The Properties of L-moments Compared to Conventional Moments

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The Properties of L-moments Compared to Conventional Moments

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Dedication

To the spirit of my father...

To my mother

To my wife

To all knowledge seekers...

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Acknowledgments

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Abstract

In this thesis, we survey the concept of L-moments. We introduce the definition of L-moments and the probability weighted moments (PWMs) and then we expressed the L-moments by the use of the probability weighted moments. Also, we established the relation between the L-moments and the order statistic. Moreover, we introduced some of the properties the L-moments especially the property that, if the mean of the distribution exists, then all of the L-moments exist and uniquely define the distribution. That is, no two distinct distributions have the same L-moments. This property is not always valid in the conventional moments. Moreover, we find the L-moments for some distributions. Later, we introduce estimation for the L-moments and probability weighted moments and then we used them in estimating the parameters of some distributions as the Uniform distribution, the Exponential distribution, Generalized Logistic distribution and Generalized Pareto Distribution. Moreover, we introduce the generalized lambda distribution (GLD) and we find the (PWMs) and L-moments for (GLD). Also, we defined the Censored Data which is divided into two cases: I-Right censoring and II-Left censoring and then we find the partial property weighted moments (PPWMs) for both cases. Finally, we find the type B PPWMs for GLD.

Key words: Order Statistics, Probability Weighted Moments, L-moments, Censored Data, Generalized Lambda Distribution Family, Partial Probability Weighted Moments.

Introduction

It is standard statistical practice to summarize a probability distribution or an observed data set by its moments or cumulant. It is also common, when fitting a parametric distribution to a data set, to estimate the parameters by equating the sample moments to those of the fitted distribution. Yet moment-based methods, although long established in statistics, are not always satisfactory. It is sometimes difficult to assess exactly what information about the shape of a distribution is conveyed by its moments of third and higher order; the numerical values of sample moments, particularly when the sample is small, can be very different from those of the probability distribution from which the sample was drawn; and the estimated parameters of distributions fitted by the method of moments are often markedly less accurate than those obtainable by other estimation procedures such as the method of maximum likelihood.
The alternative approach described here is based on quantities which we call L-moments. These are analogous to the conventional moments but can be estimated by linear combinations of order statistics. L-moments have theoretical advantages over conventional moments of being able to characterize a wider range of distributions and, when estimated from a sample, of being more robust to the presence of outliers in the data. Experience also shows that, compared with conventional moments, L-moments are less subject to bias in estimation and approximate their asymptotic normal distribution more closely in finite samples. Parameter estimates obtained from L-moments are sometimes more accurate in small samples than even the maximum likelihood estimates[17].

The origins of our work can be traced to the early 1970, when there was a growing awareness among hydrologists that annual maximum streamflow data, although commonly modeled by the Gumbel distribution, often had higher skewness than was consistent with that distribution. Moment statistics were widely used as the basis for identifying and fitting frequency distributions, but to use them effectively required knowledge of their sampling properties in small samples. A massive (for the time) computational effort using simulated data was performed by Wallis, Matalas, and Slack in 1974. It revealed some unpleasant properties of moment statistics-high bias and algebraic boundedness. Wallis and others went on to establish the phenomenon of "separation of skewness," which is that for annual maximum streamflow data "the relationship between the mean and the standard deviation of regional estimates of skewness for historical flood sequences is not compatible with the relations derived from several well-known distribution" (Matalas, Slack, and Wallis in 1975). Separation can be explained by "mixed distribution" (Wallis, Matalas, and Slack in 1977)- regional heterogeneity in our present terminology- or if the frequency distribution of stremflow has a longer tail than those of the distribution commonly used in the 1970s. In particular, the Wakeby distribution dose not exhibit the phenomenon of separation (Landwehr, Matalas, and Wallis in 1978). The Wakaby distribution was devised by H.A Thomas Jr. (personal communication to J.R. Wallis, in 1976). It is hard to estimate by conventional methods such as maximum likelihood or the method of moments, and the desirability of obtaining closed-from estimates of Wakeby parameters led Greenwood et al. (1979) to devise probability weighted moments. Probability weighted moments were found to perform well for other distributions (Landwehr, Matalas, and Wallis in 1979; Hosking, Wallis, and Wood in 1985; Hosking and Wallis in 1987) but were hard to interpret. In 1990 Hosking found that certain linear combinations of probability weighted moments, which he called "L-moments," could be interpreted as measures of the location, scale, and shape of probability distribution and formed the basis for a comprehensive theory of the description, identification, and estimation of distributions ([15], Pages xi, xii).

This thesis consists of four chapters. In the first chapter, we introduce general concepts and definitions that are related to the L-moments. The definition of the cumulative distribution function, quantile function and the probability density function are very important in chapter two. The definition of the random sample is essential in the definition of the order statistic. The concept of the estimator is useful in chapter three in the estimation of L-moment. The concepts of the nth moment, rth central moment and moment generated functions are introduced to be used in comparing between the conventional moments and L-moments. The concept of order statistics is the base for defining the L-moments. In fact, the first chapter consists of seven sections: distribution functions and probability density or mass functions, random samples, estimators, moment and moment generating functions, skewness and kurtosis, the shifted Legendre polynomials and order statistics.

Charter 2, which is the main chapter in this research, consists of nine sections. In this chapter, we define L-moments and L-moments ratios in the first section. In the second section, we define probability weight moments and we find the relationship between L-moments and probability weight moments and it will make it easier to find L-moments for some distributions. In the third section, we find the relation between L-moments and order statistic. In the fourth section, we establish some properties of L-moments. After that, we talk about L-skewness and L-kurtosis (which are considered as a special case of the L-moments) in section 2.5. In the sixth section, we write about the L-moments of a polynomial function of a random variable. In the seventh section, we write about an inversion theorem, expressing the quantile function in terms of L-moments. In the eighth section, we write about L-moments as a measure of distribution shape. Finally, in the ninth section, we find L-moments for some distributions. This section is divided into four subsections: L-moments for uniform distribution, L-moments for exponential distribution, L-moments for logistic distribution and L-moments for generalized pareto distribution. This last section is used in chapter three in estimating the parameters of some of the previous distributions.

Chapter 3, which is titled by estimation of L-moments, consists of four sections: the rth sample L-moments (which is used in estimating the parameters of some distributions), the sample probability weighted moments (which is used in chapter four in finding PP-WMs estimators for Right and Left Censoring), the rth sample L-moment ratios, and finally the parameter estimation using L-moments.

In chapter 4, we deal with the "Estimation of the Generalized Lambda Distribution from Censored Data". In the first section, we find the PWMs and L-moments for GLD. In the second section, we discus the PWMs and L-moments for Censored Data (type B for Right Censoring and Left Censoring). In the third section, we find L-moments for Censored Distributions using GLD. In the last section, we discuss the fitting of the distributions to Censored Data using GLD. In fact, chapter 4 is considered as an application of the previous chapters.

Chapter 1

Preliminaries

In this chapter, we give the basic definitions, that we think they are very important for our thesis.

In the first section, we define the cumulative distribution functions, quantile functions and the probability density functions, and these definitions are needed in chapters 2, 3 and 4.

In the second section, we define the random sample and give related examples. The importance of section two will appear in section (1.7).

In the third section, we wrote about estimators and define bias estimators. This section is necessary in chapter three.

In section four, we define the n^{th} moment and the n^{th} central moment and find the n^{th} center moment for normal distribution. After that, we define skewness and kurtosis in section five. In the sixth section we define the shifted Legendre polynomials. Finally, we introduce the order statistic and its distributions in section seven.

1.1 Distribution Functions and Probability Density or Mass Functions

In this section we define the cumulative distribution functions, quantile functions and the probability density functions. These definitions are essential in defining the L-moments, the main definition in this research.

Definition 1.1.1. ([26], Page 112) Let X be random variable defined on a sample space S with probability function P. For any real number x, the *cumulative distribution function* of X [abbreviated (*cdf*) and written $F(x)$] is the probability associated with the set of sample points in S that get mapped by X into values on the real line less than or equal to x. Formally, $F(x) := P(\{s \in S \mid X(s) \leq x\})$.

We shall normally be concerned with continuous random variables, $F(x)$ is an increasing function of x, and $0 \leq F(x) \leq 1$ for all x, for which $P(X = t) = 0$ for all t. That is; no single value has nonzero probability. In this case, $F(x)$ is a continuous function and has an inverse function.

Definition 1.1.2. ([15], Page 14) If $F(x)$ is the cumulative distribution function of X, then the inverse function of $F(x)$ is called the *quantile function* of X and is denoted by $x(F)$.

Notice that, given any $u,\ 0 < u < 1,\ x(u)$ is the unique value that satisfies

$$F(x(u)) = u.$$

Definition 1.1.3. ([10], Page 35) The *probability density function* (*pdf*) of a continuous random variable X is the function f satisfying

$$F(x) := \int_{-\infty}^{x} f(t)dt \qquad \text{for all } x.$$

Remark 1.1.1. We deduce from the above two definitions the following:

1. If X is a *discrete random variable*, then $F(x) = \sum_{y \leq x} P(X = y) = \sum_{y \leq x} f(y)$, and in this case, $f(x)$ is said to be *probability mass function* (pmf) of X.

2. If X is a *continuous random variable*, and f is a continuous function, then by the Fundamental Theorem of Calculus, $f(x) = \frac{d}{dx}F(x)$.

Definition 1.1.4. ([26], Page 131) Two random variables X and Y are said to be independent, if and only if $f_{XY}(x,y) = f_X(x)f_Y(y)$, for all x and y where $f(x,y)$ is the joint (*pdf*) or (pmf) of X and Y, and $f(x)_X$, $f(y)_Y$ are the (*pdf*) of X and Y, respectively.

Definition 1.1.5. ([10], Page 174) Let $X_1, X_2, ..., X_n$ be random variables with the joint (*pdf*) or (pmf) $f(x_1, x_2, ..., x_n)$. Let $f_i(x)$ denote the marginal (*pdf*) or (pmf) of X_i Then $X_1, X_2, ..., X_n$ are called *mutually independent* random variables if for every $(x_1, x_2, ..., x_n)$ within their range

$$f(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} f_i(x_i).$$

Definition 1.1.6. ([26], Page 154) Let X be any random variable with the marginal (*pdf*) or (pmf) $f(x)$. The expected value denoted by $E(X)$ and is given by:

$$(1)\ E(X) = \int_{-\infty}^{\infty} xf(x)\, dx; \text{ if } X \text{ is a continuous random variable, provided that } \int_{-\infty}^{\infty} |x|f(x) < \infty\ . \tag{1.1.1}$$

We may also write, via the transformation $u = F(x)$,

$$E(X) = \int_0^1 x(u)du\ .$$

(2) $E(X) = \sum_x xf(x)$ if X is a discrete random variable, provided that $\sum_x |x|f(x) < \infty$.

Example 1.1.1. *Let X be a random variable from the exponential distribution with parameter β. Then the expectation of X is given by:*

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_0^{\infty} x\Big(\frac{1}{\beta}\Big)e^{\frac{-x}{\beta}}dx \\ &= \beta. \end{aligned}$$

1.2 Random Samples

In this section, we define the random sample which is used to define the order statistics in section 1.7. Then, we give related examples.

Definition 1.2.1. ([10], Page 201) The collection of random variables $X_1, X_2, ..., X_n$ is called a *random sample* of size n from the population with (*pdf*) $f(x)$ if $X_1, X_2, ..., X_n$ are mutually independent and marginal probability density function (*pdf*) or probability mass function (pmf) of each X_i is the sample function $f(x)$.

Alternatively, $X_1, X_2, ..., X_n$ are called independent and identically distributed random variables with (*pdf*) or (pmf) $f(x)$. This is commonly abbreviated to *iid* random variables. From the above definition of a *random sample*, the *joint* (*pdf*) *or* (pmf) of the random sample $X_1, X_2, ..., X_n$ is given by

$$f(x_1, x_2, ..., x_n) = f(x_1)f(x_2)...f(x_n) = \prod_{i=1}^{n} f(x_i).$$

Example 1.2.1. *Let $X_1, X_2, ..., X_n$ be a random sample of size n from the exponential distribution with parameter (β), corresponding to the time until failure for identical circuit that one puts on the test and used until they fail. Then the* joint (*pdf*) *of the sample is:*

$$\begin{aligned} f(x_1, x_2, ..., x_n|\beta) &= f(x_1)f(x_2)...f(x_n) \\ &= \prod_{i=1}^{n} f(x_i|\beta) \\ &= \prod_{i=1}^{n} (1/\beta)e^{-\frac{x_i}{\beta}} \\ &= (1/\beta)^n e^{-\frac{1}{\beta}\sum_{i=1}^{n} x_i}. \end{aligned}$$

Now, to compute the probability of the all boards last more than 2 time units, we do the following

$$\begin{aligned} P(X_1 > 2,\ X_2 > 2, ...,\ X_n > 2) &= \prod_{i=1}^{n} P(X_i > 2) \\ &= \prod_{i=1}^{n} \int_2^{\infty} \frac{1}{\beta} e^{-x_i/\beta} dx_i \\ &= \Big(\int_2^{\infty} \frac{1}{\beta} e^{-x/\beta} dx \Big)^n \\ &= (e^{-2/\beta})^n = e^{-2n/\beta}\ . \end{aligned}$$

1.3 Estimators

In practice, it is often assumed that the distribution of some physical quantities is exactly known apart from a finite set of parameters $\theta_1, ..., \theta_p$. When needed for clarity, we write the quantile function of a distribution with p unknown parameters as $x(u; \theta_1, ..., \theta_p)$. In most applications the unknown parameters include a location parameter and a scale parameter [15].

Definition 1.3.1. ([15], Page 15) A parameter ξ of a distribution is a *location parameter* if the quantile function of the distribution satisfies

$$x(u; \xi, \theta_1, ..., \theta_p) = \xi + x(u; 0, \theta_1, ..., \theta_p).$$

Definition 1.3.2. ([15], Page 16) A parameter α of a distribution is a *scale parameter* if the quantile function of the distribution satisfies

$$x(u; \alpha, \theta_1, ..., \theta_p) = \alpha \times x(u; 1, \theta_1, ..., \theta_p).$$

or, if the distribution also has a location parameter ξ,

$$x(u; \xi, \alpha, \theta_1, ..., \theta_p) = \xi + \alpha \times x(u; 0, 1, \theta_1, ..., \theta_p).$$

Example 1.3.1. *The gamble distribution has the quantile function[15]:*

$$x(u) = \xi - \alpha \log(-\log u).$$

Since $x(u;\xi,\alpha) = (\xi) + [-\alpha\log(-\log u)] = \xi + x(u;0,\alpha)$, *then* ξ *is a location parameter. Now,* ξ *is a location parameter and* $x(u;\xi,\alpha) = \xi - \alpha\log(-\log u) = (\xi) + (\alpha)[-\log(-\log u)] = \xi + \alpha \times x(u;0,1)$, *hence* α *is a scale parameter.*

The unknown parameters are estimated from the observed data. Given a set of data, a function $\hat{\theta}$ of the data values may be chosen as an estimator of θ. The estimator $\hat{\theta}$ is a random variable and has a probability distribution. The goodness of $\hat{\theta}$ as an estimator of θ depends on how close $\hat{\theta}$ typically is to θ. The deviation of $\hat{\theta}$ from θ may be decomposed into bias - a tendency to give estimates that are consistently higher or lower than the true value - and variability - the random deviation of the estimate from the true value that occurs even for estimators that have no bias [15].

Definition 1.3.3. ([15], Page 16) $bias(\hat{\theta}) = E(\hat{\theta} - \theta)$

Definition 1.3.4. ([15], Page 16) We say that $\hat{\theta}$ is *unbiased* if $bias(\hat{\theta}) = 0$, that is if $E(\hat{\theta}) = \theta$.

1.4 Moment and Moment Generating Functions

In this section, we define the nth moment, nth central moment and also define the moment generating function. Also, we introduce a theorem that generates the moment from moment generating function and find the nth center moment for normal distribution. After that, we define skewness and kurtosis. The shape of a probability distribution has traditionally been described by the moments of the distribution.

Definition 1.4.1. ([10], Page 58) For each integer n, the nth *moment* of X, μ'_n, is

$$\mu'_n = E(X^n).$$

The nth *central moment* of X, μ_n, is

$$\mu_n = E(X-\mu)^n,$$

where $\mu = \mu'_1 = E(X)$.

The mean is the center of location of the distribution. The dispersion of the distribution about its center is measured by the *standard deviation*,

$$\sigma = \mu_2^{1/2} = \{E(X-\mu)^2\}^{1/2} \ ,$$

or the variance, $\sigma^2 = var(X)$. The *coefficient of variation* (CV), $C_v \ = \ \sigma/\mu$,

Definition 1.4.2. ([15], Page 17) Analogous quantities can be computed from a data sample $x_1, x_2, ..., x_n$. The *sample mean*

$$\bar{x} = n^{-1}\sum_{i=1}^{n} x_i$$

is the natural estimator of μ.

Definition 1.4.3. ([15], Page 17) The higher sample moments

$$m_r = n^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^r$$

are reasonable estimators of the μ_r, but are not unbiased.

Unbiased estimators are often used. In particular, σ^2, μ_3 and the fourth cumulant $\kappa_4 = \mu_4 - 3\mu_2^2$ are unbiasedly estimated by

$$\begin{aligned} s^2 &= (n-1)^{-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 \quad , \\ \tilde{m}_3 &= \frac{n^2}{(n-1)(n-2)}m_3 \\ \tilde{k}_4 &= \frac{n^2}{(n-2)(n-3)}\Big\{\Big(\frac{n+1}{n-1}\Big)m_4 - 3m_2^2\Big\} \ , \end{aligned}$$

respectively. The sample standard deviation, $s = \sqrt{s^2}$, is an estimator of σ but is not unbiased. The sample estimator of CV, is,

$$\hat{C}_v = s/\bar{x}$$

We now introduce a new function that is associated with a probability distribution, the *moment generating function* mgf. As its name suggests, the mgf can be used to generate moments.

Definition 1.4.4. ([15], Page 61) Let X be a random variable with cdf $F(X)$. *The moment generating function* (*mgf*) of X, denoted by $M_X(t)$, is

$$M_X(t) = E(e^{tX}) \ ,$$

provided that the expectation exists for t in some neighborhood of 0.

More explicitly, we can write the mgf of X as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \qquad \text{if } X \text{ is continuous}$$

or

$$M_X(t) = \sum_x e^{tx} P(X = x) \quad \text{if } X \text{ is discrete.}$$

It is very easy to see how the mgf generates moments. We summarize the result in the following theorem.

Theorem 1.4.1. *[15] If X has mgf $M_X(t)$, then $E(X^n) = M_X^{(n)}(0)$,*

where we define

$$M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\Big|_{t=0}.$$

That is, the n^{th} moment is equal to the n^{th} derivative of $M_X(t)$ evaluated at $t = 0$.

Proof. Assuming that X has (pdf) $f_X(x)$. If we can differential under the integral sign we have

$$\begin{aligned}
\frac{d}{dt}M_X(t) = \frac{d}{dt}M_X(t) &= \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\
&= \int_{-\infty}^{\infty} (\frac{d}{dt}e^{tx}) f_X(x)dx \\
&= \int_{-\infty}^{\infty} (xe^{tx}) f_X(x)dx \\
&= E(Xe^{tX}).
\end{aligned}$$

Thus,

$$\frac{d}{dt}M_X(t)|_{t=0} = E(Xe^{tX}) \mid_{t=0} = E(X).$$

Proceeding in an analogous manner, we can establish that

$$\frac{d^n}{dt^n}M_X(t)|_{t=0} = E(X^n e^{tX}) \mid_{t=0} = E(X^n).$$

□

Definition 1.4.5. ([10], Page 100) For any real number $r > 0$, the *gamma function* (of r) is given by:

$$\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx.$$

Note 1.4.2. *([10], Page 100) If r is a positive real number, then $\Gamma(r+1) = r\Gamma(r)$.*

Note 1.4.3. *([10], Page 100) For any positive integer n, $\Gamma(n) = (n-1)! \cdot$*

Example 1.4.1. *The full gamma(α, β) family, is,*

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}, 0 < x < \infty, \alpha > 0, \beta > 0,$$

where $\Gamma(\alpha)$ denotes the gamma function,

$$M_X(t) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty e^{tx}x^{\alpha-1}e^{-x/\beta}dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{\alpha-1}e^{-(\frac{1}{\beta}-t)x}dx \tag{1.4.1}$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{\alpha-1}e^{-x/(\frac{\beta}{1-\beta t})}dx.$$

Using the fact that, for any positive constants a and b,

$$f(x) = \frac{1}{\Gamma(a)b^a}x^{a-1}e^{-x/b}$$

is a pdf, we have that

$$\int_0^{\infty} \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b} dx = 1$$

and hence,

$$\int_0^{\infty} x^{a-1} e^{-x/b} dx = \Gamma(a) b^a. \tag{1.4.2}$$

Applying (1.4.2) to (1.4.1), we have

$$M_X(t) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \Gamma(\alpha) \Big(\frac{\beta}{1-\beta t}\Big)^{\alpha} = \Big(\frac{1}{1-\beta t}\Big)^{\alpha} \qquad if\ t < \frac{1}{\beta}.$$

If $t \geq \frac{1}{\beta}$, then the quantity $(1/\beta) - t$, in the integrand of (1.4.1), is nonpositive and the integral in (1.4.2) is infinite. Thus, the mgf of the gamma distribution exists only if $t < 1/\beta$

The mean of the gamma distribution is given by

$$EX = \frac{d}{dt} M_X(t)\big|_{t=0} = \frac{\alpha\beta}{(1-\beta t)^{\alpha+1}}\Big|_{t=0} = \alpha\beta.$$

Other moments can be calculated in similar manner.

Example 1.4.2. ***Central moments of the normal distribution N(0,σ^2).*** *The moment generating function for the normal distribution N(0,σ^2) is as follows:*

$$M_X(t) = e^{\frac{t^2\sigma^2}{2}}.$$

The moments are then as follows. The first central moments is

$$E(X-\mu) = \tfrac{d}{dt}\Big(e^{\frac{t^2\sigma^2}{2}}\Big)|_{t=0}$$

$$= t\sigma^2\Big(e^{\frac{t^2\sigma^2}{2}}\Big)|_{t=0}$$

$$= 0.$$

The second central moment is

$$E(X-\mu)^2 = \frac{d^2}{dt^2}\left(e^{\frac{t^2\sigma^2}{2}}\right)|_{t=0}$$

$$= \frac{d}{dt}\left(t\sigma^2\left(e^{\frac{t^2\sigma^2}{2}}\right)\right)|_{t=0}$$

$$= \left(t^2\sigma^4\left(e^{\frac{t^2\sigma^2}{2}}\right) + \sigma^2\left(e^{\frac{t^2\sigma^2}{2}}\right)\right)|_{t=0}$$

$$= \sigma^2.$$

The third central moment is

$$E(X-\mu)^3 = \frac{d^3}{dt^3}\left(e^{\frac{t^2\sigma^2}{2}}\right)|_{t=0}$$

$$= \frac{d}{dt}\left(t^2\sigma^4\left(e^{\frac{t^2\sigma^2}{2}}\right) + \sigma^2\left(e^{\frac{t^2\sigma^2}{2}}\right)\right)|_{t=0}$$

$$= \left(t^3\sigma^6\left(e^{\frac{t^2\sigma^2}{2}}\right) + 2t\sigma^4\left(e^{\frac{t^2\sigma^2}{2}}\right) + t\sigma^4\left(e^{\frac{t^2\sigma^2}{2}}\right)\right)|_{t=0}$$

$$= \left(t^3\sigma^6\left(e^{\frac{t^2\sigma^2}{2}}\right) + 3t\sigma^4\left(e^{\frac{t^2\sigma^2}{2}}\right)\right)|_{t=0}$$

$$= 0.$$

The fourth central moment is

$$E(X-\mu)^4 = \frac{d^4}{dt^4}\left(e^{\frac{t^2\sigma^2}{2}}\right)|_{t=0}$$

$$= \frac{d}{dt}\left(t^3\sigma^6\left(e^{\frac{t^2\sigma^2}{2}}\right) + 3t\sigma^4\left(e^{\frac{t^2\sigma^2}{2}}\right)\right)|_{t=0}$$

$$= \left(t^4\sigma^8\left(e^{\frac{t^2\sigma^2}{2}}\right) + 3t^2\sigma^6\left(e^{\frac{t^2\sigma^2}{2}}\right) + 3t^2\sigma^6\left(e^{\frac{t^2\sigma^2}{2}}\right) + 3\sigma^4\left(e^{\frac{t^2\sigma^2}{2}}\right)\right)|_{t=0}$$

$$= \left(t^4\sigma^8\left(e^{\frac{t^2\sigma^2}{2}}\right) + 6t^2\sigma^6\left(e^{\frac{t^2\sigma^2}{2}}\right) + 3\sigma^4\left(e^{\frac{t^2\sigma^2}{2}}\right)\right)|_{t=0} = 3\sigma^4.$$

Now, we write this Theorem because it is used in the proof of Theorem 2.4.1.

