODUCTION

Currently much attention has been focused on mesoscopic systems, in which electron transport is governed by quantum mechanics rather than classical electrodynamics. [1-5]. In this paper, we study one dimensional quantum mechanical scattering from a locally periodic potential as many artificial structures, for example, quantum wires, have one-dimensional characteristics. In section II, we present a simple and interesting method

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suggested by J. M. Vigoureux [6] for the reflection and transmission of electromagnetic waves by stratified planer structure. Using the well-known matrix formulation of reflection and transmission by a planner stratified media [7], the reflection and transmission coefficients of any number of isotropic media can be written by a simple general formula. This formula uses the so-called elementary symmetric functions of the mathematical theory of polynomials [8-10]. In section III, we apply the derived method in section II to a case of any number of quantum wells or barriers as we write the reflection and transmission coefficients in the same compact formula [11]. In section VI, we treat a sequence of $(N+1)$ identical Dirac delta function potentials spaced at regular intervals in a similar manner. Finally, we explore the results graphically and compare them with those obtained in another approach that uses Chebychev polynomials [12,13].

II- POLYNOMIAL REPRESENTATION OF REFLECTION AND TRANSMISSION COEFFICIENTS FOR THE PLANNER STRATIFIED STRUCTURE

Consider the case when the light is incident on a stack of media as shown in fig. (1). The j^{th} medium has d_j and n_j as a thickness and a refractive index respectively. The j^{th} interface located at z_j separates the two media of refractive indices n_j and n_{j+1}.

If we consider the fields at two different planes $(Z_0+\in)$ and $(Z_N+\in)$ where $\in$ is infinitely small, then $E(Z_0+\in)$ and $E(Z_N+\in)$ must be related by a 2×2 transformation matrix $[M]$. Denoting the fields in the first media by E_1 and in the last one by E_N, then we can write,

$$\Re_N = \frac{E_1^-}{E_1^+} = \frac{M_2}{M} \tag{1}$$

$$\tau_N = \frac{E_N^+}{E_1^+} = \frac{1}{M_{11}} \tag{2}$$

where $\Re_N$ and τ_N are the reflection and transmission coefficients respectively.

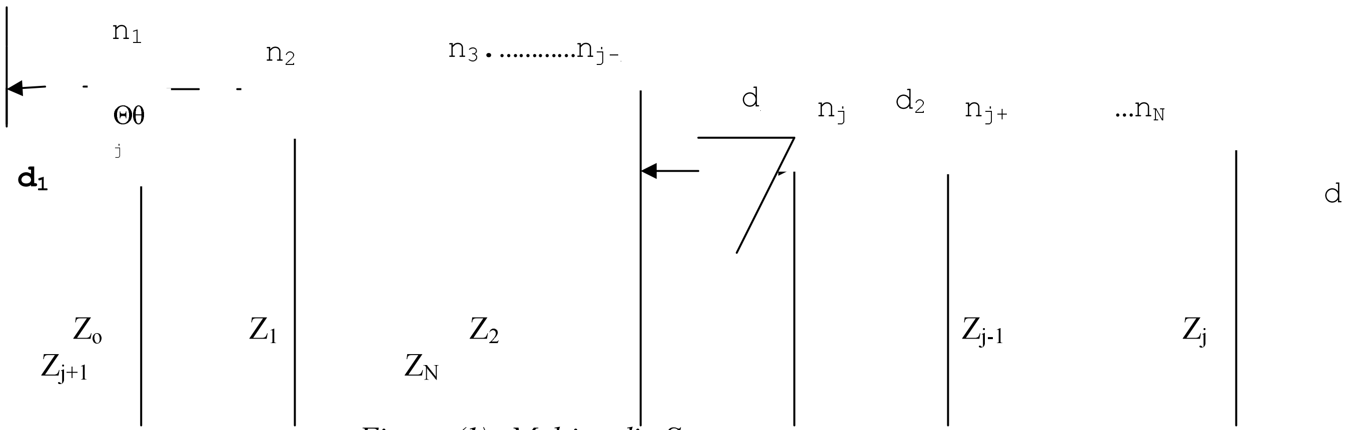


Figure (1): Multimedia Structure

The matrix [M] can be expressed as the product of interface matrices and layer matrices. The matrix [r_j] of the j[th] interface located at the plane z_j between two layers of refractive indices n_j and n_{j+1} is given by,

