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New Recursive Algorithm to Find the Failure Function of the Connected (2,2)-out-of-(*m,n*): F Linear and Circular System

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Abstract
The connected (*r,s*)-out-of-(*m,n*): F linear and circular system fails if there is an area of failures that includes at least a sub matrix (*r,s*) that contains all failed components. For example, the connected (2,2)-out-of-(*m,n*): F linear and circular system is failed if there is a 2×2 square of failed components.
In this paper, a new recursive algorithm to find the failure function of the connected (2,2)-out-of-(*m,n*): F linear and circular system is obtained. The algorithm uses the failure states of the consecutive 2-out-of-*n*: F linear and circular.

Keywords Consecutive *k*-out-of-*n*: F Linear and Circular System, The Connected (*r,s*)-out-of-(*m,n*): Linear and Circular System.

خوارزمية جديدة لإيجاد اقتران فشل النظام التتابعي "(2,2)-out-of-(*m,n*): F" الخطي والدائري

ملخص
النظام التتابعي "connected (*r,s*)-out-of-(*m,n*): F" الخطي والدائري يفشل في حالة ان يكون هناك منطقة من الفشل تتضمن عدد $r\times s$ اجزاء متصلة وفاشلة، على سبيل المثال النظام "connected (2,2)-out-of-(*m,n*): F" الخطي والدائري يفشل في حالة فشل مربع متصل مكون من 2×2 جزء فاشل.
في هذا البحث، تم ايجاد خوارزمية جديدة بطريقة الاسترجاع من رتبة اقل لايجاد اقتران فشل النظام " connected (2,2)-out-of-(*m,n*): F" الخطي والدائري، الخوارزمية تعتمد على ايجاد حالات فشل النظام التتابعي 2-out-of-*n*: F الخطي والدائري احادي البعد.

الكلمات مفتاحية: النظام التتابعي Consecutive *k*-out-of-*n*: F الخطي والدائري، النظام التتابعي connected (*r,s*)-out-of-(*m,n*): F الخطي والدائري.

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Notation:

L (C)	: Linear (Circular)
$\mathbb{I}_n^1$	$: = \{1, 2, ...n\}$.
$P(\mathbb{I}_n^1)$	: The power set of $\mathbb{I}_n^1$.
$\Theta(\Psi)$	: The functioning (failure) space of the consecutive 2-out-of-*n*: F system
$\mathbb{F}^{L(C)}(i)$	: The failure function of the connected (2,2)-out-of-(*i*,*n*): F linear (circular) subsystem.
$\mathbb{F}_i^{L(C)}(\Theta)$	: The failure function of the connected (2,2)-out-of-(*i*,*n*): F linear (circular) subsystem, when the *i*th layer (circle) represented by any $X \in \Theta_{L(C)}$.
$\mathbb{F}_i^{L(C)}(\Psi)$	The failure function of the connected (2,2)-out-of-(*i*,*n*): F linear (circular) subsystem, when the *i*th layer (circle) represented by any $X \in \Psi_{L(C)}$.
$\mathbb{F}_i^{L(C)}(X)$	: The failure function of the connected (2,2)-out-of-(*i*,*n*): F linear (circular) subsystem, when the *i*th layer (circle) represented by the set $X \subseteq \mathbb{I}_n^1$.
q_n^s	$= q(n,s) = q^s p^{n-s}$
$b(n,q)$	$= \sum_{s=0}^{n} \binom{r}{s} q_n^s$.
$p_j(q_j)$	: The reliability (failure) of the *j*th component.
$p_j^i(q_j^i)$	: The reliability (failure) of the *j*th component at the *i*th layer (circle).
$p_W^i(q_W^i)$	: The reliability (failure) of the *i*th layer (circle), when the indices failed components labeled by the set $W \subseteq \mathbb{I}_n^1$, $p_W^i = \prod_{j \notin W} p_j^i, q_W^i = \prod_{j \in W} q_j^i$.
$F(W)$	$:= P\{Z = W\} = p_W q_W \equiv q_W$.
$MN(in)$	$: = \sum_{(x_{11}, x_{12}, ..., x_{in})} \prod_{l=1}^{n} \left(\prod_{k=1}^{i} p_{kl}^{1-x_{kl}} q_{kl}^{x_{kl}} \right)$ where : $x_{lk} = 0,1$, *l*=1,…,*n*, and *k*=1,…, *i*

