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**HAMILTONIAN
FORMULATION OF THE
SUPERMEMBRANE**

By

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TO

my daughter,,
the most beautiful in my life

Mona

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0.2 Abstract

By replacing ten-dimensional pure spinors with eleven-dimensional pure spinors, the Hamiltonian formalism is developed for covariantly quantizing the $d = 10$ superparticle and superstring is extended to the $d = 11$ superparticle and supermembrane.

The Hamiltonian formulation of the supermembrane theory in eleven dimensions is given by two approaches: The first is Dirac approach where the covariant is splited into primary and secondary class constraints is exhibited, and their Dirac brackets are constructed.

The second is Güler Approach, (Hamilton-Jacobi) in which the equations of motion are obtained as total differential equations in many variables without the need to distinguish between primary and secondary constraints. These equations of motion are in exact agreement with those equations that had been obtained using Dirac's method.

ملخص

صياغة هاملتون للأغشية الفائقة

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باستبدال المغزليات من عشرة أبعاد بالمغزليات من أحد عشر بعداء، تم تطوير صياغة هاملتون لتكميم الجسيمات الفائقة بالبعد العاشر والأوتار الفائقة إلى الأغشية الفائقة من البعد الحادي عشر.

المعالجة الهاملتونية للأغشية الفائقة تم دراستها باستخدام طريقتين، طريقة ديراك وطريقة هاملتون جاكوبي (القانونية). في طريقة ديراك تم فصل القيود إلى قيود ابتدائية وثانوية، وتم إنشاء أقواس ديراك. الطريقة الثانية طريقة جولر أو (هاملتون – جاكوبي) حيث لم يلزم فيها التمييز بين القيود وتكتب معادلات الحركة كمعادلات تفاضلية تامة. تطابقت نتائج الطريقتين تطابقاً تاماً.

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Chapter 1

Introduction

1.1 Historical Remarks

In 1933, Dirac made the observation that action plays a central role in classical mechanics [1]. He considered the Lagrangian formulation of classical mechanics to be more fundamental than the Hamiltonian formulation, but that it seemed to have no important role in quantum mechanics as it was known at that time. He speculated how this situation might be rectified, and he arrived at conclusion that the propagator in quantum mechanics "corresponds to" $\exp iS/\hbar$, where S is the classical action evaluated along the classical path.

The Hamiltonian formulation of constrained system in classical mechanics was initiated by Dirac in 1933 [1,2], who set up a formalism for treating singular systems and the constraints

involved. He showed that, in the presence of constraints, the number of degrees of freedom of the dynamical system was reduced. His approach are subsequently extended to continuous systems [3].

In 1948, Feynman developed Dirac's suggestion, and succeeded in deriving a third formulation of quantum mechanics, based on the fact that the propagator can be written as a sum over all possible paths (not just the classical one) between the initial and final points. Each path contributes $\exp iS/\hbar$ to the propagator [2].

Following Dirac, There is another approach for quantizing constrained systems of classical singular theories, which was initiated by Feynman kernal [4,5], who first set up a formalism of the path integral quantization. Faddeev and Popov [6,7] handle constraints in the path integral formalism when only first-class constraints in the canonical gauge are present. The generalization of the method to theories with second-class constraints is given by Senjanovic [8]. Fradkin and Vilkovisky [9,10] rederived both results in a broader context, where they improved Faddeev's procedure mainly to include covariant constraints; also they extended this procedure to the Gressman variables. When the dynamical system possesses some second-class constraints

there exists another method given by Batalin and Fradkin [11]. The Batalin, Fradkin, Vilkovisky (BFV)- Becchi, Rouet, Stora, Tyutin (BRST) operator quantization method. In the BRST method one introduces auxiliary variables and constructs the BRST charge which is first class. The auxiliary variables are also used in the conversion scheme. Moreover, Gitman and Tyutin [12] discussed the canonical quantization of singular theories as well as the Hamiltonian formalism of gauge theories in an arbitrary gauge. An alternative approach was developed by Bukenhout, Sprague and Faddeev [13,14] without following Dirac step by step. In this formalism there is no need to distinguish between first and second-class or primary and secondary constraints, where the primary constraint is a set of relations connection between the momenta and the coordinates. The general formalism is then applied to several problems, quantization of the massive Yang-Mills field theory, Light-Cone quantization of the self interacting scalar field, and quantization of a local field theory of magnetic monopoles, etc.

In 1992, Güler developed a formalism based on Hamilton Jacobi formulation of constrained system [15,16] which has been developed to investigate the constrained systems. Several constrained systems were investigated by using the Hamilton-Jacobi

approach [17-20]. The equivalent lagrangian method is used to obtain the set of Hamilton-Jacobi Partial Differential Equation (HJPDE). In this approach, the distinction between the first- and second-class constraints is not necessary. The equations of motion are written as total differential equations in many variables, these equations of motion require the investigation of the integrability conditions [21,22]. In other words, the integrability conditions may lead to new constraints. Moreover, it is shown that gauge fixing, which is an essential procedure to study singular systems by Dirac's method, is not necessary if the Hamilton-Jacobi approach is used [23,24].

In the following two sections a brief review of the two formulations will be given

1.2 Dirac Approach

The standard quantization methods can't be applied directly to the singular Lagrangian theories. However, the basic idea of the classical treatment and the quantization of such systems were presented a long time by Dirac [1,2]. And is now widely used in investigating the theoretical models in contemporary elementary particle physics and applied in high energy physics,

especially in the gauge theories [25].

The presence of constraints in such theories makes one careful on applying Dirac's method, especially when first-class constraints arise. This is because the first-class constraints are generators of gauge transformation which lead to the gauge freedom [26].

Let us consider a system which is described by the Lagrangian

$$L \equiv L(q_i, \dot{q}_i; \tau), \quad i = 1, \dots, n. \quad (1.1)$$

The system that is described by the Lagrangian $L(q_i, \dot{q}_i, t)$ or ($L(\phi, \partial_i \phi)$ in field theory), $i = 1, \dots, n$, is singular if the Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, \dots, n, \quad (1.2)$$

has a rank $(n - r)$, $r < n$. In this case we have p momenta which are dependent of each other.

The singular system characterized by the fact that all velocities \dot{q}_i are not uniquely determined in terms of the coordinates and momenta only. In other words, not all momenta are independent, and there must exist a certain set of relations among them, of the form

$$\phi_m(p_i, q_i) = 0, \quad (1.3)$$

The q 's and the p 's are the dynamical variables of the Hamiltonian theory. They are connected by the relations (1.3) which are called primary constraints of the Hamiltonian formalism.

Since the rank of the Hessian matrix is $(n - r)$, the momenta components will be functionally dependent. The first $(n - r)$ equations of the momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad (1.4)$$

can be solved for the $(n - r)$ components of \dot{q}_i in terms of q_i as well as the first $(n - r)$ components of p_i and the last r components of \dot{q}_i .