$$[r_j] = \frac{1}{t_{j,j+}} \begin{bmatrix} 1 & r_{j,j+} \\ r_{j,j+1} & 1 \end{bmatrix} \tag{3}$$

with $r_{j,j+1}$ and $t_{j,j+1}$ are the Fresnal reflection and transmission coefficients of the ($j,j+1$) interface.
The propagation of the fields across the same layer with refractive index n_j between two interfaces located at Z_{j-1} and $Z_j = Z_{j-1}+d_j$ is given by the matrix $[\phi_j]$ which is given by,

$$[\phi_j] = \begin{pmatrix} e^{i\phi_j} & 0 \\ 0 & e^{-i\phi_j} \end{pmatrix} \tag{4}$$

where the phase shift ϕ_j is given by

$$\phi_j = \frac{2\pi}{\lambda} n_j \cos\theta_j \, d_j \tag{5}$$

In the expression (5) θ_j is the angle between direction of propagation of the wave in the layer of refractive index n_j and the perpendicular to its boundaries (z-axis). Let

$$R_j \equiv R_{j,j+1} = r_{j,j+1} e^{-2i(\phi_1+\phi_2+......+\phi_j)} . \tag{6}$$

The M-matrix is given by,

$$[M_N] = \frac{1}{\prod_{j=1}^{N} t_{j,j+1}} \begin{bmatrix} \sum_{m\geq} (\overline{S_{2m}^{N}}) e^{i(\phi_1+\phi_2+......+\phi\;\;)} & \sum_{m\geq} (\overline{S_{2m+1}^{N}}) e^{-i(\phi_1+\phi_2+......+\phi\;\;)} \\ \sum_{m\geq} S_{2m+1}^{N} e^{i(\phi_1+\phi_2+......+\phi\;\;)} & \sum_{m\geq 0} S_{2m}^{N} e^{-i(\phi_1+\phi_2+......+\phi_N)} \end{bmatrix}$$

(7)

Where,

$$S^N = 1 \qquad (8a)$$

$$S_1^N = \sum_{i=1}^{N} R_i = R_1 + R_2 + \ldots + R_N \qquad (8b)$$

$$S_2^N = \sum_{1\leq i<j\leq N} R_i \overline{R}_j = R_1 \overline{R}_2 + R_1 \overline{R}_3 + \ldots + R_1 \overline{R}_N + R_2 \overline{R}_3 + R_2 \overline{R}_4 \qquad (8c)$$

$$+\ldots + R_2 \overline{R}_N + \ldots + R_{N-1} \overline{R}_N \qquad (8d)$$

$$S_3^N = \sum_{1\leq i<j<k\leq N} R_i \overline{R}_j R_k = R_1 \overline{R}_2 R_3 + R_1 \overline{R}_2 R_4 + \ldots + R_1 \overline{R}_2 R_N + \ldots \qquad (8e)$$

$$+R_{N-2} \overline{R}_{N-1} R_N \qquad (8f)$$

$$S_P^N = \sum_{1\leq i<j<k\ldots<w\leq N} R_i \overline{R}_j R_k \overline{R}_l \ldots. \overset{0}{R}_w \qquad (P - \text{terms in each sum})$$

$$S_N^N = R_1 \overline{R}_2 R_3 \overline{R}_4 \ldots\ldots\ldots \overset{0}{R}_N$$