1. Introduction:

Despite the passage of many years since 1980, the consecutive system is still the focus of attention of many researchers, due to its wide range of applications and low expensive. Kontoleon [1] was the first who studied the system under the name "*r*-successive-out-of-*n*: F system", then Chiang and Niu [2] named the system by "the consecutive *k*-out-of-*n*: F system", and introduced many applications (e.g. Telecommunication systems with *n* relay stations, the pipeline of transmit oil system, etc. The consecutive *k*-out-of-*n*: F system consists of *n* components, and fails if any consecutive *k* components are in the failed state. It is classified according the connection between components to the linear and circular system. Despite the failure function is the complement of the reliability function; many researchers set a numerous algorithm to compute the reliability of such system, but a few researches studied the failure function, see [3-6].

In 1990, Salvia & Lasher [7] introduced the concept of the 2-dimensional version of the consecutive systems, Bohme et al. [8] gave more general definition "the connected-*X*-out-of-(*m*,*n*): F linear and circular system", it consists of (*mn*) components, the linear system is arranged as a matrix with *m* rows and *n* columns, and the circular system arranged as *m* circles and *n* rays, (the circles shared the same center, the intersections of circles and rays represent the elements, where 'ray' means a line from the center of the circles to their perimeters). The system fails if at least one area of connected components represented by the set *X* fails, (*X* =(*r*,*s*) or *X*=(*r*,*s*)-or-(*s*,*r*) $s, r \le m, n$). Further, investigations regarding the reliability of the connected-*X*-out-of-(*m*,*n*): F linear systems are given by [9-16].

Rarely paper studied the failure function of the 2 dimensional consecutive system, often these works studied the reliability function of the connected-(*r*,*s*)-out-of-(*m*,*n*): F linear systems, Yamamoto & Miyakawa [17,18], Noguchi et al. [19] & Yamamoto & Akiba [20] introduced a recursive algorithms to compute the reliability to compute the reliability of such system. It is worth mentioning that the reliability model of the connected (*r*,*s*)-out of-(*m*,*n*): F system might be used for 'Feelers for measuring temperature on reaction chamber' and 'TFT Liquid Crystal Display system with 360^o wide area [20].

In this paper, in the 2nd section, the failure and the functioning space of the consecutive 2-out-*n*: F system are represented using the set theory, which were participating actively in obtaining a new recursive algorithm to find the failure function of the connected (2,2)-out-of-(*m*,*n*): F linear & circular lattice system in the 3rd section.

The following assumptions are assumed to be satisfied by the connected (2,2)-out-of-(*m*,*n*): F linear and circular system:

1. The state of the component and the system is either " functioning" or "failed"
2. All the components are mutually statistically independent

2. The Consecutive 2-out-of-*n*: F system:

Consider the consecutive 2-out-of-*n*: F linear and circular system, $\mathbb{I}_n^1$ denotes the possible labels (indices) of the components of the system, then $P\left(\mathbb{I}_n^1\right)$ is the failure space of its components. The set $X \in P\left(\mathbb{I}_n^1\right)$ represents the system, and consists of all failed components in the system is called a "failed set" if it includes two consecutive indices of the failed components otherwise, *X* is called "a functioning set", (e.g. the set $X = \{1,3\} \subseteq \mathbb{I}_4^1$ or for simply 13, is a functioning set in the consecutive 2-out-of 4: F linear (circular) system, but the set *X*=12 is a failed set). Define the sub collections $\Theta_{L(C)}, \Psi_{L(C)}$, the set of all functioning sets (failed sets) respectively of the consecutive 2-out-of-*n*: F linear (circular) system, hence $P\left(\mathbb{I}_n^1\right) = \Theta_{L(C)} \cup \Psi_{L(C)}$, where $\Psi_{L(C)} = \left\{X \in P\left(\mathbb{I}_n^1\right) : \{r, r+1\} \subseteq X\right\}$ and $r \in \mathbb{I}_{n-1}^1$ for the linear system. (and $r \in \mathbb{I}_n^1$ $n+1 \equiv 1$, for the circular system) [21].