In other words

$$\dot{q}_a = \omega_a(q_i, p_\mu, \dot{q}_\mu), \quad (1.5)$$

$$a = 1, \dots, n - r, \quad \mu = 1, \dots, r, \quad i = 1, \dots, n$$

If these expressions for the \dot{q}_a are substituted into the last r equation of (1.4), the resulting equations will yield r relations of the form

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu} \Big|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, p_a, \dot{q}_\mu). \quad (1.6)$$

These relations indicate that the generalized momenta p_μ are dependent of p_a , which is natural result of the singular nature of the Lagrangian. Eq (1.6) can be written in the form

$$H'_\mu(q_i, p_a, \dot{q}_\nu) \equiv p_\mu + H_\mu \approx 0, \quad (1.7)$$

which are called primary constraints [1,2].

Now the usual Hamiltonian H_0 for any dynamical system is defined as

$$H_0(p_i, q_i) = p_i \dot{q}_i - L \quad (1.8)$$

(Here the Einstein summation rule is used which is a convention when repeated indices are implicitly summed over).

H_0 will not be uniquely determined, since we may add to it any linear combinations of the primary constraints H'_μ 's which are zero, so that the total Hamiltonian is [2,27]

$$H_T = H_0 + \lambda_\mu H'_\mu \quad (1.9)$$

where $\lambda_\mu(q, p)$ being some unknown coefficients, they are simply Lagrange's undetermined multipliers. Making use of the Poisson brackets, one can write the total time derivative of any function $g(q, p)$ as

$$\dot{g} \equiv \frac{dg}{d\tau} \approx \{g, H_T\} = \{g, H_0\} + \lambda_\mu \{g, H'_\mu\} \quad (1.10)$$

where Dirac's symbol (\approx) for weak equality has been used in the sense that one can't consider $H'_\mu = 0$ identically before working out the Poisson brackets. Thus the equations of motion can be written as

$$\dot{q}_i \approx \{q_i, H_T\} = \{q_i, H_0\} + \lambda_\mu \{q_i, H'_\mu\} \quad (1.11)$$

$$\dot{p}_i \approx \{p_i, H_T\} = \{p_i, H_0\} + \lambda_\mu \{p_i, H'_\mu\} \quad (1.12)$$

subject to the so-called consistency conditions. This means that the total time derivative of the primary constraints should be zero;

$$\begin{aligned} \dot{H}'_\mu &\equiv \frac{dH'_\mu}{d\tau} \approx \{H'_\mu, H_T\} \\ &= \{H'_\mu, H_0\} + \lambda_\nu \{H'_\mu, H'_\nu\} \approx 0, \quad \mu, \nu = 1, \dots, r. \end{aligned} \quad (1.13)$$

These equations may be reduced to $0 \equiv 0$, where it is identically satisfied as a result of primary constraints, else they will lead to new conditions which are called secondary constraints. Repeating this procedure as many times as needed, one arrives at a final set of constraints and specifies some of λ_μ . Such constraints are classified into two types :

- a) First-class constraints which have vanishing Poisson brackets with all other constraints.
- b) Second-class constraints which have non-vanishing Poisson brackets. The second-class constraints could be used to eliminate conjugated pairs of the p 's and q 's from the theory by expressing them as functions of the remaining p 's and q 's.

1.3 Güler Approach (Hamilton-Jacobi method)

Now we would like to discuss the constrained systems by Hamilton-Jacobi treatment [3,4], and demonstrate the fact that the gauge-fixing problem is solved naturally.

Güler has developed a completely different method to investigate singular systems [17].

The generalized momenta P_i corresponding to the generalized coordinates q_i are defined as,

$$P_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, \dots, n - r, \quad (1.14)$$

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = n - r + 1, \dots, n. \quad (1.15)$$

Since the rank of the Hess matrix is $(n - r)$, one may solve (1.14) for \dot{q}_a as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{q}_\mu, P_b) \equiv \omega_a. \quad (1.16)$$

Substituting (1.16) into (1.15), we obtain relations in q_i, P_a, \dot{q}_ν and t in the form

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu |_{\dot{q}_a = \omega_a}} = -H_\mu(q_i, \dot{q}_\nu, \dot{q}_a = \omega_a, P_a, t), \quad \nu = n - r + 1, \dots, n. \quad (1.17)$$

The canonical Hamiltonian H_0 is defined as

$$H_0 = -L(q_i, \dot{q}_\mu, \dot{q}_a = \omega_a, t) + P_a \omega_a + \dot{q}_\mu P_\mu |_{P_\nu = -H_\nu}. \quad (1.18)$$

The set of Hamilton-Jacobi partial differential equations (HJPDE) is expressed as

$$H'_{\alpha} \left(q_{\beta}; q_a; P_a = \frac{\partial S}{\partial q_a}; P_{\mu} = \frac{\partial S}{\partial q_{\mu}} \right) = 0, \quad \alpha, \beta = 0, 1, \dots, \quad (1.19)$$

where

$$H'_0 = P_0 + H_0; \quad (1.20)$$

$$H'_{\mu} = P_{\mu} + H_{\mu}. \quad (1.21)$$

with $q_0 = t$ and S being the action. The equations of motion are obtained as total differential equations in many variables such as,

$$dq_a = \frac{\partial H'_{\alpha}}{\partial P_a} dt_{\alpha}, \quad (1.22)$$

$$dP_r = \frac{\partial H'_{\alpha}}{\partial q_r} dt_{\alpha}, \quad r = 0, 1, \dots, n. \quad (1.23)$$

$$dZ = \left(-H_{\alpha} + P_a \frac{\partial H'_{\alpha}}{\partial P_a} \right) dt_{\alpha}. \quad (1.24)$$

where $Z = S(t_{\alpha}, q_a)$. These equations are integrable if and only if

$$dH'_0 = 0, \quad (1.25)$$

and

$$dH'_{\mu} = 0, \quad \mu = n - r + 1, \dots, n. \quad (1.26)$$

If the conditions (1.25) and (1.26) are not satisfied identically, we consider them as new constraints and we examine their variations. Thus repeating this procedure, one may obtain a set of constraints such that all the variations vanish, taking into account whether the system is completely (where the set of equations of motion and the action function is integrable.) or partially (where the set of equations of motion is only integrable.)

1.4 Supermembrane

In this section we review the basic properties of the supermembrane. A unified theory of the fundamental interactions involving supersymmetry (a supersymmetry is an invariant property under transformation of bosons into fermions and vice versa), in which the basic entities are two-dimensional extended objects (supermembrane). Supermembranes and other higher-dimensional objects have been found as solutions in superstring theory. A supermembrane action in eleven-dimensional spacetime with a nontrivial fermionic symmetry and manifest spacetime supersymmetry has been constructed using Dirac approach [28]. A consistent truncation of this action on a circle

has been shown to yield the action for the type IIA superstring coupled to the vectorlike $N = 2$ supergravity in ten dimensions [29]. Furthermore, it has been recently shown that the vacuum energy of the semiclassically quantized fluctuation around a topologically stabilized toroidal background is vanishing[30]. Although these developments are encouraging in quest for finding massless states in the spectrum of the supermembrane, several questions remain unanswered. In particular, not much is known about the infinite dimensional rigid symmetries of the supermembrane which are analogous to the super-Virasoro symmetries of the superstring . These symmetries are naturally expected to play an important role in the understanding of the spectrum of the supermembrane .

