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Quantization of constrained systems with higher–order Lagrangian

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Dedication

*To my parents, sisters and brothers,
To all my friends.*

OLA MOH'D SHIHADA

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ABSTRACT

QUANTIZATION OF CONSTRAINED SYSTEMS WITH HIGHER-ORDER LAGRANGIAN

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Dirac's method of discrete regular systems with higher-order Lagrangian, are studied as singular systems with first-order Lagrangian, and the equations of motion are obtained. It is shown that the Hamilton-Jacobi approach leads to the same equations of motion as obtained by Dirac's method. The second-order non-linear Lagrangian is studied as an example.

Continuous systems with higher-order Lagrangian density is treated as first-order ones, using Hamilton-Jacobi method. As applications, we investigated the effective higher-order Lagrangian of massive scalar field (Klein-Gordon theory) and massive vector field (Yang-Mills theory).

Besides, the canonical path integral quantization was obtained to quantize singular systems.

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Chapter 1

INTRODUCTION

1.1 Historical Introduction

The study of singular (constrained) systems has reached a great status in physics since the development by Dirac [1, 2] of the generalized Hamiltonian formulation. Therefore, this formalism has found a wide range of applications in field theory [3, 4, 5] and is still the main tool for the analysis for singular systems. Quantization of classical systems can be achieved by the canonical method. Physicists start to use the canonical method (Hamilton–Jacobi) because it has essential advantages, by using it one can easily control important properties of quantum theory, such as unitarity and positive definiteness of the metric. For quantization of constrained Hamiltonian systems we can use the operator method which is conventionally called Dirac’s quantization [1, 2]. To obtain the operator quantization for these systems, one takes the constraint equation as an operator whose action on the allowed Hilbert space vectors is constrained to zero.

The alternative quantization method for constrained systems is the path integral quantization. Historically, Feynman [6] first pointed out the fact that Green functions of Schrodinger equations are given as path integrals. Later, he used it as a functional tool for quantization of electrodynamics, and had a resounding success. The path integral quantization of singular theories with first class constraints in

canonical gauge was given by Faddeev and Popov [7, 8]. The generalization of the method to theories with second class constraints is given by Senjanovic[9]. Moreover, Fradkin and Vilkovisky [10, 11] considered quantization to bosonic theories with first–class constraints and its extension to include fermions in the canonical gauge. Recently, an approach based on Hamilton–Jacobi formalism was developed to study singular Lagrangian systems by Güler [12, 13], and then to obtain the path integral quantization of singular systems [14, 15, 16, 17].

The Hamilton–Jacobi (The canonical method) formulation of the singular systems enables us to obtain the equations of motion as total differential equations in many variables, which satisfy the integrability conditions.

Any physical system can be described by the Lagrangian function $L(q, \dot{q}, t)$, which is a function of n generalized coordinates q_i , n generalized velocities \dot{q}_i and parameter t , with $i = 1, \dots, n$. The Lagrangian function which contains higher derivatives of the generalized coordinates q_i , is called higher–order Lagrangian, which be written as $L(q_i, \dot{q}_i, \ddot{q}_i, \dots, q_i^{(m)})$. The Hess. matrix of higher–order Lagrangian of m –order is expressed as

$$A_{ij} = \frac{\partial^2 L}{\partial q_i^{(m)} \partial q_j^{(m)}}. \quad (1.1)$$

If the rank of (1.1) is n , the system is called regular, and singular if the rank is less than n , where n is the number of degrees of freedom.

The higher–order singular Lagrangian have been studied in many different problems of physics like general relativity, string theories, Dirac’s model of the radiating electron, which take a wide range in Refs.[18, 19, 20, 21, 22, 23].

The generalization of Hamilton’s least action principle and the Hamiltonian formulation to non degenerate Lagrangian depending on higher–order derivatives was first achieved by Ostrogradsky [24].

In the following, we discuss some methods to quantize the singular systems with first order Lagrangian.

1.1.1 Dirac's Method

The well-known method to investigate the Hamiltonian formulation of singular systems was initiated by Dirac [1, 2]. For the Lagrangian function $L(q_i, \dot{q}_i, t)$, if the rank of the Hess. matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, \dots, n, \quad (1.2)$$

is n , then the Lagrangian function is called regular, and singular if the rank is $n - r$, $r < n$.

The generalized momenta which corresponding to the generalized coordinates q_i are defined as

$$P_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, \dots, n - r, \quad (1.3)$$

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = n - r + 1, \dots, n. \quad (1.4)$$

Here, \dot{q}_i stands for the total derivative with respect to time t . The equation (1.4) enables us to write the primary constraints as

$$H'_\mu = P_\mu + H_\mu \approx 0. \quad (1.5)$$

One can define the total Hamiltonian as

$$H_T = H_o + \lambda_\mu H'_\mu, \quad (1.6)$$

where λ_μ are arbitrary functions, H'_μ are the primary constraints and H_o is the canonical (usual) Hamiltonian, which is defined as

$$H_o = p_i \dot{q}_i - L(q_i, \dot{q}_i, t), \quad i = 1, \dots, n. \quad (1.7)$$

The time variation of any function g is defined in the phase space as

$$\dot{g} = \{g, H_T\} = \{g, H_o\} + \lambda_\mu \{g, H'_\mu\}, \quad (1.8)$$

where $\{f, g\}$ called the Poisson brackets of the two functions $f(q_i, p_i)$ and $g(q_i, p_i)$ is defined as

$$\{f, g\} = \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i}. \quad (1.9)$$

Thus, the equations of motion can be written as

$$\dot{q}_a = \{q_a, H_T\} = \{q_a, H_o\} + \lambda_\mu \{q_a, H'_\mu\}, \quad (1.10)$$

$$\dot{p}_a = \{p_a, H_T\} = \{p_a, H_o\} + \lambda_\mu \{p_a, H'_\mu\}. \quad (1.11)$$

The consistency conditions, which means that the total derivative of primary constraints should be vanish, are given as

$$\dot{H}'_\mu = \{H'_\mu, H_T\} = \{H'_\mu, H_o\} + \lambda_\mu \{H'_\mu, H'_\mu\} \approx 0, \quad (1.12)$$

which may be identically satisfied with the help of primary constraints, or lead to new relations which are called secondary constraints, repeating this procedure until one arrives at a final set of constraints or specifies some of λ_μ . Primary and secondary constraints are divided into two types: *first– and second–class constraints*, the first–class constraints have vanishing Poisson brackets with all other constraints, but the second–class ones have non–vanishing Poisson brackets.

1.1.2 Hamilton–Jacobi approach

Hamilton–Jacobi approach of singular systems was developed by Güler [7, 8]. One starts from singular Lagrangian $L(q_i, \dot{q}_i, t)$. Since the rank of the Hess. matrix is $n - r, r < n$, one may solves (1.3) for \dot{q}_a as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{q}_i, p_a; t) \equiv \omega_a, \quad (1.13)$$

where ω_a is a constant. Substituting from (1.13) in (1.4), we get

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu} \Big|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, \dot{q}_\mu, P_a; t). \quad (1.14)$$

The canonical Hamiltonian H_o is defined as

$$H_o = -L(q_i, \dot{q}_\nu, \dot{q}_a; t) + P_a \omega_a + P_\mu \dot{q}_\mu \Big|_{P_\nu = -H_\nu}, \quad (1.15)$$

$$\mu, \nu = n - r + 1, \dots, n.$$

The set of Hamilton–Jacobi partial differential equations (**HJPDE**) is expressed as

$$H'_\alpha(t, q_\nu, q_a, P_i = \frac{\partial S}{\partial q_i}, P_0 = \frac{\partial S}{\partial t}) = 0, \quad (1.16)$$

where

$$H'_o = P_o + H_o, \quad (1.17)$$

$$H'_\mu = P_\mu + H_\mu. \quad (1.18)$$

The equations of motion are obtained as total differential equations in many variables as follows

$$dq_a = \frac{\partial H'_\alpha}{\partial P_a} dt_\alpha; \quad (1.19)$$

$$dP_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha; \quad (1.20)$$

$$dP_\nu = -\frac{\partial H'_\alpha}{\partial q_\nu} dt_\alpha; \quad (1.21)$$

$$dz = \left(-H_\mu + P_a \frac{\partial H'_\mu}{\partial P_a} \right) dt_\mu; \quad (1.22)$$

where $z = S(t_\mu, q_a)$ is the action.

The equations of motion (1.19 - 1.22) are integrable if and only if [12, 14, 15]

$$dH'_\alpha = 0, \quad \alpha = 0, n - r + 1, \dots, n. \quad (1.23)$$

If dH'_α is not identically zero, we have a new constraint, repeating the procedure until a complete system is obtained. The equation (1.23) is the necessary and

sufficient condition that the system (1.19 - 1.22) of total differential equations to be completely integrable. The set of equations of motion (1.19 - 1.21) may be only integrable, then we call this system as partially integrable [25].

1.1.3 The canonical path integral quantization

The Hamilton–Jacobi path integral quantization of singular systems was developed by Muslih and Güler [14, 15, 16, 17]. If the set of equations (1.19 - 1.22) is integrable, then one can solve them to obtain the trajectories of the motion in the canonical phase space coordinates as

$$q_a \equiv q_a(t, t_\mu), \quad P_a \equiv P_a(t, t_\mu), \quad \mu = 1, \dots, r, \quad a = 1, \dots, n - r. \quad (1.24)$$

In mathematics and physics, phase space is the space in which all possible states of the system as represented, with each possible state of the system corresponding to one unique point in the phase space.

In phase space, every degrees of freedom or parameter of the system is represented as an axis of a multidimensional space.

In classical mechanics, the phase space coordinates are the generalized coordinates q_i and their conjugated generalized momenta p_i . The motion of an ensemble of systems in this space is studied by classical statistical mechanics.

Moreover, the canonical action integral is integrable and can be obtained in terms of canonical variables. In this case, the path integral representation may be written as [16, 17, 26]

$$\psi(q'_a, t'_\alpha; q_a, t_\alpha) = \int_{q_a}^{q'_a} \prod_{a=1}^{n-r} Dq^a DP^a \times \exp \left\{ i \int_{t_\alpha}^{t'_\alpha} \left[-H_\alpha + P_a \frac{\partial H'_\alpha}{\partial P_a} \right] dt_\alpha \right\}; \quad (1.25)$$

$\alpha = 0, n - r + 1, \dots, n$.

One should notice that the integral (1.25) is an integration over the canonical phase space coordinates (q_a, P_a) .

1.2 Quantization of singular systems with second order Lagrangian

The singular systems with second order Lagrangian was studied in [19, 22, 23]. In this section, we will summarize the quantization of second order Lagrangian, which discussed in Ref.[19]

The Lagrangian formulation of these theories requires the configuration space formed by n generalized coordinates q_i , \dot{q}_i and \ddot{q}_i . Using the Hamilton's principle, the Euler–Lagrange equation of motion is given by

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial \ddot{q}_i} \right) = 0, \quad (1.26)$$

which are obtained from the action integral

$$S = \int L(q_i, \dot{q}_i, \ddot{q}_i) dt. \quad (1.27)$$

The passage from the Lagrangian approach to the Hamiltonian approach is achieved by introducing the generalized momenta (p_i, π_i) conjugated to the generalized coordinates (q_i, \dot{q}_i) as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \ddot{q}_i} \right), \quad (1.28)$$

$$\pi_i = \frac{\partial L}{\partial \ddot{q}_i}. \quad (1.29)$$

The phase space will be then be spanned by the canonical variables (q_i, p_i) and (\bar{q}_i, π_i) , where $(\bar{q}_i \equiv \dot{q}_i)$.

However, a valid phase space is formed if the rank of the Hess matrix

$$\frac{\partial^2 L}{\partial \ddot{q}_i \partial \ddot{q}_j}, \quad i, j = 1, \dots, n, \quad (1.30)$$

is n . Systems which have this property are called regular and their treatments are found in a standard mechanics books. Systems which have the rank less than n are called singular systems.

In the following two sections, one investigates singular systems with two different methods: Dirac's method and the canonical method.

1.2.1 Dirac method

The equivalence canonical Hamiltonian H_o in (1.7) is defined as

$$H_o = p_i \bar{q}_i + \pi_i \ddot{q}_i - L(q_i \dot{q}_i, \ddot{q}_i), \quad (1.31)$$

Thus, the extended Hamiltonian is determined as

$$H_E = H_o + \lambda_\alpha H'_\alpha, \quad (1.32)$$

where λ_α are unknown coefficients.

Due to the singular nature of the Hessian, we have α functionally independent relations of the form

$$H'_\alpha(q_i, p_i, \bar{q}_i, \pi_i) \approx 0. \quad (1.33)$$

The consistency conditions

$$\dot{H}'_\alpha = \{H'_\alpha, H_o\} + \lambda_\alpha \{H'_\alpha, H'_\beta\} \approx 0, \quad (1.34)$$

lead to the secondary constraints. Sometimes, there are some difficulties to determine the multipliers λ_μ , to remove this arbitrariness, one has to impose the external gauge fixing conditions for each first class constraints.

Fixing gauge is not always an easy task, which make one to be careful when applying Dirac's method.

1.2.2 The canonical path integral quantization for second–order Lagrangian

The canonical method [12, 13, 15, 16] has been developed to investigate singular systems using the Caratheodoy's equivalent Lagrangian method where the equations of motion are obtained as total differential equations in many variables.