Eqs. (8) define the elementary symmetric functions of the variables R_1, R_2, R_3,……., R_N, which are extensively used in the mathematical theory of polynomials, and Eq. (7) enables us to write the overall reflection and transmission coefficients of the planner stratified structure in the general form,

$$\Re_N = \frac{M_{21}}{M_{11}} = \frac{\sum_{m\geq 0} S^N_{2m+1}}{\sum_{m\geq 0} (\overline{S^N_{2m}})} \tag{9}$$

and (10)

$$\tau_N = \frac{1}{M_{11}} = \frac{\prod_{j=1}^{N} t_{j,j+1}}{\sum_{m\geq 0} (\overline{S^N_{2m}})\, e^{i(\phi_1+\phi_2+......+\phi_N)}}$$

III- SCATTERING FROM QUANTUM BARRIERS

In this section a polynomial method is presented for the scattering from a stratified potential [11]. This method treats the matter waves satisfying the Schrodinger equation from the electromagnetic theory point of view [11]. Both the layer matrix and the interface matrix are determined for a stratified potential which are used to obtain the transformation matrix of the system. A compact formula for the transmission and reflection coefficients is obtained.

The transformation of ψ from plane x' located in potential V_1 to plane x'' in potential V_N is given by a transformation matrix [M].

$$\psi(x') = [M]\ \ \psi(x'') \tag{11}$$

The medium N is assumed to be the last one so that it is justified to choose $\psi(x'') = 0$. The reflection and transmission amplitudes are given in the terms of the M-matrix elements as,

$$R = \frac{\psi_-(x')}{\psi_+(x')} = \frac{M_{21}}{M_{11}} \tag{12}$$

$$T = \frac{\psi_+(x'')}{\psi_+(x')} = \frac{1}{M_{11}} \tag{13}$$

where R and T are the reflection and transmission amplitudes respectively. Let us consider a system consisting of N successive interfaces separated by constant values of the potential V_j .

Let also the transformation matrix $[\phi_j]$ representing the propagation of the wave function through the same constant potential V_j between two planes (interfaces) located at x_{j-1} and $x = x_{1} + d$. The matrix $[\phi_j]$ is give n by

$$[\phi_j] = \begin{pmatrix} e^{-\phi_j} & 0 \\ 0 & e^{i\phi_j} \end{pmatrix} \tag{14}$$

The interface matrix $[r_j]$ relates the wave function components on both sides of the interface located at x_j. It can be defined as

$$[r] = \begin{pmatrix} \frac{k + k_{+1}}{2k} & \frac{k - k_{+1}}{2k} \\ \frac{k - k_{+1}}{2k} & \frac{k + k_{+1}}{2k_j} \end{pmatrix} \tag{15}$$

where k_j and k_{j+1} are the wave numbers in the regions where the potential is given by Vj and Vj+1 respectively. In this case, the matrix $[M]$ takes the form

$$[M] = \frac{1}{\prod_{j=1}^{N} t_{j,j+1}} \begin{pmatrix} \sum_{m\geq} \overline{(S_{2m}^{N})}\, e^{-i(\phi_1+....+\phi\ )} & \sum_{m\geq} \overline{(S_{2m+1}^{N})}\, e^{i(\phi_1+....+\phi\ )} \\ \sum_{m\geq} (S_{2m+1}^{N})\, e^{-i(\phi_1+....+\phi\ )} & \sum_{m\geq 0} (S_{2m}^{N})\, e^{i(\phi_1+....+\phi_N)} \end{pmatrix} \tag{16}$$

where the S_i^j are defined in Eqs. (8).