A natural framework for studying the Virasoro-like symmetries of an extended object is the Hamiltonian formalism . In string theory, the first class constraint given by the traceless energy-momentum tensor, $T_{++} = 0$, has the classical Poisson bracket, $\{T_{++}, T_{++}\} = T_{++}$, with no *field dependent*” structure constants”. The Fourier expansion of $T_{++} = \sum_n L_n e^{in\sigma}$ defines the Virasoro generators, L_n , which obey the usual Virasoro algebra.

In membrane theory, the first class constraints generate the

reparametrization of the three-dimensional world-volume swept by the membrane. Unlike in the string case, the Poisson bracket of two first class constraints which generate time reparametrizations involve the field dependent factor $\partial_a X^\mu \partial_b X_\mu$, where $a, b = 1, 2$ labels the spacelike coordinates of the world-volume, and X^μ are the membrane coordinates [31]. Therefore, the search for Virasoro-like symmetries will not only involve harmonic expansions in the two-dimensional membrane parameter space, but also finding an appropriate method to treat the field dependent "structure constant".

1.5 Outline

The main goal of this thesis is to find the explicit formulations of the dynamics of supermembrane theory, we will study model of supermembrane using both Dirac and Güler approach.

The organization is as follow. Chapter 2, Shows covariant quantization of the superparticle. In chapter 3, the supermembrane action and its invariances and the Hamiltonian formalism using Dirac approach is set up, and the primary and secondary constraints are discussed. Chapter 4 contains a discussion of

different models of supermembrane which is treated as a singular system to be investigated by Hamilton-Jacobi approach (or Güler approach).

Finally, Chapter 5, is devoted to conclusion.

Chapter 2

Superparticle

In extended supersymmetry there may be more than one superparticle for a given particle. For instance, with two copies of supersymmetry in four dimensions.

In zero dimensions it is possible to have supersymmetry, but no superparticle. However, this is the only situation where supersymmetry does not imply the existence of superparticle.

2.1 Covariant quantization of the $N = 1, d = 10$ superparticle

Since the $d = 11$ superparticle has a simpler action than the $d = 11$ supermembrane, it will be useful to explain how to covariantly quantize the $d = 11$ superparticle before discussing the

supermembrane. The quantization method is similar to that used in [32] for quantizing the $N = 1, d = 10$ superparticle, which will be reviewed first.

2.1.1 Standard description of the $N = 1, d = 10$ superparticle

The standard action for the $N = 1, d = 10$ superparticle is [33]

$$S = \int d\tau (P_m \dot{x}^m + e P^m P_m) \quad (2.1)$$

where

$$\Pi^m = \dot{x}^m + \frac{i}{2} \theta^\mu \gamma_{\mu\nu}^m \dot{\theta}^\nu, \quad (2.2)$$

$m = 1$ to 10 with x^{10} as the time coordinate, $\mu = 1$ to 16 , P_m is the canonical momentum for x^m , and e is the Lagrange multiplier which enforces the mass-shell condition. The gamma matrices $\gamma_{\mu\nu}^m$ and $\gamma^{m\mu\nu}$ are 16×16 symmetric matrices.

The action of (2.1) is invariant under the global $N=1$ $d=10$ spacetime-supersymmetry transformations

$$\delta\theta^\mu = \epsilon^\mu, \quad \delta x^m = \frac{i}{2} \theta^\nu \gamma_{\nu\mu}^m \epsilon, \quad \delta P_m = \delta e = 0, \quad (2.3)$$

The canonical momentum to θ^m which will be called p_μ , satisfies

$$p_\mu = \frac{\partial L}{\partial \dot{\theta}^\mu} = \frac{i}{2} P^m (\gamma_m \theta)_\mu. \quad (2.4)$$

so canonical quantization requires that physical states are annihilated by the sixteen fermionic Dirac constraints defined by

$$d_\mu = p_\mu - \frac{i}{2}P_m(\gamma^m\theta)_\mu. \quad (2.5)$$

Since $\{p_\mu, \theta^\nu\} = i\delta_\mu^\nu$, these constraints satisfy the Poisson brackets

$$\{d_\mu, d_\nu\} = -P_m\gamma_{\mu\nu}^m, \quad (2.6)$$

and since $P^m P_m = 0$ is also a constraint.

Although one cannot covariantly quantize the action of (2.1), one can classically couple the superparticle to a super-Maxwell background using the action

$$\hat{S} = \int d\tau [P_m \Pi^m + e P^m P_m + q(\dot{\theta}^\mu A_\mu(x, \theta) + \Pi^m A_m(x, \theta))] \quad (2.7)$$

where A_μ and A_m are the spinor and vector super-Maxwell gauge superfields and q is the charge of the superparticle. The action of (2.7) is invariant under spacetime supersymmetry and under the background gauge transformations $\delta A_\mu = D_\mu \Lambda$ and $\delta A_m = \partial \Lambda$ where $D_\mu = \frac{\partial}{\partial \theta^\mu} + \frac{i}{2}(\gamma^m \theta)_\mu \partial_m$.

2.1.2 Pure spinor description of the $N = 1$, $d = 10$ superparticle

Instead of using the standard superparticle action of (2.1), the pure spinor formalism for the $N = 1$, $d = 10$ superparticle uses the quadratic action [32]

$$S_{pure} = \int d\tau (P_m \dot{x}^m + p_\mu \dot{\theta}^\mu + w_\mu \dot{\lambda}^\mu - \frac{1}{2} P^m P_m) \quad (2.8)$$

where p_μ is now an independent variable [34], λ^μ is a pure spinor ghost variable satisfying

$$\lambda \gamma^m \lambda = 0 \quad m = 1 \text{ to } 10, \quad (2.9)$$

and w_μ is the canonical momentum to λ^μ which is defined up to the gauge transformation

$$\delta w_\mu = (\gamma^m \lambda)_\mu \Lambda_m \quad (2.10)$$

for arbitrary gauge parameter Λ_m .

The action of (2.8) can be written in manifestly spacetime supersymmetric notation as

$$S_{pure} = \int d\tau (P_m \Pi^m + d_\mu \dot{\theta}^\mu + w_\mu \dot{\lambda}^\mu - \frac{1}{2} P^m P_m) \quad (2.11)$$

where Π^m and d_μ are defined as in (2.2) and (2.5). Note that d_μ is defined to be invariant under spacetime supersymmetry, so p_μ should be defined to transform as $\delta p_\mu = \frac{i}{2} P_m (\gamma^m \epsilon)_\mu$ under (2.3).

2.2 Covariant quantization of the $N = 2$, $d = 10$ superparticle

Before quantizing the $d = 11$ superparticle, it will be useful to discuss the $N = 2$, $d = 10$ superparticle. Since the $N = 2$, $d = 10$ superparticle describes the zero modes of the type-II superstring, its quantization is expected a linearized version of type-II supergravity.

As will be seen in this section, there are subtleties at zero momentum with quantizing the $N = 2$, $d = 10$ superparticle which are related to subtleties with quantizing the type-II superstring.