Now we will give a brief review of the Caratheodoy's equivalent Lagrangian method. Considering the Lagrangian $L(q_i, \dot{q}_i, \ddot{q}_i, t)$, we can obtain a completely equivalent one by

$$L' = L(q_i, \dot{q}_i, \ddot{q}_i, t) - \frac{dS(q_i, \dot{q}_i, t)}{dt}, \quad (1.35)$$

Such a function $S(q_i, \dot{q}_i, t)$ must satisfy

$$\frac{\partial S}{\partial t} = -H_o, \quad (1.36)$$

$$H_o = p_i \bar{q}_i + \pi_i \ddot{q}_i - L, \quad (1.37)$$

$$p_i = \frac{\partial S}{\partial q_i}, \quad (1.38)$$

$$\pi_i = \frac{\partial S}{\partial \ddot{q}_i}. \quad (1.39)$$

If the rank of the Hess matrix $\frac{\partial^2 L}{\partial \ddot{q}_i \partial \ddot{q}_j}$ is $n - R$, $R < n$, then the generalized momenta conjugated to the generalized coordinates \bar{q}_i are defined as

$$\pi_a = \frac{\partial L}{\partial \dot{\bar{q}}_a}, \quad a = R + 1, \dots, n, \quad (1.40)$$

$$\pi_\alpha = \frac{\partial L}{\partial \dot{\bar{q}}_\alpha}, \quad \alpha = 1, \dots, R. \quad (1.41)$$

Since the rank of the Hess matrix $\frac{\partial^2 L}{\partial \dot{\bar{q}}_i \partial \dot{\bar{q}}_j}$ is $n - R$, one can solve the $n - R$ accelerations $\dot{\bar{q}}_a$ in terms of coordinates (q_i, \bar{q}_i) , the momenta π_a and $\dot{\bar{q}}_\alpha$ as follows:

$$\dot{\bar{q}}_a = \omega_a(q_i, \bar{q}_i, \pi_a, \dot{\bar{q}}_\alpha). \quad (1.42)$$

Substituting (1.42) in (1.41), one has

$$\pi_\alpha = \frac{\partial L}{\partial \dot{\bar{q}}_\alpha} \Big|_{\dot{q}_a = \omega_a(q_i, \bar{q}_i, \pi_a, \dot{\bar{q}}_\alpha)} = -H_\alpha^\pi(q_i, \bar{q}_i, p_b, \pi_a). \quad (1.43)$$

On the other hand, from (1.28) if the rank of the Hess matrix $\frac{\partial^2 L}{\partial \bar{q}_i \partial \bar{q}_j}$ is $n - r$, we can obtain a similar expression for the momenta p_γ

$$p_\gamma = -H_\gamma^p(q_i, \bar{q}_b, p_b, \pi_a), \quad \gamma = 1, \dots, r, b = r + 1, \dots, n. \quad (1.44)$$

The Hamiltonian is defined as

$$H_o = p_b \bar{q}_b + \bar{q}_\gamma p_\gamma \Big|_{p_\epsilon = -H_\epsilon^p} + \pi_a \omega_a + \bar{q}_\alpha \pi_\alpha \Big|_{p_\beta = -H_\beta^p} - L(q_i, \bar{q}_i, \dot{\bar{q}}_\alpha, \dot{q}_a = \omega_a), \quad \beta, \epsilon = 1, \dots, r. \quad (1.45)$$

Relabelling the coordinates t and q_γ as t_o and t_γ respectively, and \bar{q}_α will be called t_α , and defining the momenta as

$$P_o = \frac{\partial S}{\partial t}, \quad (1.46)$$

then, the set of Hamilton Jacobi partial differential equations [HJPDE] is

$$H'_o = P_o + H_o(t_o, t_\gamma, t_\alpha; q_b, \bar{q}_a; p_b = \frac{\partial S}{\partial q_b}; \pi_a = \frac{\partial S}{\partial \bar{q}_a}) = 0, \quad (1.47)$$

$$H'_\gamma{}^p = P_\gamma^p + H_\gamma^p(t_o, t_\gamma, t_\alpha; q_b, \bar{q}_a; p_b = \frac{\partial S}{\partial q_b}; \pi_a = \frac{\partial S}{\partial \bar{q}_a}) = 0, \quad (1.48)$$

$$H'_\alpha{}^\pi = P_\alpha^\pi + H_\alpha^\pi(t_o, t_\gamma, t_\alpha; q_b, \bar{q}_a; p_b = \frac{\partial S}{\partial q_b}; \pi_a = \frac{\partial S}{\partial \bar{q}_a}) = 0. \quad (1.49)$$

The equations of motion are obtained as total differential equations in many variables as follows:

$$dq_i = \frac{\partial H'_o}{\partial p_i} dt_o + \frac{\partial H'_\gamma{}^p}{\partial p_i} dt_\gamma + \frac{\partial H'_\alpha{}^\pi}{\partial p_i} dt_\alpha, \quad (1.50)$$

$$d\bar{q}_i = \frac{\partial H'_o}{\partial \pi_i} dt_o + \frac{\partial H'_\gamma{}^p}{\partial \pi_i} dt_\gamma + \frac{\partial H'_\alpha{}^\pi}{\partial \pi_i} dt_\alpha, \quad (1.51)$$

$$dp_i = -\frac{\partial H'_o}{\partial q_i} dt_o - \frac{\partial H'_\gamma{}^p}{\partial q_i} dt_\gamma - \frac{\partial H'_\alpha{}^\pi}{\partial q_i} dt_\alpha, \quad (1.52)$$

$$d\pi_i = -\frac{\partial H'_o}{\partial \bar{q}_i} dt_o - \frac{\partial H'_\gamma{}^p}{\partial \bar{q}_i} dt_\gamma - \frac{\partial H'_\alpha{}^\pi}{\partial \bar{q}_i} dt_\alpha, \quad (1.53)$$

$$dP_o = -\frac{\partial H'_o}{\partial t_o} dt_o - \frac{\partial H'_\gamma}{\partial t_o} dt_\gamma - \frac{\partial H'_\alpha}{\partial t_o} dt_\alpha, \quad (1.54)$$

$$\begin{aligned} dZ = & \left(-H_o + p_b \frac{\partial H'_o}{\partial p_b} + \pi_a \frac{\partial H'_o}{\partial \pi_a} \right) dt_o \\ & + \left(-H'_\gamma + p_b \frac{\partial H'_\gamma}{\partial p_b} + \pi_a \frac{\partial H'_\gamma}{\partial \pi_a} \right) dt_\gamma \\ & + \left(-H'_\alpha + p_b \frac{\partial H'_\alpha}{\partial p_b} + \pi_a \frac{\partial H'_\alpha}{\partial \pi_a} \right) dt_\alpha. \end{aligned} \quad (1.55)$$

The equations (1.50 - 1.55) are integrable if and only if

$$dH'_o = 0, \quad (1.56)$$

$$dH'_\gamma = 0, \quad (1.57)$$

$$dH'_\alpha = 0. \quad (1.58)$$

If conditions (1.56 - 1.58) are not satisfied identically, one considers them as new constraints and again tests the integrability conditions. Thus, repeating this procedure one may obtain a set of a conditions. Hence the canonical formulation leads us to obtain the set of canonical phase-space coordinates as

$$q_b \equiv q_b(t_o, t_\gamma, t_\alpha), \quad p_b \equiv p_b(t_o, t_\gamma, t_\alpha),$$

$$b = r + 1, \dots, n, \quad \gamma = 1, \dots, r, \quad \alpha = 1, \dots, R, \quad (1.59)$$

$$\bar{q}_a \equiv \bar{q}_a(t_o, t_\gamma, t_\alpha), \quad \pi_a \equiv \pi_a(t_o, t_\gamma, t_\alpha),$$

$$a = R + 1, \dots, n, \quad \gamma = 1, \dots, r, \quad \alpha = 1, \dots, R. \quad (1.60)$$

Besides the canonical action integral is obtained in terms of the canonical coordinates. The H'_o, H'_γ and H'_α can be interpreted as the infinitesimal generators of canonical transformations given by parameters t_o, t_γ and t_α respectively. In this case, the path integral quantization may be written as

$$\langle q_b, \bar{q}_a, t_\gamma, t_\alpha \mid q'_b, \bar{q}'_a, t'_\gamma, t'_\alpha \rangle = \int \prod_{b=1}^r dq^b dp^b \prod_{a=1}^R d\bar{q}^a d\pi^a \times$$

$$\begin{aligned} & \exp\left\{i \int_{t_\gamma, t_\alpha}^{t'_\gamma, t'_\alpha} \left(-H_o + p_b \frac{\partial H'_o}{\partial p_b} + \pi_a \frac{\partial H'_o}{\partial \pi_a}\right) dt_o\right. \\ & \left. + \left(-H_\gamma^p + p_b \frac{\partial H_\gamma^p}{\partial p_b} + \pi_a \frac{\partial H_\gamma^p}{\partial \pi_a}\right) dt_\gamma + \left(-H_\alpha^\pi + p_b \frac{\partial H_\alpha^\pi}{\partial p_b} + \pi_a \frac{\partial H_\alpha^\pi}{\partial \pi_a}\right) dt_\alpha\right\}. \end{aligned} \quad (1.61)$$

One should notice that the integral (1.61) is an integration over the canonical phase space coordinates $(q^b, \bar{q}^a; p^b, \pi^a)$.

Our aim of this thesis is to deal with higher–order regular Lagrangian as first–order singular one, as will be introduced in the below chapters.

In chapter two, the discrete systems of higher–order Lagrangian were discussed, using Dirac’s and Hamilton–Jacobi’s methods, then construct the canonical path integral quantization. These systems were converted to first–order Lagrangian by introducing an auxiliary fields[27]. The results that were obtained, using the Hamilton–Jacobi method are in exact agreement with those obtained using Dirac’s method in Ref. [28]. An example with second–order Lagrangian was studied, and path integral quantization was done.

In chapter three, the continuous systems with higher–order regular Lagrangian were treated as first–order singular Lagrangian. We used the Hamilton–Jacobi method; to construct the equations of motion as total differential equations. These equations are integrable under specified conditions on new coordinates (auxiliary fields).

In chapter four, as an application we considered the effective Lagrangian, which consists of free–massive field part, and the interaction part, which is a function of higher–order derivatives of fields. The obtained results using Hamilton–Jacobi method are in agreement with that in ref.[29].

In chapter five, we discussed the quantization of Yang–Mills field with an interaction term, as a second application on quantization of the continuous systems.

Chapter 2

DISCRETE SYSTEMS WITH HIGHER–ORDER LAGRANGIANS

Systems with higher–order Lagrangian have been studied with increasing interest because they appear in many relevant physical problems. Many authors' studied higher–order singular Lagrangian systems using both Dirac and Hamilton–Jacobi approaches [18, 21, 22, 23, 28, 30, 31].

In this chapter, we will discuss a model of discrete systems with higher–order Lagrangian. In section (2.1), we will be recognized to the Lagrangian formulation with an auxiliary fields. In section (2.2) the Dirac method will be used to investigate the system. Hamilton–Jacobi method will be addressed in section (2.3). The canonical path integral quantization is displayed in section (2.4). In section (2.5) an example will be discussed.

2.1 Lagrangian formulation with an auxiliary fields

In this section, we use the technique of Ref.[28] to transform the higher-order Lagrangian to a first-order one.

Let us consider the higher-order Lagrangian of discrete regular system

$$\mathcal{L} \equiv \mathcal{L}_o(x, \dot{x}, \ddot{x}, \dots, \overset{(m)}{x}), \quad (2.1)$$

with m -order.

The corresponding Euler-Lagrange equation of motion is

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}}{\partial \ddot{x}} \right) + \dots + (-1)^m \frac{d^m}{dt^m} \left(\frac{\partial \mathcal{L}}{\partial \overset{(m)}{x}} \right) = 0 \quad (2.2)$$

First, let us introduce the new auxiliary fields by making the following identification in the original Lagrangian in (2.1):

$$q_1 = \dot{x}; \quad q_2 = \ddot{x} = \dot{q}_1; \quad \dots \quad ; q_m = \overset{(m)}{x} = \dot{q}_{m-1} \quad (2.3)$$

Since the auxiliary fields q_i are not true physical degrees of freedom, the equations (2.3) are re-written as a set of m -constraints;

$$\chi_1 = (q_1 - \dot{x}) \approx 0; \quad \chi_2 = (q_2 - \dot{q}_1) \approx 0; \quad \dots \quad \chi_m = (q_m - \dot{q}_{m-1}) \approx 0 \quad (2.4)$$

So, the new form of Lagrangian with constraints is expressed as

$$\mathcal{L} = \mathcal{L}_o(x, q_1, q_2, \dots, q_m, \dot{q}_m) + \mu_1(q_1 - \dot{x}) + \mu_2(q_2 - \dot{q}_1) + \dots + \mu_m(q_m - \dot{q}_{m-1}), \quad (2.5)$$

where μ_m are multipliers. Thus the coordinates of restricted phase space is including the multipliers (x, q_i 's and μ_i 's).

This Lagrangian, unlike the original Lagrangian, has the advantage of being first-order in the sense that it is only a function of the coordinates and their first derivative.

Now we derive the equations of motion as a result of varying the coordinates of the extended phase space including the multipliers (x, q_i 's and μ_i 's),

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = 0, \quad \frac{\partial \mathcal{L}_o}{\partial x} + \dot{\mu}_1 = 0, \quad (2.6)$$

$$\frac{\partial \mathcal{L}}{\partial q_1} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \right) = 0, \quad \frac{\partial \mathcal{L}_0}{\partial q_1} + \mu_1 + \dot{\mu}_2 = 0, \quad (2.7)$$

$$\frac{\partial \mathcal{L}}{\partial q_2} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2} \right) = 0, \quad \frac{\partial \mathcal{L}_0}{\partial q_2} + \mu_2 + \dot{\mu}_3 = 0, \quad (2.8)$$

⋮

$$\frac{\partial \mathcal{L}}{\partial q_{m-1}} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_{m-1}} \right) = 0, \quad \frac{\partial \mathcal{L}_0}{\partial q_{m-1}} + \mu_{m-1} + \dot{\mu}_m = 0, \quad (2.9)$$

$$\frac{\partial \mathcal{L}}{\partial q_m} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_m} \right) = 0 \quad \mu_m = -\frac{\partial \mathcal{L}_0}{\partial q_m} + \frac{d}{dt} \left(\frac{\partial \mathcal{L}_0}{\partial \dot{q}_m} \right), \quad (2.10)$$

Removing the multipliers by successive substitution of (2.6 - 2.10), one obtain the expression

$$\frac{\partial \mathcal{L}_0}{\partial x} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}_0}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}_0}{\partial \ddot{x}} \right) + \dots + (-1)^m \frac{d^m}{dt^m} \left(\frac{\partial \mathcal{L}_0}{\partial x^{(m)}} \right) = 0, \quad (2.11)$$

which is identical to the original equation of motion.

2.2 Dirac Method

Regular systems with higher-order Lagrangian will be treated as singular systems with first order Lagrangian. Hamiltonian description plays an important role to determine the equations of motion in Dirac approach [30]. To arrive to the Hamiltonian description of the Lagrangian (2.5), we first write the canonical momenta conjugated to $\dot{x}, \dot{q}_1, \dots, \dot{q}_{m-1}, \dot{\mu}_i$ as

$$P = \frac{\partial \mathcal{L}}{\partial \dot{x}} = -\mu_1, \quad (2.12)$$

$$Q_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = -\mu_2, \quad (2.13)$$

$$Q_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = -\mu_3, \quad (2.14)$$

⋮

$$Q_{m-1} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{m-1}} = -\mu_m, \quad (2.15)$$

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\mu}_i} = 0 \quad i = 1, \dots, n. \quad (2.16)$$

By observing that the conjugated momenta in equations (2.12 - 2.16) are time independent, therefore, the set of primary constraints is

$$\Phi_1 = P + \mu_1 \approx 0, \quad (2.17)$$

$$\Phi_2 = Q_1 + \mu_2 \approx 0, \quad (2.18)$$

$$\Phi_3 = Q_2 + \mu_3 \approx 0, \quad (2.19)$$

⋮

$$\Phi_m = Q_{m-1} + \mu_m \approx 0, \quad (2.20)$$

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\mu}_i} \approx 0. \quad (2.21)$$

In fact, it is easy to show that the above constraints satisfy the following algebra:

$$\{\Phi_i, \Phi_j\} = 0, \quad \{\Pi_i, \Pi_j\} = 0, \quad (2.22)$$

$$\{\Phi_i, \Pi_j\} = \delta_{ij}, \quad (2.23)$$

which shows that these constraints are second-class constraints.