Theorem 1.4.4. *([10], Page 65) Let $F_X(x)$ and $F_Y(y)$ be two cdfs all whose moments exist. If $F_X(x)$ and $F_Y(y)$ have bounded support, then $F_X = F_Y$ for all u if and only if $EX^r = EY^r$ for all integers $r = 0, 1, 2, \ldots$*

Proof. Assume that $F_X(u) = F_Y(u)$ for all u, hence $dF_X(u) = dF_Y(u)$.
Now, for all integers $r = 0, 1, 2, \ldots,$

$$E(X^r) = \int_{-\infty}^{\infty} u^r dF_X(u) = \int_{-\infty}^{\infty} u^r dF_Y(u) = E(Y^r).$$

Conversely, assume that $EX^r = EY^r$ for all integers $r = 0, 1, 2, \ldots$, then in special case $EX^0 = EY^0$.
Conceder,

$$EX^0 = \int_{-\infty}^{\infty} u^0 dF_X(u) = \int_{-\infty}^{\infty} dF_X = F_X.$$

Similarly,

$$EY^0 = \int_{-\infty}^{\infty} u^0 dF_Y(u) = \int_{-\infty}^{\infty} dF_Y = F_Y.$$

Since $EX^0 = EY^0$, then $F_X = F_Y$. That means, $F_X(u) = F_Y(u)$, for all u. □

1.5 Skewness and Kurtosis

Skewness measures the lak of symmetry in the probability density function $f(x)$ of a distribution [10].

Definition 1.5.1. ([15], Page 17) The *skewness* is :

$$\gamma = \mu_3/\mu_2^{3/2}.$$

A distribution that's symmetric about its mean has 0 skewness. But if it has a long tail to the right and a short one to the left, then it has a positive skewness, and a negative skewness in the opposite situation.
The sample estimator of skewness is,

$$g = \tilde{m}_3/s^3 \quad [15],$$

where

$$s^2 = (n-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \ ,$$
$$\tilde{m}_3 = \frac{n^2}{(n-1)(n-2)}m_3.$$

The estimator g is biased estimators of γ. Indeed, g has algebraic bounds that depend on the sample size; for a sample of size n the bound is

$$|g| \leq n^{1/2} \ \ [15].$$

Example 1.5.1. ***The skewness of the normal distribution N(0,σ^2):***

From example 1.4.2, we have the second and the third central moments of the normal distribution N(0,σ^2) are: $\mu_2 = \sigma^2$ and $\mu_3 = 0$. Then, the skewness of the normal distribution N(0,σ^2) is:

$$\gamma = \mu_3/\mu_2^{3/2} = \frac{0}{(\sigma^2)^{3/2}} = \frac{0}{\sigma^3} = 0 \ .$$

Table 1.1: The following table gives the skewness for a number of common distributions.

Distribution	pdf, $f(x)$	Skewness
Bernoulli	$p^x q^{1-x}$	$\frac{1-2p}{\sqrt{p(1-p)}}$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(1-x)^{\beta-1}x^{\alpha}$	$\frac{2(\beta-\alpha)}{(2+\alpha+\beta)}\sqrt{\frac{1+\alpha+\beta}{\alpha\beta}}$
Binomial	$\binom{N}{x}p^x q^{N-x}$	$\frac{q-p}{\sqrt{Npq}}$
Chi-squared	$\frac{x^{r/2-1}\ e^{-x/2}}{\Gamma(\frac{1}{2}r)\ 2^{r/2}}$	$2\sqrt{\frac{2}{r}}$
Exponential	$\frac{1}{\beta}e^{-(x-\alpha)/\beta}$	2
Gamma	$\frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\ \theta^{\alpha}}$	$\frac{2}{\sqrt{\alpha}}$
Geometric distribution	$p\ q^x$	$\frac{2-p}{\sqrt{1-p}}$
Half-normal	$\frac{2\theta}{\pi}e^{-x^2\theta^2/\pi}$	$\frac{\sqrt{2}(4-n)}{(\pi-2)^{3/2}}$
Laplace	$\frac{1}{2b}e^{-\lvert x-\mu\rvert/b}$	0
Log normal	$\frac{1}{S\sqrt{2\pi}\ x}\ e^{-(\ln x-M)^2/(2S^2)}$	$\sqrt{e^{S^2}-1}(2+e^{S^2})$
Maxwell	$\sqrt{\frac{2}{\pi}}\ \frac{x^2 e^{-x^2/(2a^2)}}{a^3}$	$\frac{2\sqrt{2}(5n-16)}{(3n-8)^{3/2}}$
Negative binomial	$\binom{x+r-1}{r-1}p^r q^x$	$\frac{2-p}{\sqrt{rq}}$
Normal	$\frac{1}{\sigma\sqrt{2\pi}}\ e^{-(x-\mu)^2/(2\sigma^2)}$	0
Poisson	$\frac{\nu^n\ e^{-\nu}}{n!}$	$\nu^{-1/2}$
Rayleigh	$\frac{x\ e^{-x^2/(2s^2)}}{s^2}$	$(\pi-3)\sqrt{\frac{\pi}{2(2-\frac{1}{2}\pi)^3}}$
Student's t	$\frac{\left(\frac{r}{r+x^2}\right)^{(1+r)/2}}{\sqrt{r}\ B(\frac{1}{2}r,\frac{1}{2})}$	0
Continuous uniform	$\frac{1}{\beta-\alpha}$	0
Discrete uniform	$\frac{1}{N}$	0

Kurtosis

kurtosis is the degree of peakedness of a distribution, defined as a normalized from the fourth central moment μ_4.

Definition 1.5.2. ([15], Page 17) The *kurtosis* is

$$\kappa \ = \mu_4/\mu_2^2.$$

A fairly flat distribution with long tails has a high kurtosis, while a short tailed distribution has a low kurtosis. A normal distribution has a kurtosis of 3.
The sample estimators of kurtosis,

$$k = \tilde{k}_4/s^4 + 3 \ \ [15],$$

where

$$\begin{aligned} s^2 &= (n-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \ \ , \\ \tilde{k}_4 &= \frac{n^2}{(n-2)(n-3)}\Big\{\Big(\frac{n+1}{n-1}\Big)m_4 - 3m_2^2\Big\} \ . \end{aligned}$$

The estimator k is biased estimators of κ. Indeed k has algebraic bounds that depend on the sample size; for a sample of size n the bound is

$$k \leq n + 3 \ \ [15].$$

Example 1.5.2. ***The kurtosis of the normal distribution N(0,σ^2):***
Since the second and the fourth central moments of the normal distribution N(0,σ^2) are: $\mu_2 = \sigma^2$ and $\mu_4 = 3\sigma^4$ (see example 1.4.2). Hence, the kurtosis of the normal distribution N(0,σ^2) is:

$$\kappa \ = \mu_4/\mu_2^2 = \frac{3\sigma^4}{(\sigma^2)^2} = \frac{3\sigma^4}{\sigma^4} = 3 \ .$$

Table 1.2: The following table gives the Kurtosis for a number of common distributions.

Distribution	pdf, $f(x)$	Kurtosis
Bernoulli	$p^x q^{1-x}$	$\frac{1}{1-p} + \frac{1}{p} - 6$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(1-x)^{\beta-1}x^{\alpha}$	$\frac{6[a^3+a^2(1-2b)+b^2(1+b)-2ab(2+b)]}{ab(2+a+b)(3+a+b)}$
Binomial	$\binom{N}{x} p^x q^{N-x}$	$\frac{1-6pq}{Npq}$
Chi-squared	$\frac{x^{r/2-1}\, e^{-x/2}}{\Gamma(\frac{1}{2}r)\, 2^{r/2}}$	$\frac{12}{r}$
Exponential	$\frac{1}{\beta}e^{-(x-\alpha)/\beta}$	6
Gamma	$\frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\,\theta^{\alpha}}$	$\frac{6}{\alpha}$
Geometric distribution	$p\, q^x$	$5 - p + \frac{1}{1+p}$
Half-normal	$\frac{2\theta}{\pi}e^{-x^2\theta^2/\pi}$	$\frac{8(\pi-3)}{(\pi-2)^2}$
Laplace	$\frac{1}{2b}e^{-\lvert x-\mu\rvert/b}$	3
Log normal	$\frac{1}{S\sqrt{2\pi}\,x}\; e^{-(\ln x-M)^2/(2S^2)}$	$e^{4S^2+2e^{3S^2}+3e^{2S^2}-6}$
Maxwell	$\sqrt{\frac{2}{\pi}}\;\frac{x^2e^{-x^2/(2a^2)}}{a^3}$	$-\frac{4(96-40\pi+3\pi^2)}{(3\pi-8)^2}$
Negative binomial	$\binom{x+r-1}{r-1} p^r q^x$	$\frac{6-p(6-p)}{r(1-p)}$
Normal	$\frac{1}{\sigma\sqrt{2\pi}}\; e^{-(x-\mu)^2/(2\sigma^2)}$	3
Poisson	$\frac{\nu^n\, e^{-\nu}}{n!}$	$\frac{1}{v}$
Rayleigh	$\frac{x\, e^{-x^2/(2s^2)}}{s^2}$	$\frac{6\pi(4-\pi)-16}{(\pi-4)^2}$
Student's t	$\frac{\left(\frac{r}{r+x^2}\right)^{(1+r)/2}}{\sqrt{r}\, B(\frac{1}{2}r,\frac{1}{2})}$	$\frac{6}{r-4}$
Continuous uniform	$\frac{1}{\beta-\alpha}$	$-\frac{6}{5}$
Discrete uniform	$\frac{1}{N}$	$-\frac{6(N^2+1)}{5(N^2-1)}$

1.6 The Shifted Legendre Polynomials

The base of our thesis is to define the L-moments λ_r which depends on the r^{th} *shifted Legendre polynomial* which is related to the usual *Legendre polynomials* $P_{r-1}(F)$. So, we defined the *Legendre polynomials* and the *shifted Legendre polynomials* and we extract some relations that we use in this thesis. In addition, we show that *Legendre polynomials* and *shifted Legendre polynomials* are *eigenfunctions*.
Furthermore, we serve the Corollary that will be used to prove Theorem 2.7.1 in section (2.7).

Definition 1.6.1. ([5], Page 60) A self-adjoint differential equation of the form

$$[p(x)\ y']' + [q(x) + \lambda r(x)]\ y = 0, \tag{1.6.1}$$

on the interval $0 < x < 1$, together with the boundary conditions

$$a_1 y(0) + a_2 y'(0) = 0, \qquad b_1 y(1) + b_2 y'(1) = 0, \tag{1.6.2}$$

is called a *Sturm-Liouville eigenvalue problem.* Those values of λ for which non-trivial solutions for such problems exits, are called *eigenvalues* and the corresponding solutions are called *eigenfunctions.*

The following theorem expresses the property of *orthogonality* of the eigenfunctions with respect to the weight function r.

Theorem 1.6.1. *([31], Page 636) If y_1 and y_2 are two eigenfunctions of a Sturm-Liouville problem (1.6.1), (1.6.2) corresponding to eigenvalues λ_1 and λ_2, respectively, and $\lambda_1 \neq \lambda_2$, then*

$$\int_0^1 r(x)\ y_1(x)\ y_2(x)\ dx = 0, \quad \text{where } r(x) \text{ is weight function.} \tag{1.6.3}$$

Corollary 1.6.2. *([5], Page 61) (Eigenfunction expansion). If* $\{y_i(x)\}$ *is the set of eigenfunctions of the Sturm-Liouville eigenvalue problem:*

$$[p(x)\,y']' + [q(x) + \lambda r(x)]\,y = 0,$$
$$a_1 y(a) + a_2 y'(a) = 0, \qquad b_1 y(b) + b_2 y'(b) = 0,$$

and $f(x)$ *is a function on [a,b] such that* $f(a) = f(b) = 0$, *then*

$$f(x) = \sum_{i=0}^{\infty} c_i\, y_i(x) \tag{1.6.4}$$

$$\textit{where} \quad c_i = \frac{1}{\mu_i}\int_a^b r(x)\, f(x)\, y_i(x)\, dx, \qquad \mu_i = \int_a^b r(x)\, y_i^2(x)\, dx.$$

Definition 1.6.2. ([5], Page 83) The *Legendre's equation* is

$$(1 - x^2)\, y'' - 2xy' + n(n+1)\, y = 0 \tag{1.6.5}$$

where n is a positive integer.

One of the solutions of equation (1.6.5) is the polynomial

$$P_n(x) = F\Big[-n, n+1; 1; \frac{1-x}{2}\Big],$$

where

$$F\Big[a, b; c; x\Big] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\, dt$$

and $\Gamma(.)$ is gamma function.
$P_n(x)$ is called the n^{th} Legendre's polynomial.

The Legendre's equation can be written in the self-adjoint form

$$[(1 - x^2)\, y']' + n(n+1)\, y = 0 \tag{1.6.6}$$

Comparing equation (1.6.6) with the form (1.6.1), $p(x) = 1 - x^2,\ q(x) = 0,\ r(x) = 1,\ \lambda = n(n+1)$. Since $p(x) = 0$ for $x = -1,\ 1$, represents a Sturm-Liouville problem without explicit boundary conditions, its eigenfunctions are $P_n(x)$ with the related eigenvalues

$n(n+1)$ $(n = 0, 1, 2, \ldots)$. Hence $\{P_n(x)\}$ is an orthogonal set of polynomials over the interval $-1 \leqslant x \leqslant 1$ with weight function equal to 1, i.e.,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, \qquad m \neq n.$$

There are other approaches for establishing orthogonality of the Legendre sequence. The following is the complete statement [5]

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & m \neq n; \\ \frac{2}{2n+1}, & m = n. \end{cases}$$

Definition 1.6.3. ([15], Page 19) We define polynomials $P_r^{'}(u),\ r = 0, 1, 2, \ldots$ as follows:
(i) $P_r^{'}(u)$ is a polynomial of degree r in u.
(ii) $P_r^{'}(1) = 1$.
(iii) $\int_0^1 P_r^{'}(u)P_s^{'}(u)\,du = 0\ if\ r \neq s$.
Condition(iii) is the orthogonality condition. These conditions define the shifted Legendre polynomials Condition(" shifted", because the ordinary Legendre polynomials $P_r(u)$ are defined to be orthogonal on the interval $-1 \leq u \leq +1$, *not* $0 \leq u \leq 1$).

The $P_r^{'}(F)$ is the r^{th} shifted Legendre polynomial related to the usual Legendre polynomials $P_r^{'}(u) = P_r(2u-1)$. Shifted Legendre polynomials are orthogonal on the interval (0,1) with constant weight function $r(u) = 1$[15].

Note 1.6.3. *[18] The $P_r^*(F)$ is the r^{th} shifted Legendre polynomials, where*

$$P_r^*(F) = \sum_{m=0}^{r} p_{r,m}^*\ F^m,$$

and

$$p_{r,m}^* = (-1)^{r-m}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} r+m \\ m \end{pmatrix}.$$

Note 1.6.4. $\int_0^1 \{P_r^*(u)\}^2 du = \frac{1}{2r+1}$.

Proof. Since,

$$\int_0^1 \{P_r^*(u)\}^2 du = \int_0^1 \{P_r(2u-1)\}^2 du. \tag{1.6.7}$$

Let $z = 2u - 1$, then, $dz = 2du$ and substituting in eqn (1.6.7) we have:

$$\int_0^1 \{P_r^*(u)\}^2 du = \frac{1}{2}\int_{-1}^1 \{P_r(z)\}^2 dz = \frac{1}{2}\Big(\frac{2}{2r+1}\Big) = \frac{1}{2r+1}\cdot \qquad (1.6.8)$$

□

Note 1.6.5. $\int_0^1 P_r^*(u)du = 0$ *for* $r > 0$.

Proof. From definition of $P_r^*(u)$, we have

$$P_0^*(u) = \sum_{m=0}^{0} p_{0,m}^* \; u^m = p_{0,0}^* = (-1)^{0-0}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1.$$

Then from orthogonality condition,

$$\int_0^1 P_r^*(u)du = \int_0^1 1 \times P_r^*(u)du = \int_0^1 P_0^*(u)\; P_r^*(u)du = 0 \;\; \text{becase } r > 0 \cdot \qquad (1.6.9)$$

□

We introduce (Chebyshev's Other Inequality) since it is used in the proof of Theorem 2.8.1.

Theorem 1.6.6. *[9] (Chebyshev's Other Inequality). Let f and g be real-valued functions that are either both increasing or both decreasing on the interval (a,b) (a and b can be infinite), and let w be a function that is positive on (a,b). Then*

$$\int_a^b f(x)g(x)w(x)dx \int_a^b w(x)dx \geq \int_a^b f(x)w(x)dx \int_a^b g(x)w(x)dx.$$

Proof. We have

$$\{f(y) - f(x)\}\{g(y) - g(x)\} \geq 0 \quad \text{for any } x \text{ and } y \text{ in } (a,b),$$

so,

$$\begin{aligned} 0 &\leq \int_a^b \int_a^b \{f(y) - f(x)\}\{g(y) - g(x)\}w(x)w(y)dx\,dy \\ &= \int_a^b \int_a^b f(y)g(y)w(x)w(y)dx\,dy - \int_a^b \int_a^b f(y)g(x)w(x)w(y)dx\,dy \end{aligned}$$

$$
\begin{aligned}
&- \int_a^b \int_a^b f(x)g(y)w(x)w(y)dx\,dy + \int_a^b \int_a^b f(x)g(x)w(x)w(y)dx\,dy \\
&= \int_a^b f(y)g(y)w(y)dy \int_a^b w(x)dx - \int_a^b f(y)w(y)dy \int_a^b g(x)w(x)dx \\
&- \int_a^b f(x)w(x)dx \int_a^b g(y)w(y)dy + \int_a^b f(x)g(x)w(x)dx \int_a^b w(y)dy \\
&= 2\int_a^b f(x)g(x)w(x)dx \int_a^b w(x)dx - 2\int_a^b f(x)w(x)dx \int_a^b g(x)w(x)dx.
\end{aligned}
$$

The result follows.

□

1.7 Order Statistics

In this section, we deal with *order statistics* and related subjects. At first, we define *order statistics* and their distribution functions. Next, we give examples for *order statistics* . Then, we present some significant propositions. After that, we define the probability density function and the cumulative distribution function for an *order statistic*. We then present some related theorems.

Definition 1.7.1. ([10], Page 229) The *order statistics* of a random sample $X_1, X_2, ..., X_n$ are the sample values placed in ascending order. They are denoted by $X_{(1)}, X_{(2)}, ..., X_{(n)}$. In other words, the *order statistics* are random variables that satisfy $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$, where

$$
\begin{aligned}
X_{(1)} &:= \min_i X_i \\
X_{(2)} &:= 2^{nd}\ smallest\ X_i \\
&. \\
&. \\
X_{(n)} &:= \max_{1\leq i\leq n} X_i.
\end{aligned}
$$

Example 1.7.1. *The values* $x_1 = 0.62,\ x_2 = 0.98,\ x_3 = 0.31,\ x4 = 0.81$ *and* $x_5 = 0.53$ *are the* $n = 5$ *observed values of five independent trials of an experiment with (*pdf*)*

$f(x) = 2x,\ 0 < x < 1$. *The observed values of the* order statistics *are*

$$x_1 = 0.31 < x_2 = 0.53 < x_3 = 0.62 < x_4 = 0.81 < x_5 = 0.98.$$

Now, the next theorem gives the *cdf* of the j^{th} *order statistic.*

Theorem 1.7.1. *([10], Page 231) Let $X_1, X_2, ..., X_n$ be a random sample of size n from a distribution with pdf $f(x)$ and (cdf) $F(x)$. Then the cdf of the j^{th}* order statistic, *is given by*

$$F_j(x) = \sum_{k=j}^{n} \begin{pmatrix} n \\ k \end{pmatrix} [F(x)]^k [1 - F(x)]^{n-k}. \tag{1.7.1}$$

Example 1.7.2. *Let $X_1, X_2, ..., X_n$ be a random sample of size n from the* uniform *distribution with parameter θ. Then*

$$f(x) = \begin{cases} \frac{1}{\theta}, & 0 < x < \theta; \\ 0, & otherwise. \end{cases}$$

and

$$\begin{aligned} F(x) &= \begin{cases} 0, & x \leq 0; \\ \frac{x}{\theta}, & 0 < x < \theta; \\ 1, & x \geq \theta. \end{cases} \\ F_j(x) &= \sum_{k=j}^{n} \begin{pmatrix} n \\ k \end{pmatrix} [F(x)]^k [1 - F(x)]^{n-k} \\ &= \sum_{k=j}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \Big[\frac{x}{\theta}\Big]^k \Big[1 - \Big(\frac{x}{\theta}\Big)\Big]^{n-k}. \end{aligned}$$

Example 1.7.3. *Let $X_1, X_2, ..., X_n$ be the random sample of size n from an exponential distribution with parameter β. Then*

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}}, & x \geq 0; \\ 0, & otherwise. \end{cases}$$

So,

$$
\begin{aligned}
F_j(x) &= \sum_{k=j}^{n} \begin{pmatrix} n \\ k \end{pmatrix} [F(x)]^k [1-F(x)]^{n-k} \\
&= \sum_{k=j}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \left[1-\left(e^{-\frac{x}{\beta}}\right)\right]^k \left[e^{-\frac{x}{\beta}}\right]^{n-k}.
\end{aligned}
$$

Now, we introduce the probability density function of any *order statistic* through the following theorem.

Theorem 1.7.2. *([10], Page 232) Let $X_1, X_2, ..., X_n$ be a random sample of size n from a distribution of continuous population with $(pdf) f(x)$ and cdf $F(x)$.*
Then the (pdf) of the j^{th} order statistic is given by

$$
f_j(x) = j \begin{pmatrix} n \\ j \end{pmatrix} f(x)[F(x)]^{j-1}[1-F(x)]^{n-j}. \tag{1.7.2}
$$

Example 1.7.4. *Let $X_1, X_2, ..., X_n$ be a random sample of size n from the uniform distribution with parameter $\theta = 1$. Then by Example 1.7.2, the cdf is defined by:*

$$
F(x) = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x < 1, \\ 1, & x \geq 1. \end{cases}
$$

Now, for $0 < x < 1$, Theorem 1.7.2 yields

$$
\begin{aligned}
f_j(x) &= j \begin{pmatrix} n \\ j \end{pmatrix} f(x)[F(x)]^{j-1}[1-F(x)]^{n-j} \\
&= j \begin{pmatrix} n \\ j \end{pmatrix} x^{j-1}(1-x)^{n-j} \\
&= \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n-j+1)} x^{j-1}(1-x)^{(n-j+1)-1}.
\end{aligned}
$$

Thus, the j^{th} order statistic *has a* Beta *distribution with parameters j and $n-j+1$.*

Chapter 2

L-MOMENTS OF PROBABILITY DISTRIBUTIONS

L-moments are expectations of certain linear combinations of order statistics. They can be defined for any random variable whose mean exists and from the basis of a general theory which covers the summarization and description of theoretical probability distributions, the summarization and description of observed data samples, estimation of parameters and quantile of probability distributions, and hypothesis tests for probability distributions [17].

In the first section of this chapter, we define L-moments and L-moment ratios.
In the second section, we define probability weight moments and we find the relationship between L-moments and probability weight moments and it will make it easier to find L-moments for some distributions.
In the third section, we find the relation between L-moments and order statistic.
In the fourth section, we established some properties of L-moments. After that, we talk about L-skewness and L-kurtosis.
In the sixth section, we write about the L-moments of a polynomial function of a random variable.

In the seventh section, we write about an inversion theorem, expressing the quantile function in terms of L-moments.

In the eighth section, we write about L-moments as measure of distribution shape.

Finally, in the ninth section, we find L-moments for some distribution.

2.1 Definitions and Basic Properties

Here we introduce some basic and related definitions and properties.