This formalism directly gives an analytic expression for the reflection and transmission amplitudes of any number of interfaces.

$$R_N = \frac{\sum_{m\geq 0} S_{2m+1}^{N}}{\sum_{m\geq 0} \overline{(S_{2m}^{N})}} \tag{17}$$

$$T_N = \frac{\prod_{j=1}^{N} t_{j,j+1}\; e^{i(\phi_1+\phi_2+..........+\phi_N)}}{\sum_{m\geq 0} \overline{(S_{2m}^{N})}} \tag{18}$$

VI- SCATTERING FROM A STRING OF DELTA FUNCTION BARRIERS (DIRAC COMB)

Consider a string of $N+1$ delta function barriers, all of strength ل, evenly spaced a distance S apart, at $x = 0, S, 2S, \ldots\ldots, NS$. This system is called Dirac Comb as shown in fig. (2).
The potential of such a system is given by,

$$V(x) = \alpha \sum_{j=0}^{N} \delta\,(x - j\,S) \tag{19}$$

In a recent paper [12] Griffiths and Taussig discussed Dirac Comb and they arrived at an interesting result for the transmission coefficient. In this section it is intended to present a novel technique for this system.
The wave function at the two sides of the potential (Fig. 1) can be written in a matrix relation as

$$\psi(x < 0) = [\,M\,]\psi(NS + \varepsilon) \tag{20}$$

where the transformation matrix $[M]$ can be expressed in the ordered product as

$$[M] = \prod_{=0}^{N} [\phi]\,[r_j] \tag{21}$$

where the layer matrix $[\phi_j]$ is given in a similar way by Eq. (14)
The phase shift is given by

$$\phi_j = k_j\, d_j = k\, S \tag{22}$$

$$and \quad k = \sqrt{\frac{2mE}{\hbar^2}} \tag{23}$$

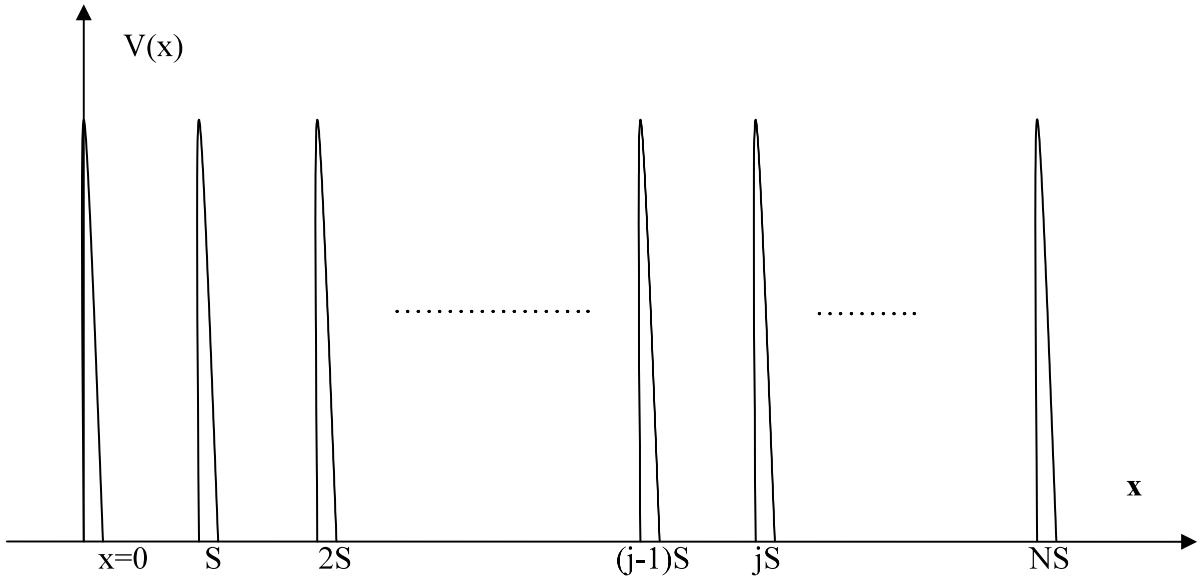


Figure (2): Dirac Comb

In order to find the interface matrix $[r_j]$, we apply the boundary conditions of the wave function. Ordinarily, both ϕ_j and $\frac{d\psi}{dx}$ are continuous, but where the potential goes to infinity, only the first of these is applied [14]. In the region *(j-1)S<x<jS*, *V(x)=0* so that the solution of Schrodinger equation is given by