The form of the basic Hamiltonian \mathcal{H}_o is

$$\mathcal{H}_o = \dot{x}P + \dot{q}_i Q^i + \mu_i \Pi_i - \mathcal{L}_o(x, q_i, \dot{q}_i) - \mu_i(q_i - \dot{q}_{i-1}), \quad (2.24)$$

$i = 1, \dots, m$. Subsequently, the form of the extended Hamiltonian \mathcal{H}_E is

$$\mathcal{H}_E = \mathcal{H}_o + \lambda_i \Phi_i + \lambda'_i \Pi_i \quad (2.25)$$

In the extended Hamiltonian (2.25), one determines the two arbitrary multipliers λ_i, λ'_i , using the consistency conditions

$$\dot{\Phi}_i = \{\Phi_i, \mathcal{H}_E\} = 0, \quad (2.26)$$

$$\dot{\Pi}_i = \{\Pi_i, \mathcal{H}_E\} = 0, \quad (2.27)$$

Then it follows trivially that

$$\lambda'_i = -\frac{\partial \mathcal{L}}{\partial q_{i-1}} - \dot{\mu}_i \quad (2.28)$$

$$\lambda_i = q_i - \dot{q}_{i-1}. \quad (2.29)$$

Substituting the multipliers (2.28) and (2.29) in equation (2.25), the new extended Hamiltonian takes the form

$$\mathcal{H}_E = \mathcal{H}_o + (q_i - \dot{q}_{i-1})(Q_{i-1} + \mu_i) - \frac{\partial \mathcal{L}}{\partial q_{i-1}} \Pi^i - \dot{\mu}_i \Pi^i, \quad (2.30)$$

which after cancellation and rearrangement of some terms, it becomes

$$\mathcal{H}_E = q_1 P + q_2 Q_1 + \cdots + q_m Q_{m-1} + \dot{q}_m Q_m - \mathcal{L}_o, \quad (2.31)$$

or

$$\mathcal{H}_E = \dot{q}_m Q_m + q_i Q_{i-1} - \mathcal{L}_o. \quad (2.32)$$

To check the validity of the above Hamiltonian, let us derive the equations of motion, following the normal procedure of evaluating Poisson brackets of the phase space variables (x, P, q_i, Q_i) with the extended Hamiltonian H_E . In doing so, one gets the following equations

$$\dot{P} = \{P, \mathcal{H}_E\} = \frac{\partial \mathcal{L}_o}{\partial x}, \quad (2.33)$$

$$\dot{Q}_1 = \{Q_1, \mathcal{H}_E\} = \frac{\partial \mathcal{L}_o}{\partial q_1} - P, \quad (2.34)$$

$$\dot{Q}_2 = \{Q_2, \mathcal{H}_E\} = \frac{\partial \mathcal{L}_o}{\partial q_2} - Q_1, \quad (2.35)$$

⋮

$$\dot{Q}_m = \{Q_m, \mathcal{H}_E\} = \frac{\partial \mathcal{L}_o}{\partial q_m} - Q_{m-1}, \quad (2.36)$$

and

$$\dot{x} = \{x, \mathcal{H}_E\} = q_1, \quad (2.37)$$

$$\dot{q}_1 = \{q_1, \mathcal{H}_E\} = q_2, \quad (2.38)$$

⋮

$$\dot{q}_{m-1} = \{P, \mathcal{H}_E\} = q_m, \quad (2.39)$$

$$\dot{q}_m = \{q_m, \mathcal{H}_E\} = \dot{q}_m. \quad (2.40)$$

Equation (2.40) is trivial, whereas the other equations are expected to appear since they are the original constraints χ_i in the Lagrangian formulation. Furthermore, one can solve the equations (2.33 - 2.40) simultaneously to reproduce the original equation of motion (2.11).

2.3 Hamilton–Jacobi Method

In this section, we will investigate the same system using the Hamilton-Jacobi method. First, we will construct the corresponding canonical Hamiltonian of the reduced form of the regular Lagrangian (2.5).

The canonical Hamiltonian (2.24) rewritten as

$$\mathcal{H}_o = -\mathcal{L}_o(x, q_1, q_2, \dots, q_m, \dot{q}_m) + \sum_{i=1}^m q_i Q_{i-1}. \quad (2.41)$$

Return to (2.2.1 - 2.2.5), one can find the set of Hamilton-Jacobi Partial Differential Equations (1.16) can be written as

$$\mathcal{H}'_o = p_o + \mathcal{H}_o = 0, \quad p_o = \frac{\partial S}{\partial t}, \quad (2.42)$$

$$\mathcal{H}'_1 = P + \mu_1 = 0, \quad P = \frac{\partial S}{\partial x}, \quad (2.43)$$

$$\mathcal{H}'_2 = Q_1 + \mu_2 = 0, \quad Q_1 = \frac{\partial S}{\partial q_1}, \quad (2.44)$$

⋮

$$\mathcal{H}'_m = Q_{m-1} + \mu_m = 0, \quad Q_{m-1} = \frac{\partial S}{\partial q_{m-1}}, \quad (2.45)$$

$$\pi_i = 0, \quad \pi = \frac{\partial S}{\partial \mu_i}. \quad (2.46)$$

Thus, the equations of motion (1.19) and (1.20) can be obtained as total differential equations in many variables as follows:

$$dq_i = \frac{\partial(p_o + \mathcal{H}_o)}{\partial p_i} dt + \frac{\partial(P + \mu_1)}{\partial p_i} dx + \dots + \frac{\partial(Q_{m-1+\mu_m})}{\partial p_i} dq_{m-1} + \frac{\partial \pi_i}{\partial p_i} d\mu_i, \quad (2.47)$$

$$dP_o = -\frac{\partial(p_o + \mathcal{H}_o)}{\partial t} dt - \frac{\partial(P + \mu_1)}{\partial t} dx - \dots - \frac{\partial(Q_{m-1+\mu_m})}{\partial t} dq_{m-1} - \frac{\partial \pi_i}{\partial t} d\mu_i, \quad (2.48)$$

$$dQ_i = -\frac{\partial(p_o + \mathcal{H}_o)}{\partial q_i} dt - \frac{\partial(P + \mu_1)}{\partial q_i} dx - \dots - \frac{\partial(Q_{m-1+\mu_m})}{\partial q_i} dq_{m-1} - \frac{\partial \pi_i}{\partial q_i} d\mu_i, \quad (2.49)$$

$$d\mu_i = \frac{\partial(p_o + \mathcal{H}_o)}{\partial \pi_i} dt + \frac{\partial(P + \mu_1)}{\partial \pi_i} dx + \dots + \frac{\partial(Q_{m-1+\mu_m})}{\partial \pi_i} dq_{m-1} + \frac{\partial \pi_i}{\partial \pi_i} d\mu_i, \quad (2.50)$$

$$d\pi_i = -\frac{\partial(p_o + \mathcal{H}_o)}{\partial \mu_i} dt - \frac{\partial(P + \mu_1)}{\partial \mu_i} dx - \dots - \frac{\partial(Q_{m-1+\mu_m})}{\partial \mu_i} dq_{m-1} - \frac{\partial \pi_i}{\partial \mu_i} d\mu_i = 0 \quad (2.51)$$

The set of equations (2.47 - 2.51) reduces to the forms

$$dx = q_1 dt, \quad (2.52)$$

$$dq_1 = q_2 dt, \quad (2.53)$$

$$dq_2 = q_3 dt, \quad (2.54)$$

⋮

$$dq_{m-1} = q_m dt, \quad (2.55)$$

$$dP = \frac{\partial L_o}{\partial x} dt, \quad (2.56)$$

$$dQ_1 = \left(\frac{\partial L_o}{\partial q_1} - P \right) dt, \quad (2.57)$$

$$dQ_2 = \left(\frac{\partial L_o}{\partial q_2} - Q_1 \right) dt, \quad (2.58)$$

⋮

$$dQ_m = \left(\frac{\partial L_o}{\partial q_m} - Q_{m-1} \right) dt. \quad (2.59)$$

These total differential equations are integrable if and only if the variations of (2.42 - 2.46) vanish identically,

$$d\mathcal{H}'_o = dp_o + d\mathcal{H}_o = 0, \quad (2.60)$$

$$d\pi_i = 0, \quad (2.61)$$

$$d\mathcal{H}'_i = dQ_i + d\mu_i = 0. \quad (2.62)$$

The condition (2.62) is satisfied under specialized conditions, which are:

$$\dot{\mu}_1 = \frac{\partial \mathcal{L}_o}{\partial x}, \quad (2.63)$$

$$\dot{\mu}_2 + \mu_1 + \frac{\partial \mathcal{L}_o}{\partial q_1} = 0, \quad (2.64)$$

$$\dot{\mu}_3 + \mu_2 + \frac{\partial \mathcal{L}_o}{\partial q_2} = 0, \quad (2.65)$$

⋮

$$\dot{\mu}_m + \mu_{m-1} + \frac{\partial \mathcal{L}_o}{\partial q_{m-1}} = 0, \quad (2.66)$$

$$\mu_m = \frac{d}{dt} \left(\frac{\partial \mathcal{L}_o}{\partial q_m} \right) - \frac{\partial \mathcal{L}_o}{\partial q_m}. \quad (2.67)$$

Solving equations (2.52 - 2.59) and (2.63 - 2.67) simultaneously, we obtain the Euler-Lagrange equation of motion as

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}}{\partial \ddot{x}} \right) + \dots + (-1)^m \frac{d^m}{dt^m} \left(\frac{\partial \mathcal{L}}{\partial x^{(m)}} \right) = 0, \quad (2.68)$$

which is the same as eq. (2.11).

2.4 The canonical path integral quantization

To obtain the path integral quantization for the model (2.5), we need the canonical integrable action $z \equiv S(x, q_1, q_2, \dots, q_m)$. From (1.22), we have

$$dz = \left(\sum_{i=1}^m q_i Q_{i-1} - \mathcal{H}_o \right) dt. \quad (2.69)$$

If the equations (2.52 - 2.59) and (2.69) are integrable, then the system is complete and the canonical path integral quantization of this system is expressed as

$$\psi(q'_a, t'_\alpha; q_a, t_\alpha) = \int \prod_{i=1}^{m-1} \prod_{j=0}^m dq_i dQ_j \times \exp \left\{ i \int \left(\sum_{i=1}^m q_i Q_{i-1} - \mathcal{H}_0 \right) dt \right\}. \quad (2.70)$$

Integrating over the momenta Q_j gives

$$\psi(q'_a, t'_\alpha; q_a, t_\alpha) = \int \prod_{j=1}^{m-1} dq_j \times \exp \left\{ i \int \mathcal{L}(x, q_1, \dots, q_m, \dot{q}_m) dt \right\}. \quad (2.71)$$

2.5 An example

As an example, let us consider the following second order Lagrangian [30],

$$\mathcal{L}(x, \dot{x}, \ddot{x}) = \frac{1}{2} a x \ddot{x}^2 - \frac{1}{2} b x \dot{x}^2, \quad (2.72)$$

with a and b are arbitrary constant parameters. To convert the Lagrangian (2.72) to first order one, let us consider

$$q_1 = \dot{x}, \quad (2.73)$$

$$q_2 = \ddot{x} = \dot{q}_1. \quad (2.74)$$

The equations (2.73) and (2.74) are re-written as a set of constraints

$$\chi_1 = q_1 - \dot{x} \approx 0, \quad (2.75)$$

$$\chi_2 = q_2 - \dot{q}_1 \approx 0. \quad (2.76)$$

Therefore, the new Lagrangian with constraints takes the form

$$\mathcal{L}(x, q_1, \dot{q}_1) = \frac{1}{2} a x \dot{q}_1^2 - \frac{1}{2} b x q_1^2 + \mu_1 (q_1 - \dot{x}) + \mu_2 (q_2 - \dot{q}_1). \quad (2.77)$$

The canonical momenta (?? - ??) are obtained as

$$P = \frac{\partial \mathcal{L}}{\partial \dot{x}} = -\mu_1, \quad (2.78)$$

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = -\mu_2 + a x \dot{q}_1, \quad (2.79)$$

$$\pi_1 = \frac{\partial \mathcal{L}}{\partial \dot{\mu}_1} = 0, \quad (2.80)$$

$$\pi_2 = \frac{\partial \mathcal{L}}{\partial \dot{\mu}_2} = 0. \quad (2.81)$$

The canonical Hamiltonian (2.41) of this system can be written as

$$\mathcal{H}_o = -\frac{1}{2}ax\dot{q}_1^2 + \frac{1}{2}xq_1^2 + q_1P + q_2Q. \quad (2.82)$$

The corresponding set of **HJPDEs** (2.42 - 2.46) is

$$\mathcal{H}'_o = p_o + \mathcal{H}_o = 0, \quad (2.83)$$

$$\mathcal{H}'_1 = P + \mu_1 = 0, \quad (2.84)$$

$$\mathcal{H}'_2 = Q + \mu_2 = 0. \quad (2.85)$$

The total differential equations of motion (2.52 - 2.59) are obtained as

$$dx = q_1 dt, \quad (2.86)$$

$$dq_1 = q_2 dt, \quad (2.87)$$

$$dP = \left(\frac{1}{2}a\dot{q}_1^2 - \frac{1}{2}bq_1^2 \right) dt, \quad (2.88)$$

$$dQ = (-bxq_1 - P) dt. \quad (2.89)$$

According to the integrability conditions (2.60 - 2.62), equations (2.86 - 2.89) are integrable if the following conditions are satisfied:

$$\dot{\mu}_1 = \frac{1}{2}bq_1^2 - \frac{1}{2}a\dot{q}_1^2, \quad (2.90)$$

$$\dot{\mu}_2 = bxq_1 + P. \quad (2.91)$$

From equation (2.67), we obtain

$$\dot{\mu}_1 = a\dot{q}_1^2 + 2aq_1\ddot{q}_2 + ax\dot{q}_1^{(3)} + bq_1^2 + bx\dot{q}_1. \quad (2.92)$$

Solving equations (2.86 - 2.92) simultaneously, we obtain

$$\frac{3}{2}a\ddot{x}^2 + \frac{1}{2}b\dot{x}^2 + bx\ddot{x} + 2ax\dot{x}^{(3)} + ax\dot{x}^{(4)} = 0, \quad (2.93)$$

which is the same as Euler–Lagrange equation of motion of the system (2.72). To obtain the path integral quantization for this system, we need the integrable canonical action. In fact

$$dz = (q_1 P + q_2 Q - \mathcal{H}_o) dt. \quad (2.94)$$

The path integral (2.70) is expressed as

$$\psi(x, q, t; x', q', t') = \int dx dq_1 dP dQ \exp \left\{ i \int (q_1 P + q_2 Q - \mathcal{H}_o) dt \right\}, \quad (2.95)$$

which is an integration over the canonical phase space coordinates (x, P) and (q_1, Q) .

Integrating over momenta, we obtain

$$\psi(x, q, t; x', q', t') = \int dx dq_1 \exp \left\{ i \int \left(\frac{1}{2} a x q_1^2 - \frac{1}{2} b x q_1^2 \right) dt \right\}. \quad (2.96)$$

Chapter 3

CONTINUOUS SYSTEMS WITH HIGHER-ORDER LAGRANGIAN DENSITY

In this chapter, we will discuss a model of continuous system with higher-order Lagrangian density. The main idea of this work is to apply Hamilton–Jacobi approach to the reduced form of the higher order Lagrangian density.

As in chapter two, we will reduce the higher-order regular Lagrangian density to the first-order singular one, by introducing an auxiliary fields [28, 27]. In section (3.1), we will offer an introduction of Hamilton’s equations of motion for higher-order Lagrangian density. In section (3.2), the Hamilton–Jacobi approach will be used to investigate the system. The canonical path integral quantization determined in section (3.3).

3.1 Introduction

Let us consider the second order Lagrangian density $\mathcal{L}(\phi, \partial_{\mu_1}\phi, \partial_{\mu_1}\partial_{\mu_2}\phi)$, where $\partial_{\mu_1}\phi = \frac{\partial\phi}{\partial x^{\mu_1}}$. We can convert it to first order singular Lagrangian density by introducing $\mathcal{Z}_{\mu_1} = \partial_{\mu_1}\phi$, and then the constraint is $\chi_{\mu_1} = \mathcal{Z}_{\mu_1} - \partial_{\mu_1}\phi \approx 0$. The new

form of the Lagrangian density is written as

$$\mathcal{L}(\phi, \partial_{\mu_1}\phi, \partial_{\mu_1}\partial_{\mu_2}\phi) = \mathcal{L}_o(\phi, \mathcal{Z}_{\mu_1}, \partial_{\mu_2}\mathcal{Z}_{\mu_1}) + \lambda_{\mu_1}(\mathcal{Z}_{\mu_1} - \partial_{\mu_1}\phi), \quad (3.1)$$

where λ_{μ} is an auxiliary field.