Definition 2.1.1. [17] Let X be a real-valued random variable with cumulative distribution $F(x)$ and quantile function $x(F)$, and let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the order statistics of a random sample of size n drawn from the distribution of X. Define the *L-moments* of X to be the quantities

$$\lambda_r \equiv r^{-1} \sum_{k=0}^{r-1} (-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix} EX_{r-k:r} \qquad r = 1, 2, ... \tag{2.1.1}$$

The L in "*L-moments*" emphasizes that λ_r is a linear function of the expected order statistics. Furthermore, as noted in [17], the natural estimator of λ_r based on an observed sample of data is a linear combination of the ordered data values. From Theorem 1.7.2, the (pdf) of the jth order statistic is given by:

$$\begin{aligned} f_j(x) &= j \begin{pmatrix} r \\ j \end{pmatrix} f(x)[F(x)]^{j-1}[1-F(x)]^{r-j} \\ &= \frac{r!}{(j-1)!(r-j)!}[F(x)]^{j-1}[1-F(x)]^{r-j} f(x). \end{aligned}$$

The expectation of an order statistic from eqn.(1.1.1) may be written as:

$$\begin{aligned} EX_{j:r} &= \int_{-\infty}^{\infty} x f_j(x)\, dx \\ &= \int_{-\infty}^{\infty} x \frac{r!}{(j-1)!(r-j)!}[F(x)]^{j-1}[1-F(x)]^{r-j} f(x)\, dx. \end{aligned}$$

Hence,

$$EX_{j:r} = \frac{r!}{(j-1)!(r-j)!} \int_0^1 x[F(x)]^{j-1}[1-F(x)]^{r-j}\, dF(x). \tag{2.1.2}$$

Lemma 2.1.1. *[11] A finite mean implies finite expectation of all order statistics.*

Proof. Assume that the mean $\mu = \int_0^1 x(u)du$ is finite. So, $x(u)$ is integrable in the interval (0,1). Since from eqsn.(2.3.2) and (2.3.3) we have:

$$\int_0^1 u^{j-1}[1-u]^{r-j}du = B(j, r-j+1) = \frac{(j-1)!\,(r-j)!}{(r!)} \quad \text{is finite,}$$

then $u^{j-1}[1-u]^{r-j}$ is integrable in the interval (0,1). Hence, $x(u)\ u^{j-1}[1-u]^{r-j}$ is integrable in the interval (0,1) (because the product of two integrable functions on any interval is an integrable function on this interval) and so

$$\int_0^1 x(u)\ u^{j-1}[1-u]^{r-j}du \quad \text{is finite.}$$

From eqn.(2.1.2),

$$EX_{j:r} = \frac{r!}{(j-1)!(r-j)!} \int_0^1 x(u)\ u^{j-1}[1-u]^{r-j}du \quad \text{is finite}\cdot$$

Therefore, a finite mean implies finite expectation of all order statistics. □

Let's rewrite the definition of the L-moment given in eqn.(2.1.1) to a simpler form that is easy in use.

Change variable $u = F(x)$. Let Q be the inverse of function F; i.e., $Q(F(x)) = x$ or $F(Q(u)) = u$:

$$EX_{r-k:r} = \frac{r!}{(j-1)!(r-j)!} \int_0^1 Q(u)\ u^{r-k-1}(1-u)^k\ du. \tag{2.1.3}$$

Substitute from eqn.(2.1.3) into eqn.(2.1.1) :

$$\lambda_r \equiv r^{-1} \sum_{k=0}^{r-1} (-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix} \frac{r!}{(r-1-k)!k!} \int_0^1 Q(u)\ u^{r-k-1}(1-u)^k\ du.$$

For convenience, consider λ_{r+1} instead of λ_r:

$$\lambda_{r+1} \equiv (r+1)^{-1} \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix} \frac{(r+1)!}{(r-k)!k!} \int_0^1 Q(u)\ u^{r-k}(1-u)^k\ du.$$

Note that $(r+1)^{-1}\ (r+1)! = r!$ and rearrange terms:

$$\lambda_{r+1} = \int_0^1 \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix}^2 u^{r-k}(1-u)^k\ Q(u)\ du. \tag{2.1.4}$$

Expand $(1-u)^k$ in powers of u :

$$\lambda_{r+1} = \int_0^1 \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix}^2 u^{r-k} \sum_{j=0}^{k} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} u^{k-j}\ Q(u)\ du$$

$$= \int_0^1 \sum_{k=0}^{r} \sum_{j=0}^{k} (-1)^j \begin{pmatrix} r \\ k \end{pmatrix}^2 \begin{pmatrix} k \\ j \end{pmatrix} u^{r-j}\ Q(u)\ du.$$

Interchange order of summation over j and k:

$$\lambda_{r+1} = \int_0^1 \sum_{j=0}^{r} \sum_{k=j}^{r} (-1)^j \begin{pmatrix} r \\ k \end{pmatrix}^2 \begin{pmatrix} k \\ j \end{pmatrix} u^{r-j}\ Q(u)\ du.$$

Reverse order of summation: set $m = r - j$, $n = r - k$:

$$\lambda_{r+1} = \int_0^1 \sum_{m=0}^{r} \sum_{n=0}^{m} (-1)^{r-m} \begin{pmatrix} r \\ r-n \end{pmatrix}^2 \begin{pmatrix} r-n \\ r-m \end{pmatrix} u^m\ Q(u)\ du$$

$$\lambda_{r+1} = \int_0^1 \sum_{m=0}^{r} (-1)^{r-m} \Big\{ \sum_{n=0}^{m} \begin{pmatrix} r \\ r-n \end{pmatrix}^2 \begin{pmatrix} r-n \\ r-m \end{pmatrix} \Big\} u^m\ Q(u)\ du. \tag{2.1.5}$$

Note that

$$\begin{pmatrix} r \\ r-n \end{pmatrix}^2 \begin{pmatrix} r-n \\ r-m \end{pmatrix} = \begin{pmatrix} r \\ n \end{pmatrix} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \tag{2.1.6}$$

(expand the binomial coefficients in terms of factorials) and that

$$\sum_{n=0}^{m}\begin{pmatrix} r \\ n \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \sum_{n=0}^{m}\begin{pmatrix} r \\ r-n \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} r+m \\ r \end{pmatrix} = \begin{pmatrix} r+m \\ m \end{pmatrix} \tag{2.1.7}$$

(second equality follows because to choose r items from $r+m$ we can choose from the first m items and $r-n$ from the remaining r items, for any n in $0, 1, ..., m$). From (2.2.5) and (2.2.6), we have:

$$\sum_{n=0}^{m}\begin{pmatrix} r \\ r-n \end{pmatrix}^2 \begin{pmatrix} r-n \\ r-m \end{pmatrix} = \begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} r+m \\ m \end{pmatrix}, \tag{2.1.8}$$

and substituting into (2.1.5) gives

$$\lambda_{r+1} = \int_0^1 \sum_{m=0}^{r}(-1)^{r-m}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} r+m \\ m \end{pmatrix} u^m \; Q(u) \; du$$

$$\lambda_{r+1} = \int_0^1 \sum_{m=0}^{r}(-1)^{r-m}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} r+m \\ m \end{pmatrix} x(F) \; F^m \; dF. \tag{2.1.9}$$

Let

$$p^*_{r,m} = (-1)^{r-m}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} r+m \\ m \end{pmatrix}, \tag{2.1.10}$$

and

$$P^*_r(F) = \sum_{m=0}^{r} p^*_{r,m} \; F^m. \tag{2.1.11}$$

Substituting (2.1.11) into (2.1.9) we have [11]:

$$\lambda_r = \int_0^1 x(F) \; P^*_{r-1}(F) \; dF, \qquad r = 1, 2, \ldots. \tag{2.1.12}$$

Example 2.1.1. *To fined λ_2, substitute $r = 2$ in eqn.(2.1.1),*

$$\begin{aligned} \lambda_2 &= \frac{1}{2}\sum_{k=0}^{1}(-1)^k \begin{pmatrix} 1 \\ k \end{pmatrix} EX_{2-k:2} \\ &= \frac{1}{2}\Big[(-1)^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} EX_{2:2} + (-1)^1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} EX_{1:2}\Big] \\ &= \frac{1}{2}\Big[EX_{2:2} - EX_{1:2}\Big] = \frac{1}{2}E(X_{2:2} - X_{1:2}). \end{aligned}$$

And we can substitute $r = 2$ *in eqn.(2.1.12),*

$$\begin{aligned}
\lambda_2 &= \int_0^1 x(F)\; P_1^*(F)\; dF \\
&= \int_0^1 x(F) \Big[\sum_{m=0}^{1} p_{1,m}^* \; F^m \Big] \; dF \qquad \textit{from} \;\; \textit{eq.n.}(2.1.11) \\
&= \int_0^1 x(F) \Big[p_{1,0}^* F^0 + p_{1,1}^* F^1 \Big] \; dF \\
&= \int_0^1 x(F) \Big[(-1)^1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1)^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} F \Big] \; dF \;\; \textit{from}\; (2.1.10) \\
&= \int_0^1 x(F)(2F - 1)\; dF.
\end{aligned}$$

Hence,

$$\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}) = \int_0^1 x.(2F-1)\, dF.$$

The first few L-moments are:

$$\begin{aligned}
\lambda_1 = EX &= \int_0^1 x.\, dF, \\
\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}) &= \int_0^1 x.(2F-1)\, dF, \\
\lambda_3 = \frac{1}{3} E(X_{3:3} - X_{2:3} + X_{1:3}) &= \int_0^1 x.(6F^2 - 6F + 1)\, dF, \\
\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}) &= \int_0^1 x.(20F^3 - 30F^2 + 12F - 1)\, dF.
\end{aligned}$$

The use of L-moments to describe probability distributions is justified by the next theorem. As shown in [17], λ_2 is a measure of the scale or dispersion of the random variable X. It is often convenient to standardize the higher moments λ_r, $r \geq 3$, so that they are independent of the units of measurement of X.

Definition 2.1.2. [18] Define the *L-moment ratios* of X to be the quantities

$$\tau_r \equiv \lambda_r / \lambda_2, \qquad r = 3, 4, ...$$

Note that [17]:

$\tau_3 = \lambda_3/\lambda_2$ is called *L-skewness*,

$\tau_4 = \lambda_4/\lambda_2$ is called *L-kurtosis.*

It is also possible to define a function of L-moments which is analogous to the coefficient of variation: this is the $L-CV$, $\tau \equiv \lambda_2/\lambda_1$. Bounds on the numerical values of the L-moment ratios and $L-CV$ is given by the following theorem.

Theorem 2.1.2. *[18] Let X be a nondegenerate random variable with finite mean. Then the L-moment ratios of X satisfy $\mid \tau_r \mid < 1$, $r \geq 3$. If in addition $X \geq 0$ almost surely, then τ, the $L-CV$ of X, satisfies $0 < \tau < 1$.*

Proof. Define $Q_r(t)$ by

$$t(1-t)Q_r(t) = \frac{(-1)^r}{r!}\frac{d^r}{dt^r}[t(1-t)]^{r+1},$$

where $Q_r(t)$ is the Jacobi polynomial $P_r^{(1,1)}(2t-1)$.

Then,

$$\begin{aligned}
\frac{d}{dt}[t(1-t)Q_r(t)] &= \frac{(-1)^r}{r!}\frac{d^{r+1}}{dt^{r+1}}[t(1-t)]^{r+1} \\
&= \frac{(-1)^r}{r!}\frac{d^{r+1}}{dt^{r+1}}\sum_{k=0}^{r+1}(-1)^k \begin{pmatrix} r+1 \\ k \end{pmatrix} t^{r+1+k} \\
&= \frac{(-1)^r}{r!}\sum_{k=0}^{r+1}(-1)^k \begin{pmatrix} r+1 \\ k \end{pmatrix} \frac{d^{r+1}}{dt^{r+1}}[t^{r+1+k}] \\
&= \frac{(-1)^r}{r!}\sum_{k=0}^{r+1}(-1)^k \begin{pmatrix} r+1 \\ k \end{pmatrix} \big(r+1+k\big)\big(r+k\big).....\big(k+1\big)t^k \\
&= -(r+1)\sum_{k=0}^{r+1}(-1)^{r-k+1}\frac{(r+1)!(r+1+k)!}{(r+1)!k!(r+1-k)!k!}t^k \\
&= -(r+1)\sum_{k=0}^{r+1}(-1)^{r-k+1}\begin{pmatrix} r+1 \\ k \end{pmatrix}\begin{pmatrix} r+1+k \\ k \end{pmatrix} t^k \\
&= -(r+1)\sum_{k=0}^{r+1}(-1)^{r-k+1}p^*_{r+1,k}t^k \\
&= -(r+1)P^*_{r+1,k}\big(t\big).
\end{aligned}$$

Hence,

$$\frac{d}{dt}[t(1-t)Q_r(t)] = -(r+1)P^*_{r+1}(t).$$

Then,

$$P^*_{r+1}(t) = \frac{-1}{r+1}\frac{d}{dt}[t(1-t)Q_r(t)].$$

Therefore,

$$P^*_{r-1}(F) = \frac{-1}{r-1}\frac{d}{dF}[F(1-F)Q_{r-2}(F)].$$

So, from eq.n.(2.1.12),

$$\lambda_r = \frac{-1}{r-1}\int x(F)\frac{d}{dF}\Big[F(1-F)Q_{r-2}(F)\Big]dF.$$

Now, integrating by parts:

$$\begin{aligned}\lambda_r &= \frac{-1}{r-1}\Big[x(F)F(1-F)Q_{r-2}(F) - \int F(1-F)Q_{r-2}(F)dx\Big] \\ &= \Big[-xF(x)\Big[1-F(x)\Big](r-1)^{-1}Q_{r-2}(F(x))\Big] + \int F(x)\Big[1-F(x)\Big](r-1)^{-1}Q_{r-2}(F(x))dx.\end{aligned}$$

Since $xF(x)\Big[1-F(x)\Big] \to 0$ as x approaches the endpoint of the distribution, then

$$\lambda_r = \int F(x)\Big[1-F(x)\Big](r-1)^{-1}Q_{r-2}(F(x))dx. \qquad (2.1.13)$$

Since

$$Q_r(t) = \frac{(-1)^r}{r!}\frac{1}{t(1-t)}\frac{d^r}{dt^r}[t(1-t)]^{r+1},$$

then $Q_t(0) = 1$. In the case $r = 2$,

$$\lambda_2 = \int F(x)\big[1-F(x)\big]dx. \qquad (2.1.14)$$

Now, $0 \le F(x) \le 1$ for all x. So,

$$\begin{aligned}|\lambda_r| &\le (r-1)^{-1}\int |F(1-F)Q_{r-2}(F)|dx \\ &= (r-1)^{-1}\int |Q_{r-2}|F(1-F)dx \\ &\le (r-1)^{-1}\int \sup_{0\le t\le 1}|Q_{r-2}(t)|F(1-F)dx \\ &= (r-1)^{-1}\sup_{0\le t\le 1}|Q_{r-2}(t)|\lambda_2.\end{aligned}$$

We have (see [30])

$$\sup_{0 \le t \le 1} |Q_r(t)| = r + 1$$

with the supremume being attained only at $t = 0$ or $t = 1$. Thus, (see [18]), $|\lambda_r| \le \lambda_2$, with equality only if $F(x)$ can take only the values 0 and 1; i.e., only if X is degenerate. Thus, a nondegenerate distribution has $|\lambda_r| \le \lambda_2$, which together with $\lambda_2 > 0$ implies $|\tau_r| < 1$.

If $X \ge 0$ almost surely, then $\lambda_1 = EX > 0$ and $\lambda_2 > 0$. So,

$$\tau = \frac{\lambda_2}{\lambda_1} > 0.$$

Furthermore, $EX_{1:2} > 0$. So,

$$\tau - 1 = (\lambda_2 - \lambda_1)/\lambda_1 = -EX_{1:2}/\lambda_1 < 0.$$

□

2.2 Probability Weighted Moments

Here we are about to have a tool by which we can easily find the L-moments for any distribution.

Definition 2.2.1. [14] The probability weighted moments (PWMs) of a random variable X with a cumulative distribution function $u = F(X)$ is the quantities

$$M_{p,r,s} = E\big\{X^p F(X)^r (1 - F(X))^s\big\} = \int_0^1 X^p F(X)^r (1 - F(X))^s dF \qquad r = 0, 1, \dots.$$

If we write a cumulative distribution function $F(X) = u$, then the quantile function is $x(u)$ and

$$M_{p,r,s} = E\big\{x(u)^p u^r (1 - u)^s\big\} = \int_0^1 x(u)^p u^r (1 - u)^s du \qquad r = 0, 1, \dots.$$

A particular useful special cases are the probability weighted moments $\alpha_r = M_{1,0,r}$ and $\beta_r = M_{1,r,0}$. For a distribution that has a quantile function $x(u)$,

$$\alpha_r = \int_0^1 x(u)(1 - u)^r du,$$

$$\beta_r = \int_0^1 x(u)u^r du. \tag{2.2.1}$$

These equations may be contrasted with the definition of the ordinary moments, which may be written as

$$E(X^r) = \int_0^1 \{x(u)\}^r \; du.$$

Conventional moments involve successively higher powers of the quantile functions $x(u)$, whereas probability weighted moments involve successively higher powers of u or $1 - u$ and may be regarded as integrals of $x(u)$ weighted by the polynomials u^r or $(1 - u)^r$ [15].

The probability weighted moments α_r and β_r have been used as the basis of methods for estimating parameters of probability distributions. However, they are difficult to interpret directly as measures of the scale and shape of a probability distribution. This information is carried in certain linear combinations of the probability weighted moments. For example, estimates of scale parameters of distributions are multiples of $\alpha_0 - 2\alpha_1$ or $2\beta_1 - \beta_0$. The skewness of a distribution can be measured by $6\beta_2 - 6\beta_1 + \beta_0$ ([15]).

L-moments are linear combination of probability-weighed moments [28], since

$$\lambda_{r+1} = \int_0^1 xP_r^*(F)dF = \int_0^1 \sum_{m=0}^{r} x(F)p_{r,m}^* F^m dF.$$

$$= \sum_{m=0}^{r} p_{r,m}^* \int_0^1 x(F)F^m dF = \sum_{m=0}^{r} p_{r,m}^* \beta_m. \tag{2.2.2}$$

From e.qn. (2.1.4), we have

$$\lambda_{r+1} = \int_0^1 \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix}^2 u^{r-k}(1-u)^k \, Q(u) \, du. \tag{2.2.3}$$

Expand u^{r-k} in powers of $(1-u)$:

$$\lambda_{r+1} = \int_0^1 \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix}^2 (1-u)^k \sum_{j=0}^{r-k} (-1)^{r-k-j} \begin{pmatrix} r-k \\ j \end{pmatrix} (1-u)^{r-k-j} \, Q(u) \, du$$

$$= \int_0^1 \sum_{k=0}^{r} \sum_{j=0}^{r-k} (-1)^{r-j} \begin{pmatrix} r \\ k \end{pmatrix}^2 \begin{pmatrix} r-k \\ j \end{pmatrix} (1-u)^{r-j} \, Q(u) \, du \cdot$$

$$= (-1)^r \int_0^1 \sum_{k=0}^{r} \sum_{j=0}^{r-k} (-1)^j \begin{pmatrix} r \\ r-k \end{pmatrix}^2 \begin{pmatrix} r-k \\ j \end{pmatrix} (1-u)^{r-j} \, Q(u) \, du \cdot$$

$$= (-1)^r \int_0^1 \sum_{k=0}^{r} \sum_{j=0}^{k} (-1)^j \begin{pmatrix} r \\ k \end{pmatrix}^2 \begin{pmatrix} k \\ j \end{pmatrix} (1-u)^{r-j} \, Q(u) \, du \cdot$$

Interchange order of summation over j and k:

$$\lambda_{r+1} = (-1)^r \int_0^1 \sum_{j=0}^{r} \sum_{k=j}^{r} (-1)^j \begin{pmatrix} r \\ k \end{pmatrix}^2 \begin{pmatrix} k \\ j \end{pmatrix} (1-u)^{r-j} \, Q(u) \, du \cdot$$

Reverse order of summation, set $m = r - j$, $n = r - k$:

$$\lambda_{r+1} = (-1)^r \int_0^1 \sum_{m=0}^{r} \sum_{n=0}^{m} (-1)^{r-m} \begin{pmatrix} r \\ r-n \end{pmatrix}^2 \begin{pmatrix} r-n \\ r-m \end{pmatrix} (1-u)^m \, Q(u) \, du$$

$$\lambda_{r+1} = (-1)^r \int_0^1 \sum_{m=0}^{r} (-1)^{r-m} \Big\{ \sum_{n=0}^{m} \begin{pmatrix} r \\ r-n \end{pmatrix}^2 \begin{pmatrix} r-n \\ r-m \end{pmatrix} \Big\} (1-u)^m \, Q(u) \, du. \quad (2.2.4)$$

Note that

$$\begin{pmatrix} r \\ r-n \end{pmatrix}^2 \begin{pmatrix} r-n \\ r-m \end{pmatrix} = \begin{pmatrix} r \\ n \end{pmatrix} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \quad (2.2.5)$$

(expand the binomial coefficients in terms of factorials) and that

$$\sum_{n=0}^{m} \begin{pmatrix} r \\ n \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \sum_{n=0}^{m} \begin{pmatrix} r \\ r-n \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} r+m \\ r \end{pmatrix} = \begin{pmatrix} r+m \\ m \end{pmatrix} \quad (2.2.6)$$

(second equality follows because to choose r items from $r+m$ we can choose from the first m items and $r-n$ from the remaining r items, for any n in $0, 1, ..., m$). From eq.n.(2.2.5) and eq.n.(2.2.6), we have:

$$\sum_{n=0}^{m} \begin{pmatrix} r \\ r-n \end{pmatrix}^2 \begin{pmatrix} r-n \\ r-m \end{pmatrix} = \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} r+m \\ m \end{pmatrix}, \quad (2.2.7)$$

and substituting into 2.4.1 gives

$$\lambda_{r+1} = (-1)^r \int_0^1 \sum_{m=0}^{r} (-1)^{r-m} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} r+m \\ m \end{pmatrix} (1-u)^m \, Q(u) \, du$$

$$\lambda_{r+1} = (-1)^r \int_0^1 \sum_{m=0}^{r} (-1)^{r-m} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} r+m \\ m \end{pmatrix} x(F) \, (1-F)^m \, dF. \quad (2.2.8)$$

$$\lambda_{r+1} = (-1)^r \sum_{m=0}^{r} p^*_{r,m} \int_0^1 x(F) \, (1-F)^m \, dF \quad (2.2.9)$$

$$\lambda_{r+1} = (-1)^r \sum_{m=0}^{r} p^*_{r,m} \alpha_m.$$

Hence,

$$\lambda_{r+1} = \sum_{m=0}^{r} p^*_{r,m}\ \beta_m = (-1)^r \sum_{m=0}^{r} p^*_{r,m}\ \alpha_m. \tag{2.2.10}$$

For example, the first four L-moments are related to the PWMs as follows [25]:

$$\begin{aligned} \lambda_1 &= \beta_0 = \alpha_0, \\ \lambda_2 &= 2\beta_1 - \beta_0 = \alpha_0 - 2\alpha_1, \\ \lambda_3 &= 6\beta_2 - 6\beta_1 + \beta_0 = \alpha_0 - 6\alpha_1 + 6\alpha_2, \\ \lambda_4 &= 20\beta_3 - 30\beta_2 + 12\beta_1 + \beta_0 = \alpha_0 - 12\alpha_1 + 30\alpha_2 - 20\alpha_3. \end{aligned} \tag{2.2.11}$$

2.3 Relation of L-moments with Order Statistic

From (1.7.1), the cdf of r^{th} order statistic is given by:

$$F_r(x) = \sum_{k=r}^{n} \binom{n}{k} F(x)^k \big[1 - F(x)\big]^{n-k}. \tag{2.3.1}$$

Definition 2.3.1. ([10], Page 107) We define the *Beta function* $B(a, b)$ as follows:

$$B(a, b) = \int_0^1 t^{a-1}(1 - t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}\ . \tag{2.3.2}$$

Note 2.3.1. *If a, b are positive integers, then from Note 1.4.3 we can write*

$$B(a, b) = \frac{(a - 1)!(b - 1)!}{(a + b - 1)!}\ . \tag{2.3.3}$$

Definition 2.3.2. [25] The *incomplete Beta function* $I_x(a, b)$ is defined via the *Beta function* B(a,b) as follows:

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1}(1 - t)^{b-1} dt. \tag{2.3.4}$$

Theorem 2.3.2. *The expression*

$$F_r(x) = \sum_{k=r}^{n} \binom{n}{k} F(x)^k \big[1 - F(x)\big]^{n-k}.$$

can be written in terms of an incomplete Beta function as:

$$F_r(x) = r\begin{pmatrix} n \\ r \end{pmatrix}\int_0^{F(x)} u^{r-1}(1-u)^{n-r}du = I_{F(x)}(r, n-r+1).$$

Proof. **Claim**:

$$\sum_{k=a}^{n}\begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^x t^{a-1}(1-t)^{b-1}dt,$$

where $n = a+b-1$, Γ is the gamma function and $0 < x < 1$.