3. The Proposed Algorithm:

Consider the connected-(2,2)-out-of-(*m*,*n*): F linear (circle) system, and let $\mathbb{I}_n^1$ be the label (indices) of the components for any layer (circle) of the system, this implies that the failure space of the components for any layer (circle) is $P\left(\mathbb{I}_n^1\right) = \Theta_{L(C)} \cup \Psi_{L(C)}$. The system fails, if the event "an area of failures includes a square of 2×2 connected failed components" occurs, i.e. $X, Y \in P\left(\mathbb{I}_n^1\right)$ represent any 2 consecutive layers (circles), such that $X \cap Y \in \Psi_{L(C)}$ (see figure 1.*a*).

For *i*= 0,1,…, *m*-1, define the random variable Z_{i+1} on $P\left(\mathbb{I}_n^1\right)$ such that $\left\{Z_{i+1} = X\right\}$ the event "the *i*+1th layer (circle) has the indices of the failed components represented by $X \in P\left(\mathbb{I}_n^1\right)$ ", let $A_{i+1}^{L(C)}(X)$ the event "the connected (2,2)-out-of-(*i*+1,*n*): F linear (circular) subsystem is in the failure state, when *X* represents the *i*+1th layers (circles)", and $A_{i+1}^{L(C)}(X,Y)$ the event "the connected (2,2)-out-of-(*i*+1,*n*): F linear (circular) subsystem is in the failure state, when the sets $X, Y \in P\left(\mathbb{I}_n^1\right)$ represent the *i*+1th and *i*th layers (circles) respectively".

Now, consider the connected (2,2)-out-of-(*i*+1,*n*): F linear (circular) system in the failure state, and the sets $X, Y \in P\left(\mathbb{I}_n^1\right)$ represent the *i*+1th and *i*th layers (circles) respectively, then we have the following cases:

1. If $X \in \Theta_{L(C)}$, then for all $Y \in P\left(\mathbb{I}_n^1\right)$ we have $X \cap Y \notin \Psi_{L(C)}$, which implies that, *X* is not a part of the failure in (*i*+1,*n*) subsystem, hence, the failure is as a result of the (*i*,*n*) subsystem, i.e. $A_{i+1}^{L(C)}(X) = \{Z_{i+1} = X\} \cap \bigcup_{Y \in P\left(\mathbb{I}_n^1\right)} A_i^{L(C)}(Y)$.

2. If $X \in \Psi_{L(C)}$, then we have 2 cases:

2.1. For all $Y \in \Theta_{L(C)}$, we have $X \cap Y \notin \Psi_{L(C)}$, which implies that, X is not a part of the failure of the (i+1, n) subsystem, and failure is as a result of the (i,n) subsystem.

2.2. If $Y \in \Psi_{L(C)}$, we have the following 2 subcases:

2.2.1. If $X \cap Y \notin \Psi_{L(C)}$ then X is not a part of the failure in (i+1, n) subsystem, the failure is as a result of the (i,n) subsystem where Y in the i^{th} layer, (see figure 1.b).

2.2.2. If $X \cap Y \in \Psi_{L(C)}$, then the event of failure occurs, hence no matter what are the states of the components from the 1^{st} to the i-1^{th} layer (circle), (see figure 1.a).

According the cases (2, 2.1, 2.2, 2.2.1, 2.2.2), we have for all $X \in \Psi_{L(C)}$,

$$A_{i+1}^{L(C)}(X) = \{Z_{i+1} = X\} \cap \left\{ \left(\bigcup_{Y \in \Theta_{L(C)}} A_i^{L(C)}(Y) \right) \cup \left(\bigcup_{Y \in \Psi'_{L(C)}(X)} A_i^{L(C)}(Y) \right) \cup \left(\bigcup_{Y \in \Psi_{L(C)}(X)} A_{i+1}^{L(C)}(X,Y) \right) \right\}$$

where

$\Psi'_{L(C)}(X) = \{Y \in \Psi_{L(C)} : X \cap Y \notin \Psi_{L(C)}\}$ and

$\Psi_{L(C)}(X) = \{Y \in \Psi_{L(C)} : X \cap Y \in \Psi_{L(C)}\}$, note that $\Psi'_{L(C)}(\mathbb{I}_n^1) = \{\ \}$, $\Psi_{L(C)}(\mathbb{I}_n^1) = \Psi_{L(C)}$.