To construct the corresponding Hamiltonian density, let us define the momenta

$$\pi_{\mu_1} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1}\phi)} = -\lambda_{\mu_1}, \quad (3.2)$$

$$\pi_{\mu_1, \mu_2} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu_2}\mathcal{Z}_{\mu_1})} = \frac{\partial \mathcal{L}_o}{\partial(\partial_{\mu_2}\mathcal{Z}_{\mu_1})}. \quad (3.3)$$

Therefore, the canonical Hamiltonian density is

$$\mathcal{H} = \mathcal{Z}_{\mu_1}\pi_{\mu_1} + (\partial_{\mu_2}\mathcal{Z}_{\mu_1})\pi_{\mu_1, \mu_2} - \mathcal{L}_o(\phi, \mathcal{Z}_{\mu_1}, \partial_{\mu_2}\mathcal{Z}_{\mu_1}). \quad (3.4)$$

The Hamilton's equations of motion are

$$\frac{d\phi}{dx_{\mu_1}} = \frac{\partial \mathcal{H}}{\partial \pi_{\mu_1}} = \mathcal{Z}_{\mu_1}, \quad (3.5)$$

$$\frac{d\mathcal{Z}_{\mu_1}}{dx_{\mu_2}} = \frac{\partial \mathcal{H}}{\partial \pi_{\mu_1, \mu_2}} = \partial_{\mu_2}\mathcal{Z}_{\mu_1}, \quad (3.6)$$

$$\frac{d\pi_{\mu_1}}{dx_{\mu_1}} = -\frac{\partial \mathcal{H}}{\partial \phi} = \frac{\partial \mathcal{L}_o}{\partial \phi}, \quad (3.7)$$

$$\frac{d\pi_{\mu_1, \mu_2}}{dx_{\mu_2}} = -\frac{\partial \mathcal{H}}{\partial \mathcal{Z}_{\mu_1}} = \frac{\partial \mathcal{L}_o}{\partial \mathcal{Z}_{\mu_1}} - \pi_{\mu_1}. \quad (3.8)$$

One can solve equations (3.5 - 3.8) simultaneously, and get

$$\frac{\partial \mathcal{L}_o}{\partial \phi} - \partial_{\mu_1} \left(\frac{\partial \mathcal{L}_o}{\partial(\partial_{\mu_1}\phi)} \right) + \partial_{\mu_1}\partial_{\mu_2} \left(\frac{\partial \mathcal{L}_o}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi)} \right) = 0, \quad (3.9)$$

which is the Euler–Lagrange equation of motion with second order Lagrangian.

Similarly, for the third order Lagrangian density $\mathcal{L}(\phi, \partial_{\mu_1}\phi, \partial_{\mu_1}\partial_{\mu_2}\phi, \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi)$ after converting it to first order singular Lagrangian density by introducing

$$\mathcal{Z}_{\mu_1} = \partial_{\mu_1}\phi, \quad \chi_{\mu_1} = (\mathcal{Z}_{\mu_1} - \partial_{\mu_1}\phi) \approx 0, \quad (3.10)$$

$$\mathcal{Z}_{\mu_1,\mu_2} = \partial_{\mu_1}\partial_{\mu_2}\phi = \partial_{\mu_2}\mathcal{Z}_{\mu_1}, \quad \chi_{\mu_1,\mu_2} = (\mathcal{Z}_{\mu_1,\mu_2} - \partial_{\mu_2}\mathcal{Z}_{\mu_1}) \approx 0. \quad (3.11)$$

The new form of the Lagrangian density is written as

$$\begin{aligned} \mathcal{L}(\phi, \partial_{\mu_1}\phi, \partial_{\mu_1}\partial_{\mu_2}\phi, \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi) &= \mathcal{L}_o(\phi, \mathcal{Z}_{\mu_1}, \mathcal{Z}_{\mu_1,\mu_2}, \partial_{\mu_3}\mathcal{Z}_{\mu_1,\mu_2}) + \\ &\lambda_{\mu_1}(\mathcal{Z}_{\mu_1} - \partial_{\mu_1}\phi) + \lambda_{\mu_1,\mu_2}(\mathcal{Z}_{\mu_1,\mu_2} - \partial_{\mu_2}\mathcal{Z}_{\mu_1}). \end{aligned} \quad (3.12)$$

One can make the Legendre transformation and determines the corresponding canonical Hamiltonian density of (3.12), and get

$$\mathcal{H} = \mathcal{Z}_{\mu_1}\pi_{\mu_1} + \mathcal{Z}_{\mu_1,\mu_2}\pi_{\mu_1,\mu_2} + (\partial_{\mu_3}\mathcal{Z}_{\mu_1,\mu_2})\pi_{\mu_1,\mu_2,\mu_3} - \mathcal{L}_o(\phi, \mathcal{Z}_{\mu_1}, \mathcal{Z}_{\mu_1,\mu_2}, \partial_{\mu_3}\mathcal{Z}_{\mu_1,\mu_2}). \quad (3.13)$$

The corresponding Hamilton's equations of motion can be written as

$$\frac{d\phi}{dx_{\mu_1}} = \frac{\partial\mathcal{H}}{\partial\pi_{\mu_1}} = \mathcal{Z}_{\mu_1}, \quad (3.14)$$

$$\frac{d\mathcal{Z}_{\mu_1}}{dx_{\mu_2}} = \frac{\partial\mathcal{H}}{\partial\pi_{\mu_1,\mu_2}} = \mathcal{Z}_{\mu_1,\mu_2}, \quad (3.15)$$

$$\frac{d\mathcal{Z}_{\mu_1,\mu_2}}{dx_{\mu_3}} = \frac{\partial\mathcal{H}}{\partial\pi_{\mu_1,\mu_2,\mu_3}} = \partial_{\mu_3}\mathcal{Z}_{\mu_1,\mu_2}, \quad (3.16)$$

$$\frac{d\pi_{\mu_1}}{dx_{\mu_1}} = -\frac{\partial\mathcal{H}}{\partial\phi} = \frac{\partial\mathcal{L}_o}{\partial\phi}, \quad (3.17)$$

$$\frac{d\pi_{\mu_1,\mu_2}}{dx_{\mu_2}} = -\frac{\partial\mathcal{H}}{\partial\mathcal{Z}_{\mu_1}} = \frac{\partial\mathcal{L}_o}{\partial\mathcal{Z}_{\mu_1}} - \pi_{\mu_1}, \quad (3.18)$$

$$\frac{d\pi_{\mu_1, \mu_2, \mu_3}}{dx_{\mu_3}} = -\frac{\partial \mathcal{H}}{\partial \mathcal{Z}_{\mu_1, \mu_2}} = \frac{\partial \mathcal{L}_o}{\partial \mathcal{Z}_{\mu_1, \mu_2}} - \pi_{\mu_1, \mu_2}. \quad (3.19)$$

One can solve (3.14 - 3.19) simultaneously, and get the equation of motion, which is

$$\frac{\partial \mathcal{L}_o}{\partial \phi} - \partial_{\mu_1} \left(\frac{\partial \mathcal{L}_o}{\partial (\partial_{\mu_1} \phi)} \right) + \partial_{\mu_1} \partial_{\mu_2} \left(\frac{\partial \mathcal{L}_o}{\partial (\partial_{\mu_1} \partial_{\mu_2} \phi)} \right) - \partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \left(\frac{\partial \mathcal{L}_o}{\partial (\partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \phi)} \right) = 0. \quad (3.20)$$

Therefore, the Euler–Lagrange equation of motion of higher-order Lagrangian density, with order n takes the form

$$\frac{\partial \mathcal{L}_o}{\partial \phi} - \partial_{\mu_1} \left(\frac{\partial \mathcal{L}_o}{\partial (\partial_{\mu_1} \phi)} \right) + \cdots + (-1)^n \partial_{\mu_1} \cdots \partial_{\mu_n} \left(\frac{\partial \mathcal{L}_o}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_n} \phi)} \right) = 0. \quad (3.21)$$

3.2 Hamilton-Jacobi Method

The higher–order regular Lagrangian density $\mathcal{L}(\phi, \partial_{\mu_1} \phi, \dots, \partial_{\mu_1} \cdots \partial_{\mu_n} \phi)$ can be reduced to first order singular Lagrangian density according to sec. (3.1)

$$\begin{aligned} \mathcal{L}(\phi, \partial_{\mu_1} \phi, \dots, \partial_{\mu_1} \cdots \partial_{\mu_n} \phi) &= \mathcal{L}_o(\phi, \mathcal{Z}_{\mu_1}, \dots, \mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}, \partial_{\mu_n} \mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}) + \\ &\lambda_{\mu_1} (\mathcal{Z}_{\mu_1} - \partial_{\mu_1} \phi) + \cdots + \lambda_{\mu_1, \dots, \mu_n} (\mathcal{Z}_{\mu_1, \dots, \mu_n} - \partial_{\mu_n} \mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}) \end{aligned} \quad (3.22)$$

The canonical momenta of the Lagrangian density (3.22) read as

$$\pi_{\mu_1} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu_1} \phi)} = -\lambda_{\mu_1}, \quad (3.23)$$

$$\pi_{\mu_1, \mu_2} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu_2} \mathcal{Z}_{\mu_1})} = -\lambda_{\mu_1, \mu_2}, \quad (3.24)$$

$$\pi_{\mu_1, \mu_2, \mu_3} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu_3} \mathcal{Z}_{\mu_1, \mu_2})} = -\lambda_{\mu_1, \mu_2, \mu_3}, \quad (3.25)$$

⋮

$$\pi_{\mu_1, \dots, \mu_{n-1}} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu_{n-1}} \mathcal{Z}_{\mu_1, \dots, \mu_{n-2}})} = -\lambda_{\mu_1, \dots, \mu_{n-1}}, \quad (3.26)$$

$$\pi_{\mu_1, \dots, \mu_n} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu_n} \mathcal{Z}_{\mu_1, \dots, \mu_{n-1}})} = -\lambda_{\mu_1, \dots, \mu_n}, \quad (3.27)$$

$$\pi_{\mu_1, \dots, \mu_{n+1}} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu_n} \lambda_{\mu_1, \dots, \mu_n})} = 0. \quad (3.28)$$

Therefore, the canonical Hamiltonian density can be written as

$$\begin{aligned} \mathcal{H}_\alpha &= \mathcal{Z}_{\mu_1} \pi_{\mu_1} + \mathcal{Z}_{\mu_1, \mu_2} \pi_{\mu_1, \mu_2} + \dots + \mathcal{Z}_{\mu_1, \dots, \mu_n} \pi_{\mu_1, \dots, \mu_n} \\ &\quad - \mathcal{L}_o(\phi, \mathcal{Z}_{\mu_1}, \mathcal{Z}_{\mu_1, \mu_2}, \dots, \mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}, \partial_{\mu_n} \mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}). \end{aligned} \quad (3.29)$$

The set of Hamilton-Jacobi Partial Differential Equations (**HJPDE**) is

$$\mathcal{H}'_\alpha = \pi_\alpha + \mathcal{H}_\alpha = 0, \quad \pi_\alpha = \frac{\partial S}{\partial x^\alpha} = -\mathcal{H}_\alpha, \quad (3.30)$$

$$\mathcal{H}'_{\nu_1} = \pi_{\nu_1} + \lambda_{\nu_1} = 0, \quad \pi_{\nu_1} = \frac{\partial S}{\partial \phi}, \quad (3.31)$$

$$\mathcal{H}'_{\nu_1, \nu_2} = \pi_{\nu_1, \nu_2} + \lambda_{\nu_1, \nu_2} = 0, \quad \pi_{\nu_1, \nu_2} = \frac{\partial S}{\partial \mathcal{Z}_{\nu_1}}, \quad (3.32)$$

⋮

$$\mathcal{H}'_{\nu_1, \dots, \nu_{n-1}} = \pi_{\nu_1, \dots, \nu_{n-1}} + \lambda_{\nu_1, \dots, \nu_{n-1}} = 0, \quad \pi_{\nu_1, \dots, \nu_{n-1}} = \frac{\partial S}{\partial \mathcal{Z}_{\nu_1, \dots, \nu_{n-2}}}, \quad (3.33)$$

$$\mathcal{H}'_{\nu_1, \dots, \nu_n} = \pi_{\nu_1, \dots, \nu_n} + \lambda_{\nu_1, \dots, \nu_n} = 0, \quad \pi_{\nu_1, \dots, \nu_n} = \frac{\partial S}{\partial \mathcal{Z}_{\nu_1, \dots, \nu_{n-1}}}, \quad (3.34)$$

$$\mathcal{H}'_{\nu_1, \dots, \nu_{n+1}} = \pi_{\nu_1, \dots, \nu_{n+1}} = 0, \quad \pi_{\nu_1, \dots, \nu_n} = \frac{\partial S}{\partial \lambda_{\nu_1, \dots, \nu_n}}. \quad (3.35)$$

The set of total differential equations of motion (1.19,1.20) is

$$\begin{aligned} d\phi &= \frac{\partial(\pi_\alpha + \mathcal{H}_\alpha)}{\partial \pi_{\mu_1}} dx_\alpha + \frac{\partial(\pi_{\nu_1} + \lambda_{\nu_1})}{\partial \pi_{\mu_1}} \Big|_{\pi_{\nu_1} = -\lambda_{\nu_1}} d\phi + \frac{\partial(\pi_{\nu_1, \nu_2} + \lambda_{\nu_1, \nu_2})}{\partial \pi_{\mu_1}} d\mathcal{Z}_{\nu_1} \\ &\quad + \dots + \frac{\partial(\pi_{\nu_1, \dots, \nu_n} + \lambda_{\nu_1, \dots, \nu_n})}{\partial \pi_{\mu_1}} d\mathcal{Z}_{\nu_1, \dots, \nu_{n-1}} + \frac{\partial(\pi_{\nu_1, \dots, \nu_{n+1}})}{\partial \pi_{\mu_1}} d\lambda_{\mu_1, \dots, \mu_n}, \end{aligned} \quad (3.36)$$

$$\begin{aligned} d\mathcal{Z}_{\mu_1} &= \frac{\partial(\pi_\alpha + \mathcal{H}_\alpha)}{\partial \pi_{\mu_1, \mu_2}} dx_\alpha + \frac{\partial(\pi_{\nu_1} + \lambda_{\nu_1})}{\partial \pi_{\mu_1, \mu_2}} d\phi + \frac{\partial(\pi_{\nu_1, \nu_2} + \lambda_{\nu_1, \nu_2})}{\partial \pi_{\mu_1, \mu_2}} \Big|_{\pi_{\nu_1, \nu_2} = -\lambda_{\nu_1, \nu_2}} d\mathcal{Z}_{\nu_1} \\ &\quad + \dots + \frac{\partial(\pi_{\nu_1, \dots, \nu_n} + \lambda_{\nu_1, \dots, \nu_n})}{\partial \pi_{\mu_1, \mu_2}} d\mathcal{Z}_{\nu_1, \dots, \nu_{n-1}} + \frac{\partial(\pi_{\nu_1, \dots, \nu_{n+1}})}{\partial \pi_{\mu_1, \mu_2}} d\lambda_{\mu_1, \dots, \mu_n}, \end{aligned} \quad (3.37)$$