Proof of the claim: First, we want to find a formula for

$$\int_0^x t^{a-1}(1-t)^{b-1}dt.$$

Integrating by partes, let's put $u = (1-t)^{b-1}$, $dv = t^{a-1}dt$, then

$$du = -(b-1)(1-t)^{b-2}dt, \qquad v = \frac{t^a}{a}.$$

So,

$$\int_0^x t^{a-1}(1-t)^{b-1}dt = \frac{t^a(1-t)^{b-1}}{a}\Big|_0^x + \int_0^x \frac{t^a}{a}(b-1)(1-t)^{b-2}dt.$$

Hence,

$$\int_0^x t^{a-1}(1-t)^{b-1}dt = \frac{x^a(1-x)^{b-1}}{a} + \frac{(b-1)}{a}\int_0^x t^a(1-t)^{b-2}dt. \tag{2.3.5}$$

Now, by formula (2.3.5), we have:

$$\int_0^x t^{a-1}(1-t)^{b-1}dt = \frac{x^a(1-x)^{b-1}}{a} + \frac{(b-1)}{a}\int_0^x t^a(1-t)^{b-2}dt$$

$$= \frac{x^a(1-x)^{b-1}}{a} + \frac{(b-1)}{a}\Big[\frac{x^{a+1}(1-x)^{b-2}}{a+1} + \frac{(b-2)}{a+1}\int_0^x t^{a+1}(1-t)^{b-3}dt\Big]$$

$$= \frac{x^a(1-x)^{b-1}}{a} + \frac{(b-1)x^{a+1}(1-x)^{b-2}}{a(a+1)} + \frac{(b-1)(b-2)}{a(a+1)}\int_0^x t^{a+1}(1-t)^{b-3}dt$$

$$= \frac{x^a(1-x)^{b-1}}{a} + \frac{(b-1)x^{a+1}(1-x)^{b-2}}{a(a+1)}$$

$$+ \frac{(b-1)(b-2)}{a(a+1)}\Big[\frac{x^{a+2}(1-x)^{b-3}}{a+2} + \frac{(b-3)}{a+2}\int_0^x t^{a+2}(1-t)^{b-4}dt\Big]$$

$$= \frac{x^a(1-x)^{b-1}}{a} + \frac{(b-1)x^{a+1}(1-x)^{b-2}}{a(a+1)} + \frac{(b-1)(b-2)x^{a+2}(1-x)^{b-3}}{a(a+1)(a+2)}$$

$$+ \frac{(b-1)(b-2)(b-3)}{a(a+1)(a+2)} \int_0^x t^{a+2}(1-t)^{b-4}dt$$

$$= \frac{x^a(1-x)^{b-1}}{a} + \frac{(b-1)x^{a+1}(1-x)^{b-2}}{a(a+1)} + \frac{(b-1)(b-2)x^{a+2}(1-x)^{b-3}}{a(a+1)(a+2)}$$

$$+ \ ... + \frac{(b-1)(b-2)(b-3)}{a(a+1)(a+2)} \int_0^x t^{a+2}(1-t)^{b-4}dt.$$

Therefore,

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1-t)^{b-1}dt = \frac{(a+b-1)!}{(a-1)!(b-1)!} \int_0^x t^{a-1}(1-t)^{b-1}dt \quad \text{(from Note 1.4.3)}$$

$$= \frac{(a+b-1)!}{(a-1)!(b-1)!}\Big[\frac{x^a(1-x)^{b-1}}{a} + \frac{(b-1)x^{a+1}(1-x)^{b-2}}{a(a+1)} + \frac{(b-1)(b-2)x^{a+2}(1-x)^{b-3}}{a(a+1)(a+2)}$$

$$+ \ ... + \frac{(b-1)(b-2)(b-3)}{a(a+1)(a+2)} \int_0^x t^{a+2}(1-t)^{b-4}dt\Big]$$

$$= \frac{(a+b-1)!}{(a-1)!(b-1)!} \times \frac{x^a(1-x)^{b-1}}{a} + \frac{(a+b-1)!}{(a-1)!(b-1)!} \times \frac{(b-1)x^{a+1}(1-x)^{b-2}}{a(a+1)}$$

$$+ \frac{(a+b-1)!}{(a-1)!(b-1)!} \times \frac{(b-1)(b-2)x^{a+2}(1-x)^{b-3}}{a(a+1)(a+2)}$$

$$+ \ ... + \frac{(a+b-1)!}{(a-1)!(b-1)!} \times \frac{(b-1)(b-2)(b-3)}{a(a+1)(a+2)} \int_0^x t^{a+2}(1-t)^{b-4}dt$$

$$= \frac{(a+b-1)! \ x^a \ (1-x)^{b-1}}{a! \ (b-1)!} + \frac{(a+b-1)! \ x^{a+1} \ (1-x)^{b-2}}{(a+1)! \ (b-2)!}$$

$$+ \frac{(a+b-1)! \ x^{a+2} \ (1-x)^{b-3}}{(a+2)! \ (b-3)!} + \frac{(a+b-1)!}{(b-4)! \ (a+2)!} \int_0^x t^{a+2}(1-t)^{b-4}dt$$

$$= \frac{(a+b-1)!\, x^a\, (1-x)^{b-1}}{a!\, (b-1)!} + \frac{(a+b-1)!\, x^{a+1}\, (1-x)^{b-2}}{(a+1)!\, (b-2)!}$$

$$+ \frac{(a+b-1)!\, x^{a+2}\, (1-x)^{b-3}}{(a+2)!\, (b-3)!} + ... + \frac{(a+b-1)!}{(a+b-2)!\, (b-b)!} \int_0^x t^{a+b-2}(1-t)^{b-b} dt$$

$$= \frac{(a+b-1)!\, x^a\, (1-x)^{b-1}}{a!\, (b-1)!} + \frac{(a+b-1)!\, x^{a+1}\, (1-x)^{b-2}}{(a+1)!\, (b-2)!}$$

$$+ \frac{(a+b-1)!\, x^{a+2}\, (1-x)^{b-3}}{(a+2)!\, (b-3)!} + ... + \frac{(a+b-1)!\, x^{a+b-1}\, (1-x)^{0}}{(a+b-1)!\, 0!}$$

$$= \begin{pmatrix} a+b-1 \\ a \end{pmatrix} x^a (1-x)^{b-1} + \begin{pmatrix} a+b-1 \\ a+1 \end{pmatrix} x^{a+1}(1-x)^{b-2}$$

$$+ \begin{pmatrix} a+b-1 \\ a+2 \end{pmatrix} x^{a+2}(1-x)^{b-3} + ... + \begin{pmatrix} a+b-1 \\ a+b-1 \end{pmatrix} x^{a+b-1}(1-x)^{0}$$

$$= \sum_{k=a}^{a+b-1} \begin{pmatrix} a+b-1 \\ k \end{pmatrix} x^k\, (1-x)^{a+b-1-k}$$

$$= \sum_{k=a}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k\, (1-x)^{n-k}, \quad \text{where} \;\; n = a+b-1.$$

This completes the proof of the claim.

Now, we want to show that

$$F_r(x) = r \begin{pmatrix} n \\ r \end{pmatrix} \int_0^{F(x)} u^{r-1}(1-u)^{n-r} du = I_{F(x)}(r, n-r+1).$$

From eqn.(2.3.4), we have

$$I_{F(x)}(r, n-r+1) = \frac{1}{B(r, n-r+1)} \int_0^{F(x)} u^{r-1}(1-u)^{n-r} du. \qquad (2.3.6)$$

Indeed; note that:

$$\frac{1}{B(r, n-r+1)} = \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} = \frac{\Gamma(r+n-r+1)}{\Gamma(r)\Gamma(n-r+1)}.$$

Substitute in eqn.(2.3.6),

$$I_{F(x)}(r, n-r+1) = \frac{\Gamma(r+n-r+1)}{\Gamma(r)\Gamma(n-r+1)} \int_0^{F(x)} u^{r-1}(1-u)^{n-r}du.$$

By the Claim we have:

$$I_{F(x)}(r, n-r+1) = \sum_{k=r}^{n} \begin{pmatrix} n \\ k \end{pmatrix} F(x)^k \big[1-F(x)\big]^{n-k}.$$

From eq.n.(2.3.1), we have:

$$I_{F(x)}(r, n-r+1) = F_r(x).$$

□

We want to connect the *cdf* of the j^{th} *order statistic* with the probability wight moments.

The probability density function of $X_{r:n}$ is given by Theorem 1.7.2 as follows:

$$f_r(x) = r \begin{pmatrix} n \\ r \end{pmatrix} F^{r-1}(x)\big[1-F(x)\big]^{n-r} f(x). \tag{2.3.7}$$

Now, the expected value of r^{th} order statistics can be obtained as

$$E[X_{r:n}] = \int_{-\infty}^{\infty} x f_r(x)dx. \tag{2.3.8}$$

Substituting from e.qn.(2.3.7) into (2.3.8) and introducing a transformation, $u = F(x)$ or $x = F^{-1}(u)$, $0 \leq u \leq 1$, leads to:

$$E[X_{r:n}] = r \begin{pmatrix} n \\ r \end{pmatrix} \int_0^1 x(u)u^{r-1}(1-u)^{n-r}du\cdot \tag{2.3.9}$$

Note that, $x(u)$ denotes the quantile function of a random variable. The expectation of the maximum and minimum of a sample of size n can be easily obtained from eq.n.(2.3.9) by setting $r = n$ and $r = 1$, respectively as follows:

$$E[X_{n,n}] = n \int_0^1 x(u)u^{n-1}du, \qquad (2.3.10)$$

and

$$E[X_{1:n}] = n \int_0^1 x(u)(1-u)^{n-1}du. \qquad (2.3.11)$$

The probability weighted moments (PWMs)of a random variable was formally defined by :

$$M_{i,j,k} = E[x(u)^i u^j (1-u)^k] = \int_0^1 x(u)^i u^j (1-u)^k du.$$

The following two forms of PWMs are particularly simple and useful:

$$\alpha_k = M_{1,0,k} = \int_0^1 x(u)(1-u)^k du \qquad (k = 0, 1, ...n) \qquad (2.3.12)$$

and

$$\beta_k = M_{1,k,0} = \int_0^1 x(u)u^k du \qquad (k = 0, 1, ...n). \qquad (2.3.13)$$

Comparing eq.ns (2.3.12) and (2.3.13), with e.qns (2.3.10) and (2.3.11), it can be seen that α_k and β_k, respectively, are related to the expectation of the minimum and maximum in a sample of size k as follows:

$$\alpha_k = \frac{1}{k+1} E[X_{1:k+1}],$$

$$\beta_k = \frac{1}{k+1} E[X_{k+1:k+1}] \quad (k \geq 1). \qquad (2.3.14)$$

In fact, PWMs are the normalized expectations of maximum/minimum of k random observations. The normalization is done by the sample of size k itself. From eq.n.(2.3.10), we notice that:

$$E[X_{n:n}] = n\beta_{n-1}.$$

From e.qn.(2.3.13), we have:

$$\beta_{n-1} = \int_{-\infty}^{\infty} xF^{n-1}(x)f(x)dx.$$

So,

$$E[X_{n:n}] = \int_{-\infty}^{\infty} xnf(x)F^{n-1}(x)dx.$$

2.4 Properties of L-moments

The L-moments λ_1 and λ_2, the $L-CV,\ \tau$, and L-moment ratios τ_3 and τ_4 are most useful quantities for summarizing probability distributions. Their most important properties are the following (proofs are given in [17], [18]):

1. *Existence.* If the mean of the distribution exists, then all of the L-moments exist.

2. *Uniqueness.* If the mean of the distribution exists, then the L-moments uniquely define the distribution. That is; no two distinct distributions have the same L-moment. Properties 1 and 2 are proved in the next theorem.

   Theorem 2.4.1. *[1] (i) The L-moments $\lambda_r,\ r = 1, 2, \ldots$ of a real-valued random variable X exists if and only if X has a finite mean.*
   (ii) A distribution whose mean exists is characterized by its L-moments $\lambda_r : r = 1, 2, \ldots$.

   Proof. We know that a finite mean implies a finite expectation of all order statistics (see Lemma 2.1.1). Since the L-moments $\lambda_r,\ r = 1, 2, \ldots$ are a linear functions of the expected order statistics, then the L-moments $\lambda_r,\ r = 1, 2, \ldots$ exist.
   Conversely, if the L-moments $\lambda_r,\ r = 1, 2, \ldots$ of a real-valued random variable X exist, then the mean$= \lambda_1$ exists.
   For part (ii), we first show that a distribution is characterized by the set

   $$\{EX_{r:r}, \quad r = 1, 2, \ldots\}.$$

   Let X and Y be random variables with cumulative distribution functions F and G and quantile functions $x(u)$ and $y(u)$, respectively. Let
   $\xi_r^{(X)} \equiv EX_{r:r} = r \int_0^1 x\ F(x)^{r-1}\ dF(x), \qquad \xi_r^{(Y)} \equiv EY_{r:r} = r \int_0^1 x\ G(x)^{r-1}\ dG(x).$
   Then,

   $$\xi_{r+2}^{(X)} - \xi_{r+1}^{(X)} \;\; = \;\; \int_0^1 x(r+2)u^{r+1} - (r+1)u^r\ du$$

$$
\begin{aligned}
&= \int_0^1 u^r . u(1-u)\, dx(u) \qquad\qquad \text{integrating by parts,}\\
&= \int_0^1 u^r .\, dz_X(u),
\end{aligned}
$$

where $z_X(u)$, defined by $dz_X(u) = u(1-u)\, dx(u)$, is an increasing function on (0,1). If $\xi_r^{(X)} = \xi_r^{(Y)},\ r = 1, 2, \ldots$, then

$$\int_0^1 u^r .\, dz_X(u) = \int_0^1 u^r .\, dz_Y(u).$$

Thus, z_X and z_Y are distributions which have the same moments on the finite interval (0,1). Consequently, by Theorem 1.4.4, $z_X(u) = z_Y(u)$. Hence, $dz_X = dz_Y$. That means, $u(1-u)\, dx(u) = u(1-u)\, dy(u)$. Since $u(1-u) \neq 0$, then $dx(u) = dy(u)$. This implies that $x(u) = y(u)$, and so $F = G$.

Conversely, if $F = G$, then $x(u) = y(u)$ and

$$\xi_r^{(X)} = EX_{r:r} = r\int_0^1 xF(x)^{r-1}\, dF(x) = r\int_0^1 xG(x)^{r-1}\, dG(x) = EY_{r:r} = \xi_r^{(Y)}.$$

We have shown that a distribution with finite mean is characterized by the set $\{\xi_r : r = 1, 2, ...\}$.

Now, we want to show that a distribution with finite mean is characterized by its L-moments $\lambda_r : r = 1, 2, \ldots$.

Recall eq.n. (2.1.12),

$$\lambda_r = \int_0^1 x(F) P^*_{r-1}(F) dF, \qquad\qquad r = 1, 2, ...,$$

$$P^*_r(F) = \sum_{k=0}^{r} p^*_{r,k} F^k, \qquad p^*_{r,k} = (-1)^{r-k} \begin{pmatrix} r \\ k \end{pmatrix} \begin{pmatrix} r+k \\ k \end{pmatrix}.$$

Since

$$P^*_{r-1}(F) = \sum_{k=0}^{r-1} p^*_{r-1,k} F^k,$$

then,

$$
\begin{aligned}
\lambda_r &= \int_0^1 x(F)\Big[\sum_{k=0}^{r-1} p^*_{r-1,k}\ F^k\Big]\ dF \\
&= \int_0^1 \sum_{k=0}^{r-1} p^*_{r-1,k}\ x(F)\ F^k\ dF = \sum_{k=0}^{r-1} \int_0^1 p^*_{r-1,k}\ x(F)\ \{F(x)\}^k\ dF \\
&= \sum_{k=0}^{r-1} p^*_{r-1,k} \int_0^1 x(F)\ \{F(x)\}^k\ dF = \sum_{k=1}^{r} p^*_{r-1,k-1} \int_0^1 x(F)\ \{F(x)\}^{k-1}\ dF \\
&= \sum_{k=1}^{r} p^*_{r-1,k-1} k^{-1} \underbrace{k \int_0^1 x(F)\ \{F(x)\}^{k-1}\ dF}_{\xi_k} = \sum_{k=1}^{r} p^*_{r-1,k-1}\ k^{-1} \xi_k.
\end{aligned}
$$

From [18], we have:

$$
\xi_r = \sum_{k=1}^{r} \frac{(2k-1)\ r!\ (r-1)!}{(r-k)!\ (r-1+k)!} \lambda_k\ .
$$

Thus, a given set of λ_r determines a unique set of $\{\xi_r : r = 1, 2, ...\}$, since a distribution with finite mean is characterized by the set $\{\xi_r : r = 1, 2, ...\}$. Therefor, a distribution whose mean exists is characterized by its L-moments $\lambda_r : r = 1, 2, \ldots$.

□

Thus, a distribution may be specified by its L-moments even if some of its conventional moments do not exist([18]).

3. *Terminology*([15], Page 24)

   - λ_1 is the L-location or mean of the distribution.
   - λ_2 is the L-scale.
   - τ is the L-CV
   - τ_3 is the L-skewness.
   - τ_4 is the L-kurtosis.

4. *Numerical values*

   - λ_1 can take any value, because $\lambda_1 = E(X)$ and X may be positive or negative.

- $\lambda_2 \geq 0$, because $\lambda_2 = E(X_{2:2} - X_{2:1})$ and $X_{2:2} \geq X_{2:1}$.
- For any distribution that takes only positive values, $0 \leq \tau < 1$, this is proved in Theorem 2.1.2.
- L-moment ratios satisfy $|\tau| < 1$ for all $r \geq 3$. This is proved in Theorem 2.1.2.
- Tighter bounds can be found for individual τ_r quantities. For example, bounds for τ_4 given τ_3 are
$$\frac{1}{4}(5\tau_3^2 - 1) \leq \tau_4 < 1 \quad ([15])\cdot$$
- For a distribution that takes only positive values, bounds for τ_3 given τ are $2\tau - 1 \leq \tau_3 < 1 \quad ([18])\cdot$

5. *Linear transformation.* Let X and Y be random variables with L-moments λ_r and λ_r^*, respectively, and suppose that $Y = aX + b$, $a > 0$. Then,
(I) $\lambda_1^* = a\lambda_1 + b$;
(II) $\lambda_2^* = a\lambda_2$;
(III) $\tau_r^* = \tau_r$, $\quad r \geq 3$.

Proof. (I) Assume that X and Y are random variables with cumulative distribution functions F and G and quantile functions $x(u)$ and $y(u)$.
Let $u = G_Y(y)$. Then,
$$y = G_Y^{-1}(u) = y(u). \tag{2.4.1}$$
Since
$$\begin{aligned} u &= G_Y(y) = P(Y \leq y) = P(aX + b \leq y) \\ &= P\Big(aX \leq y - b\Big) = P\Big(X \leq \frac{y-b}{a}\Big) \quad \text{(because } a > 0) \\ &= F_X\Big(\frac{y-b}{a}\Big), \end{aligned}$$
then $F_X^{-1}(u) = \frac{y-b}{a}\cdot$ So, $x(u) = \frac{y-b}{a}\cdot$ Hence,
$$y = ax(u) + b. \tag{2.4.2}$$

From eq.n.(2.4.1) and eq.n(2.4.2) we have:

$$y(u) = ax(u) + b. \tag{2.4.3}$$

Then, from eq.n.(2.4.3) we have:

$$\begin{aligned} \lambda_1(Y) &= \int_0^1 y(u)du = \int_0^1 [ax(u) + b]du \\ &= a\int_0^1 x(u)du + b = a\lambda_1(X) + b. \end{aligned}$$

(II)

$$\begin{aligned} \lambda_2(Y) &= \int_0^1 y(u)(2u-1)du \\ &= \int_0^1 [ax(u) + b](2u-1)du \\ &= a\int_0^1 x(u)(2u-1)du + b\int_0^1 (2u-1)du \\ &= a\lambda_2(X). \end{aligned}$$

(III) From eq.n.(2.1.12),

$$\begin{aligned} \lambda_r(Y) &= \int_0^1 y(F)P^*_{r-1}(F)d\,F, \qquad r = 3, 4 \ldots \\ &= \int_0^1 (ax(F) + b)P^*_{r-1}(F)d\,F \\ &= a\int_0^1 x(F)P^*_{r-1}(F)d\,F + b\int_0^1 P^*_{r-1}(F)\,dF, \quad \text{by eq.n(1.6.9)} \\ &= a\lambda_r(X). \end{aligned}$$

Hence, for all $r \geq 3$,

$$\begin{aligned} \tau_r(Y) &= \lambda_r(Y)/\lambda_2(Y) \\ &= a\lambda_r(X)/a\lambda_2(X) \\ &= \tau_r(X). \end{aligned}$$

□

6. *Symmetry.* Let X be a symmetric random variable with mean μ. That is; $P(X \geq \mu + x) = P(X \leq \mu - x)$ for all x. Then, all of the odd-order L-moment ratios of X are zero. That is; $\tau_r = 0,\ r = 3, 5, ...$([15], Page 24).

Proof. Assume X has the cumulative distribution function $F_X(x)$ and the quantile function $x(u)$.

Claim: If X is a symmetric random variable with mean μ then,

$$x(u) = -x(1-u).$$

Proof of the claim: Conceder, X is a symmetric random variable with mean μ, that means, $P(X \geq \mu + x) = P(X \leq \mu - x)$, then $P(X \geq x) = P(X \leq -x)$.
Now, $F_X(-x) = P(X \leq -x) = P(X \geq x) = 1 - P(X \leq x) = 1 - F_X(x)$.
Hence, $-x(u) = F_X^{-1}[F_X(-x)] = F_X^{-1}[1 - F_X(x)] = F_X^{-1}(1-u) = x(1-u)$.
Therefor, $x(u) = -x(1-u)$. This completes the proof of the claim.
Now we want to calculate $\lambda_{2r+1}, r = 1, 2, 3\ldots$ for a symmetric random variable X, then we want to fined it's $\tau_{2r+1}, r = 1, 2, 3\ldots$
Replace $2r+1$ with r in eqn.(2.1.12) we have

$$\lambda_{2r+1} = \int_0^1 x(u)\ P^*_{2r}(u)\ du = \int_0^1 \sum_{m=0}^{2r} p^*_{2r,m}\ x(u)\ u^m du. \tag{2.4.4}$$

And replace $2r+1$ with $r+1$ in eqn.(2.2.9) we have

$$\begin{aligned}
\lambda_{2r+1} &= (-1)^{2r+1} \sum_{m=0}^{2r} p^*_{2r,m} \int_0^1 x(u)\ (1-u)^m\ du \\
&= -\int_0^1 \sum_{m=0}^{2r} p^*_{2r,m}\ x(u)\ (1-u)^m\ du \\
&= -\int_0^1 \sum_{m=0}^{2r} p^*_{2r,m}\ -x(1-u)\ (1-u)^m\ du, \quad \text{by the claim} \\
&= -\int_0^1 \sum_{m=0}^{2r} p^*_{2r,m}\ x(1-u)\ (1-u)^m\ d(1-u)
\end{aligned}$$

$$= -\int_0^1 \sum_{m=0}^{2r} p^*_{2r,m}\ x(z)\ z^m\ dz, \quad \text{where } z = 1-u. \tag{2.4.5}$$

From eqn's (2.4.4) and (2.4.5) we have $\lambda_{2r+1} = -\lambda_{2r+1}$, that means, $\lambda_{2r+1} = 0$ for all $r = 1, 2, 3, \ldots$.

Then,

$$\tau_{2r+1} = \frac{\lambda_{2r+1}}{\lambda_2} = \frac{0}{\lambda_2} = 0.$$

□

2.5 L-skewness and L-kurtosis

The main features of a probability distribution should be well-summarized by the following four measures: the mean or L-location (λ_1), the L-scale λ_2, the L-skewness τ_3 and the L-kurtosis τ_4. We now consider these measures, particulary τ_3 and τ_4 in more details.

The L-moment measure of location is the mean, λ_1. This is a well-established and familiar quantity which needs no further description or justification here [17].

The L-scale λ_2 is also long established in statistic, for it is, apart from a scalar multiple, the expectation of Gini's mean difference statistic. To compare λ_2 with the more familiar scale measure σ, the standard deviation, write

$$\lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2}), \qquad \sigma_2 = \frac{1}{2}E(X_{2:2} - X_{1:2})^2$$

Both quantities measure the difference between two randomly drawn elements of a distribution, but σ_2 gives relatively more weight to the largest difference [17].

λ_2 is used to obtain scale-free higher-order descriptive measure, $\tau_k = \frac{\lambda_k}{\lambda_2}$, $k \geq 3$ called L-moment ratios, very conveniently for practical use and interpretation.

Table 2.1 shows the L-skewness for some common distributions. The L-skewness τ_3 is a dimensionless analogue of λ_3. By theorem 2.4.1, τ_3 takes values between -1 and $+1$.

$$\tau_3 = \frac{EX_{3:3} - 2EX_{2:3} + EX_{1:3}}{EX_{3:3} - EX_{1:3}}$$

shows that τ_3 is similar in form to a measure of skewness.

Table 2.1: L-skewness of some common distribution

Distribution	L-skewness
Uniform	0
Exponential	$\frac{1}{3}$
Gumble	0.1699
Logistic	0
Normal	0
Generalized Pareto	$\tau_3 = (1-k)/(3+k)$
Generalized extreme value	$2(1-3^{-k})/(1-2^{-k})-3$
Generalized logistic	$-k$

Table 2.2: L-kurtosis of some common distributions

Distribution	L-kurtosis
Uniform	0
Exponintial	$\frac{1}{6}$
Gumble	0.1504
Logistic	$\frac{1}{6}$
Normal	0.1226
Generalized Pareto	$(1-k)(2-k)/(3+k)(4+k)$
Generalized extreme value	$(1-6.2^{-k}+10.3^{-k}-5.4^{-k})/(1-2^{-k})$
Generalized logistic	$(1+5k^2)/6$

L-kurtosis, τ_4, is equally difficult to interpret uniquely and is best thought of as a measure similar to κ but giving less weight to the extreme tails of the distribution [17]. Table 2.2 shows the L-kurtosis for some common distributions:

2.6 L-moments of a Polynomial Function of Random Variables

In this section, we find the k^{th} PWMs of a random variable $Y = X^m$ and we apply this relation to the standard normal distribution and to the exponential distribution.