2.2.1 Standard description of the $N = 2$, $d = 10$ superparticle

The standard action for the $N = 2$, $d = 10$ superparticle is

$$S = \int d\tau (P_m \dot{\Pi}^m + e P^m P_m) \quad (2.12)$$

where

$$\Pi^m = x^m + \frac{i}{2} \theta_L^\mu \gamma_{\mu\nu}^m \dot{\theta}_L^\nu + \frac{i}{2} \theta_R^{\hat{\mu}} \gamma_{\hat{\mu}\hat{\nu}}^m \dot{\theta}_R^{\hat{\nu}}, \quad (2.13)$$

$m = 1$ to 10 , $\mu = 1$ to 16 , and $(\theta_L^\mu, \theta_R^{\hat{\mu}})$ are the type-II fermionic superspace variables. For the type-IIA superparticle, μ and $\hat{\mu}$

denote spinors and opposite chirality, while for the type-IIB superparticle, μ and $\hat{\mu}$ denote spinors of the same chirality.

The action of (2.12) is invariant under the global $N = 2$, $d = 10$ spacetime-supersymmetry transformations

$$\delta\theta_L^\mu = \epsilon_L^\mu, \quad \delta\theta_R^{\hat{\mu}} = \epsilon_R^{\hat{\mu}}, \quad \delta x^m = \frac{i}{2}(\theta_L\gamma^m\epsilon_L + \theta_R\gamma^m\epsilon_R), \quad \delta P_m = \delta e = 0, \quad (2.14)$$

The canonical momenta to θ_L^μ and $\theta_R^{\hat{\mu}}$, which will be called $p_{L\mu}$ and $p_{R\hat{\mu}}$, satisfy

$$p_{L\mu} = \frac{\partial L}{\partial \dot{\theta}_L^\mu} = \frac{i}{2}P^m(\gamma_m\theta_L)_\mu, \quad p_{R\hat{\mu}} = \frac{\partial L}{\partial \dot{\theta}_R^{\hat{\mu}}} = \frac{i}{2}P^m(\gamma_m\theta_R)^{\hat{\mu}}, \quad (2.15)$$

so canonical quantization requires that physical states are annihilated by the 32 fermionic Dirac constraints defined by

$$d_{L\mu} = p_{L\mu} - \frac{i}{2}P_m(\gamma^m\theta_L)_\mu, \quad d_{R\hat{\mu}} = p_{R\hat{\mu}} - \frac{i}{2}P_m(\gamma^m\theta_R)^{\hat{\mu}}. \quad (2.16)$$

Since $\{p_{L\mu}, \theta_L^\nu\} = -i\delta_\mu^\nu$ and $\{p_{R\hat{\mu}}, \theta_R^{\hat{\nu}}\} = -i\delta_{\hat{\mu}}^{\hat{\nu}}$, these constraints satisfy the Poisson brackets

$$\{d_{L\mu}, d_{L\nu}\} = -P_m\gamma_{\mu\nu}^m, \quad \{d_{R\hat{\mu}}, d_{R\hat{\nu}}\} = -P_m\gamma_{\hat{\mu}\hat{\nu}}^m, \quad \{d_{L\mu}, d_{R\hat{\nu}}\} = 0, \quad (2.17)$$

and since $P^m P_m = 0$ is also a constraint, 16 of the 32 Dirac constraint, are first-class and 16 are second-class.

2.2.2 Pure spinor description of the $N = 2$, $d = 10$ superparticle

Instead of using the standard $N = 2$ superparticle action of (2.12), the pure spinor formalism for the $N = 2$, $d = 10$ superparticle uses the quadratic action

$$S_{pure} = \int d\tau (P_m \dot{x}^m + p_{L\mu} \dot{\theta}_L^\mu + p_{R\hat{\mu}} \dot{\theta}_R^{\hat{\mu}} + w_{L\mu} \dot{\lambda}_L^\mu + w_{R\hat{\mu}} \dot{\lambda}_R^{\hat{\mu}} - \frac{1}{2} P^m P_m) \quad (2.18)$$

where $p_{L\mu}$ and $p_{R\hat{\mu}}$ are now independent variable, λ_L^μ and $\lambda_R^{\hat{\mu}}$ are pure spinor ghost variable satisfying.

$$\lambda_L \gamma^m \lambda_L = 0 \quad \lambda_R \gamma^m \lambda_R = 0 \quad m = 1 \text{ to } 10. \quad (2.19)$$

and $w_{L\mu}$ and $w_{R\hat{\mu}}$ are defined up to the gauge transformations

$$\delta w_{L\mu} = (\gamma^m \lambda_L)_\mu \Lambda_{Lm}, \quad \delta w_{R\hat{\mu}} = (\gamma^m \lambda_R)_{\hat{\mu}} \Lambda_{Rm}, \quad (2.20)$$

for arbitrary gauge parameter Λ_{Lm} and Λ_{Rm} . The action of (2.18) can be written in manifestly spacetime supersymmetric notation as

$$S_{pure} = \int d\tau (P_m \Pi^m + d_{L\mu} \dot{\theta}_L^\mu + d_{R\hat{\mu}} \dot{\theta}_R^{\hat{\mu}} + w_{L\mu} \dot{\lambda}_L^\mu + w_{R\hat{\mu}} \dot{\lambda}_R^{\hat{\mu}} - \frac{1}{2} P^m P_m) \quad (2.21)$$

where $\Pi^m, d_{L\mu}$ and $d_{R\hat{\mu}}$ are defined as in (2.13) and (2.16).

2.3 Covariant quantization of d = 11 superparticle

In this section, the d = 11 superparticle will be covariantly quantized of linearized d = 11 supergravity.

2.3.1 Standard description of the d = 11 superparticle

The standard action for the d = 11 superparticle is

$$S = \int d\tau (P_c \dot{x}^c + e P^c P_c) \quad (2.22)$$

where

$$\Pi^c = \dot{x}^c + \frac{i}{2} \theta^\alpha \Gamma_{\alpha\beta}^c \dot{\theta}^\beta, \quad (2.23)$$

c = 1 to 11 with x^{10} as the time coordinate, and $\alpha = 1$ to 32. The d = 11 gamma matrices $\Gamma_{\alpha\beta}^c$ are 32×32 symmetric matrices. In d = 11, spinor indices can be raised and lowered using the antisymmetric metric tensor $C^{\alpha\beta}$ and its inverse $C_{\alpha\beta}^{-1}$. For example, $\Gamma^{c\alpha\beta} = C^{\alpha\delta} \Gamma_{\delta}^{c\beta} = C^{\alpha\delta} C^{\beta\gamma} \Gamma_{\delta\gamma}^c$.

The action of (2.22) is invariant under the global d = 11 spacetime-supersymmetry transformations

$$\delta\theta^\alpha = \epsilon^\alpha, \quad \delta x^c = \frac{i}{2} \theta \Gamma^c \epsilon, \quad \delta P_c = \delta e = 0, \quad (2.24)$$

The canonical momentum to θ^α , which will be called p_α , satisfies

$$p_\alpha = \frac{\partial L}{\partial \dot{\theta}^\alpha} = \frac{i}{2} P^c (\Gamma_c \theta)_\alpha, \quad (2.25)$$

so canonical quantization requires that physical states are annihilated by the 32 fermionic Dirac constraints defined by

$$d_\alpha = p_\alpha - \frac{i}{2} P_c (\Gamma^c \theta)_\alpha. \quad (2.26)$$

Since $\{p_\alpha, \theta^\beta\} = -\delta_\alpha^\beta$, these constraints satisfy the Poisson brackets

$$\{d_\alpha, d_\beta\} = -P_c \Gamma_{\alpha\beta}^c, \quad (2.27)$$

and since $P^c P_c = 0$ is also a constraint, 16 of the 32 Dirac constraints are first-class and 16 are second-class.