⋮

$$\begin{aligned}
d\mathcal{Z}_{\mu_1, \dots, \mu_{n-1}} &= \frac{\partial(\pi_\alpha + \mathcal{H}_\alpha)}{\partial\pi_{\mu_1, \dots, \mu_n}} dx_\alpha + \frac{\partial(\pi_{\nu_1} + \lambda_{\nu_1})}{\partial\pi_{\mu_1, \dots, \mu_n}} d\phi + \frac{\partial(\pi_{\nu_1, \nu_2} + \lambda_{\nu_1, \nu_2})}{\partial\pi_{\mu_1, \dots, \mu_n}} d\mathcal{Z}_{\nu_1} \\
&+ \dots + \frac{\partial(\pi_{\nu_1, \dots, \nu_n} + \lambda_{\nu_1, \dots, \nu_n})}{\partial\pi_{\mu_1, \dots, \mu_n}} \Big|_{\pi_{\nu_1, \dots, \nu_n} = -\lambda_{\nu_1, \dots, \nu_n}} d\mathcal{Z}_{\nu_1, \dots, \nu_{n-1}} \\
&+ \frac{\partial(\pi_{\nu_1, \dots, \nu_{n+1}})}{\partial\pi_{\mu_1, \dots, \mu_n}} d\lambda_{\mu_1, \dots, \mu_n}, \tag{3.38}
\end{aligned}$$

$$\begin{aligned}
d\pi_{\mu_1} &= -\frac{\partial(\pi_\alpha + \mathcal{H}_\alpha)}{\partial\phi} dx_\alpha - \frac{\partial(\pi_{\nu_1} + \lambda_{\nu_1})}{\partial\phi} d\phi - \frac{\partial(\pi_{\nu_1, \nu_2} + \lambda_{\nu_1, \nu_2})}{\partial\phi} d\mathcal{Z}_{\nu_1} - \dots - \\
&\frac{\partial(\pi_{\nu_1, \dots, \nu_n} + \lambda_{\nu_1, \dots, \nu_n})}{\partial\phi} d\mathcal{Z}_{\nu_1, \dots, \nu_{n-1}} - \frac{\partial(\pi_{\nu_1, \dots, \nu_{n+1}})}{\partial\phi} d\lambda_{\mu_1, \dots, \mu_n}, \tag{3.39}
\end{aligned}$$

$$\begin{aligned}
d\pi_{\mu_1, \mu_2} &= -\frac{\partial(\pi_\alpha + \mathcal{H}_\alpha)}{\partial\mathcal{Z}_{\mu_1}} dx_\alpha - \frac{\partial(\pi_{\nu_1} + \lambda_{\nu_1})}{\partial\mathcal{Z}_{\mu_1}} d\phi - \frac{\partial(\pi_{\nu_1, \nu_2} + \lambda_{\nu_1, \nu_2})}{\partial\mathcal{Z}_{\mu_1}} d\mathcal{Z}_{\nu_1} \\
&- \dots - \frac{\partial(\pi_{\nu_1, \dots, \nu_n} + \lambda_{\nu_1, \dots, \nu_n})}{\partial\mathcal{Z}_{\mu_1}} d\mathcal{Z}_{\nu_1, \dots, \nu_{n-1}} - \frac{\partial(\pi_{\nu_1, \dots, \nu_{n+1}})}{\partial\mathcal{Z}_{\mu_1}} d\lambda_{\mu_1, \dots, \mu_n}, \tag{3.40}
\end{aligned}$$

$$\begin{aligned}
d\pi_{\mu_1, \mu_2, \mu_3} &= -\frac{\partial(\pi_\alpha + \mathcal{H}_\alpha)}{\partial\mathcal{Z}_{\mu_1, \mu_2}} dx_\alpha - \frac{\partial(\pi_{\nu_1} + \lambda_{\nu_1})}{\partial\mathcal{Z}_{\mu_1, \mu_2}} d\phi - \frac{\partial(\pi_{\nu_1, \nu_2} + \lambda_{\nu_1, \nu_2})}{\partial\mathcal{Z}_{\mu_1, \mu_2}} d\mathcal{Z}_{\nu_1} \\
&- \dots - \frac{\partial(\pi_{\nu_1, \dots, \nu_n} + \lambda_{\nu_1, \dots, \nu_n})}{\partial\mathcal{Z}_{\mu_1, \mu_2}} d\mathcal{Z}_{\nu_1, \dots, \nu_{n-1}} - \frac{\partial(\pi_{\nu_1, \dots, \nu_{n+1}})}{\partial\mathcal{Z}_{\mu_1, \mu_2}} d\lambda_{\mu_1, \dots, \mu_n}, \tag{3.41}
\end{aligned}$$

⋮

$$\begin{aligned}
d\pi_{\mu_1, \dots, \mu_n} &= -\frac{\partial(\pi_\alpha + \mathcal{H}_\alpha)}{\partial\mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}} dx_\alpha - \frac{\partial(\pi_{\nu_1} + \lambda_{\nu_1})}{\partial\mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}} d\phi - \frac{\partial(\pi_{\nu_1, \nu_2} + \lambda_{\nu_1, \nu_2})}{\partial\mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}} d\mathcal{Z}_{\nu_1} \\
&- \dots - \frac{\partial(\pi_{\nu_1, \dots, \nu_n} + \lambda_{\nu_1, \dots, \nu_n})}{\partial\mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}} d\mathcal{Z}_{\nu_1, \dots, \nu_{n-1}} - \frac{\partial(\pi_{\nu_1, \dots, \nu_{n+1}})}{\partial\mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}} d\lambda_{\mu_1, \dots, \mu_n}. \tag{3.42}
\end{aligned}$$

$$\begin{aligned}
d\pi_{\mu_1, \dots, \mu_{n+1}} &= -\frac{\partial(\pi_\alpha + \mathcal{H}_\alpha)}{\partial\lambda_{\mu_1, \dots, \mu_n}} dx_\alpha - \frac{\partial(\pi_{\nu_1} + \lambda_{\nu_1})}{\partial\lambda_{\mu_1, \dots, \mu_n}} d\phi - \frac{\partial(\pi_{\nu_1, \nu_2} + \lambda_{\nu_1, \nu_2})}{\partial\lambda_{\mu_1, \dots, \mu_n}} d\mathcal{Z}_{\nu_1} \\
&- \dots - \frac{\partial(\pi_{\nu_1, \dots, \nu_n} + \lambda_{\nu_1, \dots, \nu_n})}{\partial\lambda_{\mu_1, \dots, \mu_n}} d\mathcal{Z}_{\nu_1, \dots, \nu_{n-1}} - \frac{\partial(\pi_{\nu_1, \dots, \nu_{n+1}})}{\partial\lambda_{\mu_1, \dots, \mu_n}} d\lambda_{\mu_1, \dots, \mu_n}. \tag{3.43}
\end{aligned}$$

The equations (3.36 - 3.43) are reduced to

$$d\phi = \mathcal{Z}_{\mu_1} dx_{\mu_1}, \tag{3.44}$$

$$d\mathcal{Z}_{\mu_1} = \mathcal{Z}_{\mu_1, \mu_2} dx_{\mu_2}, \tag{3.45}$$

$$d\mathcal{Z}_{\mu_1, \mu_2} = \mathcal{Z}_{\mu_1, \mu_2, \mu_3} dx_{\mu_3}, \quad (3.46)$$

⋮

$$d\mathcal{Z}_{\mu_1, \dots, \mu_{n-1}} = (\partial_{\mu_n} \mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}) dx_{\mu_n}, \quad (3.47)$$

$$d\pi_{\mu_1} = \frac{\partial \mathcal{L}_o}{\partial \phi} dx_{\mu_1}, \quad (3.48)$$

$$d\pi_{\mu_1, \mu_2} = \left(\frac{\partial \mathcal{L}_o}{\partial \mathcal{Z}_{\mu_1}} - \pi_{\mu_1} \right) dx_{\mu_2}, \quad (3.49)$$

⋮

$$d\pi_{\mu_1, \dots, \mu_n} = \left(\frac{\partial \mathcal{L}_o}{\partial \mathcal{Z}_{\mu_1, \dots, \mu_{n-1}}} - \pi_{\mu_1, \dots, \mu_{n-1}} \right) dx_{\mu_n}, \quad (3.50)$$

$$d\pi_{\mu_1, \dots, \mu_{n+1}} = 0. \quad (3.51)$$

The equations (3.44 - 3.51) are integrable if and only if the integrability conditions (1.23) are identically satisfied. The variation of relations (3.30 - 3.35) respectively are

$$d\mathcal{H}'_{\alpha} = d\pi_{\alpha} + d\mathcal{H}_{\alpha} = 0, \quad (3.52)$$

$$d\mathcal{H}'_{\nu_1} = d\pi_{\nu_1} + d\lambda_{\nu_1} = 0, \quad (3.53)$$

$$d\mathcal{H}'_{\nu_1, \nu_2} = d\pi_{\nu_1, \nu_2} + d\lambda_{\nu_1, \nu_2} = 0, \quad (3.54)$$

⋮

$$d\mathcal{H}'_{\nu_1, \dots, \nu_n} = d\pi_{\nu_1, \dots, \nu_n} + d\lambda_{\nu_1, \dots, \nu_n} = 0, \quad (3.55)$$

$$d\mathcal{H}'_{\nu_1, \dots, \nu_{n+1}} = d\pi_{\nu_1, \dots, \nu_{n+1}} = 0. \quad (3.56)$$

Relation (3.52) vanishes identically, but relations (3.53 - 3.56) are satisfied under the following conditions:

$$\frac{d}{dx_{\nu_1}} \lambda_{\nu_1} = -\frac{\partial \mathcal{L}_o}{\partial \phi}, \quad (3.57)$$

$$\frac{d}{dx_{\nu_2}} \lambda_{\nu_1, \nu_2} = \pi_{\nu_1} - \frac{\partial \mathcal{L}_o}{\partial \mathcal{Z}_{\nu_1}}, \quad (3.58)$$

⋮

$$\frac{d}{dx_{\nu_{n-1}}} \lambda_{\nu_1, \dots, \nu_{n-1}} = \pi_{\nu_1, \dots, \nu_{n-2}} - \frac{\partial \mathcal{L}_o}{\partial \mathcal{Z}_{\nu_1, \dots, \nu_{n-2}}}, \quad (3.59)$$

$$\frac{d}{dx_{\nu_n}} \lambda_{\nu_1, \dots, \nu_n} = \pi_{\nu_1, \dots, \nu_{n-1}} - \frac{\partial \mathcal{L}_o}{\partial \mathcal{Z}_{\nu_1, \dots, \nu_{n-1}}}. \quad (3.60)$$

$$\pi_{\nu_1, \dots, \nu_n} = \frac{\partial \mathcal{L}_o}{\partial \mathcal{Z}_{\nu_1, \dots, \nu_n}}. \quad (3.61)$$

Solving (3.44 - 3.51) simultaneously, one obtains

$$\frac{\partial \mathcal{L}_o}{\partial \phi} - \partial_{\mu_1} \left(\frac{\partial \mathcal{L}_o}{\partial \mathcal{Z}_{\mu_1}} \right) + \partial_{\mu_1} \partial_{\mu_2} \left(\frac{\partial \mathcal{L}_o}{\partial \mathcal{Z}_{\mu_1, \mu_2}} \right) + \dots + (-1)^n \partial_{\mu_1} \dots \partial_{\mu_n} \left(\frac{\partial \mathcal{L}_o}{\partial \mathcal{Z}_{\mu_1, \dots, \mu_n}} \right) = 0, \quad (3.62)$$

or

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu_1} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu_1} \phi)} \right) + \dots + (-1)^n \partial_{\mu_1} \dots \partial_{\mu_n} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu_1} \dots \partial_{\mu_n} \phi)} \right) = 0, \quad (3.63)$$

which is the same form as (3.21).

3.3 The canonical path integral quantization

To obtain the canonical path integral quantization for the continuous Lagrangian density (3.22), we determine the action $z \equiv S(x_\mu, \phi, Z_{\nu_1}, Z_{\nu_1, \nu_2}, \dots, Z_{\nu_1, \dots, \nu_{n-1}})$, one can obtain

$$dz = \left[-\mathcal{H}_\mu + \pi_{\nu_1} \frac{\partial \mathcal{H}_\mu}{\partial \pi_{\nu_1}} + \pi_{\nu_1, \nu_2} \frac{\partial \mathcal{H}_\mu}{\partial \pi_{\nu_1, \nu_2}} + \dots + \pi_{\nu_1, \dots, \nu_{n-1}} \frac{\partial \mathcal{H}_\mu}{\partial \pi_{\nu_1, \dots, \nu_{n-1}}} \right] dx_\mu, \quad (3.64)$$

this equation rewritten as

$$dz = \left[-\mathcal{H}_\mu + \pi_{\nu_1} \mathcal{Z}_{\nu_1} + \pi_{\nu_1, \nu_2} \mathcal{Z}_{\nu_1, \nu_2} + \dots + \pi_{\nu_1, \dots, \nu_{n-1}} \mathcal{Z}_{\nu_1, \dots, \nu_{n-1}} \right] dx_\mu, \quad (3.65)$$

If the equations (3.44 - 3.51, 3.65) are integrable, then the correspondence path integral quantization is determined as

$$\langle out|S|in \rangle = \int \prod_{\lambda, \epsilon=1}^N d\mathcal{Z}_\epsilon d\pi_\lambda \times \exp\{i \int [-\mathcal{H}_\mu + \pi_{\nu_1} \mathcal{Z}_{\nu_1} + \pi_{\nu_1, \nu_2} \mathcal{Z}_{\nu_1, \nu_2} + \cdots + \pi_{\nu_1, \dots, \nu_{n-1}} \mathcal{Z}_{\nu_1, \dots, \nu_{n-1}}] dx_\mu\}, \quad (3.66)$$

which is an integration over the canonical phase space coordinates $(\mathcal{Z}_\epsilon, \pi_\lambda)$.

Chapter 4

THE EFFECTIVE LAGRANGIAN MASSIVE OF KLEIN–GORDON (SCALAR FIELDS) THEORY

In chapter three, we quantized the continuous systems with higher–order Lagrangian density, using the Hamilton–Jacobi approach. In this chapter, we consider an application of it. The effective higher–order Lagrangian density of massive scalar field will be quantized, using the Hamilton–Jacobi method, after reducing it to first order singular Lagrangian density [32].

Section (4.1), is an introduction to systems with effective Lagrangian of massive scalar field. The Hamilton–Jacobi method is used to investigate the system in section(4.2). In section (4.3), we calculate the canonical path integral quantization.

4.1 Introduction

The Lagrangian density of massive scalar field may be written in the form [33]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi), \quad (4.1)$$

where

$$V(\varphi) = \frac{1}{2}\mathcal{M}^2\varphi^2 - \mathcal{H}_{int}, \quad (4.2)$$

with \mathcal{M} is the mass of the particle, φ is a scalar field and \mathcal{H}_{int} is the interaction Hamiltonian.

If $\mathcal{H}_{int} = 0$, then the Lagrangian of a free spin-0 is recovered, and (4.1) takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}\mathcal{M}^2\varphi^2, \quad (4.3)$$

which is the classical Lagrangian.