The k^{th} PWMs of a random variable X with quantile function $x(u)$ is given from eq.n.(2.2.1) by:

$$\beta_k = \int_0^1 x(u)u^k du.$$

The quantile function of the random variable $Y = X^m$ follows from a transformation $y(u) = x^m(u)$: Since

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(X^m \le y) = P(X \le y^{\frac{1}{m}}) = F_X(y^{\frac{1}{m}}) \\ F_Y^{-1}(u) &= y, \qquad F_X^{-1}(u) = y^{\frac{1}{m}}. \end{aligned}$$

Then,

$$y(u) = F_Y^{-1}(u) = y = (F_X^{-1}(u))^m = x^m(u).$$

Therefore, the k^{th} PWMs of X^m is given by:

$$\beta_k = \int_0^1 [F_X^{-1}(u)]^m u^k du.$$

In particular, if X is a standard normally distributed variable, then the following PWM's is:

$$\beta_k = \int_0^1 [\Phi^{-1}(u)]^m u^k du$$

and can be calculated numerically as shown in Table 2.3.

Table 2.3: [25] Matrix B with numerical evaluations of $\beta_k = \int_0^1 (\Phi^{-1}(u))^m u^k du$

	X^0	X^1	X^2	X^3	X^4	X^5
β_0	1	0	1	0	3	0
β_1	1/2	0.282	0.5	0.705	1.5	3.032
β_2	1/3	0.282	0.425	0.705	1.400	3.032
β_3	1/4	0.257	0.388	0.675	1.350	2.969
β_4	1/5	0.233	0.360	0.650	1.305	2.907
β_5	1/6	0.211	0.337	0.618	1.266	2.848

In particular, if X is exponentially distributed, then the following PWM's can be written as:

$$\beta_k = \int_0^1 [\xi - \alpha \ln(1-u)]^m u^k du$$

and can be calculated numerically as shown in Table 2.4.
L-moments are linear combinations of the PWMs, from eq.n.s(2.2.11)

$$\begin{aligned}
\lambda_1 &= \beta_0, \\
\lambda_2 &= -\beta_0 + 2\beta_1, \\
\lambda_3 &= \beta_0 - 6\beta_1 + 6\beta_2, \\
\lambda_4 &= -\beta_0 + 12\beta_1 - 30\beta_2 + 20\beta_3.
\end{aligned}$$

Then, L-moments are given by the matrix multiplication $\lambda = AB$, in which

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -6 & 6 & 0 \\ -1 & 12 & -30 & 20 \end{pmatrix}, \quad B = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}.$$

Furthermore, L-moments are linear combinations of observations and therefore the L-moments of the summation of two random variables is given by the summation of the L-moments of the random variables separately[25].

Table 2.4: [25] Matrix B with numerical evaluation of $\beta_k = \int_0^1 (\xi - \alpha \log(1-u))^m u^k du$

	X^0	X^1	X^2	X^3	X^4
β_0	1	$\xi + \alpha$	$\xi^2 + 2\alpha^2 + 2\xi\alpha$	$\xi^3 + 6\alpha^3 + 6\xi\alpha^2 + 3\xi^2 + 3\xi^2\alpha$	$4\xi^3\alpha + 12\xi^2\alpha^2 + 24\xi\alpha^3 + \xi^4 + 24\alpha^4$
β_1	1/2	$1/2\xi + 3/4\alpha$	$3/2\xi\alpha + 7/4\xi^2 + 1/2\xi^2$	$21/4\xi\alpha^2 + 1/2\xi^3 +45/8\alpha^3 + 9/4\xi^2\alpha$	$1/2\xi^4 + 93/4\alpha^4 + 3\xi^3\alpha +21/2\xi^2\alpha^2 + 45/2\xi\alpha^3$
β_2	1/3	$1/3\xi + 11/18\alpha$	$11/9\xi\alpha + 85/54\alpha^2 +1/3\xi^2$	$11/6\xi^2\alpha + 575/108\alpha^3 + 1/3\xi^3 + 85/18\xi\alpha^2$	$575/27\xi\alpha^3 + 3661/162\alpha^4 + 22/9\xi^3\alpha + 85/9\xi^2\alpha^2 + 1/3\xi^4$
β_3	1/4	$1/4\xi + 25/48\alpha$	$25/24\xi\alpha + 415/288\alpha^2 +1/4\xi^2$	$1/4\xi^3 + 25/16\xi^2\alpha + 5845/ 1152\alpha^3 + 415/96\xi\alpha^2$	$415/48\xi^2\alpha^2 + 5845/288\xi\alpha^3 + 76111/3456\alpha^4 + 25/12\xi^3\alpha + 1/4\xi^4$
β_4	1/5	$1/2\xi + 137/300\alpha$	$1/5\xi^2 + 137/ 150\xi\alpha + 12019/ 9000\alpha^2$	$1/5\xi^3 + 137/100\xi^2\alpha + 12019/ 3000\xi\alpha^2 + 874853/180000\alpha^3$ $58067611/2700000\alpha^4$	$137/75\xi^3\alpha + 12019/1500\xi^2\alpha^2 +874853/45000\xi\alpha^3 + 1/5\xi^4 + 18000\xi^2\alpha^3 + 3673451957/$
β_5	1/6	$1/2\xi + 49/120\alpha$	$49/60\xi\alpha + 13489/ 10800\alpha^2 + 1/6\xi^2$	$49/40\xi^2\alpha + 1/6\xi^3 + 13489/ 3600\xi\alpha^2 + 336581/72000\alpha^3$	$1/6\xi^4 + 68165041/3240000\alpha^4 + 49/30\xi^3\alpha + 13489/1800\xi^2\alpha^2 +336581/18000\xi\alpha^3$

	X^5
β_0	$120\xi\alpha^4 + 5\xi^4 + 20\xi^3\alpha^2 + 60\xi^2\alpha^3 + 120\alpha^5 + \xi^5$
β_1	$465/4\xi\alpha^4 + 15/4\xi^4\alpha + 2\xi^3\alpha^2 + 225/4\xi^2\alpha^3 + 945/8\alpha^5 + 1/2\xi^5$
β_2	$18305/162\xi\alpha^4\alpha + 425/27\xi^3\alpha^2 + 2875/54\xi^2\alpha^3 +113155/972\xi^5 + 1/3\xi^5$
β_3	$3805555/3456\xi\alpha^4 + 125/48\xi^4\alpha + 2075/144\xi^3\alpha^2 + 29225/576\xi^2\alpha^3 + 4762625/41472\alpha^5 + 1/4\xi^5$
β_4	$58067611/540000\xi\alpha^4 + 137/ 60\xi^4\alpha + 12019/900\xi^3\alpha^2 + 874853/ 18000\xi^2\alpha^3 + 3673451957/ 32400000\alpha^5 + 1/5\xi^5$
β_5	$68165041/648000\xi\alpha^4 + 49/ 24\xi^4\alpha + 13489/1080\xi^3\alpha^2 + 336581/ 7200\xi^2\alpha^3 + 483900263/ 432000\alpha^5 + 1/6\xi^5$

2.7 Approximating a Quantile Function

In this section we introduce a theorem of a special importance. From this theorem that approximates a quantile function $x(F)$, we can find any distribution when we know it's L-moments and this is by finding its quantile function $x(F)$. After that, we can find the cumulative distribution function $F(x)$ to this distribution and then, we can get its probability distribution function $f(x)$.

Theorem 2.7.1. *[27]. Let X be a real-valued continuous random variable with finite variance, quantile function $x(F)$ and L-moment λ_r, $r \geq 1$. Then the representation*

$$x(F) = \sum_{r=1}^{\infty} (2r-1)\lambda_r P^*_{r-1}(F), \qquad 0 < F < 1,$$

is convergent in mean square, i.e.,

$$R_s(F) \equiv x(F) - \sum_{r=1}^{s} (2r-1)\lambda_r P^*_{r-1}(F),$$

the remainder after stopping the infinite sum after s terms, satisfies

$$\int_0^1 \{R_s(F)\}^2 dF \ \rightarrow \ 0 \qquad as \ s \ \rightarrow \ \infty \ .$$

Proof. The shifted Legendre polynomials $P^*_{r-1}(F)$ are a natural choice as the basis of the approximation because they are orthogonal on $0 < F < 1$ with constant weight function $r(F) = 1$. We can say that $\{P^*_{r-1}(F) : r = 1, 2, 3 \ldots\}$ is the set of eigenfunctions, and $x(F)$ is a function on [0,1], (see section 1.6).

Now, we can apply e.qn. (1.6.4) in Corollary 1.6.2 to fined $x(F)$ as follows:

$$x(F) = \sum_{r=0}^{\infty} c_r \ P^*_{r-1}(F),$$

$$\text{where} \quad c_r = \frac{1}{\mu_r} \int_0^1 r(F) \ u(F) \ P^*_{r-1}(F) \ dF,$$

$$\mu_r = \int_0^1 r(F) \ \{P^*_{r-1}(F)\}^2 \ dF = \int_0^1 \ \{P^*_{r-1}(F)\}^2 \ dF.$$

Since $r(F) = 1$ and by eq.n.(1.6.8),

$$\int_0^1 \{P_r^*(F)\}^2 \, dF = \frac{1}{2r+1} \; ,$$

then,

$$\mu_r = \int_0^1 \{P_{r-1}^*(F)\}^2 \, dF = \frac{1}{2(r-1)+1} = \frac{1}{2r-1} \; \cdot$$

So,

$$c_r = \Big(\frac{1}{2r-1}\Big)^{-1} \int_0^1 \; u(F) \; P_{r-1}^*(F) \; dF = (2r-1)\lambda_r.$$

Hence,

$$x(F) = \sum_{r=0}^{\infty} (2r-1)\lambda_r \; P_{r-1}^*(F). \tag{2.7.1}$$

Now,

$$\int_0^1 \{R_s(F)\}^2 dF \;\; = \;\; \int_0^1 \Big\{x(F) - \sum_{r=1}^{s} (2r-1)\lambda_r P_{r-1}^*(F)\Big\}^2 dF.$$

By e.qn.(2.7.1) and as $s \to \infty$,

$$\int_0^1 \{R_s(F)\}^2 dF = \int_0^1 \Big\{x(F) - \sum_{r=1}^{\infty} (2r-1)\lambda_r P_{r-1}^*(F)\Big\}^2 dF = 0\cdot$$

□

Example 2.7.1. *We can apply Theorem 2.7.1 to find the quantile function $x(F)$ of the uniform distribution from it's L-moments. Since, from subsection 3.9.1, the L-moments for the uniform distribution are: $\lambda_1 = \frac{1}{2}(\alpha+\beta)$, $\lambda_2 = \frac{1}{6}(\beta-\alpha)$ and $\lambda_r = 0$, $\quad r = 3, 4, 5 \ldots$, then,*

$$\begin{aligned} x(F) &= \sum_{r=1}^{\infty} (2r-1)\lambda_r P_{r-1}^*(F) \\ &= \sum_{r=1}^{2} (2r-1)\lambda_r P_{r-1}^*(F) \qquad (\textit{because } \lambda_r = 0, \textit{for all } r > 2) \\ &= \lambda_1 P_0^*(F) + 3\lambda_2 P_1^*(F) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(\alpha+\beta) + 3 \times \frac{1}{6}(\beta-\alpha)[2F-1] \\
&= \frac{1}{2}(\alpha+\beta) - \frac{1}{2}(\beta-\alpha) + (\beta-\alpha)F \\
&= \frac{1}{2}(\alpha+\beta-\beta+\alpha) + (\beta-\alpha)F \\
&= \alpha + (\beta-\alpha)F\cdot
\end{aligned}
$$

Since the cumulative distribution function $F(x)$ is the inverse function of quantile function $x(F)$, then we can find the cumulative distribution function of the uniform distribution $F(x)$ by

$$
F(x) = \begin{cases} 0, & x < \alpha; \\ (x-\alpha)/(\beta-\alpha), & \alpha \le x < \beta; \\ 1, & x \ge \beta. \end{cases}
$$

and we can get the probability distribution function of the uniform distribution $f(x)$ by

$$
f(x) = F'(x) = 1/(\beta-\alpha).
$$

2.8 L-moments as Measures of Distributional Shape

In [12], Oja has defined intuitively reasonable criteria for one probability distribution on the real line to be located further to the right (more dispersed, more skew, kurtotic) than another. Areal-valued function of a distribution that preserves the partial ordering of distributions implied by these criteria may then reasonably be called a “measure of location” (dispersion, skewness, kurtosis). The following theorem shows that τ_3 and τ_4 are, by Oja’s criteria, measures of skewness and kurtosis respectively.

Definition 2.8.1. [24] Let $S \subset R^n$ be a nonempty convex set. Function $f : S \to R$ is said to be convex on S if for any $x_1, x_2 \, \epsilon S$ and all $0 \le \alpha \le 1$, we have

$$
f(\alpha x_1 + (1-\alpha)x_2) \le \alpha f(x_1) + (1-\alpha) f(x_2).
$$

Theorem 2.8.1. *[12] Let X and Y be continues real-valued random variables with cumulative distribution functions F and G respectively, and L-moment $\lambda_r(X)$ and $\lambda_r(Y)$*

respectively.

(i) If $Y = aX + b$*, then* $\lambda_1(Y) = a\lambda_1(X) + b$*,* $\lambda_2(Y) = |a|\lambda_2(X)$ *,* $\tau_3(Y) = \tau_3(X)$*,* $\tau_4(Y) = \tau_4(X)$*.*

(ii) Let $\triangle(x) = G^{-1}(F(x)) - x$*. If* $\triangle(x) \geq 0$ *for all* x*, then* $\lambda_1(Y) \geq \lambda_1(X)$*. If* $\triangle(x)$ *is convex, then* $\tau_3(Y) \geq \tau_3(X)$*.*

Proof. (i) Since $\lambda_1(Y) = E(Y) = E(aX + b) = aE(X) + b = a\lambda_1(X) + b$
and since $Y = (aX + b)$, then
$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P(aX \leq y - b)$, we have two cases:
case(1): $a > 0$, then $a = |a|$ and $F_Y(y) = P(X \leq \frac{y-b}{|a|}) = F_X(\frac{y-b}{|a|})$,
case(2): $a < 0$, then $a = -|a|$ and $F_Y(y) = P(-X \leq \frac{y-b}{|a|}) = P(X \geq -\frac{y-b}{|a|})$
$= P(X \leq \frac{y-b}{|a|}) = F_X(\frac{y-b}{|a|})$.

Let $u = F_Y(y) = F_X(\frac{y-b}{|a|})$. Then,

$$y = F_Y^{-1}(u) \tag{2.8.1}$$

and $\frac{y-b}{|a|} = F_X^{-1}(u)$. Now,

$$y = |a|F_X^{-1}(u) + b. \tag{2.8.2}$$

From eq.n.(2.8.1) and eq.n.(2.8.2), we have: $F_Y^{-1}(u) = |a|F_X^{-1}(u) + b$.
So,

$$y(u) = |a|x(u) + b. \tag{2.8.3}$$

Therefore,

$$\begin{aligned}
\lambda_2(Y) &= \int_0^1 y(u)(2u-1)du \\
&= \int_0^1 [|a|x(u) + b](2u-1)du \qquad \text{from eq.n.(2.8.3)} \\
&= |a| \int_0^1 x(u)(2u-1)du + b\int_0^1 (2u-1)du \\
&= |a|\lambda_2^{(X)} + b(u^2 - u)\Big|_0^1 = |a|\lambda_2(X) \cdot
\end{aligned}$$

Claim:

$$\lambda_r(Y) = |a|\lambda_r(X) + b\sum_{k=0}^{r-1} \frac{p^*_{r-1,k}}{k+1} \ .$$

Proof of the claim: From eq.n.(2.1.12),

$$\begin{aligned}
\lambda_r(Y) &= \int_0^1 y(u)P^*_{r-1}(u)du \\
&= \int_0^1 [|a|x(u)+b]P^*_{r-1}(u)du \qquad \text{from eq.n.(2.8.3)} \\
&= |a|\int_0^1 x(u)P^*_{r-1}(u)du + b\int_0^1 P^*_{r-1}(u)du \\
&= |a|\lambda_r(X) + b\int_0^1 \Big[\sum_{k=0}^{r-1} p^*_{r-1,k}u^k\Big]du \\
&= |a|\lambda_r(X) + b\sum_{k=0}^{r-1} p^*_{r-1,k}\Big[\int_0^1 u^k du\Big] \\
&= |a|\lambda_r(X) + b\sum_{k=0}^{r-1} \frac{p^*_{r-1,k}}{k+1} \ .
\end{aligned}$$

So, from the claim, we have:

$$\begin{aligned}
\lambda_3(Y) &= |a|\lambda_3(X) + b\sum_{k=0}^{2} \frac{p^*_{2,k}}{k+1} \\
&= |a|\lambda_3(X) + b\Big[\frac{p^*_{2,0}}{1} + \frac{p^*_{2,1}}{2} + \frac{p^*_{2,2}}{3}\Big] \\
&= |a|\lambda_3(X) + b\Big[\Big(-1\Big)^2\begin{pmatrix}2\\0\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} + \frac{1}{2}\Big(-1\Big)^1\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} \\
&+ \frac{1}{3}\Big(-1\Big)^0\begin{pmatrix}2\\2\end{pmatrix}\begin{pmatrix}4\\2\end{pmatrix}\Big] \\
&= |a|\lambda_3(X) + b\Big[1 - \frac{1}{2}\times 2\times 3 + \frac{1}{3}\times 1\times 6\Big] \\
&= |a|\lambda_3(X) + b\Big[1 - 3 + 2\Big] = |a|\lambda_3(X).
\end{aligned}$$

Therefore,

$$\begin{aligned} \tau_3(Y) &= \lambda_3(Y)/\lambda_2(Y) = |a|\lambda_3(X)/|a|\lambda_2(X) \\ &= \lambda_3(X)/\lambda_2(X) = \tau_3(X), \end{aligned}$$

and

$$\begin{aligned} \lambda_4(Y) &= |a|\lambda_4(X) + b\sum_{k=0}^{3}\frac{p^*_{3,k}}{k+1} \\ &= |a|\lambda_4(X) + b\Big[\frac{p^*_{3,0}}{1} + \frac{p^*_{3,1}}{2} + \frac{p^*_{3,2}}{3} + \frac{p^*_{3,3}}{4}\Big] \\ &= |a|\lambda_4(X) + b\Big[\Big(-1\Big)^3\begin{pmatrix}3\\0\end{pmatrix}\begin{pmatrix}3\\0\end{pmatrix} + \frac{1}{2}\Big(-1\Big)^2\begin{pmatrix}3\\1\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix} \\ &+ \frac{1}{3}\Big(-1\Big)^1\begin{pmatrix}3\\2\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix} + \frac{1}{4}\Big(-1\Big)^0\begin{pmatrix}3\\3\end{pmatrix}\begin{pmatrix}6\\3\end{pmatrix}\Big] \\ &= |a|\lambda_4(X) + b\Big[-1 + \frac{1}{2}\times 3\times 4 - \frac{1}{3}\times 3\times 10 + \frac{1}{4}\times 1\times 20\Big] \\ &= |a|\lambda_4(X) + b\Big[-1 + 6 - 10 + 5\Big] = |a|\lambda_4(X)\cdot \end{aligned}$$

Now,

$$\begin{aligned} \tau_4(Y) &= \lambda_4(Y)/\lambda_2(Y) = |a|\lambda_4(X)/|a|\lambda_2(X) \\ &= \lambda_4(X)/\lambda_2(X) = \tau_4(X)\cdot \end{aligned}$$

To show that $\lambda_1(Y) \geq \lambda_1(X)$, let $y = G^{-1}(F(x))$. Since $\triangle\ (x) \geq 0$ for all x, then $G^{-1}(F) \geq x(F)$ for all F.

Hence, $y(u) \geq x(u)$ for all u. Therefore,

$$\int_0^1 y(u)du \geq \int_0^1 x(u)du.$$

From eq.n.(2.1.12), $\lambda_1(Y) \geq \lambda_1(X)\cdot$

For τ_3, we want to show that $\tau_3(Y) \geq \tau_3(X)$. Assume that the probability density functions of X and Y are respectively, f and g and let $r(x) = f(x)/g\{G^{-1}(F(x))\}$. Because

$\triangle (x)$ is convex, $r(x) = d \triangle (x)/dx + 1$ is increasing. Since $y = G^{-1}(F(x)),\ G(y) = F(x)$. This implies that

$$\frac{dG(y)}{dx} = \frac{dF(x)}{dx}, \quad \text{and so,} \quad g(y)\frac{dy}{dx} = f(x).$$

So,

$$dy = \Big[f(x)/g(y)\Big]dx \ = \Big[f(x)/g\{G^{-1}(F(x))\}\Big]\ dx = r(x)\ dx.$$

Now, by eq.n.(2.1.14), and from $F(x) = G(y), \ \ dy = r(x)\ dx$ we have:

$$\lambda_2(Y) = \int_{-\infty}^{\infty} G(y)\{1 - G(y)\}\ dy = \int_{-\infty}^{\infty} F(x)\{1 - F(x)\}r(x)dx.$$

Similarly,

$$\lambda_3(Y) = \int_{-\infty}^{\infty} G(y)\{1 - G(y)\}\{2G(y) - 1\}\ dy = \int_{-\infty}^{\infty} F(x)\{1 - F(x)\}\{2F(x) - 1\}r(x)\ dx.$$

Thus, $\lambda_2(X)\lambda_2(Y)\tau_3(Y) - \tau_3(X) = \lambda_3(Y)\lambda_2(X) - \lambda_3(X)\lambda_2(Y)$ which can be written as:

$$\int_{-\infty}^{\infty} F(x)\{1 - F(x)\}\{2F(x) - 1\}r(x)dx. \int_{-\infty}^{\infty} F(x)\{1 - F(x)\}dx$$
$$- \int_{-\infty}^{\infty} F(x)\{1 - F(x)\}\{2F(x) - 1\}. \int_{-\infty}^{\infty} F(x)\{1 - F(x)\}r(x) \qquad (2.8.4)$$

wherein $F(x)\{1-F(x)\}$ is a positive function of x and $2F(x)-1$ and $r(x)$ are increasing. Chebyshev's Other Inequality for integrals (see Theorem 1.6.6) implies that (2.8.4) is positive. Because $\lambda_2(X)\lambda_2(Y) > 0$, it follows that $\tau_3(Y) \geq \tau_3(X)$. $\square$

2.9 L-moments for some Distributions

In this section, we find the first four L-moments for some distributions and this will be used in chapter three in estimating the parameters of some distributions.

This section is divided into four subsections: L-moments for uniform distribution, L-moments for exponential distribution, L-moments for logistic distribution and L-moments for generalized pareto distribution. In Table 2.5, we introduce the first four L-moments for some distributions.

Table 2.5: [17] L-moments of some common distributions

Distribution	$F(x)$ or $x(F)$	L-moments
Uniform	$x = \alpha + (\beta - \alpha)F$	$\lambda_1 = \frac{1}{2}(\beta + \alpha),\ \lambda_2 = \frac{1}{6}(\beta - \alpha),\ \tau_3 = 0,\ \tau_4 = 0$
Exponential	$x = \xi - \alpha \log(1 - F)$	$\lambda_1 = \xi + \alpha,\ \lambda_2 = \frac{1}{2}\alpha,\ \tau_3 = \frac{1}{3},\ \tau_4 = \frac{1}{6}$
Gumble	$x = \xi - \alpha \log(-logF)$	$\lambda_1 = \xi + \gamma\alpha,\ \lambda_2 = \alpha \log 2,\ \tau_3 = 0.1699,\ \tau_4 = 0.1504$
Logistic	$x = \xi - \alpha \log\{F/(1 - F)\}$	$\lambda_1 = \xi,\ \lambda_2 = \alpha,\ \tau_3 = 0,\ \tau_4 = \frac{1}{6}$
Normal	$F = \Phi\left(\frac{x-\mu}{\sigma}\right)$	$\lambda_1 = \mu,\ \lambda_2 = \pi^{-\frac{1}{2}}\sigma,\ \tau_3 = 0,\ \tau_4 = 0.1226$
Generalized Pareto	$x = \xi + \alpha\{1 - (1 - F)^k\}/k$	$\lambda_1 = \xi + \alpha\{1 - \Gamma(1 + k)\}/k,\ \lambda_2 = \alpha/(1 + k)(2k + 1),$ $\tau_3 = (1 - k)/(3 + k),\ \tau_4 = (1 - k)(2 - k)/(3 + k)(4 + k)$
Generalized extreme value	$x = \xi + \alpha\{1 - (-\log F)^k\}/k$	$\lambda_1 = \xi + \alpha\{1 - \Gamma(1 + k)\}/k,\ \lambda_2 = \alpha(1 - 2^{-k})\Gamma(1 + k)/k,$ $\tau_3 = 2(1 - 3^{-k})/(1 - 2^{-k}) - 3,$ $\tau_4 = (1 - 6.2^{-k} + 10.3^{-k} - 5.4^{-k})/(1 - 2^{-k})$
Generalized logistic	$x = \xi + \alpha[1 - \{(1 - F)/F\}^k]/k$	$\lambda_1 = \xi + \alpha\{1 - \Gamma(1 + k)\Gamma(1 - k)\}/k,\ \lambda_2 = \alpha\Gamma(1 + k)(1 - k),$ $\tau_3 = -k,\ \tau_4 = (1 + 5k^2)/6$
Long-normal	$F = \Phi\left(\frac{log(x-\xi)-\mu}{\sigma}\right)$	$\lambda_1 = \xi + exp(\eta + \sigma^2/2),\ \lambda_2 = exp(\eta + \sigma^2/2)erf(\sigma/2),$ $\tau_3 = 6\pi^{-1/2} \int_0^{\sigma/2} erf(x/\sqrt{3})exp(-x^2)dx/erf(\sigma/2)$
Gamma	$F = \beta^{-\alpha} \int_0^x t^{\alpha-1}exp(-t/\beta)\, dt/\Gamma(\alpha)$	$\lambda_1 = \alpha\beta,\ \lambda_2 = \pi^{-1/2}\beta\Gamma(\alpha + \frac{1}{2})/\Gamma(\alpha),\ \tau_3 = 6I_{1/3}(\alpha, 2\alpha) - 3$

2.9.1 L-moments for Uniform Distribution

In this subsection, we find the L-moments for the uniform distribution.