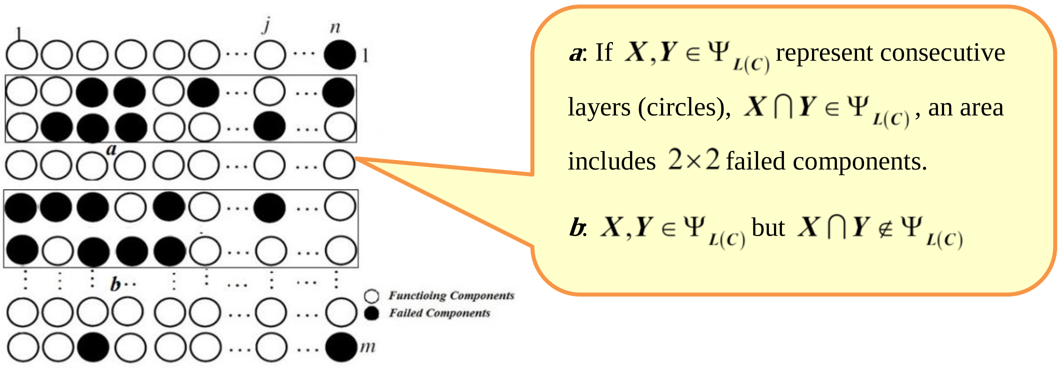

Figure 1 *Example of a connected-(2,2)-out-of-(m,n):F lattice system failures.*

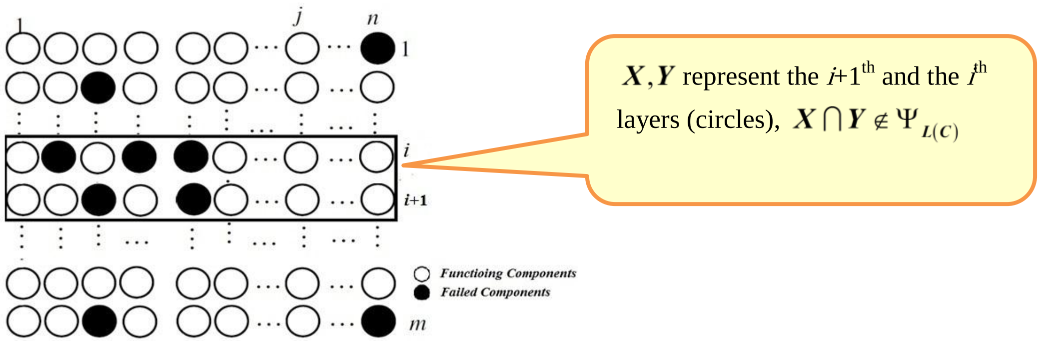

Figure 2 *Example of a functioning connected-(2,2)-out-of-(m,n):F lattice system.*

Theorem 3.1: In the connected (2,2)-out-of-(m,n): F linear (circular) lattice system

1. $\mathbb{F}^{L(C)}(1) = 0$.

2. For i=1,2,…,m-1, $\mathbb{F}_{i+1}^{L(C)}(\Theta) = F(\Theta_{L(C)})\mathbb{F}^{L(C)}(i)$

3. For i=1,2,…,m-1, and for all $X \in \Psi_{L(C)}$

$$F_{i+1}^{L(C)}(X)=F(X)\times\begin{bmatrix}F_i^{L(C)}(\Theta)+\sum_{Y\in\Psi'(X)}\mathbb{F}_i^{L(C)}(Y)\\ +MN\left((i-1)n\right)\sum_{Y\in\Psi(X)}F(Y)\end{bmatrix}$$

and then $\mathbb{F}_{i+1}^{L(C)}(\Psi)=\sum_{X\in\Psi^{L(C)}}\mathbb{F}_{i+1}^{L(C)}(X)$. If the components are independent and identically distributed, then

$$\mathbb{F}_{i+1}^{L(C)}(X)=F(X)\times\begin{bmatrix}\mathbb{F}_i^{L(C)}(\Theta)+\sum_{Y\in\Psi'(X)}\mathbb{F}_i^{L(C)}(X)\\ +b\left((i-1)n,q\right)\sum_{Y\in\Psi(X)}F(Y)\end{bmatrix}$$