The Lagrangian density $\mathcal{L}(\varphi, \partial^\mu\varphi)$ satisfies the Euler–Lagrange field equation

$$\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0. \quad (4.4)$$

Recalling that $(\partial_\mu\varphi)(\partial^\mu\varphi) = g_{\mu\nu}(\partial^\nu\varphi)(\partial^\mu\varphi)$, where μ and ν are dummy indices, and $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ is the metric tensor. (see appendix A)

The corresponding Euler–Lagrange equation of (4.3) is

$$(\partial^\mu\partial_\mu + \mathcal{M}^2)\varphi = 0, \quad (4.5)$$

which is the Klein–Gordon equation.

Otherwise, if $\mathcal{H}_{int} \neq 0$, it means that \mathcal{H}_{int} describes interactions of the field with itself, or with other fields. For simplicity, sometimes it is assumed to be independent of the field derivatives, so that the momentum variables remain as those of the free fields. In contrast, if one considers that \mathcal{H}_{int} depends on higher–field derivatives for constrained fields, then the Lagrangian density becomes effective Lagrangian density.

Effective Lagrangian density containing arbitrary interactions of massive vector (it will appear in the next chapter) and scalar (in this chapter) fields will be studied.

The effective Lagrangian of massive Klein–Gordon theory takes the form [29]

$$\mathcal{L}_{eff} = \mathcal{L}_o + \varepsilon\mathcal{L}_I = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}\mathcal{M}^2\varphi^2 + \varepsilon\mathcal{L}_I(\varphi, \partial^\mu\varphi, \dots, \partial^{\mu_1}\dots\partial^{\mu_n}\varphi), \quad (4.6)$$

where \mathcal{L}_o represents a free–massive Klein–Gordon theory and \mathcal{L}_I contains the effective interactions which depends on the derivatives of the scalar fields up to order

(n). These interactions are governed by the coupling constant ε with $\varepsilon \ll 1$.

As we see before, the corresponding equation of motion (3.21) of higher-order Lagrangian density $\mathcal{L}_I(\varphi, \partial^{\mu_1}\varphi, \dots, \partial^{\mu_1} \dots \partial^{\mu_n}\varphi)$ is

$$\frac{\partial \mathcal{L}_I}{\partial \phi} - \partial^{\mu_1} \left(\frac{\partial \mathcal{L}_I}{\partial (\partial^{\mu_1} \phi)} \right) + \dots + (-1)^n \partial^{\mu_1} \dots \partial^{\mu_n} \left(\frac{\partial \mathcal{L}_I}{\partial (\partial^{\mu_1} \dots \partial^{\mu_n} \phi)} \right) = 0, \quad (4.7)$$

Combining (4.5) with (4.7), one can obtain the equation of motion of the effective Lagrangian density (4.6), which is

$$\begin{aligned} \mathcal{M}^2 \varphi + \partial^{\mu_1} \partial_{\mu_1} \varphi - \varepsilon \left[\frac{\partial \mathcal{L}_I}{\partial \varphi} - \partial^{\mu_1} \left(\frac{\partial \mathcal{L}_I}{\partial (\partial^{\mu_1} \varphi)} \right) + \dots + (-1)^n \partial^{\mu_1} \dots \partial^{\mu_n} \left(\frac{\partial \mathcal{L}_I}{\partial (\partial^{\mu_1} \dots \partial^{\mu_n} \varphi)} \right) \right] \\ = 0. \end{aligned} \quad (4.8)$$

4.2 Hamilton–Jacobi Method

To convert the higher order interaction Lagrangian density (4.6) to first-order, let us introduce the following substitutions:

$$\mathcal{Z}^{\mu_1} = \partial^{\mu_1} \varphi, \quad \chi^{\mu_1} = \mathcal{Z}^{\mu_1} - \partial^{\mu_1} \varphi \approx 0, \quad (4.9)$$

$$\mathcal{Z}^{\mu_1, \mu_2} = \partial^{\mu_1} \partial^{\mu_2} \varphi = \partial^{\mu_2} \mathcal{Z}^{\mu_1}, \quad \chi^{\mu_1, \mu_2} = \mathcal{Z}^{\mu_1, \mu_2} - \partial^{\mu_2} \mathcal{Z}^{\mu_1} \approx 0, \quad (4.10)$$

$$\mathcal{Z}^{\mu_1, \mu_2, \mu_3} = \partial^{\mu_3} \mathcal{Z}^{\mu_1, \mu_2}, \quad \chi^{\mu_1, \mu_2, \mu_3} = \mathcal{Z}^{\mu_1, \mu_2, \mu_3} - \partial^{\mu_3} \mathcal{Z}^{\mu_1, \mu_2} \approx 0, \quad (4.11)$$

\vdots

$$\mathcal{Z}^{\mu_1, \dots, \mu_n} = \partial^{\mu_n} \mathcal{Z}^{\mu_1, \dots, \mu_{n-1}}, \quad \chi^{\mu_1, \dots, \mu_n} = \mathcal{Z}^{\mu_1, \dots, \mu_n} - \partial^{\mu_n} \mathcal{Z}^{\mu_1, \dots, \mu_{n-1}} \approx 0. \quad (4.12)$$

Therefore, the new first-order singular Lagrangian density takes the form

$$\begin{aligned} \mathcal{L}_{red} = \frac{1}{2} (\partial^\nu \varphi g_{\mu\nu}) (\partial^\mu \varphi) - \frac{1}{2} \mathcal{M}^2 \varphi^2 + \varepsilon \mathcal{L}_{oI}(\varphi, \mathcal{Z}^{\mu_1}, \mathcal{Z}^{\mu_1, \mu_2}, \dots, \partial^{\mu_n} \mathcal{Z}^{\mu_1, \dots, \mu_n}) \\ + \varepsilon \alpha^{\mu_1} (\mathcal{Z}^{\mu_1} - \partial^{\mu_1} \varphi) + \dots + \varepsilon \alpha^{\mu_1, \dots, \mu_n} (\mathcal{Z}^{\mu_1, \dots, \mu_n} - \partial^{\mu_n} \mathcal{Z}^{\mu_1, \dots, \mu_{n-1}}), \end{aligned} \quad (4.13)$$

The canonical momenta (1.3) are

$$\pi^{\mu_1} = \frac{\partial \mathcal{L}_{red}}{\partial (\partial^{\mu_1} \varphi)} = \partial^{\mu_1} \varphi - \varepsilon \alpha^{\mu_1}, \quad (4.14)$$

$$\pi^{\mu_1, \mu_2} = \frac{\partial \mathcal{L}_{red}}{\partial (\partial^{\mu_2} \mathcal{Z}^{\mu_1})} = -\varepsilon \alpha^{\mu_1, \mu_2}, \quad (4.15)$$

$$\pi^{\mu_1, \mu_2, \mu_3} = \frac{\partial \mathcal{L}_{red}}{\partial (\partial^{\mu_3} \mathcal{Z}^{\mu_1, \mu_2})} = -\varepsilon \alpha^{\mu_1, \mu_2, \mu_3}, \quad (4.16)$$

⋮

$$\pi^{\mu_1, \dots, \mu_n} = \frac{\partial \mathcal{L}_{red}}{\partial (\partial^{\mu_n} \mathcal{Z}^{\mu_1, \dots, \mu_{n-1}})} = -\varepsilon \alpha^{\mu_1, \dots, \mu_n}, \quad (4.17)$$

$$\pi^{\mu_1, \dots, \mu_{n+1}} = \frac{\partial \mathcal{L}_{red}}{\partial (\partial^{\mu_n} \alpha^{\mu_1, \dots, \mu_n})} = 0. \quad (4.18)$$

Thus, the canonical Hamiltonian density is

$$\begin{aligned} \mathcal{H} = & -\frac{1}{2}(\partial^{\mu_1} \varphi)(\partial_{\mu_1} \varphi) + \frac{1}{2} \mathcal{M}^2 \varphi^2 - \varepsilon \mathcal{L}_{oI}(\varphi, \mathcal{Z}^{\mu_1}, \mathcal{Z}^{\mu_1, \mu_2}, \dots, \partial^{\mu_n} \mathcal{Z}^{\mu_1, \dots, \mu_n}) \\ & + \mathcal{Z}^{\mu_1} \pi^{\mu_1} + \mathcal{Z}^{\mu_1, \mu_2} \pi^{\mu_1, \mu_2} + \dots + \mathcal{Z}^{\mu_1, \dots, \mu_n} \pi^{\mu_1, \dots, \mu_n}. \end{aligned} \quad (4.19)$$

The set of Hamilton–Jacobi Partial Differential Equations (**HJPE**) (1.16) is

$$\mathcal{H}^\alpha = \pi^\alpha + \mathcal{H}^\alpha, \quad \pi^\alpha = \frac{\partial S}{\partial x_\alpha} = -\mathcal{H}^\alpha, \quad (4.20)$$

$$\mathcal{H}^{\mu_1} = \pi^{\mu_1} + \varepsilon \alpha^{\mu_1} - \partial^{\mu_1} \varphi = 0, \quad \pi^{\mu_1} = \frac{\partial S}{\partial \phi}, \quad (4.21)$$

$$\mathcal{H}^{\mu_1, \mu_2} = \pi^{\mu_1, \mu_2} + \varepsilon \alpha^{\mu_1, \mu_2} = 0, \quad \pi^{\mu_1, \mu_2} = \frac{\partial S}{\partial \mathcal{Z}^{\mu_1}}, \quad (4.22)$$

⋮

$$\mathcal{H}^{\mu_1, \dots, \mu_n} = \pi^{\mu_1, \dots, \mu_n} + \varepsilon \alpha^{\mu_1, \dots, \mu_n} = 0, \quad \pi^{\mu_1, \dots, \mu_n} = \frac{\partial S}{\partial \mathcal{Z}^{\mu_1, \dots, \mu_{n-1}}}, \quad (4.23)$$

$$\mathcal{H}^{\mu_1, \dots, \mu_{n+1}} = \pi^{\mu_1, \dots, \mu_{n+1}} = 0, \quad \pi^{\mu_1, \dots, \mu_{n+1}} = \frac{\partial S}{\partial \alpha^{\mu_1, \dots, \mu_n}}. \quad (4.24)$$

Therefore, the equations of motion (1.19,1.20) are

$$d\varphi = \mathcal{Z}^{\mu_1} dx^{\mu_1}, \quad (4.25)$$

$$d\mathcal{Z}^{\mu_1} = \mathcal{Z}^{\mu_1, \mu_2} dx^{\mu_2}, \quad (4.26)$$

$$d\mathcal{Z}^{\mu_1, \mu_2} = \mathcal{Z}^{\mu_1, \mu_2, \mu_3} dx^{\mu_3}, \quad (4.27)$$

⋮

$$d\mathcal{Z}^{\mu_1, \dots, \mu_{n-1}} = \mathcal{Z}^{\mu_1, \dots, \mu_n} dx^{\mu_n}, \quad (4.28)$$

$$d\mathcal{Z}^{\mu_1, \dots, \mu_n} = (\partial^{\mu_n} \mathcal{Z}^{\mu_1, \dots, \mu_n}) dx^{\mu_n}, \quad (4.29)$$

$$d\pi^{\mu_1} = \left[\varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial \varphi} - \mathcal{M}^2 \varphi \right] dx^{\mu_1}, \quad (4.30)$$

$$d\pi^{\mu_1, \mu_2} = \left[(\partial_{\mu_1} \varphi) + \varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial \mathcal{Z}^{\mu_1}} - \pi^{\mu_1} \right] dx^{\mu_2}, \quad (4.31)$$

⋮

$$d\pi^{\mu_1, \dots, \mu_{n-1}} = \left[\varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial \mathcal{Z}^{\mu_1, \dots, \mu_{n-2}}} - \pi^{\mu_1, \dots, \mu_{n-2}} \right] dx^{\mu_{n-1}}, \quad (4.32)$$

$$d\pi^{\mu_1, \dots, \mu_n} = \left[\varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial \mathcal{Z}^{\mu_1, \dots, \mu_{n-1}}} - \pi^{\mu_1, \dots, \mu_{n-1}} \right] dx^{\mu_n}, \quad (4.33)$$

$$d\pi^{\mu_1, \dots, \mu_{n+1}} = 0. \quad (4.34)$$

These equations are integrable if and only if the variations of (4.25 - 4.34) vanish, that is

$$d\mathcal{H}'^\alpha = d\pi^\alpha + d\mathcal{H}^\alpha = 0, \quad (4.35)$$

$$d\mathcal{H}'^{\mu_1} = d\mathcal{H}^{\mu_1} + \varepsilon d\alpha^{\mu_1} - d(\partial^{\mu_1} \varphi) = 0, \quad (4.36)$$

$$d\mathcal{H}'^{\mu_1, \mu_2} = d\pi^{\mu_1, \mu_2} + \varepsilon d\alpha^{\mu_1, \mu_2} = 0, \quad (4.37)$$

⋮

$$d\mathcal{H}'^{\mu_1, \dots, \mu_n} = d\pi^{\mu_1, \dots, \mu_n} + \varepsilon d\alpha^{\mu_1, \dots, \mu_n} = 0, \quad (4.38)$$

$$d\mathcal{H}'^{\mu_1, \dots, \mu_{n+1}} = d\pi^{\mu_1, \dots, \mu_{n+1}} = 0. \quad (4.39)$$

The equation (4.35) vanishes identically, but the equations (4.36 - 4.39) are vanishing under the following conditions:

$$d\alpha^{\mu_1} = -\frac{1}{\varepsilon} \left[\left(\varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial \varphi} - \mathcal{M}^2 \varphi \right) dx^{\mu_1} - \mathcal{Z}^{\mu_1, \mu_2} dx^{\mu_2} \right], \quad (4.40)$$

$$d\alpha^{\mu_1, \mu_2} = \frac{1}{\varepsilon} \left[\pi^{\mu_1} - (\partial_{\mu_1} \varphi) - \varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial \mathcal{Z}^{\mu_1}} \right] dx^{\mu_2}, \quad (4.41)$$

⋮

$$d\alpha^{\mu_1, \dots, \mu_n} = \left[\frac{1}{\varepsilon} \pi^{\mu_1, \dots, \mu_{n-1}} - \frac{\partial \mathcal{L}_{oI}}{\partial \mathcal{Z}^{\mu_1, \dots, \mu_{n-1}}} \right] dx^{\mu_n}, \quad (4.42)$$

$$\pi^{\mu_1, \dots, \mu_n} = \varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial \mathcal{Z}^{\mu_1, \dots, \mu_n}}. \quad (4.43)$$

One can solve (4.25 - 4.34) simultaneously to obtain

$$0 = \mathcal{M}^2 \varphi + \partial^{\mu_1} \partial_{\mu_1} \varphi - \varepsilon \left[\frac{\partial \mathcal{L}_I}{\partial \varphi} - \partial^{\mu_1} \left(\frac{\partial \mathcal{L}_I}{\partial (\partial^{\mu_1} \varphi)} \right) + \dots + (-1)^n \partial^{\mu_1} \dots \partial^{\mu_n} \left(\frac{\partial \mathcal{L}_I}{\partial (\partial^{\mu_1} \dots \partial^{\mu_n} \varphi)} \right) \right]. \quad (4.44)$$

which is the same as equation (4.8), and the results obtained in ref.[29].