2.3.2 Pure spinor description of the $d = 11$ superparticle

The action of (2.22) reduces to the standard type-IIA $N=2$ superparticle action of (2.12) where $\theta_L^\mu = \frac{1}{\sqrt{2}}(1 + \Gamma^{11})_\alpha^\mu \theta^\alpha$ and $\theta_L^{\hat{\mu}} = \frac{1}{\sqrt{2}}(1 - \Gamma^{11})_{\hat{\alpha}}^{\hat{\mu}} \theta^\alpha$. This suggests construction a new pure spinor version of the $d = 11$ superparticle action which instead reduces at $P_{11} = 0$ to the type-IIA $N = 2$ superparticle action of (2.18). This pure spinor version of the $d = 11$ superparticle

action will be defined as the quadratic action

$$S_{pure} = \int d\tau (P_c \dot{x}^c + p_\alpha \dot{\theta} + w_\alpha \dot{\lambda}^\alpha - \frac{1}{2} P^c P_c) \quad (2.28)$$

where p_α is the canonical momentum to λ^α is an SO(10,1) pure spinor ghost variable satisfying

$$\lambda \Gamma^c \lambda = 0 \quad c = 1 \text{ to } 11, \quad (2.29)$$

and w_α is the canonical momentum to λ^α which is defined up to the gauge transformation

$$\delta w_\alpha = (\Gamma^c \lambda)_\alpha \Lambda_c \quad (2.30)$$

for arbitrary gauge parameter Λ_c .

With the exception of the d = 11 pure spinor constraint of (2.29), the action of (2.28) reduces when $P_{11} = 0$ to the type-IIA N=2 superparticle action of (2.18) where

$$\theta_L^\mu = \frac{1}{\sqrt{2}} (1 + \Gamma^{11})_\alpha^\mu \theta^\alpha, \quad \theta_R^{\hat{\mu}} = \frac{1}{\sqrt{2}} (1 - \Gamma^{11})_{\hat{\alpha}}^{\hat{\mu}} \theta^\alpha. \quad (2.31)$$

$$p_{L\mu} = \frac{1}{\sqrt{2}} (1 - \Gamma^{11})_\mu^\alpha p_\alpha, \quad p_{R\hat{\mu}} = \frac{1}{\sqrt{2}} (1 + \Gamma^{11})_{\hat{\mu}}^\alpha p_\alpha. \quad (2.32)$$

$$\lambda_L^\mu = \frac{1}{\sqrt{2}} (1 + \Gamma^{11})_\alpha^\mu \lambda^\alpha, \quad \lambda_R^{\hat{\mu}} = \frac{1}{\sqrt{2}} (1 - \Gamma^{11})_{\hat{\alpha}}^{\hat{\mu}} \lambda^\alpha. \quad (2.33)$$

$$w_{L\mu} = \frac{1}{\sqrt{2}} (1 - \Gamma^{11})_\mu^\alpha w_\alpha, \quad w_{R\hat{\mu}} = \frac{1}{\sqrt{2}} (1 + \Gamma^{11})_{\hat{\mu}}^\alpha w_\alpha. \quad (2.34)$$

However, the d = 11 pure spinor constraint $\lambda \Gamma^c \lambda = 0$ does not reduce to the N = 2, d = 11 pure spinor constraint $\lambda_L \gamma^m \lambda_L = \lambda_R \gamma^m \lambda_R = 0$ of (2.19).

Chapter 3

Dirac Approach of the supermembrane

3.1 The supermembrane action and its symmetries

The action for closed supermembrane is given by [28]

$$I = \int d^3\xi \left[-\frac{1}{2} \sqrt{-g} g^{ij} \Pi_i^\mu \Pi_{j\mu} + \frac{1}{2} \sqrt{-g} - \varepsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C B_{CBA} \right] \quad (3.1)$$

where $\Pi_i^A = (\Pi_i^\mu, \Pi_i^\alpha)$ with

$$\Pi_i^\mu = \partial_i X^\mu - i\bar{\psi} \Gamma^\mu \partial_i \psi, \quad (3.2)$$

$$\Pi_i^\alpha = \partial_i \psi^\alpha \quad (3.3)$$

and the membrane tension is set equal to unity. $\xi^i = (\tau, \sigma, \rho)$ where $(i = 1, 2, 3)$ are the coordinates, and g_{ij} is the metric

of the world-volume, Ψ^α is a 32-component Majorana spinor. (X^μ, ψ^α) are the coordinates of the eleven-dimensional super-space.

The super 3-form B is such that $dB = H$, with all components of H vanishing except $H_{\mu\nu\alpha\beta} = -i/3(\Gamma_{\mu\nu})_{\alpha\beta}$ [31]. Solving for B , one finds

$$B_{\mu\nu\rho} = 0, \quad B_{\mu\nu\alpha} = -\frac{1}{6}i(\Gamma_{\mu\nu}\psi)_\alpha, \quad (3.4)$$

$$B_{\mu\alpha\beta} = -\frac{1}{6}(\Gamma_{\mu\nu}\psi)_\alpha(\Gamma^\nu\psi)_\beta, \quad (3.5)$$

$$B_{\alpha\beta\gamma} = -\frac{1}{6}i(\Gamma_{\mu\nu}\psi)_\alpha(\Gamma^\mu\psi)_\beta(\Gamma^\nu\psi)_\gamma, \quad (3.6)$$

where $(\Gamma\psi)_\alpha^\mu = \Gamma_{\alpha\beta}^\mu\psi^\beta$. Substituting (3.4)-(3.6) into (3.1), one obtains

$$I = -\frac{1}{2} \int d^3\xi [\sqrt{-g}g^{ij}\Pi_i^\mu\Pi_{j\mu} - \sqrt{-g} + i\varepsilon^{ijk}\bar{\psi}\Gamma_{\mu\nu}\partial_i\psi \\ \times (\Pi_j^\mu\Pi_k^\nu + i\Pi_j^\mu\bar{\psi}\Gamma^\mu\partial_k\psi - \frac{1}{3}\bar{\psi}\Gamma^\mu\partial_j\psi\bar{\psi}\Gamma^\nu\partial_k\psi)]. \quad (3.7)$$

3.2 The covariant Hamiltonian formalism

In the Hamiltonian formulation of reparametrization invariant systems, it is convenient to parametrize the metric in terms of a shift vector N^a , and a lapse function N as follows [35].

$$g_{00} = -N^2 + \gamma_{ab}N^aN^b, \quad g_{0a} = g_{a0} = \gamma_{ab}N^b, \\ g_{ab} = \gamma_{ab}, \quad \sqrt{-g} = N\sqrt{\gamma}. \quad (3.8)$$

$$\begin{aligned}
g^{00} &= -N^{-2}, & g^{0a} &= g^{a0} = N^a N^{-2}, \\
g^{ab} &= \gamma^{ab} - N^a N^b N^{-2}.
\end{aligned} \tag{3.9}$$