The uniform distribution has the probability density function([15], Page 191):

$$f(x) = 1/(\beta - \alpha),$$

and has the quantile function[17]:

$$x(F) = \alpha + (\beta - \alpha)F.$$

We are about to find the first four L-moments of the uniform distribution. Before doing so, we have to determine the first for PWMs of the the uniform distribution.

$$\begin{aligned} \beta_r &= \int_0^1 x(F)F^r dF, \qquad r = 0, 1, 2, \ldots \\ &= \int_0^1 \Big[\alpha + (\beta - \alpha)F\Big]F^r dF = \int_0^1 \alpha F^r dF + \int_0^1 (\beta - \alpha)F^{r+1}dF \\ &= \alpha\frac{F^{r+1}}{r+1}\Big|_0^1 + \frac{(\beta - \alpha)}{r+2}F^{r+2}\Big|_0^1 = \frac{\alpha}{r+1} + \frac{\beta - \alpha}{r+2}. \end{aligned}$$

Then,

$$\beta_r = \frac{\alpha}{r+1} + \frac{\beta - \alpha}{r+2} \tag{2.9.1}$$

and,

$$\begin{aligned}
\lambda_1 &= \beta_0 = \alpha + \frac{\beta - \alpha}{2} = \frac{1}{2}(\beta - \alpha) \\
\lambda_2 &= 2\beta_1 - \beta_0 = 2\Big[\frac{\alpha}{2} + \frac{\beta - \alpha}{3}\Big] - \frac{\alpha + \beta}{2} = \frac{1}{6}(\beta - \alpha) \\
\lambda_3 &= 6\beta_2 - 6\beta_1 + \beta_0 = 6\Big[\frac{\alpha}{3} + \frac{\beta - \alpha}{4}\Big] - 6\Big[\frac{\alpha}{2} + \frac{\beta - \alpha}{3}\Big] + \frac{\alpha + \beta}{2} = 0 \\
\lambda_4 &= 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0 \\
&= 20\Big[\frac{\alpha}{4} + \frac{\beta - \alpha}{5}\Big] - 30\Big[\frac{\alpha}{3} + \frac{\beta - \alpha}{4}\Big] + 12\Big[\frac{\alpha}{2} + \frac{\beta - \alpha}{3}\Big] - \Big[\frac{\alpha + \beta}{2}\Big] = 0 \cdot
\end{aligned}$$

Hence,

$$\begin{aligned}
\tau_3 &= \lambda_3/\lambda_2 = 0 \\
\tau_4 &= \lambda_4/\lambda_2 = 0 \cdot
\end{aligned}$$

Now, we write Note 2.9.1 and Note 2.9.2 because it is used in find the others L-moments of the uniform distribution.

Proposition 2.9.1. $\sum_{m=0}^{r} \frac{p^*_{r,m}}{m+1} = 0, \quad \textit{for all} \;\; r \geq 1.$

Proof. From eq.n.(1.6.9) we have: $\int_0^1 P^*_r(F)dF = 0, \quad \text{for all r} \geq 1.$

Since

$$\begin{aligned}
\int_0^1 P^*_r(F)dF &= \int_0^1 \sum_{m=0}^{r} p^*_{r,m} F^m dF, \qquad \text{from eq.n.(2.1.11)} \\
&= \sum_{m=0}^{r} p^*_{r,m} \int_0^1 F^m dF \\
&= \sum_{m=0}^{r} p^*_{r,m} \frac{F^{m+1}}{m+1}\Big|_0^1 \\
&= \sum_{m=0}^{r} \frac{p^*_{r,m}}{m+1}.
\end{aligned}$$

Therefor,

$$\sum_{m=0}^{r} \frac{p^*_{r,m}}{m+1} = 0, \quad \text{for all r} \geq 1. \tag{2.9.2}$$

□

Proposition 2.9.2. $\sum_{m=0}^{r} \frac{p^*_{r,m}}{m+2} = 0, \quad \textit{for all } r \geq 2.$

Proof. Since $P^*_1(F)$ is orthogonal with $P^*_r(F)$ for all $r \geq 2$ on the interval (0.1), then, for all $r \geq 2$ we have: $\int_0^1 P^*_1(F)P^*_r(F)dF = 0$. Now,

$$\begin{aligned} \int_0^1 P^*_1(F)P^*_r(F)dF &= \int_0^1 \Big[\sum_{m=0}^{1} p^*_{r,m}F^m\Big]P^*_r(F)dF, \qquad \text{from eq.n.(2.1.11)} \\ &= \int_0^1 \Big[p^*_{1,0} + p^*_{1,1}F\Big]P^*_r(F)dF \\ &= \int_0^1 \Big[2F-1\Big]P^*_r(F)dF, \qquad \text{from eq.n.(2.1.10)} \end{aligned}$$

$$\begin{aligned} &= 2\int_0^1 FP^*_r(F)dF - \int_0^1 P^*_r(F)dF \\ &= 2\int_0^1 FP^*_r(F)dF \\ &= 2\int_0^1 F\Big(\sum_{m=0}^{r} p^*_{r,m}F^m\Big)dF, \qquad \text{from eq.n.(2.1.11)} \\ &= 2\sum_{m=0}^{r} p^*_{r,m}\int_0^1 F^{m+1}dF \\ &= 2\sum_{m=0}^{r} p^*_{r,m}\frac{F^{m+2}}{m+2}\Big|_0^1 \\ &= 2\sum_{m=0}^{r} \frac{p^*_{r,m}}{m+2}. \end{aligned}$$

Hence,

$$\sum_{m=0}^{r} \frac{p^*_{r,m}}{m+2} = 0 \quad \text{for all r} \geq 2. \tag{2.9.3}$$

□

We are about to find the others L-moments for the uniform distribution.

From eq.n.(2.2.2) and for all $r \geq 2$ we have:

$$\begin{aligned}
\lambda_{r+1} &= \sum_{m=0}^{r} p^*_{r,m}\beta_m \\
&= \sum_{m=0}^{r} p^*_{r,m}\Big[\frac{\alpha}{m+1} + \frac{\beta-\alpha}{m+2}\Big], \qquad \text{from eq.n.(2.9.1)} \\
&= \alpha\sum_{m=0}^{r}\frac{p^*_{r,m}}{m+1} + (\beta-\alpha)\sum_{m=0}^{r}\frac{p^*_{r,m}}{m+2} \\
&= 0, \qquad \text{from eqs.n.(2.9.2), (2.9.3).}
\end{aligned}$$

That means, the L-moments for the uniform distribution, $\lambda_r = 0$ for all $r \geq 3$.

2.9.2 L-moments for Exponential Distribution

In this subsection, we find the first four L-moments for the exponential distribution. The exponential distribution has the cumulative distribution function([15], Page 192):

$$F(x) = 1 - \exp\{-(x-\xi)/\alpha\}, \quad \text{where } \xi \leq x < \infty.$$

Firstly we want to find the quantile function of the exponential distribution. So, replace $x(F)$ with x, and F with $F(x)$ we have:
$F = 1 - \exp\{-(x(F)-\xi)/\alpha\}$, then $1 - F = \exp\{-(x(F)-\xi)/\alpha\}$,
hence, $\ln(1-F) = -(x(F)-\xi)/\alpha$. Therefor, $x(F) = \xi - \alpha\ln(1-F)$.

secondly, we want to find the r^{th} PWM for the exponential distribution:

$$\begin{aligned}
\beta_r &= \int_0^1 x(F)\; F^r d\,F \qquad\qquad r = 0,1,2,....... \\
&= \int_0^1 \Big[\xi - \alpha\ln(1-F)\Big]F^r dF = \frac{\xi}{r+1} - \alpha\int_0^1 F^r\ln(1-F)d\,F\cdot
\end{aligned}$$

Now, we will find

$$\int_0^1 F^r\ln(1-F)d\,F.$$

Integrating by parts, we get:

$$\int_0^1 F^r\ln(1-F)d\,F = \frac{F^{r+1}}{r+1}\ln(1-F)\Big|_0^1 + \frac{1}{r+1}\int_0^1\frac{F^{r+1}}{1-F}d\,F$$

$$= \frac{1}{r+1}\int_0^1 \frac{F^{r+1}}{1-F}d\,F\cdot \qquad (2.9.4)$$

Let $z = 1 - F$. Then, $dz = -dF$, $F = 1 - z$.

So,

$$F^{r+1} = (1-z)^{r+1} = \sum_{k=0}^{r+1}(-1)^k \begin{pmatrix} r+1 \\ k \end{pmatrix} z^k.$$

Therefore,

$$\frac{F^{r+1}}{1-F} = \sum_{k=0}^{r+1}(-1)^k \begin{pmatrix} r+1 \\ k \end{pmatrix} z^{k-1} = \frac{1}{z} + \sum_{k=1}^{r+1} \begin{pmatrix} r+1 \\ k \end{pmatrix} z^{k-1}.$$

From eq.n (2.9.4),

$$\begin{aligned}
\int_0^1 F^r \ln(1-F)dF &= \frac{-1}{r+1}\int_0^1 \Big[\frac{1}{z} + \sum_{k=1}^{r+1}(-1)^k \begin{pmatrix} r+1 \\ k \end{pmatrix} z^{k-1}\Big]dz \\
&= \frac{-1}{r+1}\Big[\ln z + \sum_{k=1}^{r+1}(-1)^k \frac{1}{k}\begin{pmatrix} r+1 \\ k \end{pmatrix} z^k\Big]\Big|_1^0 \\
&= \frac{1}{r+1}\Big[\sum_{k=1}^{r+1}(-1)^k \frac{1}{k}\begin{pmatrix} r+1 \\ k \end{pmatrix}\Big]
\end{aligned}$$

$$= \frac{1}{r+1}\sum_{k=1}^{r+1}(-1)^k \frac{1}{k}\begin{pmatrix} r+1 \\ k \end{pmatrix}. \qquad (2.9.5)$$

Hence,

$$\begin{aligned}
\beta_r &= \frac{\xi}{r+1} - \frac{\alpha}{r+1}\sum_{k=1}^{r+1}(-1)^k \frac{1}{k}\begin{pmatrix} r+1 \\ k \end{pmatrix} \\
\beta_0 &= \xi - \alpha(-1) = \xi + \alpha.
\end{aligned}$$

So,

$$
\begin{aligned}
\lambda_1 &= \beta_0 = \xi + \alpha \\
\beta_1 &= \frac{\xi}{2} - \frac{\alpha}{2}\sum_{k=1}^{2}(-1)^k \frac{1}{k}\begin{pmatrix} 2 \\ k \end{pmatrix} = \frac{\xi}{2} - \frac{\alpha}{2}\Big[-\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix}\Big] \\
&= \frac{\xi}{2} + \frac{3\alpha}{4}\ .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lambda_2 &= 2\beta_1 - \beta_0 = 2\Big(\frac{\xi}{2} + \frac{3\alpha}{4}\Big) - \Big(\xi + \alpha\Big) = \frac{\alpha}{2} \\
\beta_2 &= \frac{\xi}{3} - \frac{\alpha}{3}\sum_{k=1}^{2}(-1)^k \frac{1}{k}\begin{pmatrix} 3 \\ k \end{pmatrix} = \frac{\xi}{3} - \frac{\alpha}{3}\Big[-\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3 \\ 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 3 \\ 3 \end{pmatrix}\Big] \\
&= \frac{\xi}{3} + \frac{11\alpha}{18}\ .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\lambda_3 &= 6\beta_2 - 6\beta_1 + \beta_0 \\
&= 6\Big(\frac{\xi}{3} + \frac{11\alpha}{18}\Big) - 6\Big(\frac{\xi}{2} + \frac{3\alpha}{4}\Big) + \xi + \alpha = \frac{\alpha}{6}\ .
\end{aligned}
$$

Then,

$$
\begin{aligned}
\tau_3 &= \lambda_3/\lambda_2 = \frac{1}{3} \\
\beta_3 &= \frac{\xi}{4} - \frac{\alpha}{4}\sum_{k=1}^{4}(-1)^k \frac{1}{k}\begin{pmatrix} 4 \\ k \end{pmatrix} = \frac{\xi}{4} - \frac{\alpha}{4}\Big[-\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 4 \\ 3 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 4 \\ 4 \end{pmatrix}\Big] \\
&= \frac{\xi}{4} + \frac{25\alpha}{48}\ .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lambda_4 &= 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0 \\
&= 20\Big(\frac{\xi}{4} + \frac{25\alpha}{48}\Big) - 30\Big(\frac{\xi}{3} + \frac{11\alpha}{18}\Big) + 12\Big(\frac{\xi}{2} + \frac{3\alpha}{4}\Big)\Big(\xi + \alpha\Big) \\
&= 5\xi + \frac{125}{12}\alpha - 10\xi - \frac{55}{3}\alpha + 6\xi + 9\alpha - \xi - \alpha = \frac{\alpha}{12} \\
\tau_4 &= \lambda_4/\lambda_2 = \frac{1}{6}\ .
\end{aligned}
$$

2.9.3 L-moments for Logistic Distribution

In this subsection, we find the first four L-moments for the logistic distribution. The logistic distribution has the probability density function([15], Page 196):

$$f(x) = \frac{\alpha^{-1}e^{-(1-k)y}}{(1+e^{-y})^2}, \quad \text{where } y = -k^{-1}\log\{1-k(x-\xi)/\alpha\},$$

and has the quantile function[17]:

$$x(F) = \xi + \alpha \ln\left\{F/(1-F)\right\} = \xi + \alpha \ln F - \alpha \ln(1-F).$$

Now,

$$\beta_r = \int_0^1 x(F)F^r dF = \int_0^1 \Big[\xi + \alpha \ln F - \alpha \ln(1-F)\Big] F^r dF$$
$$= \frac{\xi}{r+1} + \alpha \int_0^1 F^r \ln F dF - \alpha \int_0^1 F^r \ln(1-F) dF. \tag{2.9.6}$$

Now, we will find

$$\int_0^1 F^r \ln F dF.$$

Integrating by parts, we get:

$$\int_0^1 F^r \ln F dF = \frac{F^{r+1}\ln F}{r+1}\Big|_0^1 - \frac{1}{r+1}\int_0^1 F^r dF = \frac{-1}{(r+1)} \cdot \tag{2.9.7}$$

Substituting eq.n.(2.9.5) and eq.n.(2.9.7) in eq.n.(2.9.6), we get:

$$\beta_r = \frac{\xi}{r+1} - \frac{\alpha}{(r+1)^2} - \frac{\alpha}{r+1}\sum_{k=1}^{r+1}(-1)^k \frac{1}{k}\begin{pmatrix} r+1 \\ k \end{pmatrix}.$$

Then,

$$\beta_0 = \xi - \alpha - \alpha(-1) = \xi.$$

Therefor,

$$\begin{aligned}
\lambda_1 &= \beta_0 = \xi \\
\beta_1 &= \frac{\xi}{2} - \frac{\alpha}{4} - \frac{\alpha}{2}(\frac{-3}{2}) = \frac{\xi}{2} + \frac{\alpha}{2} \\
\lambda_2 &= 2\beta_1 - \beta_0 = 2\Big(\frac{\xi}{2} + \frac{\alpha}{2}\Big) - \xi = \alpha \\
\beta_2 &= \frac{\xi}{3} - \frac{\alpha}{9} + \frac{11\alpha}{18} = \frac{\xi}{3} + \frac{\alpha}{2}\ . \\
\lambda_3 &= 6\beta_2 - 6\beta_1 + \beta_0 = 6\Big(\frac{\xi}{3} + \frac{\alpha}{2}\Big) - 6\Big(\frac{\xi}{2} + \frac{\alpha}{2}\Big) + \xi \\
&= 2\xi + 3\alpha - 3\xi - 3\alpha + \xi = 0 \\
\tau_3 &= \lambda_3/\lambda_2 = \frac{0}{\alpha} = 0 \\
\beta_3 &= \frac{\xi}{4} - \frac{\alpha}{16} = \frac{25\alpha}{48} = \frac{\xi}{4} + \frac{11\alpha}{24}
\end{aligned}$$

$$\begin{aligned}
\lambda_4 &= 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0 \\
&= 20\Big(\frac{\xi}{4} + \frac{11\alpha}{24}\Big) - 30\Big(\frac{\xi}{3} + \frac{\alpha}{2}\Big) + 12\Big(\frac{\xi}{2} + \frac{\alpha}{2}\Big) - \xi \\
&= 5\xi + \frac{55}{6}\alpha - 10\xi - 15\alpha + 6\xi + 6\alpha - \xi = \frac{1}{6}\alpha \\
\tau_4 &= \lambda_4/\lambda_2 = \frac{1}{6}\ .
\end{aligned}$$

2.9.4 L-moments for Generalized Pareto

We are about to find - as in the previous sections - the first four L-moments for the generalized pareto distribution.

The generalized pareto distribution has the probability density function([15], Page 194):

$$f(x) = \alpha^{-1} e^{-(1-k)[-k^{-1}\log\{1-k(x-\xi)/\alpha\}]}\ ,$$

and has the quantile function[17]:

$$x(F) = \xi + \alpha\big\{1 - (1-F)^k\big\}/k = \xi + \frac{\alpha}{k} - \frac{\alpha}{k}(1-F)^k.$$

Now, we will find β_r for the generalized pareto distribution:

$$\beta_r = \int_0^1 x(F)F^r = \int_0^1 \Big(\xi + \frac{\alpha}{k}\Big)F^r dF - \frac{\alpha}{k}\int_0^1 \Big(1-F\Big)^k F^r dF$$

$$= \frac{1}{r+1}\Big(\xi + \frac{\alpha}{k}\Big) - \frac{\alpha}{k}\int_0^1 (1-F)^k F^r dF. \tag{2.9.8}$$

Let $u = 1 - F \Longrightarrow du = -dF \qquad , F = 1 - u$. Then,

$$\int_0^1 (1-F)^k F^r dF = -\int_1^0 u^k (1-u)^r du = -\int_1^0 u^k \sum_{j=0}^{r} (-1)^j \begin{pmatrix} r \\ j \end{pmatrix} u^j du$$

$$= -\sum_{j=0}^{r} (-1)^j \begin{pmatrix} r \\ j \end{pmatrix} \int_1^0 u^{k+j} du = \sum_{j=0}^{r} (-1)^j \frac{1}{k+j+1} \begin{pmatrix} r \\ j \end{pmatrix}. \tag{2.9.9}$$

Substituting from eq.n.(2.9.9) in e.qn.(2.9.8) we get :

$$\beta_r \;=\; \frac{1}{r+1}\Big(\xi + \frac{\alpha}{k}\Big) - \frac{\alpha}{k}\sum_{j=0}^{r} (-1)^j \frac{1}{k+j+1} \begin{pmatrix} r \\ j \end{pmatrix}$$

Now,

$$\beta_0 \;=\; \xi + \frac{\alpha}{k} - \frac{\alpha}{k}\Big(\frac{1}{k+1}\Big) = \xi + \frac{\alpha}{k} - \frac{\alpha}{k(k+1)}$$
$$= \; \xi + \frac{\alpha}{k+1}.$$

Thus,

$$\lambda_1 \;=\; \beta_0 = \xi + \frac{\alpha}{k+1}.$$

Furthermore,

$$\beta_1 \;=\; \frac{1}{2}\Big(\xi + \frac{\alpha}{k}\Big) - \frac{\alpha}{k}\Big[\frac{1}{k+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{k+2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\Big]$$
$$= \; \frac{1}{2}\xi + \frac{\alpha}{2k} - \frac{\alpha}{k}\Big[\frac{1}{k+1} - \frac{1}{k+21}\Big] = \frac{1}{2}\xi + \frac{\alpha(k+3)}{2(k+1)(k+2)}\,.$$

So,

$$\begin{aligned}\lambda_2 &= 2\beta_1 - \beta_0 = 2\Big[\frac{1}{2}\xi + \frac{\alpha(k+3)}{2(k+1)(k+2)}\Big]\\ &= \frac{\alpha}{(k+1)(k+2)}\,.\end{aligned}$$

Moreover,

$$\begin{aligned}\beta_2 &= \frac{1}{3}\Big(\xi + \frac{\alpha}{k}\Big) - \frac{\alpha}{k}\sum_{j=0}^{2}(-1)^j\frac{1}{k+j+1}\begin{pmatrix}2\\ j\end{pmatrix}\\ &= \frac{1}{3}\xi + \frac{1}{3}\frac{\alpha}{k} - \frac{\alpha}{k}\Big[\frac{1}{k+1}\begin{pmatrix}2\\ 0\end{pmatrix} - \frac{1}{k+1}\begin{pmatrix}2\\ 1\end{pmatrix} + \frac{1}{k+3}\begin{pmatrix}2\\ 2\end{pmatrix}\Big]\\ &= \frac{1}{3} + \frac{1}{3}\frac{\alpha}{k} - \frac{\alpha}{k}\Big[\frac{1}{k+1} - \frac{2}{k+2} + \frac{1}{k+3}\Big]\\ &= \frac{1}{3}\xi + \frac{1}{3}\frac{\alpha}{k} - \frac{2\alpha}{k(k+1)(k+2)(k+3)}\\ &\quad \frac{1}{3}\xi + \frac{k^2+6k+11}{3(k+1)(k+2)(k+3)}.\end{aligned}$$

Then,

$$\begin{aligned}\lambda_3 &= 6\beta_2 - 6\beta_1 + \beta_0\\ &= 6\Big[\frac{1}{3}\xi + \frac{k^2+6k+11}{3(k+1)(k+2)(k+3)}\Big] - 6\Big[\frac{1}{2}\xi + \frac{\alpha(k+3)}{2(k+1)(k+2)}\Big] + \xi + \frac{\alpha}{k+1}\\ &= 2\xi + \frac{2k^2+12k+22}{(k+1)(k+2)(k+3)} - 3\xi - \frac{3\alpha(k+3)}{(k+1)(k+2)} + \xi + \frac{\alpha}{k+1}\\ &= \frac{2k^2+12k+22-3(k+3)^2+(k+2)(k+3)}{(k+1)(k+2)(k+3)}\;\alpha\\ &= \frac{1-k}{(k+1)(k+2)(k+3)}\;\alpha\cdot\end{aligned}$$

Also,

$$\tau_3 = \lambda_3/\lambda_2 = \frac{1-k}{(k+1)(k+2)(k+3)}\;\alpha\Big/\frac{\alpha}{(k+1)(k+2)} = \frac{1-k}{k+3}.$$

Finally,

$$
\begin{aligned}
\beta_3 &= \frac{1}{4}\Big(\xi + \frac{\alpha}{k}\Big) - \frac{\alpha}{k}\sum_{j=0}^{3}(-1)^j \frac{1}{k+1+j}\begin{pmatrix} 3 \\ j \end{pmatrix} \\
&= \frac{1}{4}\xi + \frac{\alpha}{4k} - \frac{\alpha}{k}\Big[\frac{1}{k+1}\begin{pmatrix} 3 \\ 0 \end{pmatrix} - \frac{1}{k+2}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{1}{k+3}\begin{pmatrix} 3 \\ 2 \end{pmatrix} - \frac{1}{k+4}\begin{pmatrix} 3 \\ 3 \end{pmatrix}\Big] \\
&= \frac{1}{4}\xi + \frac{\alpha}{4k} - \frac{\alpha}{k}\Big[\frac{1}{k+1} - \frac{3}{k+2} + \frac{3}{k+3} - \frac{1}{k+4}\Big] \\
&= \frac{1}{4}\xi + \frac{1}{4}\frac{\alpha}{k} - \frac{\alpha}{k}\Big[\frac{6}{(k+1)(k+2)(k+3)(k+4)}\Big] \\
&= \frac{1}{4}\xi + \frac{k^3+10k^2+35k+50}{4(k+1)(k+2)(k+3)(k+4)}\ \alpha \cdot
\end{aligned}
$$

So,

$$
\begin{aligned}
\lambda_4 &= 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0 \\
&= 20\Big[\frac{1}{4}\xi + \frac{k^3+10k^2+35k+50}{4(k+1)(k+2)(k+3)(k+4)}\ \alpha\Big] - 30\Big[\frac{1}{3}\xi + \frac{k^2+6k+11}{3(k+1)(k+2)(k+3)}\Big] \\
&+ 12\Big[\frac{1}{2}\xi + \frac{\alpha(k+3)}{2(k+1)(k+2)}\Big] - \Big[\xi + \frac{\alpha}{k+1}\Big] \\
&= 5\xi + \frac{5k^3+50k^2+175k+250}{(k+1)(k+2)(k+3)(k+4)}\alpha - 10\xi - \frac{10k^2+60k+110}{(k+1)(k+2)(k+3)(k+4)}\alpha \\
&+ 6\xi + \frac{6(k+3)\alpha}{(k+1)(k+2)} - \xi - \frac{\alpha}{k+1} \\
&= \frac{k^2-3k+2}{(k+1)(k+2)(k+3)(k+4)}\ \alpha = \frac{(1-k)(2-k)}{(k+1)(k+2)(k+3)(k+4)}\ \alpha \cdot
\end{aligned}
$$

Therefor,

$$
\tau_4 = \lambda_4/\lambda_2 = \frac{(1-k)(2-k)}{(k+1)(k+2)(k+3)(k+4)}\ \alpha \Big/ \frac{\alpha}{(k+1)(k+2)} = \frac{(1-k)(2-k)}{(k+3)(k+4)}\ .
$$

Chapter 3

ESTIMATION OF L-MOMENTS

In this chapter, we introduce estimation for L-moments, probability weighted moments and L-moment ratios. At the end of this chapter, we introduce the estimation of parameters using L-moments for any distribution with finite means and we find estimations for parameters for some distributions as the Uniform distribution, the Exponential Distribution, Generalized Logistic distribution and Generalized Pareto Distribution.