4.3 The canonical path integral quantization

To obtain the canonical path integral quantization for the continuous Lagrangian density (4.12), we consider the action $z \equiv S(x^\mu, \varphi, Z^{\nu_1}, Z^{\nu_1, \nu_2}, \dots, Z^{\nu_1, \dots, \nu_{n-1}})$, one can determine

$$dz = \left[-\mathcal{H}^\mu + \pi_{\nu_1} \frac{\partial \mathcal{H}^\mu}{\partial \pi^{\nu_1}} + \pi^{\nu_1, \nu_2} \frac{\partial \mathcal{H}^\mu}{\partial \pi^{\nu_1, \nu_2}} + \dots + \pi^{\nu_1, \dots, \nu_{n-1}} \frac{\partial \mathcal{H}^\mu}{\partial \pi^{\nu_1, \dots, \nu_{n-1}}} \right] dx^\mu, \quad (4.45)$$

this equation rewritten as

$$dz = [-\mathcal{H}^\mu + \pi^{\nu_1} \mathcal{Z}^{\nu_1} + \pi^{\nu_1, \nu_2} \mathcal{Z}^{\nu_1, \nu_2} + \dots + \pi^{\nu_1, \dots, \nu_{n-1}} \mathcal{Z}^{\nu_1, \dots, \nu_{n-1}}] dx^\mu, \quad (4.46)$$

If the equations (4.24 - 4.33, 4.46) are integrable, then the correspondence path integral quantization is determined as

$$\langle out|S|in \rangle = \int \prod_{\lambda, \epsilon=1}^N d\mathcal{Z}^\epsilon d\pi^\lambda \times \exp i \int [-\mathcal{H}^\mu + \pi^{\nu_1} \mathcal{Z}^{\nu_1} + \pi^{\nu_1, \nu_2} \mathcal{Z}^{\nu_1, \nu_2} + \dots + \pi^{\nu_1, \dots, \nu_{n-1}} \mathcal{Z}^{\nu_1, \dots, \nu_{n-1}}] dx^\mu, \quad (4.47)$$

which is an integration over the canonical phase space coordinates $(\mathcal{Z}^\epsilon, \pi^\lambda)$.

Chapter 5

THE EFFECTIVE LAGRANGIAN OF YANG-MILLS (MASSIVE VECTORS) FIELDS

As we explained in the previous chapter, the effective Lagrangian consists of two parts, which are: the Lagrangian of free fields and the interaction term with higher-order derivatives associated with the coupling constant between fields. The effective Lagrangian of Yang-Mills fields is another application on continuous systems, which is the subject here.

In the following, we shall quantize the effective higher-order Lagrangian density of Yang-Mills fields, using Hamilton-Jacobi approach, after reducing it to first-order singular Lagrangian density [32].

In section (5.1), we offer an introduction to the effective Lagrangian density of Yang-Mills fields. The Hamilton-Jacobi method will be used to investigate the system in section (5.2). The canonical path integral quantization is determined in section (5.3).

5.1 Introduction

Yang and Mills theories is one of the most important subjects in high energy physics, this type of theories became popular only in the last years.

In 1954, Yang and Mills published on the isotopic SU(2) invariance of the proton-neutron system. At the time the idea was not fully recognized since it had some unsatisfying properties. In the late 1960s these problems were solved when the full quantized field theory was developed and today quantum Yang-Mills theory is one of the cornerstones of theoretical physics. However, the mathematical foundations remain unclear. For this reason, the Clay Mathematics institute formulated the Millennium Prize problem. A solution of this problem would require a rigorous mathematical formulation of quantum Yang-Mills theory.

A generalized field tensor $\mathbf{F}_{\mu\nu}$ defined by

$$\mathbf{F}_{\mu\nu} = F_{\mu\nu}^j \tau_j = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \quad (5.1)$$

$$F_{\mu\nu}^j = \partial_\mu A_\nu^j - \partial_\nu A_\mu^j - gf_{ikl} A_\mu^k A_\nu^l, \quad (5.2)$$

plays a role in electromagnetic theory.

The Lagrangian density is invariant under a local non-Abelian gauge transformation provided it is of the form

$$\mathcal{L} = -\frac{1}{4} \mathbf{F}_{\mu\nu} \cdot \mathbf{F}^{\mu\nu} + \mathcal{L}_{matter} + \mathcal{L}_{int}(\Psi_j, D_\mu \Psi^j), \quad (5.3)$$

where the factor $-\frac{1}{4}$ is conventional and

$$\mathbf{F}_{\mu\nu} = (F_{\mu\nu}^1, F_{\mu\nu}^2, \dots, F_{\mu\nu}^N), \quad (5.4)$$

$$\mathbf{F}_{\mu\nu} \cdot \mathbf{F}^{\mu\nu} = F_{\mu\nu}^j F_j^{\mu\nu}. \quad (5.5)$$

Although, $\mathbf{F}_{\mu\nu}$ is not gauge invariant, $\mathbf{F}_{\mu\nu} \cdot \mathbf{F}^{\mu\nu}$ is gauge invariant. The term \mathcal{L}_{matter} is the gauge invariant Lagrangian density of the free matter field Ψ , while the coupling \mathcal{L}_{int} between the gauge and the matter field is a function of the fields and their

covariant derivatives [33]. In our model, we dealt with the effective Lagrangian that contains the interaction Lagrangian density with higher derivatives of the field. That form is [29]

$$\mathcal{L}_{eff} = \mathcal{L}_o + \varepsilon \mathcal{L}_I = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + \frac{1}{2} M^2 A_a^\mu A_\mu^a + \varepsilon \mathcal{L}_I(A_a^\mu, \partial^\nu A_a^\mu, \dots, \partial^{\nu_1} \dots \partial^{\nu_n} A_a^\mu), \quad (5.6)$$

where \mathcal{L}_o represents a massive Yang–Mills theory, A_μ is the covariant vector field (potential), and the effective interaction term \mathcal{L}_I contains the deviation from the Yang–Mills interactions which involves derivatives up to the order N and which are proportional to ε with $\varepsilon \ll 1$. The generalized field strength tensor $F_a^{\mu\nu}$ defined by

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f_{abc} A_b^\mu A_c^\nu, \quad (5.7)$$

where g represents to interaction coupling.

The general form of Euler–Lagrange equation of motion with n order is [5]

$$\frac{\partial \mathcal{L}}{\partial A_d^\epsilon} - \partial^\lambda \left(\frac{\partial \mathcal{L}}{\partial (\partial^\lambda A_d^\epsilon)} \right) + \dots + (-1)^n \partial^{\lambda_1} \dots \partial^{\lambda_n} \left(\frac{\partial \mathcal{L}}{\partial (\partial^{\lambda_1} \dots \partial^{\lambda_n} A_d^\epsilon)} \right) \quad (5.8)$$

The Euler-Lagrange equation of motion of the effective Lagrangian (5.6) is derived as

$$0 = D^\lambda F_{\epsilon\lambda}^d + M^2 A_\epsilon^d + \varepsilon \left[\frac{\partial \mathcal{L}_I}{\partial A_d^\epsilon} - \partial^\lambda \left(\frac{\partial \mathcal{L}_I}{\partial (\partial^\lambda A_d^\epsilon)} \right) + \dots + (-1)^n \partial^{\lambda_1} \dots \partial^{\lambda_n} \left(\frac{\partial \mathcal{L}_I}{\partial (\partial^{\lambda_1} \dots \partial^{\lambda_n} A_d^\epsilon)} \right) \right], \quad (5.9)$$

where $F_{\epsilon\lambda}^d = \partial_\epsilon A_\lambda^d - \partial_\lambda A_\epsilon^d + g f^{def} A_\epsilon^e A_\lambda^f$.

We will quantize the higher–order effective interactions of massive vector fields, using the Hamilton–Jacobi method.

5.2 Hamilton–Jacobi Method

The higher–order effective Lagrangian density (5.6) can be reduced to first order singular one.

First, rewrite eq. (5.6) as

$$\begin{aligned} \mathcal{L}_{eff} = & -\frac{1}{4}(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu + gf_{abc}A_b^\mu A_c^\nu)g_{\mu\lambda}g_{\nu\epsilon}g^{ad}g^{be}g^{cf}(\partial^\lambda A_d^\epsilon - \partial^\epsilon A_d^\lambda \\ & + gf_{def}A_e^\lambda A_f^\epsilon) + \frac{1}{2}M^2 A_a^\mu g_{\lambda\mu}g^{ad}A_d^\lambda + \varepsilon\mathcal{L}_I(A_a^\mu, \partial^\nu A_a^\mu, \dots, \partial^{\nu_1} \dots \partial^{\nu_n} A_a^\mu), \end{aligned} \quad (5.10)$$

To convert the effective Lagrangian density (5.10) to first order one, let us introduce the following substitution [32]

$$B_a^{\mu,\nu_1} = \partial^\nu A_a^\mu, \quad \chi_a^{\mu,\nu_1} = (B_a^{\mu,\nu_1} - \partial^\nu A_a^\mu) \approx 0, \quad (5.11)$$

$$B_a^{\mu,\nu_1,\nu_2} = \partial^{\nu_1} \partial^{\nu_2} A_a^\mu = \partial^{\nu_2} B_a^{\mu,\nu_1}, \quad \chi_a^{\mu,\nu_1,\nu_2} = (B_a^{\mu,\nu_1,\nu_2} - \partial^{\nu_2} B_a^{\mu,\nu_1}) \approx 0, \quad (5.12)$$

⋮

$$B_a^{\mu,\nu_1,\dots,\nu_{n-1}} = \partial^{\nu_{n-1}} B_a^{\mu,\nu_1,\dots,\nu_{n-2}}, \quad \chi_a^{\mu,\nu_1,\dots,\nu_{n-1}} = (B_a^{\mu,\nu_1,\dots,\nu_{n-1}} - \partial^{\nu_{n-1}} B_a^{\mu,\nu_1,\dots,\nu_{n-2}}) \approx 0, \quad (5.13)$$

$$B_a^{\mu,\nu_1,\dots,\nu_n} = \partial^{\nu_n} B_a^{\mu,\nu_1,\dots,\nu_{n-1}}, \quad \chi_a^{\mu,\nu_1,\dots,\nu_n} = (B_a^{\mu,\nu_1,\dots,\nu_n} - \partial^{\nu_n} B_a^{\mu,\nu_1,\dots,\nu_{n-1}}) \approx 0. \quad (5.14)$$

Therefore, the reduced form of the effective Lagrangian density can be written as

$$\begin{aligned} \mathcal{L}_{red} = & -\frac{1}{4}(B^{\nu_1,\mu_1} - B^{\mu_1,\nu_1} + gf_{abc}A_b^\mu A_c^\nu)g_{\lambda\mu}g_{\nu\epsilon}g^{ad}g^{be}g^{cf}(B_d^{\epsilon,\lambda} - B_d^{\lambda,\epsilon} + \\ & gf_{def}A_e^\lambda A_f^\epsilon) + \frac{1}{2}M^2 A_a^\mu g_{\lambda\mu}g^{ad}A_d^\lambda + \varepsilon[\mathcal{L}_{oI}(A_a^\mu, B_a^{\mu,\nu_1}, \dots, B_a^{\mu,\nu_1,\dots,\nu_{n-1}}, \\ & \partial^{\nu_n} B_a^{\mu,\nu_1,\dots,\nu_{n-1}}) + C_a^{\mu,\nu_1}(B_a^{\mu,\nu_1} - \partial^\nu A_a^\mu) + C_a^{\mu,\nu_1,\nu_2}(B_a^{\mu,\nu_1,\nu_2} - \partial^{\nu_2} B_a^{\mu,\nu_1}) \\ & + \dots + C_a^{\mu,\nu_1,\dots,\nu_n}(B_a^{\mu,\nu_1,\dots,\nu_n} - \partial^{\nu_n} B_a^{\mu,\nu_1,\dots,\nu_{n-1}})], \end{aligned} \quad (5.15)$$

where $C_a^{\mu,\nu_1,\dots,\nu_m}$ are Lagrange multipliers.

The corresponding canonical momenta are

$$\pi_a^{\mu,\nu_1} = \frac{\partial \mathcal{L}_{red}}{\partial (\partial^\nu A_a^\mu)} = -\varepsilon C_a^{\mu,\nu_1}, \quad (5.16)$$

$$\pi_a^{\mu,\nu_1,\nu_2} = \frac{\partial \mathcal{L}_{red}}{\partial(\partial^{\nu_2} B_a^{\mu,\nu_1})} = -\varepsilon C_a^{\mu,\nu_1,\nu_2}, \quad (5.17)$$

$$\pi_a^{\mu,\nu_1,\nu_2,\nu_3} = \frac{\partial \mathcal{L}_{red}}{\partial(\partial^{\nu_3} B_a^{\mu,\nu_1,\nu_2})} = -\varepsilon C_a^{\mu,\nu_1,\nu_2,\nu_3}, \quad (5.18)$$

⋮

$$\pi_a^{\mu,\nu_1,\dots,\nu_n} = \frac{\partial \mathcal{L}_{red}}{\partial(\partial^{\nu_n} B_a^{\mu,\nu_1,\dots,\nu_{n-1}})} = -\varepsilon C_a^{\mu,\nu_1,\dots,\nu_n}, \quad (5.19)$$

$$\pi_a^{\mu,\nu_1,\dots,\nu_{n+1}} = \frac{\partial \mathcal{L}_{red}}{\partial(\partial^{\nu_n} C_a^{\mu,\nu_1,\dots,\nu_n})} = 0. \quad (5.20)$$

Using the canonical momenta, one can obtain the canonical Hamiltonian density

$$\begin{aligned} \mathcal{H}^\alpha &= \frac{1}{4}(B_a^{\nu_1,\mu_1} - B_a^{\mu_1,\nu_1} + g f_{abc} A_b^\mu A_c^\nu) g_{\lambda\mu} g_{\nu\epsilon} g^{ad} g^{be} g^{cf} (B_d^{\epsilon,\lambda} - B_d^{\lambda,\epsilon} + g f_{def} A_e^\lambda A_f^\epsilon) \\ &\quad - \frac{1}{2} M^2 A_a^\mu g_{\lambda\mu} g^{ad} A_d^\lambda - \varepsilon \mathcal{L}_{oI}(A_a^\mu, B_a^{\mu,\nu_1}, \dots, B_a^{\mu,\nu_1,\dots,\nu_n}, \partial^{\nu_n} B_a^{\mu,\nu_1,\dots,\nu_n}) + B_a^{\mu,\nu_1} \pi_a^{\mu,\nu_1} \\ &\quad + B_a^{\mu,\nu_1,\nu_2} \pi_a^{\mu,\nu_1,\nu_2} + \dots + B_a^{\mu,\nu_1,\dots,\nu_n} \pi_a^{\mu,\nu_1,\dots,\nu_n}. \end{aligned} \quad (5.21)$$

The set of Hamilton Jacobi Partial Differential Equations (HJPDE) reads

$$\mathcal{H}'^\alpha = \mathcal{H}^\alpha + \pi^\alpha \approx 0, \quad \pi^\alpha = \frac{\partial S}{\partial x^\alpha}, \quad (5.22)$$

$$\mathcal{H}'_e^{\lambda,\epsilon_1} = \pi_e^{\lambda,\epsilon_1} + \varepsilon C_e^{\lambda,\epsilon_1} \approx 0, \quad \pi_e^{\lambda,\epsilon_1} = \frac{\partial S}{\partial A_e^\lambda}, \quad (5.23)$$

$$\mathcal{H}'_e^{\lambda,\epsilon_1,\epsilon_2} = \pi_e^{\lambda,\epsilon_1,\epsilon_2} + \varepsilon C_e^{\lambda,\epsilon_1,\epsilon_2} \approx 0, \quad \pi_e^{\lambda,\epsilon_1,\epsilon_2} = \frac{\partial S}{\partial B_e^{\lambda,\epsilon_1}}, \quad (5.24)$$