4. For i=1,2, …,m, the failure function of the connected-(2,2)-out-of-(i,n): F linear (circular) system is
$$\mathbb{F}^{L(C)}(i)=\mathbb{F}_i^{L(C)}(\Theta)+\mathbb{F}_i^{L(C)}(\Psi).$$

Proof:

1. We need at least 2 consecutive layers (circles) to achieve failure of the system, hence $\mathbb{F}_1^{L(C)}(X)=0:\forall X\in P\left(\mathbb{I}_n^1\right)$ and then $\mathbb{F}^{L(C)}(1)=\sum_{X\in P\left(\mathbb{I}_n^1\right)}\mathbb{F}_1^{L(C)}(X)=0$
2. For i=1,2,…,m-1, and for all $X\in\Theta_{L(C)}$

$$\mathbb{F}_{i+1}^{L(C)}(X)=P\left\{A_{i+1}^{L(C)}(X)\right\}=$$
$$=P\left\{\{Z_{i+1}=X\}\cap\bigcup_{Y\in P\left(\mathbb{I}_n^1\right)}A_i^{L(C)}(Y)\right\}$$
$$=P\left\{\bigcup_{Y\in P\left(\mathbb{I}_n^1\right)}\left\{\{Z_{i+1}=X\}\cap A_i^{L(C)}(Y)\right\}\right\}$$
$$=\sum_{Y\in P\left(\mathbb{I}_n^1\right)}P\left\{\{Z_{i+1}=X\}\cap A_i^{L(C)}(Y)\right\}$$
$$=P\{Z_{i+1}=X\}\sum_{Y\in P\left(\mathbb{I}_n^1\right)}P\left\{A_i^{L(C)}(Y)\right\}$$
$$=F(X)\sum_{Y\in P\left(\mathbb{I}_n^1\right)}\mathbb{F}_i^{L(C)}(Y)=F(X)\mathbb{F}^{L(C)}(i)$$

and then

$$\mathbb{F}_{i+1}^{L(C)}(\Theta)=P\left\{\bigcup_{X\in\Theta_{L(C)}}A_{i+1}^{L(C)}(X)\right\}=$$
$$=\sum_{X\in\Theta_{L(C)}}P\left\{A_{i+1}^{L(C)}(X)\right\}=\sum_{X\in\Theta_{L(C)}}\mathbb{F}_{i+1}^{L(C)}(X)$$
$$=\sum_{X\in\Theta_{L(C)}}F(X)\,\mathbb{F}^{L(C)}(i)=F\left(\Theta_{L(C)}\right)\mathbb{F}^{L(C)}(i)$$

3. For i=1,2,…,m-1, and for i=1,2, …, m-1 and for all $X\in\Psi_{L(C)}$

$$\mathbb{F}_{i+1}^{L(C)}(X)=P\left\{A_{i+1}^{L(C)}(X)\right\}=$$
$$=P\left\{\begin{array}{l}\{Z_{i+1}=X\}\cap\\ \left\{\begin{array}{l}\left(\bigcup_{Y\in\Theta_{L(C)}}A_i^{L(C)}(Y)\right)\cup\\ \left(\bigcup_{Y\in\Psi'_{L(C)}(X)}A_i^{L(C)}(Y)\right)\cup\\ \left(\bigcup_{Y\in\Psi_{L(C)}(X)}A_{i+1}^{L(C)}(X,Y)\right)\end{array}\right\}\end{array}\right\}$$
$$=P\left\{\begin{array}{l}\left\{\bigcup_{Y\in\Theta_{L(C)}}\{Z_{i+1}=X\}\cap A_i^{L(C)}(Y)\right\}\cup\\ \left\{\bigcup_{Y\in\Psi'_{L(C)}(X)}\{Z_{i+1}=X\}\cap A_i^{L(C)}(Y)\right\}\cup\\ \left\{\bigcup_{Y\in\Psi_{L(C)}(X)}\{Z_{i+1}=X\}\cap A_{i+1}^{L(C)}(X,Y)\right\}\end{array}\right\}$$
$$\mathbb{F}_{i+1}^{L(C)}(X)$$
$$=\sum_{Y\in\Theta_{L(C)}}P\left\{\{Z_{i+1}=X\}\cap A_i^{L(C)}(Y)\right\}$$
$$+\sum_{Y\in\Psi'_{L(C)}(X)}P\left\{\{Z_{i+1}=X\}\cap A_i^{L(C)}(Y)\right\}$$
$$+\sum_{Y\in\Psi_{L(C)}(X)}P\left\{\{Z_{i+1}=X\}\cap\{Z_i=Y\}\right\}$$