3.1 The r^{th} Sample L-moments

L-moments have been defined for a probability distributions, but in practice must often be estimated from a finite sample. Estimation is based on a sample of size n, arranged in ascending order [13]. A sample of size 2 contains two observations in ascending order $x_{1:2}$ and $x_{2:2}$. The difference between the two observations $(x_{1:2} - x_{2:2})$ is a measure of the scale of the distribution. A sample of size 3 contains three observations in ascending order $x_{1:3}$, $x_{2:3}$ and $x_{3:3}$. The difference between the two observation $(x_{2:3} - x_{1:3})$ and the difference between the two observation $(x_{3:3} - x_{2:3})$ can be subtracted from each other to have a measure of the skewness of the distribution. This leads to: $(x_{3:3} - x_{2:3}) - (x_{2:3} - x_{1:3}) = x_{3:3} - 2x_{2:3} + x_{1:3}$. A sample of size 4 contains four observations in ascending order $x_{1:4}$, $x_{2:4}$, $x_{3:4}$ and $x_{4:4}$. A measure for the kurtosis of the distribution given by: $(x_{4:4} - x_{1:4}) - 3(x_{3:4} - x_{2:4})$. In short: the above linear combinations of the elements of the order sample contain information about the location, scale, skewness and kurtosis of the distribution from which the sample was drawn [25].

A naturale way to generalize the above approach to sample of size n, is to take all possible sub-samples of size 2 and then take the average of the differences, i.e.; $(x_{1:2} - x_{2:2})/2$:

$$\ell_2 = \frac{1}{2}\begin{pmatrix} n \\ 2 \end{pmatrix}^{-1} \sum\sum_{i>j}(x_{i:n} - x_{j:n}).$$

Furthermore, the skewness and kurtosis are similarly obtained as:

$$\begin{aligned} \ell_3 &= \frac{1}{3}\begin{pmatrix} n \\ 3 \end{pmatrix}^{-1} \sum\sum\sum_{i>j>k}(x_{i:n} - 2x_{j:n} + x_{k:n}), \\ \ell_4 &= \frac{1}{4}\begin{pmatrix} n \\ 4 \end{pmatrix}^{-1} \sum\sum\sum\sum_{i>j>k>l}(x_{i:n} - 3x_{j:n} + 3x_{k:n} - x_{i:n}). \end{aligned}$$

Definition 3.1.1. ([17]) Let $x_1, x_2, ..., x_n$ be the sample and $x_{1:n} \leq x_{2:n} \leq ... \leq x_{n:n}$ the order sample, and define the r^{th} *sample L-moments* to be

$$\ell_r = \begin{pmatrix} n \\ r \end{pmatrix}^{-1} \sum\sum \cdots \sum_{1\leq i_1<i_2<...<i_r\leq n} r^{-1}\sum_{k=0}^{r-1}(-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix} x_{i_{r-k}:n}, \qquad r = 1, 2, ..., n.$$

In particular,

$$\begin{aligned} \ell_1 &= n^{-1}\sum_i x_i \\ \ell_2 &= \frac{1}{2}\begin{pmatrix} n \\ 2 \end{pmatrix}^{-1} \sum\sum_{i>j}(x_{i:n} - x_{j:n}), \\ \ell_3 &= \frac{1}{3}\begin{pmatrix} n \\ 3 \end{pmatrix}^{-1} \sum\sum\sum_{i>j>k}(x_{i:n} - 2x_{j:n} + x_{k:n}), \\ \ell_4 &= \frac{1}{4}\begin{pmatrix} n \\ 4 \end{pmatrix}^{-1} \sum\sum\sum\sum_{i>j>k>l}(x_{i:n} - 3x_{j:n} + 3x_{k:n} - x_{i:n}). \end{aligned}$$

Sample L-moments have been used to find the estimation of parameters using L-moments for any distribution with finite means. The statistic $\ell_1 = n^{-1}\sum_i x_i$ is the sample mean. The sample L-scale, ℓ_2, is a scalar multiple of Gini's mean difference

$$G = \begin{pmatrix} n \\ 2 \end{pmatrix}^{-1} \sum\sum_{i>j} (x_{i:n} - x_{j:n}).$$

Direct estimators

In [21] wang provides direct estimators of L-moment which eliminate the need for introducing **PWMs**. The estimation procedure follows closely the definition of L-moments by covering all possible combinations in more efficient way. For the sample value $x_{i:n}$ there are $(i-1)$ values $\leq x_{i:n}$ and $(n-1)$ values $\geq x_{(i:n)}$, and for each subsample of size r, the number of values drawn from each of these categories are considered. The first four direct estimators are given by:

$$\begin{aligned}
\ell_1 &= \begin{pmatrix} n \\ 1 \end{pmatrix}^{-1} \sum_{i=1}^{n} x_{(i:n)} \\
\ell_2 &= \frac{1}{2}\begin{pmatrix} n \\ 2 \end{pmatrix}^{-1} \sum_{i=1}^{n} \left[\begin{pmatrix} i-1 \\ 1 \end{pmatrix} - \begin{pmatrix} n-i \\ 1 \end{pmatrix} \right] x_{(i:n)} \\
\ell_3 &= \frac{1}{3}\begin{pmatrix} n \\ 3 \end{pmatrix}^{-1} \sum_{i=1}^{n} \left[\begin{pmatrix} i-1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} i-1 \\ 1 \end{pmatrix}\begin{pmatrix} n-i \\ 1 \end{pmatrix} + \begin{pmatrix} n-i \\ 2 \end{pmatrix} \right] x_{(i:n)} \\
\ell_4 &= \begin{pmatrix} n \\ 4 \end{pmatrix}^{-1} \sum_{i=1}^{n} \left[\begin{pmatrix} i-1 \\ 3 \end{pmatrix} - 3\begin{pmatrix} i-1 \\ 2 \end{pmatrix}\begin{pmatrix} n-i \\ 1 \end{pmatrix} \right. \\
&+ \left. 3\begin{pmatrix} i-1 \\ 1 \end{pmatrix}\begin{pmatrix} n-i \\ 2 \end{pmatrix} - \begin{pmatrix} n-i \\ 3 \end{pmatrix} \right] x_{(i:n)}.
\end{aligned}$$

3.2 The Sample Probability Weighted Moments

In this section, we introduce estimation probability weighted moments and its relation with the estimation of L-moments.

Definition 3.2.1. [4] The Sample probability weighted moments or probability weighted moments estimators (PWMs estimators), computed for data values $x_{1:n}, x_{2:n}, ..., x_{n:n}$, arranged in increasing order, are given by:

$$b_r = n^{-1} \begin{pmatrix} n-1 \\ r \end{pmatrix}^{-1} \sum_{j=r+1}^{n} \begin{pmatrix} j-1 \\ r \end{pmatrix} x_{j:n}, \quad r = 0, 1, 2 \cdots . \tag{3.2.1}$$

This may alternatively be written as

$$\begin{aligned} b_0 &= n^{-1} \sum_{j=1}^{n} x_{j:n}, \\ b_1 &= n^{-1} \sum_{j=1}^{n} \frac{(j-1)}{(n-1)} x_{j:n}, \\ b_2 &= n^{-1} \sum_{j=1}^{n} \frac{(j-1)(j-2)}{(n-1)(n-2)} x_{j:n}, \end{aligned}$$

and in general [6] :

$$b_r = n^{-1} \sum_{j=r+1}^{n} \frac{(j-1)(j-2)...(j-r)}{(n-1)(n-2)...(n-r)} x_{j:n}.$$

Remark 3.2.1. [17] The Sample probability weighted moments b_r is an unbiased estimator of the probability weighted moment β_r.

Proof. Since,

$$\begin{aligned} b_r &= n^{-1} \sum_{j=r+1}^{n} \frac{(j-1)(j-2)...(j-r)}{(n-1)(n-2)...(n-r)} x_{j:n} \\ &= \sum_{j=r+1}^{n} \frac{(j-1)(j-2)...(j-r)}{n(n-1)(n-2)...(n-r)} x_{j:n}, \end{aligned}$$

then,

$$E(b_r) = \sum_{j=r+1}^{n} \frac{(j-1)(j-2)...(j-r)}{n(n-1)(n-2)...(n-r)} E(X_{j:n}). \tag{3.2.2}$$

Forme eqn.(2.1.2), we have

$$EX_{j:n} = \frac{n!}{(j-1)!(n-j)!} \int x[F(x)]^{j-1}[1-F(x)]^{n-j}\, dF(x).$$

Hence, substituting $EX_{j:n}$ in eq.n.(3.2.2), we have

$$\begin{aligned}
E(b_r) &= \sum_{j=r+1}^{n} \frac{(j-1)(j-2)...(j-r)}{n(n-1)(n-2)...(n-r)} \times \frac{n!}{(j-1)!(n-j)!} \int_0^1 x[F(x)]^{j-1}[1-F(x)]^{n-j}\, dF(x) \\
&= \int_0^1 x\Big[\sum_{j=r+1}^{n} \frac{(n-(r+1))!}{((j-(r+1))!\ (n-j)!}\ [F(x)]^{j-1}[1-F(x)]^{n-j}\Big]\, dF(x) \\
&= \int_0^1 x\Big[\sum_{j=r+1}^{n} \begin{pmatrix} n-r-1 \\ j-r-1 \end{pmatrix}\ [F(x)]^{j-1}[1-F(x)]^{n-j}\Big]\, dF(x) \\
&= \int_0^1 x\Big[\sum_{j=r+1}^{n} \begin{pmatrix} n-(r+1) \\ j-(r+1) \end{pmatrix}\ [F(x)]^{j-1}[1-F(x)]^{n-j}\Big]\, dF(x).
\end{aligned}$$

Let $k = r+1$. So,

$$\begin{aligned}
E(b_r) &= \int_0^1 x\Big[\sum_{j=k}^{n} \begin{pmatrix} n-k \\ j-k \end{pmatrix}\ [F(x)]^{j-1}[1-F(x)]^{n-j}\Big]\, dF(x) \\
&= \int_0^1 x\Big[\sum_{j-k=0}^{n} \begin{pmatrix} n-k \\ j-k \end{pmatrix}\ [F(x)]^{(j-k)+k-1}[1-F(x)]^{n-(j-k)-k}\Big]\, dF(x) \\
&= \int_0^1 x\Big[\sum_{m=0}^{n} \begin{pmatrix} n-k \\ m \end{pmatrix}\ [F(x)]^{m+k-1}[1-F(x)]^{n-m-k}\Big]\, dF(x), \quad \text{where } m = j-k \\
&= \int_0^1 xF^{k-1}\Big[\sum_{m=0}^{n-k} \begin{pmatrix} n-k \\ m \end{pmatrix}\ [F(x)]^{m}[1-F(x)]^{n-k-m}\Big]\, dF(x) \\
&= \int_0^1 xF^{k-1}\Big[F(x)+1-F(x)\Big]^{n-k}\, dF(x) \\
&= \int_0^1 xF^{k-1} dF(x).
\end{aligned}$$

Since $k = r+1$, then from eq.n.(2.2.1) we have

$$E(b_r) = \int_0^1 x(F)\ F^r dF(x) = \beta_r. \tag{3.2.3}$$

That means, b_r is an unbiased estimator of the probability weighted moment β_r. □

Note 3.2.1. *[17] The sample L-moments ℓ_r are linear combination of PWMs estimators, b_r.*

To see this,

$$\ell_r = \sum_{k=0}^{r-1} p^*_{r-1,k}\, b_k, \qquad r = 1, 2, ..., n; [3] \tag{3.2.4}$$

where,

$$p^*_{r,k} = (-1)^{r-k} \begin{pmatrix} r \\ k \end{pmatrix} \begin{pmatrix} r+k \\ k \end{pmatrix}.$$

The coefficients $p^*_{r,k}$ are those of the shifted Legendre polynomials.

Remark 3.2.2. [17]The sample L-moments ℓ_r is an unbiased estimator of L-moments λ_r.

Proof. From eqn.(3.2.4), we have

$$\begin{aligned} E(\ell_r) &= \sum_{k=0}^{r-1} p^*_{r-1,k}\, E(b_k), \qquad r = 1, 2, ..., n \\ &= \sum_{k=0}^{r-1} p^*_{r-1,k}\, \beta_r, \quad \text{from eq.n.(3.2.3)} \\ &= \lambda_r, \quad \text{from eq.n.(2.2.10)} \cdot \end{aligned}$$

That means, ℓ_r is an unbiased estimator of L-moments λ_r. □

The first four r^{th} sample L-moments follow from PWMs estimator are [3]:

$$\begin{aligned} \ell_1 &= b_0 \\ \ell_2 &= 2b_1 - b_0 \\ \ell_3 &= 6b_2 - 6b_1 + b_0, \quad \text{and} \\ \ell_4 &= 20b_3 - 30b_2 + 12b_1 - b_0. \end{aligned}$$

Sample L-moments may be used similarly to (conventional) sample moments: they summarize the basic properties-location, scale, skewness, kurtosis-of a data set, they estimate the corresponding properties of the probability distribution from which the data were sampled and they may be used to estimate the parameters of the data were sampled and they may be used to estimate the parameters of the underlying distribution[17].

3.3 The r^{th} Sample L-moment Ratios

By dividing the higher-order r^{th} sample L-moments by the dispersion measure, we obtain the r^{th} sample L-moment ratios:

Definition 3.3.1. [16] Define the r^{th} *sample L-moment ratios* to be the quantities

$$t_r = \ell_r/\ell_2, \qquad r = 3, 4, 5, ..., \tag{3.3.1}$$

t_r is a natural estimator of τ_r ([15], Page 28).
These are dimensionless quantities, independent of the units of measurement of the data; $t_3 = \ell_3/\ell_2$ is a measure of skewness, $t_4 = \ell_3/\ell_2$ is a measure of kurtosis these are respectively the L-skewness and L-kurtosis. They take values between -1 and $+1$ (exception: some evenorder L-moment ratios computed from very small samples can be less than -1). The L-moments analogue of the coefficient of variation (standard deviation divided by the mean), is the sample L-CV, defined by:

$$t = \ell_2/\ell_1 \tag{3.3.2}$$

t is natural estimators of τ. The estimators t_r and t are not unbiased ([15], Page 28).
The quantities ℓ_1, ℓ_2 (or t), t_3, and t_4 are useful summary statistics of a sample of data. They can be used to identify the distribution from which a sample was drawn. They can also be used to estimate parameters when fitting a distribution to a sample, by equating the sample and population L-moments [19].
As an example we calculate them for six sets of annual maximum windspeed data taken from simiu, Changery and Filliben (1979). The data are tabled in Table 3.1.
The sample L-moments can be calculated using eqn.(3.2.1), eq.n.(3.3.2), eq.n.(3.2.4) and the sample L-moment ratios can be calculated using eq.n.(3.3.2). The results are given in Table 3.2 ([15], Page 30).

Table 3.1: Annual maximum windspeed data, in miles per hour, for six sites in the eastern United States.

Macon, Ga., 1950-1977.									
32	32	34	37	37	38	40	40	40	42
42	42	43	44	45	45	46	48	49	50
51	51	51	53	53	58	58	60		
Brownsville, Tex., 1943-1977									
32	33	34	34	35	36	37	37	38	38
39	39	40	40	41	41	42	42	43	43
43	44	44	46	46	48	48	49	51	53
53	53	56	63	66					
Port Arthur. Tex., 1953-1977									
39	43	44	44	45	45	45	46	47	49
51	51	51	51	54	55	55	57	57	60
61	63	66	67	81					
Montgomery, Ala., 1950-1977.									
34	36	36	37	38	40	40	40	40	40
43	43	43	43	46	46	46	46	47	47
48	49	51	51	51	52	60	77		
Key West, Fla., 1958-1976.									
35	35	36	36	36	38	42	43	43	46
48	48	52	55	58	64	78	86	90	
Corpus Chisti, Tex., 1943-1976.									
44	44	44	44	45	45	45	45	46	46
46	47	48	48	48	48	48	49	50	50
50	51	52	55	57	58	60	60	66	67
70	71	77	128						

Table 3.2: L-moments of the annual maximum windspeed data in Table(3.1)

Site	n	ℓ_1	ℓ_2	t	t_3	t_4
Macon	28	45.04	4.46	0.0990	0.0406	0.0838
Brownsvill	35	43.63	4.49	0.1030	0.1937	0.1509
Port Arthur	25	53.08	5.25	0.0989	0.2086	0.1414
Montgomery	28	45.36	4.34	0.0958	0.2316	0.2490
Key West	19	51.00	9.29	0.1821	0.3472	0.1245
Corpus Christi	34	54.47	6.70	0.1229	0.5107	0.3150

Table 3.3: Bais of sample L-CV

	τ				
τ_3	0.1	0.2	0.3	0.4	0.5
0.0	-0.001	0.000	0.003	0.009	0.020
0.1	-0.001	-0.001	0.001	0.005	0.014
0.2	-0.001	-0.002	-0.001	0.001	0.008
0.3	-0.001	-0.003	-0.005	-0.004	0.000
0.4	-0.002	-0.006	-0.010	-0.012	-0.011
0.5	-0.003	-0.011	-0.018	-0.025	-0.027

Note: Results are for samples of size 20 from a generalized extreme value distribution with L-CV τ and L-skewness τ_3

The bias of the sample L-CV, t, is negligible in sample of size 20 or more. For example, Table 3.3 gives the bias of t for samples of size 20 from a generalized extreme value distribution ([15], Page 28).

3.4 Parameter Estimation Using L-moments

A common problem in statistics is the estimation, from a random sample of size n, of a probability distribution whose specification involves a finite number, p, of unknown parameters. Analogously the usual method of moments, the method of L-moments obtains parameter estimates by equating the first p sample L-moments to the corresponding population quantities. Examples of parameter estimators derived using this method are given in Table 3.4 [17]

Method of L-moments [7]:

Let $F(x)$ be a distribution function associated with a random variable X and let $x(F) : (0, 1) \to \mathbb{R}$ be its quantile function. The r^{th} L-moment of X is given by eq.n.(2.1.12),

$$\lambda_r = \int_0^1 x(F)\; P^*_{r-1}(F)\; dF, \qquad r = 1, 2, \ldots .$$

where

$$p^*_{r,m} = (-1)^{r-m} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} r+m \\ m \end{pmatrix},$$

and

$$P^*_r(F) = \sum_{m=0}^{r} p^*_{r,m}\; F^m.$$

Theorem 2.4.1 gives the following justification for using L-moments:

a) μ_1 (mean) is finite if and only if λ_r exists for all $r = 1, 2, \ldots$;

b) a distribution $F(x)$ with finite mean μ_1 is uniquely characterized by λ_r for all $r = 1, 2, \ldots$.

L-moments can be used to estimate a finite number of parameters $\theta \in \Theta$ that identify a member of a family of distributions. Suppose $\{F(x, \theta) : \theta \in \Theta \subset \mathbb{R}^P\}$, P a natural number, is a family of distributions which is known up to θ. A sample $\{x_i\}_{i=1}^n$ is available

Table 3.4: [17]Parameter estimation via L-moments for some common distributions

Distribution	Estimators
Exponential	(ξ known) $\hat{\alpha} = l_1$
Gumble	$\hat{\alpha} = l_2/\log 2,\ \hat{\xi} = l_1 - \gamma\hat{\alpha}$
Logistic	$\hat{\alpha} = l_2,\ \hat{\xi} = l_1$
Normal	$\hat{\sigma} = \pi^{1/2} l_2,\ \hat{\mu} = l_1$
Generalized Pareto	(ξ known) $\hat{k} = l_1/l_2 - 2,\ \hat{\alpha} = (1+\hat{k}) l_1$
Generalized extreme value	$z = 2/(3+t_3) - \log 2\ \log 3,\ \hat{k} \approx 7.8590z + 2.9554z^2$ $\hat{\alpha} = l_2\hat{k}/(1-2^{-\hat{k}})\Gamma(1+\hat{k}),\ \hat{\xi} = l_1 + \hat{\alpha}\ \{\Gamma(1+\hat{k}) - 1\}/\hat{k}$
Generalized logistic	$\hat{k} = -t_3,\ \hat{\alpha} = l_2/\Gamma(1+\hat{k})\Gamma(1-\hat{k}),\ \hat{\xi} = l_1 + (l_2 - \hat{\alpha})/\hat{k}$
Log-normal	$z = \sqrt{(8/3)\Phi^{-1}\left(\frac{1+t_3}{2}\right)}$, $\hat{\sigma} \approx 0.999281z - 0.006118z^3 + 0.000127z^5$, $\hat{\mu} = \log\{l_2/erf(\sigma/2)\} - \hat{\sigma}^2/2,\ \hat{\xi} = l_1 - \exp(\hat{\mu} + \hat{\sigma}^2/2)$
Gamma	(ξ known) $t = l_2/l_1;$ if $0 < t < \frac{1}{2}$ then $z = \pi t^2$ and $\hat{\alpha} \approx (1 - 0.3080z)/(z - 0.05812z^2 + 0.01765z^3)$; if $\frac{1}{2} \le t < 1$ then $z = 1 - t$ and $\hat{\alpha} \approx (0.7213z - 0.5947z^2)/(1 - 2.1817z + 1.2113z^2)$; $\hat{\beta} = l_1/\hat{\alpha}$

γ is Euler's constant; Φ^{-1} is the inverse standard normal distribution function.

and the objective is to estimate θ. Since, $\lambda_r, r = 1, 2, 3, \ldots$ uniquely characterizes F, θ may be expressed as a function of λ_r. Hence, if estimators $\hat{\lambda}_r = \ell_r$ are available, we may obtain $\hat{\theta}(\hat{\lambda_1}, \hat{\lambda_2}, \ldots)$. From eq.n.(2.2.10), $\lambda_r = \sum_{m=0}^{r} p^*_{r-1,m}\, \beta_m$ where $\beta_m = \int_0^1 x(u)u^m du$. Given the sample, we define $x_{k,n}$ to be the k^{th} smallest element of the sample, such that $x_{1,n} \leq x_{2,n} \leq \ldots \leq x_{n,n}$. An unbiased estimator of β_r is

$$\hat{\beta}_r = b_r = n^{-1} \sum_{j=r+1}^{n} \frac{(j-1)(j-2)...(j-r)}{(n-1)(n-2)...(n-r)}\, x_{j:n}$$

and so $\hat{\lambda}_r = \sum_{m=0}^{r} p^*_{r-1,m}\, \hat{\beta}_m$ [7].

Here are some examples for parameter estimation using method of L-moments:

Example 3.4.1. ***Uniform Distribution****:*

From Table 2.5, we have the first and the second L-moments for the Uniform distribution are $\lambda_1 = \frac{1}{2}(\alpha + \beta)$ *and* $\lambda_2 = \frac{1}{6}(-\alpha + \beta)$, *then we have two equations*

$$2\lambda_1 = \alpha + \beta, \tag{3.4.1}$$

$$6\lambda_6 = -\alpha + \beta. \tag{3.4.2}$$

By solving eq.n.(3.4.1) *and* (3.4.2) *we have:*

$\beta = \lambda_1 + 3\lambda_2, \quad \alpha = \lambda_1 - 3\lambda_2.$

Hence, $\hat{\beta} = \hat{\lambda}_1 + 3\hat{\lambda}_2 = \ell_1 + 3\ell_2, \quad \hat{\alpha} = \hat{\lambda}_1 - 3\hat{\lambda}_2 = \ell_1 - 3\ell_2.$

Example 3.4.2. ***Exponential Distribution****:*

From Table 2.5, we have the first L-moments for exponential distribution is: $\lambda_1 = \xi + \alpha$. *If* $\xi = 0$, *then* $\lambda_1 = \alpha$ *and hence* $\hat{\alpha} = \hat{\lambda}_1 = \ell_1$.