$$\psi_I(x) = Ae^{ikx} + Be^{-ikx}, \qquad (j-1)S < x < jS \tag{24}$$

In a similar manner in the region *jS<x<(j+1)S*, it is given by

$$\psi_{II}(x) = Ce^{ikx} + De^{-ikx}, \qquad jS < x < (j+1)S \tag{25}$$

The continuity of the wave function at the join (interface) gives

$$\begin{aligned} &\psi_I(x = jS) = \psi_{II}(x = jS) \\ &Ae^{ikjs} + Be^{-ikjs} = Ce^{ikjs} + De^{-ikjs} \end{aligned} \tag{26}$$

The second boundary condition tells nothing because it is the exceptional case where *V* is infinite at the join (*x=jS*). The derivative of the wave function is discontinuous and the delta function must determine this discontinuity. The idea is simple, by writing Schrodinger equation in the neighborhood of $x=jS$ as,

$$\Re_N = \frac{21}{11} = \frac{\sum_{m \ge 0} {}^{N}_{2m} + 1}{\sum_{m \ge 0} \overline{{}^{N}_{2m}}} \quad and \quad \tau_N = \frac{1}{M_{11}} = \frac{\prod_{j=1}^{N} t_{j,j+1}}{\sum_{m \ge 0} (\overline{S^{N}_{2m}})\, e^{i(\phi_1+\phi_2+\ldots+\phi_N)}} \tag{27}$$

where the potential is given by $V(x)=\alpha\delta(x-jS)$

Integrating Eq. (27) from $(jS-\varepsilon)$ to $(jS+\varepsilon)$ where ε is infinitesimal, we get

$$-\frac{\hbar^2}{2m}\int_{jS-\varepsilon}^{jS+\varepsilon}\frac{d^2\psi}{dx^2}dx+\alpha\int_{jS-\varepsilon}^{jS+\varepsilon}\delta(x-jS)\psi(x)dx=E\int_{jS-\varepsilon}^{jS+\varepsilon}\psi(x)dx \tag{28}$$

The first integral gives $\frac{d\psi}{dx}$, evaluated at the two end points. The second integral gives the wave function evaluated at the point x=jS. The third integral is zero in the limit $,\varepsilon \to 0$ since it is the area of a strip with vanishing width and finite height. Thus,

$$-\frac{\hbar^2}{2m}\left.\frac{d\psi}{dx}\right|_{jS-}^{jS+0}+\alpha\psi(jS)=0 \tag{29}$$

or

$$\left[\left.\frac{d\psi}{dx}\right|_{jS+}-\left.\frac{d\psi}{dx}\right|_{jS-0}\right]=\frac{2m\alpha}{\hbar^2}\psi(jS) \tag{30}$$

$$\Delta\left(\frac{d\psi}{dx}\right)=\frac{2m\alpha}{\hbar^2}\psi(jS) \tag{31}$$

where $\Delta\frac{d\psi}{dx}$ is the difference in the wave function derivative at both sides of the joining point ($x=jS$).

From Eq. (24), we have

$$\frac{d\psi}{dx}=ikAe^{x}-ikBe^{-x}\quad for\ x<jS \qquad ,so$$

$$\left.\frac{d\psi}{dx}\right|_{-0}=ik[Ae^{ikjS}-Be^{-ikjS}] \tag{32}$$

And from Eq. (25)

$$\frac{d\psi}{dx} = ikC\,e^{x} \quad - ikDe^{-x} \quad for\ x > jS$$

so

$$\left.\frac{d\psi}{dx}\right|_{+0} = ik[C\,e^{ikjS} - D\,e^{-ikjsS}\ ] \tag{33}$$

Substituting from Eq. (32) and (33) into Eq. (31) and using Eq. (24) to find $\psi(jS)$ gives,

$$ik[C\,e^{ikjS} - D\,e^{-ikjS} - A\,e^{ikjS} + B\,e^{-ikjS}\ ] = \frac{2m\alpha}{\hbar^2}[\,A\,e^{ikjS} + B\,e^{-ikjS}\ ] \tag{34}$$