$$\mathbb{F}_{i+1}^{L(C)}(X) = P\{Z_{i+1} = X\} \times \left[\begin{array}{c} \sum_{Y \in \Theta_{L(C)}} P\left\{A_i^{L(C)}(Y)\right\} + \sum_{Y \in \Psi'_{L(C)}(X)} P\left\{A_i^{L(C)}(Y)\right\} \\ + MN((i-1)n) \sum_{Y \in \Psi_{L(C)}(X)} P\{Z_i = Y\} \end{array}\right]$$

$$\mathbb{F}_{i+1}^{L(C)}(X) = F(X) \times \left[\begin{array}{c} \mathbb{F}_i^{L(C)}(\Theta) + \sum_{Y \in \Psi'_{L(C)}(X)} \mathbb{F}_i(Y) \\ + MN((i-1)n) \sum_{Y \in \Psi_{L(C)}(X)} F(Y) \end{array}\right]$$

and hence $\mathbb{F}_{i+1}^{L(C)}(\Psi) = \sum_{X \in \Psi^{L(C)}} \mathbb{F}_{i+1}^{L(C)}(X)$

If the components are independent and identically distributed, then $MN((i-1)n) = b((i-1)n, q)$

4. The failure function of the connected-(2,2)-out-of-(i,n): F linear (circular) system is

$$\mathbb{F}^{L(C)}(i) = \sum_{X \in P(\mathbb{I}_n^1)} \mathbb{F}_i^{L(C)}(X)$$
$$= \sum_{X \in \Theta_{L(C)}} \mathbb{F}_i^{L(C)}(X) + \sum_{X \in \Psi_{L(C)}} \mathbb{F}_i^{L(C)}(X)$$
$$= \mathbb{F}_i^{L(C)}(\Theta) + \mathbb{F}_i^{L(C)}(\Psi)$$

Example: Compute the reliability of the connected (2,2)-out-of-(3,4): F circular system, when the components are independent and identically distributed