⋮

$$\mathcal{H}'_e^{\lambda,\epsilon_1,\dots,\epsilon_n} = \pi_e^{\lambda,\epsilon_1,\dots,\epsilon_n} + \varepsilon C_e^{\lambda,\epsilon_1,\dots,\epsilon_n} \approx 0, \quad \pi_e^{\lambda,\epsilon_1,\dots,\epsilon_n} = \frac{\partial S}{\partial B_e^{\lambda,\epsilon_1,\dots,\epsilon_{n-1}}}, \quad (5.25)$$

$$\mathcal{H}'_e^{\lambda,\epsilon_1,\dots,\epsilon_{n+1}} = \pi_e^{\lambda,\epsilon_1,\dots,\epsilon_{n+1}} \approx 0, \quad \pi_e^{\lambda,\epsilon_1,\dots,\epsilon_{n+1}} = \frac{\partial S}{\partial C_e^{\lambda,\epsilon_1,\dots,\epsilon_n}}. \quad (5.26)$$

The set of total differential equations of motion is then

$$dA_d^{\epsilon_1} = \frac{\partial(\pi^\alpha + \mathcal{H}^\alpha)}{\partial \pi_d^{\epsilon,\lambda_1}} dx^\alpha + \frac{\partial(\pi_e^{\lambda,\lambda_1} + \varepsilon C_e^{\lambda,\epsilon_1})}{\partial \pi_d^{\epsilon,\lambda_1}} dA_e^{\lambda_1} + \frac{\partial(\pi_e^{\lambda,\epsilon_1,\epsilon_2} + \varepsilon C_e^{\lambda,\epsilon_1,\epsilon_2})}{\partial \pi_d^{\epsilon,\lambda_1}} dB_e^{\lambda,\epsilon_1} +$$

$$\begin{aligned}
& - \frac{\partial(\pi_e^{\lambda, \epsilon_1, \epsilon_2} + \varepsilon C_e^{\lambda, \epsilon_1, \epsilon_2})}{\partial(\partial^{\lambda_n} B_d^{\epsilon, \lambda_1, \dots, \lambda_{n-1}})} dB_e^{\lambda, \epsilon_1} - \frac{\partial(\pi_e^{\lambda, \epsilon_1, \epsilon_2, \epsilon_3} + \varepsilon C_e^{\lambda, \epsilon_1, \epsilon_2, \epsilon_3})}{\partial(\partial^{\lambda_n} B_d^{\epsilon, \lambda_1, \dots, \lambda_{n-1}})} dB_e^{\lambda, \epsilon_1, \epsilon_2} - \dots - \\
& \frac{\partial(\pi_e^{\lambda, \epsilon_1, \dots, \epsilon_n} + \varepsilon C_e^{\lambda, \epsilon_1, \dots, \epsilon_n})}{\partial(\partial^{\lambda_n} B_d^{\epsilon, \lambda_1, \dots, \lambda_{n-1}})} dB_e^{\lambda, \epsilon_1, \dots, \epsilon_n}, \tag{5.33}
\end{aligned}$$

The total differential equations of motion (5.27 - 5.2) are reduced to

$$dA_d^\epsilon = B_d^{\epsilon, \lambda_1} dx^{\lambda_1}, \tag{5.34}$$

$$dB_d^{\epsilon, \lambda_1} = B_d^{\epsilon, \lambda_1, \lambda_2} dx^{\lambda_2}, \tag{5.35}$$

$$dB_d^{\epsilon, \lambda_1, \lambda_2} = B_d^{\epsilon, \lambda_1, \lambda_2, \lambda_3} dx^{\lambda_3}, \tag{5.36}$$

⋮

$$dB_d^{\epsilon, \lambda_1, \dots, \lambda_{n-1}} = B_d^{\epsilon, \lambda_1, \dots, \lambda_{n-1}, \lambda_n} dx^{\lambda_n}, \tag{5.37}$$

$$d\pi_d^{\epsilon, \lambda_1} = \left(M^2 A_{\epsilon_1}^d + \varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial A_d^{\epsilon_1}} \right) dx^{\lambda_1}, \tag{5.38}$$

$$d\pi_d^{\epsilon, \lambda_1, \lambda_2} = \left(-F_{\epsilon \lambda}^d + \varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial B_d^{\epsilon, \lambda_1}} - \pi_d^{\epsilon, \lambda_1} \right) dx^{\lambda_2}, \tag{5.39}$$

⋮

$$d\pi_d^{\epsilon, \lambda_1, \dots, \lambda_n} = \left[\varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial B_d^{\epsilon, \lambda_1, \dots, \lambda_{n-1}}} - \pi_d^{\epsilon, \lambda_1, \dots, \lambda_{n-1}} \right] dx^{\lambda_n}, \tag{5.40}$$

$$d\pi_d^{\epsilon, \lambda_1, \dots, \lambda_{n+1}} = \varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial(\partial^{\lambda_n} B_d^{\epsilon, \lambda_1, \dots, \lambda_n})} dx^{\lambda_{n+1}}. \tag{5.41}$$

The equations (5.34 - 5.41) are integrable if and only if the integrability conditions are identically satisfied. That is the variation of relations (5.22 - 5.26) vanish identically.

$$d \mathcal{H}^\alpha = d \mathcal{H}^\alpha + d\pi^\alpha \approx 0, \tag{5.42}$$

$$d \mathcal{H}_e^{\lambda, \epsilon} = d\pi_e^{\lambda, \epsilon} + \varepsilon d C_e^{\lambda, \epsilon} \approx 0, \tag{5.43}$$

$$d \mathcal{H}_e^{\lambda, \epsilon_1, \epsilon_2} = d\pi_e^{\lambda, \epsilon_1, \epsilon_2} + \varepsilon d C_e^{\lambda, \epsilon_1, \epsilon_2} \approx 0, \tag{5.44}$$

⋮

$$d \mathcal{H}_e^{\lambda, \epsilon_1, \dots, \epsilon_n} = d\pi_e^{\lambda, \epsilon_1, \dots, \epsilon_n} + \varepsilon d C_e^{\lambda, \epsilon_1, \dots, \lambda_n} \approx 0, \quad (5.45)$$

$$d \mathcal{H}_e^{\lambda, \epsilon_1, \dots, \epsilon_{n+1}} = d\pi_e^{\lambda, \epsilon_1, \dots, \epsilon_{n+1}} \approx 0. \quad (5.46)$$

Relation (5.42) vanishes identically, but relations (5.43 - 5.46) are satisfied under the following conditions:

$$dC_e^{\lambda, \epsilon_1} = -\frac{1}{\varepsilon} \left[M^2 A_{\lambda_1}^e + \varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial A_e^{\lambda_1}} \right] dx^{\epsilon_1}, \quad (5.47)$$

$$dC_e^{\lambda, \epsilon_1, \epsilon_2} = -\frac{1}{\varepsilon} \left[F_{\lambda \epsilon}^e + \varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial B_e^{\lambda, \epsilon_1}} - \pi_e^{\lambda, \epsilon_1} \right] dx^{\epsilon_2}, \quad (5.48)$$

⋮

$$dC_e^{\lambda, \epsilon_1, \dots, \epsilon_n} = -\frac{1}{\varepsilon} \left[\varepsilon \frac{\partial \mathcal{L}_{oI}}{\partial B_e^{\lambda, \epsilon_1, \dots, \epsilon_{n-1}}} - \pi_e^{\lambda, \epsilon_1, \dots, \epsilon_{n-1}} \right] dx^{\epsilon_n}, \quad (5.49)$$

$$d\pi_e^{\lambda, \epsilon_1, \dots, \epsilon_{n+1}} = 0. \quad (5.50)$$

Solving equations (5.34 - 5.41) simultaneously, one get

$$\begin{aligned} D^{\lambda_1} F_{\epsilon \lambda}^d + M^2 A_\epsilon^d + \varepsilon \left[\frac{\partial \mathcal{L}_{oI}}{\partial A_d^\epsilon} - \frac{d}{dx^{\lambda_1}} \left(\frac{\partial \mathcal{L}_{oI}}{\partial B_d^{\epsilon_1}} \right) + \frac{d}{dx^{\lambda_1}} \frac{d}{dx^{\lambda_2}} \left(\frac{\partial \mathcal{L}_{oI}}{\partial B_d^{\epsilon_1, \epsilon_2}} \right) + \dots + \right] \\ \varepsilon \left[(-1)^n \frac{d}{dx^{\lambda_1}} \dots \frac{d}{dx^{\lambda_n}} \left(\frac{\partial \mathcal{L}_{oI}}{\partial B_d^{\epsilon_1, \dots, \epsilon_n}} \right) \right] = 0, \end{aligned} \quad (5.51)$$

Equation (5.51) can be rewritten as

$$\begin{aligned} D^\lambda F_{\epsilon \lambda}^d + M^2 A_\epsilon^d + \varepsilon \left[\frac{\partial \mathcal{L}_I}{\partial A_d^\epsilon} - \partial^\lambda \left(\frac{\partial \mathcal{L}_I}{\partial (\partial^\lambda A_d^\epsilon)} \right) + \dots + \right] \\ \varepsilon \left[(-1)^n \partial^{\lambda_1} \dots \partial^{\lambda_n} \left(\frac{\partial \mathcal{L}_I}{\partial (\partial^{\lambda_1} \dots \partial^{\lambda_n} A_d^\epsilon)} \right) \right] = 0. \end{aligned} \quad (5.52)$$

which is the same as (5.9).

5.3 The canonical path integral quantization

To obtain the canonical path integral quantization for the continuous Lagrangian density (5.15), we let the action $z \equiv S(x^\mu, A_a^\mu, B_a^{\mu, \nu_1}, B_a^{\mu, \nu_1, \nu_2}, \dots, B_a^{\mu, \nu_1, \dots, \nu_{n-1}})$, one can determine

$$dz = \left[-\mathcal{H}^\mu + \pi_a^{\mu, \nu_1} \frac{\partial \mathcal{H}^\mu}{\partial \pi_a^{\mu, \nu_1}} + \pi_a^{\mu, \nu_1, \nu_2} \frac{\partial \mathcal{H}^\mu}{\partial \pi_a^{\mu, \nu_1, \nu_2}} + \dots + \pi_a^{\mu, \nu_1, \dots, \nu_{n-1}} \frac{\partial \mathcal{H}^\mu}{\partial \pi_a^{\mu, \nu_1, \dots, \nu_{n-1}}} \right] dx^\mu. \quad (5.53)$$

This equation rewritten as

$$dz = [-\mathcal{H}^\mu + \pi_a^{\mu, \nu_1} B_a^{\mu, \nu_1} + \pi_a^{\mu, \nu_1, \nu_2} B_a^{\mu, \nu_1, \nu_2} + \dots + \pi_a^{\mu, \nu_1, \dots, \nu_{n-1}} B_a^{\mu, \nu_1, \dots, \nu_{n-1}}] dx^\mu, \quad (5.54)$$

If the equations (5.34 - 5.41, 5.54) are integrable, then the correspondence path integral quantization is determined as

$$\begin{aligned} \langle out|S|in \rangle = & \int \prod_{\lambda, \epsilon=1}^N dB_a^{\lambda, \epsilon} d\pi_a^{\epsilon, \lambda} \times \exp i \int [-\mathcal{H}^\mu + \pi_a^{\mu, \nu_1} B_a^{\mu, \nu_1} + \pi_a^{\mu, \nu_1, \nu_2} B_a^{\mu, \nu_1, \nu_2} + \dots + \\ & \pi_a^{\mu, \nu_1, \dots, \nu_{n-1}} B_a^{\mu, \nu_1, \dots, \nu_{n-1}}] dx^\mu, \end{aligned} \quad (5.55)$$

which is an integration over the canonical phase space coordinates $(B_a^{\lambda, \epsilon}, \pi_a^{\epsilon, \lambda})$.

Chapter 6

CONCLUSION

The aim of this thesis, is to study the reduced form of higher-order singular Lagrangian systems, using the Hamiltonian formulation (Dirac's and Hamilton-Jacobi methods). Both Dirac's and Hamilton-Jacobi formulations were represented to study the systems of higher-order Lagrangian. The reduction form (first-order singular Lagrangian) of higher-order (regular) one, with order m , obtained by introducing an auxiliary fields, the equations of them are re-written as a set of m -constraints. In Dirac's method, one introduces a primary constraints to construct the total Hamiltonian, which is the primary constraints multiplied by Lagrange multiplier added to canonical (usual) Hamiltonian. The consistency conditions are checked on the primary constraints, where they are classified into two types: first- and second-class constraints. The first-class constraints have vanishing Poisson brackets, but the second-class constraints have non-vanishing Poisson brackets. The equations of motion are obtained as total derivatives in terms of Poisson brackets. Sometimes, there are some difficulties to identify the Lagrange multipliers, one can put a gauge fixing, which is not an easy task.

In Hamilton-Jacobi formulation, which developed by Güler, the equations of mo-

tion are written as total differential equations in many variables. These equations must satisfy the integrability conditions. If the integrability conditions are not identically satisfied, then these will be continued until we obtain a complete system. The singular systems have two types of integrable systems, completely integrable systems were the integrability conditions are identically satisfied, and partially integrable systems were the integrability conditions are not satisfied [25].

The Hamilton–Jacobi path integral quantization of constrained systems was developed by Muslih [16, 17, 19]. This formulation leads to obtain a set of HJPDE’s in many variables. these equations are integrable if the corresponding system of partial differential equations is a Jacobi system. In this case, the path integral is obtained as an integration over the canonical phase space coordinates.

The advantages of using Hamilton-Jacobi method is that we have no difference between first- and second-class constraints and we don’t need gauge fixing term, because the gauge variables are separated in the process of constructing an integrable system of total differential equations. Besides the integrable action function provided by the formalism can be used in the process of the path integral quantization method of the constrained systems.

Appendix

Appendix A

The metric tensor

The vector x^k in space can be transformed as $x^k = x^k(q^1, \dots, q^n)$, where x^k represents to old frame, and q^i is the new one. Any differential length element of space time can be represented as

$$(ds)^2 = g_{ij}dq^i dq^j, \quad (\text{A.1})$$

where g_{ij} is the metric tensor for the space, and defined as [33]

$$g_{ij} = \frac{\partial x^k}{\partial q^i} \frac{\partial x^k}{\partial q^j}. \quad (\text{A.2})$$

These coefficients are functions of variables q^n , and they are symmetric with respect to the indices i and j .

This tensor called the metric tensor because all essential metric properties of Euclidean space are completely determined by this tensor.

The metric tensor of special relativity is diagonal:

$$g_{ij} = g^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{A.3})$$

A.1 Some properties of the metric tensor

1. In flat space-time the covariant and the contravariant metric tensors are identical

$$g_{ij} = g^{ij} \quad (\text{A.4})$$

2. The contraction of the covariant and the contravariant metric tensors on one index yields the four dimensional Kronecker delta,

$$g_{ik}g^{kj} = \delta_i^j, \quad (\text{A.5})$$

where $\delta_i^j = 1$ for $i, j = 0, 1, 2, 3$ and $\delta_i^j = 0$ if $i \neq j$

3. The metric tensors convert the covariant tensor to contravariant one, or vice versa. For example: to transform between covariant and contravariant vector components

$$x_i = g_{ij}x^j \quad (\text{A.6})$$

$$x^i = g^{ij}x_j \quad (\text{A.7})$$

For a general tensor, index raising or lowering is accomplished by a contraction of the form

$$F_{..i..} = g^{ij}F_{..j..}, \quad F_{..i..} = g_{ij}F_{..j..} \quad (\text{A.8})$$

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