Here, $\gamma_{ab}(a, b = 1, 2)$ is a 2-metric, $\gamma^{ab}\gamma_{bc} = \delta_c^a$, and all variables depend on τ, σ and ρ . In terms of these variables the action (3.7) is readily found to be

$$\begin{aligned}
I = \int d^3\xi & \left[\frac{1}{2} \sqrt{\gamma} N^{-1} \Pi_0^\mu \Pi_{0\mu} - \sqrt{\gamma} N^a N^{-1} \Pi_0^\mu \Pi_{a\mu} \right. \\
& \left. - \frac{1}{2} \sqrt{\gamma} (\gamma^{ab} - N^a N^b N^{-1}) \Pi_a^\mu \Pi_{b\mu} + \frac{1}{2} \sqrt{\gamma} N + 3\varepsilon^{ab} \Pi_0^A \Pi_a^B \Pi_b^C B_{CBA} \right],
\end{aligned} \tag{3.10}$$

where $\varepsilon^{12} = -\varepsilon^{21} = 1$.

The canonical variables are $(X^\mu, \Psi^\alpha, N, N^a, \gamma^{ab})$ and their conjugate momenta, $(P_\mu, P_\alpha, \Pi, \Pi_a, \Pi_{ab})$, which are given

from eq. (3.2) to be $\Pi_i^\mu = \partial_i X^\mu - i\bar{\Psi}\Gamma^\mu\partial_i\Psi$

and replacing i by zero, we get $\Pi_0^\mu = \partial_0 X^\mu - i\bar{\Psi}\Gamma^\mu\partial_0\Psi$

and

$$\Pi_0^\mu = \dot{X}^\mu - i\bar{\psi}\Gamma^\mu\partial_0\psi \tag{3.11}$$

where $\partial_0 X^\mu = \dot{X}^\mu$

substituting eq (3.11) into eq (3.10) we get the Lagrangian density

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2}\sqrt{\gamma}N^{-1}(\dot{X}^\mu - i\bar{\psi}\Gamma^\mu\partial_0\psi)\Pi_{0\mu} - \sqrt{\gamma}N^aN^{-1}\Pi_{a\mu}(\dot{X}^\mu - i\bar{\psi}\Gamma^\mu\partial_0\psi) \\
& - \frac{1}{2}\sqrt{\gamma}(\gamma^{ab} - N^aN^bN^{-1})\Pi_a^\mu\Pi_{b\mu} + \frac{1}{2}\sqrt{\gamma}N + 3\varepsilon^{ab}\Pi_0^A\Pi_a^B\Pi_b^C B_{CBA}
\end{aligned} \tag{3.12}$$

Using the definition of canonical momenta (1.3), we get

$$\begin{aligned}
p_\mu &= \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} \\
&= \frac{1}{2}\sqrt{\gamma}N^{-1}\Pi_{0\mu} - \sqrt{\gamma}N^aN^{-1}\Pi_{a\mu} + 3\varepsilon^{ab}\frac{\partial\dot{X}^c}{\partial\dot{X}^\mu}\Pi_a^A\Pi_b^B B_{BAC} \\
&= \frac{1}{2}\sqrt{\gamma}N^{-1}\Pi_{0\mu} - \sqrt{\gamma}N^aN^{-1}\Pi_{a\mu} + 3\varepsilon^{ab}S_\mu^c\Pi_a^A\Pi_b^B B_{BAC} \\
&= \frac{1}{2}\sqrt{\gamma}N^{-1}\Pi_{0\mu} - \sqrt{\gamma}N^aN^{-1}\Pi_{a\mu} + 3\varepsilon^{ab}\Pi_a^A\Pi_b^B B_{BA\mu}
\end{aligned}$$

$$p_\mu = \frac{1}{2}\sqrt{\gamma}N^{-1}\Pi_{0\mu} - \sqrt{\gamma}N^aN^{-1}\Pi_{a\mu} + S_\mu, \tag{3.13}$$

where $S_\mu = 3\varepsilon^{ab}\Pi_a^A\Pi_b^B B_{BA\mu}$

Similarly

$$P_\alpha = -\frac{\partial\mathcal{L}}{\partial\dot{\psi}^\alpha} = -i(\bar{\psi}\Gamma^\mu)_\alpha P_\mu - S_\alpha, \tag{3.14}$$

where $S_\alpha = 3\varepsilon^{ab}\Pi_a^A\Pi_b^B B_{BA\alpha}$, the other momenta are

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{N}} = 0, \tag{3.15}$$

$$\Pi_a = \frac{\partial\mathcal{L}}{\partial\dot{N}^a} = 0, \tag{3.16}$$

$$\Pi_{ab} = \frac{\partial\mathcal{L}}{\partial\dot{\gamma}^{ab}} = 0, \tag{3.17}$$

3.2.1 The primary and secondary constraints

We can solve for the velocity \dot{X}^μ from (3.13), but we cannot solve for the velocities $\dot{\psi}^\alpha, \dot{N}, \dot{N}^a$ and $\dot{\gamma}^{ab}$ from (3.14)-(3.17).

Therefore we have the primary constraints

$$F_\alpha = P_\alpha + i(\bar{\psi}\Gamma^\mu)_\alpha P_\mu + S_\alpha \simeq 0, \quad (3.18)$$

$$\Omega = \Pi \simeq 0, \quad (3.19)$$

$$\Omega_a = \Pi_a \simeq 0, \quad (3.20)$$

$$\Omega_{ab} = \Pi_{ab} \simeq 0, \quad (3.21)$$

These constraints are "weakly zero", meaning that they may have nonvanishing Poisson bracket with some canonical variables [2.35]. We must now require that the constraints (3.18)-(3.21) are maintained in time, i.e. their Poisson bracket with the Hamiltonian is weakly zero. The Hamiltonian can be obtained using (1.7) as

$$\begin{aligned} H &= \int d\sigma d\rho (P_\mu \dot{X}^\mu + P_\alpha \dot{\psi}^\alpha - \mathcal{L}) + \Sigma^a F^a + \Sigma \Omega + \Sigma^a \Omega_a + \Sigma^{ab} \Omega_{ab} \\ &= \int d\sigma d\rho \left[\frac{N}{2\sqrt{\gamma}} (P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2} N \sqrt{\gamma} \gamma^{ab} \Pi_a^\mu \Pi_{b\mu} - \frac{1}{2} N \sqrt{\gamma} \right. \\ &\quad \left. + N^a \Pi_a^\mu (P_\mu - S_\mu) + \Sigma^a F^a + \Sigma \Omega + \Sigma^a \Omega_a + \Sigma^{ab} \Omega_{ab} \right], \end{aligned} \quad (3.22)$$

where the Σ 's are Lagrange multipliers. The Poisson brackets of two arbitrary function, A and B, of the canonical variables is defined by [28]

$$\begin{aligned} \{A, B\} = \int d\sigma d\rho \left[& (-1)^{A+1} \frac{\partial A}{\partial \psi^a} \frac{\partial B}{\partial P_a} + (-1)^{AB+B} \frac{\partial B}{\partial \psi^a} \frac{\partial A}{\partial P_a} \right. \\ & + \left(\frac{\partial A}{\partial X^\mu} \frac{\partial B}{\partial P_\mu} + \frac{\partial A}{\partial N} \frac{\partial B}{\partial \Pi} + \frac{\partial A}{\partial N^a} \frac{\partial B}{\partial \Pi_a} + \frac{\partial A}{\partial \gamma^{ab}} \frac{\partial B}{\partial \Pi_{ab}} \right. \\ & \left. \left. - (-1)^{AB} (A \longleftrightarrow B) \right) \right], \end{aligned} \tag{3.23}$$

where the grading $A=0$ for bosons, and $A=1$ for fermions .