$$\mathbb{I}_4^1 = \{1,2,3,4\}$$
$$P(\mathbb{I}_4^1) = \left\{\begin{array}{l} \varnothing,1,2,3,4,12,13,14,23,24,34,123, \\ 124,134,234,1234 \end{array}\right\}$$
$$\Theta_C = \{\varnothing,1,2,3,4,13,24\}$$
$$\Psi_C = \{12,23,34,14,123,234,134,124,1234\}$$
$$\Psi_C(12) = \{12,123,124,1234\}$$
$$\Psi'_C(12) = \{23,34,14,234,134\}$$
$$\Psi_C(123) = \{12,23,123,234,124,1234\}$$
$$\Psi'_C(123) = \{34,14,134\}$$
$$\Psi_C(1234) = \Psi_C \qquad \Psi'_C(1234) = \{\ \}$$
$$\mathbb{F}_1^C(X) = 0 : \forall X \in \Theta_C \Rightarrow \mathbb{F}_1^C(\Theta) = 0$$
$$\mathbb{F}_1^C(X) = 0 : \forall X \in \Psi_C \Rightarrow \mathbb{F}_1^C(\Psi) = 0$$
$$\mathbb{F}^C(1) = \mathbb{F}_1^C(\Theta) + \mathbb{F}_1^C(\Psi) = 0$$
$$\mathbb{F}_2^C(12) = \mathbb{F}_2^C(23) = \mathbb{F}_2^C(34) = \mathbb{F}_2^C(14)$$
$$\mathbb{F}_2^C(12) = F(12) \times \left[\underbrace{\mathbb{F}_1^C(\Theta)}_{0} + b(0,q) \sum_{Y \in \Psi(12)} F(Y) + \underbrace{\sum_{Y \in \Psi'(12)} \mathbb{F}_1^C(X)}_{0}\right]$$
$$= q_4^2\left[F(12) + F(123) + F(124) + F(1234)\right]$$
$$= q_4^2\left[q_4^2 + 2q_4^3 + q_4^4\right] = q_8^4 + 2q_8^5 + q_8^6$$
$$\mathbb{F}_2^C(123) = \mathbb{F}_2^C(234) = \mathbb{F}_2^C(134) = \mathbb{F}_2^C(124)$$
$$\mathbb{F}_2^C(123) = F(123) \times \left[\underbrace{\mathbb{F}_1^C(\Theta)}_{0} + b(0,q) \sum_{Y \in \Psi(123)} F(Y) + \underbrace{\sum_{Y \in \Psi'(123)} \mathbb{F}_1^C(X)}_{0}\right]$$
$$= q_4^3\left[\begin{array}{r} F(12) + F(23) + F(123) + \\ F(124) + F(1234) \end{array}\right]$$
$$= q_4^3\left[2q_4^2 + 3q_4^3 + q_4^4\right] = 2q_8^5 + 3q_8^6 + q_8^7$$
$$\mathbb{F}_2^C(1234) = F(1234) \times \left[\underbrace{\mathbb{F}_1^C(\Theta)}_{0} + b(0,q) \sum_{Y \in \Psi(1234)} F(Y) + \underbrace{\sum_{Y \in \Psi'(1234)} \mathbb{F}_1^C(X)}_{0}\right]$$
$$= q_4^4\left[\begin{array}{l} F(12) + F(23) + F(34) + F(14) \\ +F(123) + F(234) + F(134) + F(124) \\ +F(1234) \end{array}\right]$$
$$= q_4^4\left[4F(12) + 4F(123) + F(1234)\right]$$
$$= q_4^4\left[4q_4^2 + 4q_4^3 + q_4^4\right] = 4q_8^6 + 4q_8^7 + q_8^8$$
$$\mathbb{F}_2^C(\Psi) = \sum_{X \in \Psi_{L(C)}} \mathbb{F}_2^C(X)$$
$$= 4\mathbb{F}_2^C(12) + 4\mathbb{F}_2^C(123) + \mathbb{F}_2^C(1234)$$
$$= 4q_8^4 + 16q_8^5 + 20q_8^6 + 8q_8^7 + q_8^8$$
$$\mathbb{F}_2^C(\Theta) = F(\Theta)\mathbb{F}_2^C(1) = 0$$
$$\mathbb{F}^C(2) = \mathbb{F}_2^C(\Theta) + \mathbb{F}_2^C(\Psi) = 0 + \mathbb{F}_2^C(\Psi)$$
$$= 4q_8^4 + 16q_8^5 + 20q_8^6 + 8q_8^7 + q_8^8$$