To simplify, let

$$\beta = \frac{m\ \alpha}{\hbar^2 k} \tag{35}$$

Rearranging Eq. (34) and using Eq. (35),we get

$$A(\ 1 - 2i\ \beta\ )\ e^{ikjS} - B\ (1 + 2i\beta\ \ )e^{-ikjS} = Ce^{ikjS} - De^{-ikjS} \tag{36}$$

Solving Eqs. (26) with Eq. (36) we get,

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 + i\beta & i\ \beta\ e^{-2ikjS} \\ -i\ \beta\ e^{2ikjS} & 1 - i\beta \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \tag{37}$$

In a polar form we can write

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sqrt{1+\beta^2}\,e^{\theta} & \beta e^{(\frac{\pi}{2}-2kjS)} \\ \beta e^{-(\frac{\pi}{2}-2kjS)} & \sqrt{1+\beta^2}\,e^{-i\theta} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \tag{38}$$

where

$$\theta = tan^{-1}\beta, \tag{39}$$

Eq. (38) can be rewritten as

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sqrt{1+\beta^2} & \beta \\ \beta & \sqrt{1+\beta^2} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \tag{40}$$

Let

$$[m] = \begin{pmatrix} \sqrt{1+\beta^2} & \beta \\ \beta & \sqrt{1+\beta^2} \end{pmatrix} \tag{41}$$

Introducing the reflection and transmission coefficients at the (j, j+1) potential interface

$$r_j \equiv r_{j,j+} = \frac{m}{m} = \frac{\beta}{\sqrt{1+\beta}} \tag{42}$$

$$t_j \equiv t_{j,j+1} = \frac{1}{m_{11}} = \frac{1}{\sqrt{1+\beta^2}} \tag{43}$$

Eq. (34) can be written with the use of Eqs. 42 and 43 as,

$$\begin{pmatrix} A \\ B \end{pmatrix} = \sqrt{1+\beta^2} \begin{pmatrix} 1 & \frac{\beta}{\sqrt{1+\beta^2}} \\ \frac{\beta}{\sqrt{1+\beta^2}} & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

or

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{t_{j,j+1}} \begin{pmatrix} 1 & r_{j,j+1} \\ r_{j,j+1} & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \tag{44}$$

From Eq. (44) and using the fact that

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}_{jS-} = \left[r_j \right] \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}_{jS+0} \tag{45}$$

we can define the interface matrix $[r_j]$ as

$$\left[r_j \right] = \sqrt{1+\beta^2} \begin{pmatrix} 1 & \frac{\beta}{\sqrt{1+\beta^2}} \\ \frac{\beta}{\sqrt{1+\beta^2}} & 1 \end{pmatrix} \tag{46}$$

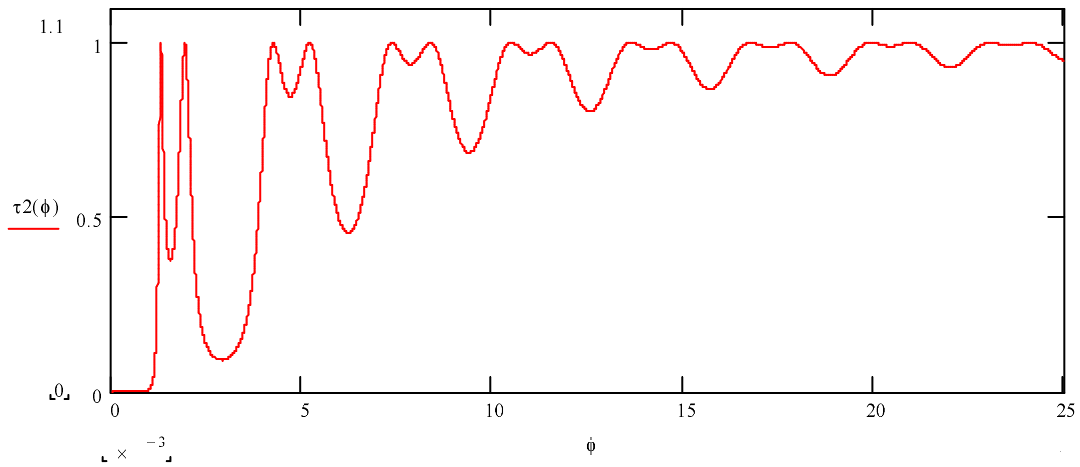
The knowledge of the layer matrix $[\phi_j]$ and the interface matrix $[r_j]$ enables us to write the M-matrix as given by Eq. (16). So that the transmission amplitude of the system consisting on (*N+1*)-spikes is given by,