(Bosons: which have an integer-valued spin 0,1,2,... example as photon.

fermions: which have a half integer spin $\frac{1}{2}, \frac{3}{2}, \dots$ example as electron).

Requiring that the Hamiltonian(3.22), has weakly vanishing Poisson brackets with the primary constraints Ω, Ω_a and Ω_{ab} , defined in (3.19)-(3.21), one readily finds the secondary constraints,

$$\begin{aligned}
\phi = \{\Omega, H\}_{\Pi, N} &= \frac{\partial\Omega}{\partial\Pi} \frac{\partial H}{\partial N} - \frac{\partial\Omega}{\partial N} \frac{\partial H}{\partial\Pi} \\
&= \frac{1}{2} \frac{1}{\sqrt{\gamma}} (P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2} \sqrt{\gamma} \gamma^{ab} \Pi_a^\mu \Pi_{b\mu} - \frac{1}{2} \sqrt{\gamma} \\
&= \frac{1}{2} (P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2} \gamma \gamma^{ab} \Pi_a^\mu \Pi_{b\mu} - \frac{1}{2} \gamma \simeq 0,
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
\phi_a \{\Omega_a, H\}_{\Pi_a, N^a} &= \frac{\partial\Omega_a}{\partial\Pi_a} \frac{\partial H}{\partial N^a} - \frac{\partial\Omega_a}{\partial N^a} \frac{\partial H}{\partial\Pi_a} \\
&= \Pi_a^\mu (P_\mu - S_\mu) \simeq 0,
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\phi_{ab} = \{\Omega_{ab}, H\}_{\Pi_{ab}, N^{ab}} &= \frac{\partial\Omega_{ab}}{\partial\Pi_{ab}} \frac{\partial H}{\partial N^{ab}} - \frac{\partial\Omega_{ab}}{\partial N^{ab}} \frac{\partial H}{\partial\Pi_{ab}} \\
&= \Pi_a^\mu \Pi_{b\mu} - \gamma_{ab} \simeq 0.
\end{aligned} \tag{3.26}$$

The Poission bracket $\{F_a, H\}$ is far more complicated.

3.2.2 The total Hamiltonian

Multiplying the secondary constraints ϕ , ϕ_a and ϕ_{ab} with the Lagrange multipliers Λ , Λ^a and Λ^{ab} , respectively, and adding them to the Hamiltonian (3.22), we obtain the total Hamilto-

nian

$$\begin{aligned}
H' &= \int d\sigma d\rho \left[\frac{N}{2\sqrt{\gamma}} (P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2} N \sqrt{\gamma} \gamma^{ab} \Pi_a^\mu \Pi_{b\mu} \right. \\
&\quad - \frac{1}{2} N \sqrt{\gamma} + N^a \Pi_a^\mu (P_\mu - S_\mu) \\
&\quad + \Sigma^a F^a + \Sigma \Omega + \Sigma^a \Omega_a + \Sigma^{ab} \Omega_{ab} \\
&\quad \left. + \Lambda \phi + \Lambda^a \phi_a + \Lambda^{ab} \phi_{ab} \right], \\
&= \int d\sigma d\rho \left[\left(\frac{N}{\sqrt{\gamma}} + \Lambda \right) \phi + (N^a + \Lambda^a) \phi_a + \Sigma \Omega + \Sigma^a \Omega_a + \Sigma_{+\alpha} F_+^\alpha \right. \\
&\quad \left. + (\Lambda^{ab} \phi_{ab} + \Sigma^{ab} \Omega_{ab} + \Sigma_{-\alpha} F_-^\alpha) \right].
\end{aligned} \tag{3.27}$$

One should now verify that the secondary constraints (3.24)-(3.26) are also maintained in time, that is their Poisson bracket with H' vanishing weakly. We find that $\{\Omega_{ab}, H'\} \simeq 0$ enables us to solve for the Lagrange multiplier Σ_{ab} , while the requirement that $\{\phi_{ab}, H'\} \simeq 0$ enables us to determine the Lagrange multiplier Λ_{ab} . The Poisson bracket of the hamiltonian H' with the remaining constraints are weakly zero. Thus, there are no new (tertiary) constraints and the constraints (3.18)-(3.21) and (3.24)-(3.26) form a complete set.

Note that in the Hamiltonian (3.27), the Lagrange multipliers $\Lambda, \Lambda_a, \Sigma, \Sigma_a,$ and Σ_{+a} are still undetermined. This is a consequence of the reparametrization and fermionic invariances of

the theory.

According to Dirac, the equations of motion reads as:

$$\dot{X}_\mu = \partial_0 X_\mu = \{X_\mu, H_T\} = \frac{2P_\mu}{\sqrt{\gamma}N^{-1}} + i\Psi\Gamma_\mu\dot{\Psi} + 2N^a\Pi_{a\mu} - \frac{2S_\mu}{\sqrt{\gamma}N^{-1}} \quad (3.28)$$

$$\dot{P}_\mu = \{P_\mu, H_T\} = 0, \quad (3.29)$$

$$\dot{P}_\alpha = \{P_\alpha, H_T\} = 0, \quad (3.30)$$

$$\dot{\Pi} = \{\Pi, H_T\} = \frac{1}{2\sqrt{\gamma}}(P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2}\sqrt{\gamma}\gamma^{ab}\Pi_a^\mu\Pi_{b\mu} - \frac{1}{2}\sqrt{\gamma}, \quad (3.31)$$

$$\dot{\Pi}_a = \{\Pi_a, H_T\} = \Pi_a^\mu(P_\mu - S_\mu), \quad (3.32)$$

and

$$\dot{\Pi}_{ab} = \{\Pi_{ab}, H_T\} = \frac{1}{2}N\Pi_a^\mu\Pi_{b\mu} \quad (3.33)$$

Chapter 4

Hamilton-Jacobi Formulation of Submembrane

4.1 Hamilton-Jacobi Formulation of Submembrane in Four Dimensions

In (3.10) we proposed the following supermembrane action

$$I = \int d^3\xi \left[\frac{1}{2} \sqrt{\gamma} N^{-1} \Pi_0^\mu \Pi_{0\mu} - \sqrt{\gamma} N^a N^{-1} \Pi_0^\mu \Pi_{a\mu} - \frac{1}{2} \sqrt{\gamma} (\gamma^{ab} - N^a N^b N^{-1}) \Pi_a^\mu \Pi_{b\mu} + \frac{1}{2} \sqrt{\gamma} N + 3\varepsilon^{ab} \Pi_0^A \Pi_a^B \Pi_b^C B_{CBA} \right]. \quad (4.1)$$

The Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \sqrt{\gamma} N^{-1} \Pi_0^\mu \Pi_{0\mu} - \sqrt{\gamma} N^a N^{-1} \Pi_0^\mu \Pi_{a\mu} - \frac{1}{2} \sqrt{\gamma} (\gamma^{ab} - N^a N^b N^{-1}) \Pi_a^\mu \Pi_{b\mu} + \frac{1}{2} \sqrt{\gamma} N + 3\varepsilon^{ab} \Pi_0^A \Pi_a^B \Pi_b^C B_{CBA}. \quad (4.2)$$

The canonical momenta defined in (1.14) and (1.15) take the forms

$$P_\mu = \frac{\partial \mathcal{L}}{\partial(\partial^0 X^\mu)} = \frac{1}{2}\sqrt{\gamma}N^{-1}\Pi_{0\mu} - \sqrt{\gamma}N^aN^{-1}\Pi_{a\mu} + S_\mu, \quad (4.3)$$

$$p_\alpha = -\frac{\partial \mathcal{L}}{\partial(\partial^0 \Psi^\alpha)} = -i(\bar{\Psi}\Gamma^\mu)_\alpha P_\mu - S_\alpha = -H_\alpha, \quad (4.4)$$

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial^0 N)} = 0 = -H_\Pi, \quad (4.5)$$

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial(\partial^0 N^a)} = 0 = -H_a, \quad (4.6)$$

$$\Pi_{ab} = \frac{\partial \mathcal{L}}{\partial(\partial^0 \gamma^{ab})} = 0 = -H_{ab}, \quad (4.7)$$

We can solve (4.3) for \dot{X}_μ in terms of P_μ and other coordinates as

$$\dot{X}_\mu = \partial_0 X_\mu = \frac{2P_\mu}{\sqrt{\gamma}N^{-1}} + i\Psi\Gamma_\mu\dot{\Psi} + 2N^a\Pi_{a\mu} - \frac{2S_\mu}{\sqrt{\gamma}N^{-1}} \quad (4.8)$$

Now , we introduce the Hamiltonian density H_0 as

$$\begin{aligned} \mathcal{H}_0 &= P_\mu(\partial^0 X^\mu) + P_\alpha(\partial^0 \Psi^\alpha) + \Pi(\partial^0 N) + \Pi_a(\partial^0 N^a) + \Pi_{ab}(\partial^0 N^{ab}) - \mathcal{L} \\ &= \frac{N}{2\sqrt{\gamma}}(P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2}N\sqrt{\gamma}\gamma^{ab}\Pi_a^\mu\Pi_{ab} - \frac{1}{2}N\sqrt{\gamma} \\ &\quad + N^a\Pi_a^\mu(P_\mu - S_\mu), \end{aligned} \quad (4.9)$$

and the canonical Hamiltonian may be written as

$$\begin{aligned}
H_0 &= \int d\sigma d\rho \mathcal{H}_0 \\
&= \int d\sigma d\rho \left[\frac{N}{2\sqrt{\gamma}} (P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2} N \sqrt{\gamma} \gamma^{ab} \Pi_a^\mu \Pi_{ab} - \frac{1}{2} N \sqrt{\gamma} \right. \\
&\quad \left. + N^a \Pi_a^\mu (P_\mu - S_\mu) \right],
\end{aligned} \tag{4.10}$$

The set of HJPDEs defined in (1.19) reads as

$$\begin{aligned}
\mathcal{H}'_0 &= P_0 + \frac{N}{2\sqrt{\gamma}} (P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2} N \sqrt{\gamma} \gamma^{ab} \Pi_a^\mu \Pi_{ab} - \frac{1}{2} N \sqrt{\gamma} \\
&\quad + N^a \Pi_a^\mu (P_\mu - S_\mu),
\end{aligned} \tag{4.11}$$

$$\mathcal{H}'_\alpha = P_\alpha + i(\bar{\Psi}\Gamma^\mu)_\alpha P_\mu + S_\alpha = 0, \tag{4.12}$$

$$\mathcal{H}'_\Pi = \Pi = 0, \tag{4.13}$$

$$\mathcal{H}'_a = \Pi_a = 0, \tag{4.14}$$

and

$$\mathcal{H}'_{ab} = \Pi_{ab} = 0 \tag{4.15}$$

Using (1.22) and (1.23), the set of HJPDE (4.11)-(4.15) leads to the following total differential equations :

$$dX_\mu = \left(\frac{2P_\mu}{\sqrt{\gamma}N^{-1}} + i\Psi\Gamma_\mu\dot{\Psi} + 2N^a\Pi_{a\mu} - \frac{2S_\mu}{\sqrt{\gamma}N^{-1}} \right) d\tau \tag{4.16}$$

$$dP_\mu = 0, \quad (4.17)$$

$$dP_\alpha = 0, \quad (4.18)$$

$$d\Pi = \left(\frac{1}{2\sqrt{\gamma}}(P_\mu - S_\mu)(P^\mu - S^\mu) + \frac{1}{2}\sqrt{\gamma}\gamma^{ab}\Pi_a^\mu\Pi_{b\mu} - \frac{1}{2}\sqrt{\gamma} \right) d\tau, \quad (4.19)$$

$$d\Pi_a = (\Pi_a^\mu(P_\mu - S_\mu))d\tau, \quad (4.20)$$

and

$$d\Pi_{ab} = \left(\frac{1}{2}N\Pi_a^\mu\Pi_{b\mu} \right) d\tau \quad (4.21)$$

Chapter 5

Conclusion

In this thesis supermembrane theory are discussed within the framework of two methods: Dirac's and the Hamilton-Jacobi. The two methods, represent the Hamiltonian treatment of the supermembrane theory.

They are thought to have about the same length scale as superstrings, i.e. 10^{-35} m. At the present time there is no experimental evidence for supermembranes. Supermembranes and other higher-dimensional objects have been found as solutions in superstring theory.

In chapter 2 the covariant quantization of the superparticle theory is discussed, presents several quantization of superparticle, such as covariant quantization of the N=1, d=10 in standard description and pure spinor description, covariant quantization of the N=2, d=10 in standard description and pure spinor description, and standard description and pure spinor description of $d = 11$.

The supermembrane action and its symmetries was studied in

chapter 3. The covariant Hamiltonian formalism are obtained by using Dirac approach is set up, the total Hamiltonian by adding the primary constraints, multiplied by Lagrangian multipliers, to the usual Hamiltonian. I have computed the algebra of constraints in the eleven-dimensional supermembrane theory. We have shown explicitly how to separate the primary and secondary constraints. The equation of motion, obtained using Poisson brackets, are in ordinary differential equations forms.

Chapter 4 contains a discussion of different models of supermembrane which treated as a singular system to be investigated by Hamilton-Jacobi approach (or Güler approach). In other words, the equations of motion are not ordinary differential equations but total differential ones in many variables.

The final results of the two approach are found the same, and the Hamilton-Jacobi approach simpler than Dirac's approach.

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