$$\mathbb{F}_3^C(12) = F(12) \times$$
$$\left[\underbrace{\mathbb{F}_2^C(\Theta)}_{0} + \left[b(4,q) \sum_{Y \in \Psi(12)} F(Y) \right] + \sum_{Y \in \Psi'(12)} \mathbb{F}_2^C(X) \right]$$
$$\mathbb{F}_3^C(12) = q_4^2 \left[\begin{array}{l} 0 + \left(q_4^2 + 2q_4^3 + q_4^4\right) b(4,q) \\ + \left[\begin{array}{r} \mathbb{F}_2^C(23) + \mathbb{F}_2^C(34) + \mathbb{F}_2^C(14) \\ + \mathbb{F}_2^C(234) + \mathbb{F}_2^C(134) \end{array} \right] \end{array} \right]$$
$$= q_4^2 \left[\begin{array}{l} 0 + \left(q_4^2 + 2q_4^3 + q_4^4\right) b(4,q) \\ \qquad + 3\mathbb{F}_2^C(12) + 2\mathbb{F}_2^C(123) \end{array} \right]$$
$$= q_{12}^4 + 6q_{12}^5 + 18q_{12}^6 + 30q_{12}^7 + 24q_{12}^8 + 8q_{12}^9 + q_{12}^{10}$$
$$\mathbb{F}_3^C(123) = F(123) \times$$
$$\left[\underbrace{\mathbb{F}_2^C(\Theta)}_{0} + b(4,q) \sum_{Y \in \Psi(123)} F(Y) + \sum_{Y \in \Psi'(123)} \mathbb{F}_2^C(X) \right]$$
$$= q_4^3 \left[\begin{array}{l} 0 + \left(2q_4^2 + 3q_4^3 + q_4^4\right) b(4,q) \\ + \left[\mathbb{F}_2^C(34) + \mathbb{F}_2^C(14) + \mathbb{F}_2^C(134) \right] \end{array} \right]$$
$$= q_4^3 \left[\begin{array}{l} 0 + \left(2q_4^2 + 3q_4^3 + q_4^4\right) b(4,q) \\ \qquad + 2\mathbb{F}_2^C(12) + \mathbb{F}_2^C(123) \end{array} \right]$$
$$\mathbb{F}_3^C(123) = 2q_{12}^5 + 11q_{12}^6 + 27q_{12}^7 + 36q_{12}^8 + 25q_{12}^9 + 8q_{12}^{10} + q_{12}^{11}$$
$$\mathbb{F}_3^C(1234) = F(1234) \times$$
$$\left[\underbrace{\mathbb{F}_2^C(\Theta)}_{0} + b(4,q) \sum_{Y \in \Psi(1234)} F(Y) + \sum_{Y \in \Psi'(1234)} \mathbb{F}_2^C(X) \right]$$
$$\mathbb{F}_3^C(1234) = q_4^4 \left[0 + \left(4q_4^2 + 4q_4^3 + q_4^4\right) b(4,q) + 0 \right]$$
$$= q_4^4 \left[0 + \left(4q_4^2 + 4q_4^3 + q_4^4\right) b(4,q) + 0 \right]$$
$$= 4q_{12}^6 + 20q_{12}^7 + 41q_{12}^8 + 44q_{12}^9 + 26q_{12}^{10} + 8q_{12}^{11} + q_{12}^{12}$$
$$\mathbb{F}_3^C(\Psi) = 4\mathbb{F}_3^C(12) + 4\mathbb{F}_3^C(123) + \mathbb{F}_3^C(1234)$$
$$= 4q_{12}^4 + 32q_{12}^5 + 120q_{12}^6 + 248q_{12}^7 + 281q_{12}^8 + 176q_{12}^9 + 62q_{12}^{10} + 12q_{12}^{11} + q_{12}^{12}$$
$$\mathbb{F}_3^C(\Theta) = F(\Theta)\mathbb{F}^C(2)$$
$$= \left(q_4^0 + 4q_4^1 + 2q_4^2\right)\left(4q_8^4 + 16q_8^5 + 20q_8^6 + 8q_8^7 + q_8^8\right)$$
$$= 4q_{12}^4 + 32q_{12}^5 + 92q_{12}^6 + 120q_{12}^7 + 73q_{12}^8 + 20q_{12}^9 + 2q_{12}^{10}$$

$$\mathbb{F}^C(3) = \mathbb{F}_3^C(\Theta) + \mathbb{F}_3^C(\Psi)$$
$$= 8q_{12}^4 + 64q_{12}^5 + 212q_{12}^6 + 368q_{12}^7 + 354q_{12}^8 + 196q_{12}^9 + 64q_{12}^{10} + 12q_{12}^{11} + q_{12}^{12}$$
$$= 8p^8q^4 + 64p^7q^5 + 212p^6q^6 + 368p^5q^7 + 354p^4q^8 + 196p^3q^9 + 64p^2q^{10} + 12pq^{11} + q^{12}$$

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