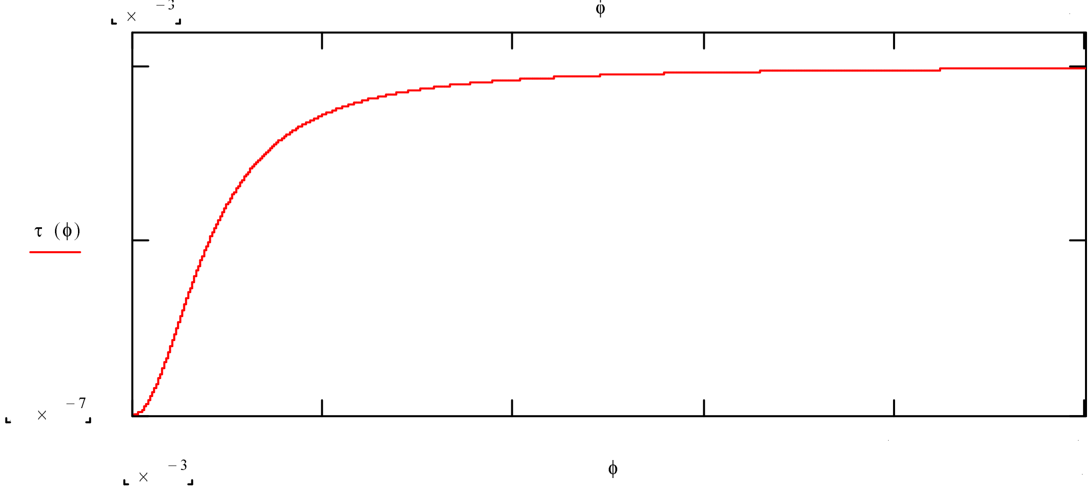
$$T_N = \frac{1}{M_{11}} = \frac{\prod_{j=}^{N} t_{j,j+1} e^{-i(\varphi_0+\varphi_1+\ldots+\varphi_N)}}{\sum_{m\geq 0} (\overline{S_{2m}^{N}})} \tag{47}$$

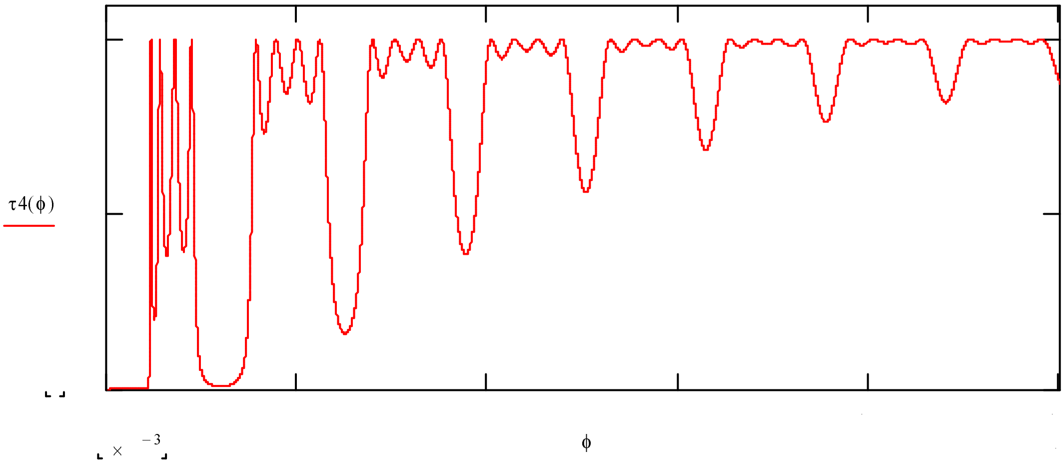
The transmission probability (τ_N), is given by

$$\tau_N = T_N T_N^{*} \tag{48}$$

In figure (2), the transmission probability is plotted as a function of the energy (not with *E* explicitly but with □ which is function of *E)* using the formula given by Eq. (47) for *N=0* (one barrier), *N=2* (three barriers), *N=4* (five barriers), and *N=8* (nine barriers).







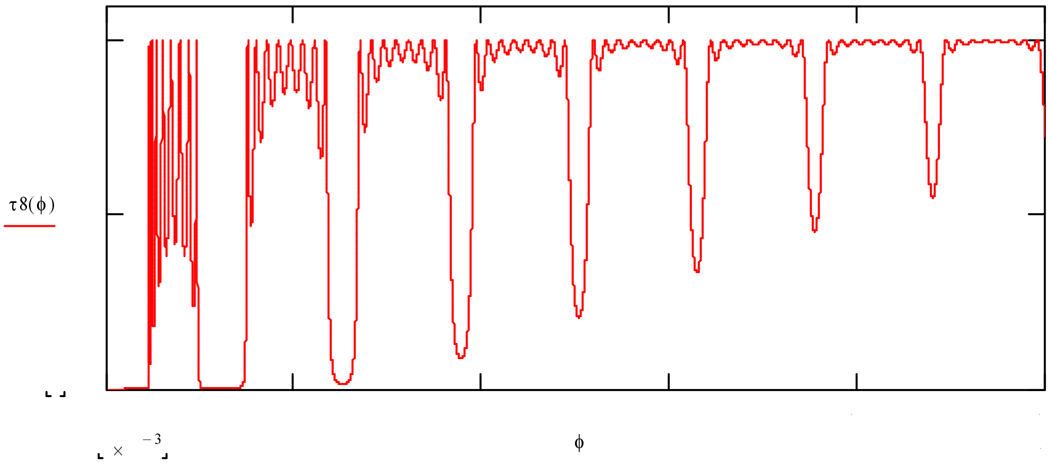


Figure (3): The transmission probability as a function of the energy

V- DISCUSSION AND CONCLUSION

We have discussed an alternative method for studying the reflection and the transmission from Dirac Comb. This formulation is called a polynomial one because of the use of the elementary symmetric functions which are extensively used in the mathematical theory of polynomials. The polynomial formulation allowed us to write an analytic expression for the reflection and the transmission coefficients.

Actually, among others, this method is considered as an easy method to handle since it avoids the matrix products. Approximately, all other approaches for that system adopted the product of matrices, which is very difficult for large N. One of the most important features of this method is treating the quantum scattering problem of Dirac Comb from the electromagnetic theory point of approach.

Comparison with the recent approach used by Griffiths and Taussig [12] shows that this approach is efficient, fast, and highly accurate. It is clear that figure (3) is in complete agreement with the results obtained in [12,13] which imply that the new method derived for Dirac comb in section IV satisfies the results of other approaches.

Our general novel technique can also be used to investigate and handle other periodic structures such as quantum wires with serial stubs or even more disordered structures with defect stubs and defect segments as will be discussed in future papers. We hope that the present work will facilitate more theoretical interests on various complicated mesoscopic structures, as this technique can be implemented easily and even on an IBM PC/AT computer and complete calculation in a short time.

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