Example 3.4.3. ***Generalized Logistic Distribution***

From Table 2.5, we have the first and the second L-moments and the third ratio L-moments for the Generalized Logistic distribution (respectively) are: $\lambda_1 = \xi + \alpha\{1 -$

$\Gamma(1+k)\Gamma(1-k)\}/k$, $\lambda_2 = \alpha\Gamma(1+k)(1-k)$, $\tau_3 = -k$. Then, $k = -\tau_3 = -\frac{\lambda_3}{\lambda_2}$. So $\hat{k} = -\frac{\hat{\lambda}_2}{\hat{\lambda}_3} = -\frac{\ell_3}{\ell_2}$ $\cdot$ Since $\alpha = \lambda_2\Gamma(1+k)(1-k)$, then $\hat{\alpha} = \hat{\lambda}_2\Gamma(1+\hat{k})(1-\hat{k})$. Now, $\xi = \lambda_1 - \alpha\{1-\Gamma(1+k)\Gamma(1-k)\}/k$. Since, $\Gamma(1+k)(1-k) = \frac{\lambda_2}{\alpha}$, then $\xi = \lambda_1 - \alpha\{1-\frac{\lambda_2}{\alpha}\}/k = \lambda_1 - \frac{\alpha}{k} + \frac{\lambda_2}{k} = \lambda_1 + (\lambda_2 - \alpha)/k$ $\cdot$ Therefor, $\hat{\xi} = \hat{\lambda}_1 + (\hat{\lambda}_2 - \hat{\alpha})/k = \ell_1 + (\ell_2 - \hat{\alpha})/\hat{k}$ $\cdot$

Example 3.4.4. ***Generalized Pareto Distribution***

From Table 2.5, we have the first and the second L-moments for the Generalized pareto distribution are: $\lambda_1 = \xi + \alpha/(1+k)$, $\lambda_2 = \alpha/(1+k)(2+k)$. If $\xi = 0$, then $\lambda_1 = \alpha/(1+k)$. So, $\alpha = (1+k)\lambda_1$. Hence, $\hat{\alpha} = (1+\hat{k})\hat{\lambda}_1 = (1+\hat{k})\ell_1$.

Since, $\lambda_2 = \alpha/(1+k)(2+k) = \frac{\alpha}{(1+k)} \times \frac{1}{(2+k)} = \lambda_1/(k+2)$, then $(k+2)\lambda_2 = \lambda_1$.

So, $k = \frac{\lambda_1}{\lambda_2} - 2$. Therefor, $\hat{k} = \frac{\hat{\lambda}_1}{\hat{\lambda}_2} - 2 = \frac{\ell_1}{\ell_2} - 2$.

Chapter 4

Estimation of the Generalized Lambda Distribution from Censored Data

The Generalized Lambda Distribution GLD is a four parameter family of distributions, consisting of a wide variety of curve shapes. The expressions for the PWMs and L-moments of it help us to find out the same for any univariate continuous (both complete and censored) distribution. Another advantage of the use of GLD is that the expressions for the PWMs and L-moments both for complete and censored data do not change, with respect to changes in the form of the distribution, except for the values of the parameters. This makes both analysis and decision making much simpler [23].

In this chapter, we deal with the "Estimation of the Generalized Lambda Distribution from Censored Data". In the first section, we find the PWMs and L-moments for GLD. In the second section, we discus the PWMs and L-moments for Censored Data (type B for Right Censoring and Left Censoring). In the third section, we find L-moments for Censored Distributions using GLD. In the last section, we discuss the fitting of the distributions to Censored Data using GLD.

4.1 The Family of Generalized Lambda Distribution

The Generalized Lambda Distribution, GLD, is a family of distributions that can take on a very wide range of shapes within one distributional form. The GLD has a number of different applications. Its main use has been in fitting distributions to empirical data, and in the computer generation of different distributions[23].
In this section we introduced the definitions of the quantile function and the probability density function of the Generalized Lambda Distribution, GLD.

Definition 4.1.1. [2] Distributions belonging to the Generalized Lambda Distribution GLD family are specified in terms of their quantile function given by

$$x(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}, \tag{4.1.1}$$

where $0 \leq u \leq 1$ and $u = P(X \leq x) = F(x)$.

λ_1, λ_2 are respectively the location and scale parameters and λ_3, λ_4 are the shape parameters which jointly determines skewness and kurtosis [23].

Definition 4.1.2. [2] The probability density function of the Generalized Lambda Distribution GLD is given by

$$f(x) = \frac{\lambda_2}{\lambda_3 u^{(\lambda_3 - 1)} + \lambda_4 (1-u)^{(\lambda_4 - 1)}},$$

where $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$ as given above.

4.2 PWMs and L-moments for GLD

In this section, we find -in general- that the probability weighted moments (PWMs) is the quantities $M_{k,r,s}$ for GLD, and in special case we find that (PWMs) is $\beta_r = M_{1,r,0}$ for GLD which enabled us to calculate the L-moments for GLD.
Recalling Definition(2.2.1), we have the probability weighted moments (PWMs) of a random variable X with a cumulative distribution function $F(X)$ and quantile function $x(F)$

is the quantities

$$M_{k,r,s} = E\{X^k[F(X)]^r[1-F(X)]^s\} = \int_0^1 X^k[F(X)]^r[1-F(X)]^s dF,$$

where $k,\ r,\ s$ are real numbers.

A particular useful special cases are the probability weighted moments $\alpha_r = M_{1,0,r}$ and $\beta_r = M_{1,r,0}$. For a distribution that has a quantile function $x(u)$,

$$\alpha_r = \int_0^1 x(u)(1-u)^r du,$$

$$\beta_r = \int_0^1 x(u)u^r du. \tag{4.2.1}$$

Lemma 4.2.1.

$$B(n+1,a) = \frac{n!}{\prod_{j=0}^{n}(a+j)},\ \textit{where } n \textit{ is nonnegative integer.} \tag{4.2.2}$$

Proof. From eq.n.(2.3.2) we have:

$$\begin{aligned} B(n+1,a) &= \frac{\Gamma(n+1)\Gamma(a)}{\Gamma(a+n+1)} \\ &= \frac{n!\,\Gamma(a)}{\Gamma(a+n+1)}, \quad \text{from eq.n.(1.4.3)} \\ &= \frac{n!\,\Gamma(a)}{(a+n)(a+n-1)\cdots(a+1)a\Gamma(a)}, \quad \text{from eq.n.(1.4.2)} \\ &= \frac{n!}{(a+0)(a+1)\cdots(a+(n-1))(a+n)} \\ &= \frac{n!}{\prod_{j=0}^{n}(a+j)}. \end{aligned}$$

□

In the next proposition, we find the quantities $M_{k,r,s}$ of GLD.

Proposition 4.2.2. *[23] The PWM $M_{k,r,s}$ of a GLD$(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ family with quantile function $x(u)$ is given by*

$$M_{k,r,s} = \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \lambda_1^{k-i}\lambda_2^{-i} \sum_{j=0}^{i}(-1)^j \begin{pmatrix} i \\ j \end{pmatrix} B(\lambda_3(i-j)+r+1, \lambda_4 j+s+1),$$

where $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, (Beta function).

Proof.

$$\begin{aligned}
M_{k,r,s} &= E\{X^K[F(x)]^r[1-F(x)]^s\} \\
&= \int_0^1 [x(u)]^k u^r[1-u]^s\,du \\
&= \int_0^1 \Big[\lambda_1 + \frac{u^{\lambda_3}-(1-u)^{\lambda_4}}{\lambda_2}\Big]^k u^r[1-u]^s\,du \\
&= \int_0^1 \Big[\sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix}\Big(\frac{u^{\lambda_3}-(1-u)^{\lambda_4}}{\lambda_2}\Big)^i \lambda_1^{k-i}\Big] u^r[1-u]^s\,du \\
&= \sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix}\lambda_1^{k-i}\lambda_2^{-i}\int_0^1 \Big(u^{\lambda_3}-(1-u)^{\lambda_4}\Big)^i u^r[1-u]^s\,du \\
&= \sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix}\lambda_1^{k-i}\lambda_2^{-i}\int_0^1 \Big[\sum_{j=0}^{i}\begin{pmatrix} i \\ j \end{pmatrix}\Big(-(1-u)^{\lambda_4}\Big)^j (u^{\lambda_3})^{i-j}\Big] u^r[1-u]^s\,du \\
&= \sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix}\lambda_1^{k-i}\lambda_2^{-i}\sum_{j=0}^{i}(-1)^j\begin{pmatrix} i \\ j \end{pmatrix}\int_0^1 (u^{\lambda_3})^{i-j}((1-u)^{\lambda_4})^j u^r[1-u]^s\,du \\
&= \sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix}\lambda_1^{k-i}\lambda_2^{-i}\sum_{j=0}^{i}(-1)^j\begin{pmatrix} i \\ j \end{pmatrix}\int_0^1 u^{\lambda_3(i-j)+r}(1-u)^{\lambda_4 j+s}\,du, \quad \text{from eq.n.(2.3.2)} \\
&= \sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix}\lambda_1^{k-i}\lambda_2^{-i}\sum_{j=0}^{i}(-1)^j\begin{pmatrix} i \\ j \end{pmatrix} B(\lambda_3(i-j)+r+1, \lambda_4 j+s+1)\cdot
\end{aligned}$$

□

$M_{0,r,0}$, $M_{0,0,s}$ and $M_{0,r,s}$ do not involve any parameters of the distribution and hence are of no practical use. From Definition (2.2.1), the quantities $M_{k,0,0} = E(X^k)$, $(k = 1, 2, ...)$ are the usual noncentral moments of X. For GLD it is given as

$$M_{k,0,0} = \sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix}\lambda_1^{k-i}\lambda_2^{-i}\sum_{j=0}^{i}(-1)^j\begin{pmatrix} i \\ j \end{pmatrix} B(\lambda_3(i-j)+1, \lambda_4 j+1).$$

Similarly, $$M_{k,r,0} = \sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix}\lambda_1^{k-i}\lambda_2^{-i}\sum_{j=0}^{i}(-1)^j\begin{pmatrix} i \\ j \end{pmatrix} B(\lambda_3(i-j)+r+1, \lambda_4 j+1).$$

$$M_{k,0,s} = \sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix}\lambda_1^{k-i}\lambda_2^{-i}\sum_{j=0}^{i}(-1)^j\begin{pmatrix} i \\ j \end{pmatrix} B(\lambda_3(i-j)+1, \lambda_4 j+s+1).$$

It is to be noted that

$$
\begin{aligned}
M_{1,0,r} &= \sum_{i=0}^{1}\begin{pmatrix}1\\ i\end{pmatrix}\lambda_1^{1-i}\lambda_2^{-i}\sum_{j=0}^{i}(-1)^j\begin{pmatrix}i\\ j\end{pmatrix}B(\lambda_3(i-j)+1,\lambda_4 j+s+1)\\
&= \lambda_1 B(1,r+1)+\frac{1}{\lambda_2}\sum_{j=0}^{i}(-1)^j\begin{pmatrix}i\\ j\end{pmatrix}B(\lambda_3(i-j)+1,\lambda_4 j+s+1)\\
&= \lambda_1 B(1,r+1)+\frac{1}{\lambda_2}\big[B(\lambda_3+1,r+1)-B(1,\lambda_4+r+1)\big]\ \text{(by Lemma (4.2.1))}\\
&= \frac{\lambda_1}{r+1}+\frac{1}{\lambda_2}\Big[B(\lambda_3+1,r+1)-\frac{1}{\lambda_4+r+1}\Big]\cdot
\end{aligned}
$$

$$
\begin{aligned}
M_{1,r,0} &= \sum_{i=0}^{1}\begin{pmatrix}1\\ i\end{pmatrix}\lambda_1^{1-i}\lambda_2^{-i}\sum_{j=0}^{1}(-1)^j\begin{pmatrix}i\\ j\end{pmatrix}B(\lambda_3(i-j)+r+1,\lambda_4 j+1)\\
&= \lambda_1 B(r+1,1)+\frac{1}{\lambda_2}\sum_{j=0}^{1}(-1)^j\begin{pmatrix}i\\ j\end{pmatrix}B(\lambda_3(i-j)+r+1,\lambda_4 j+1)\\
&= \lambda_1 B(r+1,1)+\frac{1}{\lambda_2}\big[B(\lambda_3+r+1,1)-B(r+1,\lambda_4+1)\big]\\
&= \frac{\lambda_1}{r+1}+\frac{1}{\lambda_2}\Big[\frac{1}{\lambda_3+r+1}-B(r+1,\lambda_4+1)\Big]\cdot
\end{aligned}
$$

Hence

$$M_{1,r,0}=\frac{\lambda_1}{r+1}+\frac{1}{\lambda_2}\Big[\frac{1}{\lambda_3+r+1}-B(r+1,\lambda_4+1)\Big] \tag{4.2.3}$$

$$=\frac{\lambda_1}{r+1}+\frac{1}{\lambda_2}\Big[\frac{1}{\lambda_3+r+1}-\frac{r!}{\prod_{j=0}^{r}(\lambda_4+j+1)}\Big]\ \text{(by Lemma (4.2.1))}\cdot$$

Here we consider $M_{1,r,0}$ only and denote it as β_r. From eq.n.(4.2.3) we get the r^{th} PWM β_r of a $GLD(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ family as

$$\beta_r=\frac{\lambda_1}{r+1}+\frac{1}{\lambda_2}\Big[\frac{1}{\lambda_3+r+1}-B(r+1,\lambda_4+1)\Big]\cdot \tag{4.2.4}$$

Now, we are going to find $\beta_0,\ \beta_1,\ \beta_2,\ \beta_3$ of GLD so that we can get the first four L-moments for the GLD.

Putting $r = 0,\ 1,\ 2,\ 3$ in eq.n.(4.2.4), we get

$$\beta_0 = \lambda_1 + \frac{1}{\lambda_2}\Big[\frac{1}{\lambda_3+1} - B(1,\lambda_4+1)\Big] \tag{4.2.5}$$

$$\beta_1 = \frac{\lambda_1}{2} + \frac{1}{\lambda_2}\Big[\frac{1}{\lambda_3+2} - B(2,\lambda_4+1)\Big] \tag{4.2.6}$$

$$\beta_2 = \frac{\lambda_1}{3} + \frac{1}{\lambda_2}\Big[\frac{1}{\lambda_3+3} - B(3,\lambda_4+1)\Big] \tag{4.2.7}$$

$$\beta_3 = \frac{\lambda_1}{4} + \frac{1}{\lambda_2}\Big[\frac{1}{\lambda_3+4} - B(4,\lambda_4+1)\Big]\cdot \tag{4.2.8}$$

Recalling Definition 2.1.1, we have the L-moments of X to be the quantities

$$L_r \equiv r^{-1}\sum_{k=0}^{r-1}(-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix} EX_{r-k:r} \qquad r = 1, 2, ...$$

(Here we used the symbol L instead of λ to distinguish between λ's of the GLD and that of the L-moments).

We can write L-moments from eq.n.(2.2.2) as :

$$L_{r+1} = \sum_{m=0}^{r} p^*_{r,m}\beta_m, \quad \text{where } p^*_{r,m} = (-1)^{r-m}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} r+m \\ m \end{pmatrix}. \tag{4.2.9}$$

For example the first four L-moments are related to the PWMs as:

$$L_1 = \beta_0,$$

$$L2 = 2\beta_1 - \beta_0,$$

$$L_3 = 6\beta_2 - 6\beta_1 + \beta_0,$$

$$L_4 = 30\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0.$$

By giving appropriate values of $L_1,\ L_2,\ L_3$, and L_4 corresponding to various distributions, in eq.n.(4.2.4), we can approximate the values of their PWMs and hence L-moments from eq.n. (4.2.9). The expressions for $L_1,\ L_2,\ L_3$, and L_4 of some distributions are given in [17] and the numerical values of them obtained by direct calculation are compared with the values obtained from GLD and are given in Table 4.1. Uniform(0,1), Exponential(3), Normal(0,1), Pareto(1,5), Logistic(0,1) and Gumbel(0,1) are approximated

Table 4.1: [23] Comparison of L-moments

Distribution	$x(u)$	L-moments		
		Theoretical	Numerical value	
			Direct	using GLD
Uniform (α, β)	$x = \alpha +$ $(\beta - \alpha)u$	$\frac{1}{2}(\alpha+\beta)$	0.5	0.5
		$\frac{1}{6}(\beta-\alpha)$	0.1667	0.1667
		0	0	0
		0	0	0
$Exp(\alpha)$	$x = -\alpha \log(1-u)$	α	3	2.9993
		$\alpha/2$	1.5	1.5013
		$1/3$	0.3333	0.3313
		$1/6$	0.1667	0.1670
		μ	0	0
Normal (μ, σ)	$x = \mu + \sigma\phi^{-1}(u)$	$\pi^{-1/2}\sigma$	0.5642	0.5638
		0	0	0
		$30\pi^{-1}\tan^{-1}\sqrt{2} - 9$	0.1226	0.1245
		$\alpha/(1+k)$	0.25	0.25
Pareto (α, k)	$x = \frac{\alpha[1-(1-u)^k]}{k}$	$\alpha/(1+k)(2+k)$	0.1389	0.1389
		$(1-k)/(3+k)$	0.4286	0.4286
		$(1-k)(2-k)/(3+k)(4+k)$	0.2481	0.2481
		ξ	0	0
Logistic (ξ, α)	$x = \xi +$ $\alpha \log\left(\frac{u}{1-u}\right)$	α	1	0.9989
		0	0	0
		$1/6$	0.1667	0.1668
		$\xi + \gamma\alpha$	0.5772	0.5775
Gumbel (ξ, α)	$x = \xi -$ $\alpha \log(-\log u)$	$\alpha \log 2$	0.6931	0.6905
		0.1699	0.1699	0.1742
		0.1504	0.1504	0.16
		(γ is Euler's constant)		

respectively by $GLD[0.5, 2, 1, 1]$, $GLD[0.02100, -0 : 0003603; -0.4072 * 10^{-5}, -0.001076]$, $GLD[0, 0.1975, 0.1349, 0.1349]$, $GLD[0, -1, 7.34512 * 10^{-12}, -0.2]$,
$GLD[0, -0.0003637, -0.0003630, -0.0003637]$ and
$GLD[-0.1857, 0.02107, 0.006696, 0.02326]$. In column 4 the values in 1^{st}, 2^{nd}, 3^{rd} and 4^{th} rows against each distribution give the numerical values of L_1, L_2, L_3, and L_4 respectively of that distribution. The tabled values clearly justify the use of GLD for computing the PWMs and L-moments of unimodal continuous distributions [23].

4.3 PWMs and L-moments for Type I and II Singly Censored Data

This section consists of two subsections. In the first subsection, we introduce the definition of type B right censoring, and then we find the type B PPWMs - which will be defined later in this section - of a right-censoring for GLD. In the other hand, we give the definition of the type B' left censoring, and then we find the type B' PPWMs of a left-censoring for GLD. This will be used later in section 4.5 to estimate the parameters of GLD.

Definition 4.3.1. [23] Observed data sets containing values above or below the analytical threshold of measuring equipment are referred to as *censored data*.

Such data are frequently encountered in quality and quantity monitoring applications of water, soil, and air samples [28].
In right censoring, the censored observations are greater than the measurement threshold

4.3.1 Case I-Right Censoring

The order statistics of a complete sample of n observations are denoted by the following: $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$.
Type B right censoring occurs when m of these values are observed ($m \leq n$) and the

remaining $n-m$ are censored above a known threshold T:

$$\underbrace{X_{1:n} \leq X_{2:n} \leq ... \leq X_{m:n}}_{m \text{ observed}} \leq \underbrace{T \leq X_{m+1:n} \leq X_{m+2:n} \leq ... \leq X_{n:n}}_{n-m \text{ censored}}.$$

The censoring threshold T is the random variable in type B censoring and m is fixed [28].

Example 4.3.1. *During the T hours of test we observe r failures (where r can be any number from 0 to n). The (exact) failure times are $t_1, t_2, ..., t_r$ and there are $(n-r)$ units that survived the entire T-hour test without failing. Note that T is fixed in advance and r is random, since we don't know how many failures will occur until the test is run. Note also that we assume the exact times of failure are recorded when there are failures.*

Type B PPWM, b_r^B, is equal to the PWM of the "completed sample," where the censored observations above the censoring threshold T are set equal to the censoring threshold. It is a well established fact that replacing the censored observations with a fixed value such as the measurement threshold leads to a significant bias in the resulting statistics such as the mean, the median, or a quantile [28].

Definition 4.3.2. [23] The type B PPWM of a right-censored distribution is the ordinary PWM of a (complete) distribution with quantile function

$$y^B(u) = \begin{cases} x(u), & 0 < u < c; \\ x(c), & c \leq u < 1. \end{cases} \tag{4.3.1}$$

where $T = x(c)$ is the censoring threshold satisfying $P(X \leq T) = c$.

Note 4.3.1. *[20] The type B PPWMs, of a right-censored distribution is given by the relation:*

$$\beta_r^B = \int_0^c u^r x(u) du + \frac{1-c^{r+1}}{r+1} x(c) \cdot \tag{4.3.2}$$

Proof. The type B PPWMs is obtained from substitution of eq.n.(4.3.1) into eq.n.(4.2.1) leading to

$$\beta_r^B = \int_0^1 x(u) u^r du$$

$$
\begin{aligned}
&= \int_0^1 u^r y^B(u)\, du \\
&= \int_0^c u^r y^B(u)\, du + \int_c^1 u^r y^B(u)\, du \\
&= \int_0^c u^r x(u)\, du + \int_c^1 u^r x(c)\, du
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^c u^r x(u)\, du + x(c) \int_c^1 u^r\, du \\
&= \int_0^c u^r x(u)\, du + x(c) \Big[\frac{u^{r+1}}{r+1}\Big]_c^1 \\
&= \int_0^c u^r x(u)\, du + \frac{1 - c^{r+1}}{r+1} x(c).
\end{aligned}
$$

□

Definition 4.3.3. [23] Define the *incomplete beta function* $\beta_c(m, n)$ as:

$$\beta_c(m, n) = \int_{u \le c} u^{m-1}(1-u)^{n-1} du, \quad \text{for } 0 < c < 1; \quad u > 0.$$

Note 4.3.2. *We can write the incomplete beta function* $\beta_c(m, n)$ *as:*

$$\beta_c(m, n) = \int_0^c u^{m-1}(1-u)^{n-1} du. \tag{4.3.3}$$

Note 4.3.3. *The incomplete beta function* $\beta_c(m, n) = B(m, n)\ I_c(m, n)$.

Proof. From eq.n.(2.3.4) we have:

$$
\begin{aligned}
I_c(m, n) &= \frac{1}{B(m, n)} \int_0^c u^{m-1}(1-u)^{n-1} du \\
&= \frac{1}{B(m, n)}\ \beta_c(m, n), \qquad \text{from } eq.n.(4.3.3).
\end{aligned}
$$

Hence, $\beta_c(m, n) = B(m, n)\ I_c(m, n)$. □

Lemma 4.3.4.

$$\beta_c(r+1, a) = \frac{r!}{\prod_{j=0}^{r}(a+j)} - \sum_{j=0}^{r} \frac{r!}{(r-j)!} \frac{c^{(r-j)}\ (1-c)^{(a+j)}}{\prod_{i=0}^{j}(a+i)} \tag{4.3.4}$$

Proof.

$$\begin{aligned}\beta_c(r+1,a) &= \int_0^c u^r(1-u)^{a-1}du \\ &= \int_0^1 u^r(1-u)^{a-1}du - \int_c^1 u^r(1-u)^{a-1}du \\ &= B(r+1,a) - \int_c^1 u^r(1-u)^{a-1}du\end{aligned}$$

$$= \frac{r!}{\prod_{j=0}^{r}(a+j)} - \int_c^1 u^r(1-u)^{a-1}du, \quad \text{from eq.n.(4.2.2)} \tag{4.3.5}$$

Now, we want to find $\int_c^1 u^r(1-u)^{a-1}du$, using integrating by parts:
Let $\quad z = u^r, \qquad dv = (1-u)^{a-1}du.$
Then, $dz = ru^{r-1}du, \qquad v = -\frac{(1-u)^a}{a}.$
So,

$$\int_c^1 u^r(1-u)^{a-1}du = -\frac{u^r(1-u)^a}{a}\Big|_{u=c}^{u=1} - \int_c^1 ru^{r-1}\Big[-\frac{(1-u)^a}{a}\Big]du.$$

Hence,

$$\int_c^1 u^r(1-u)^{a-1}du = \frac{c^r(1-c)^a}{a} + \frac{r}{a}\int_c^1 u^{r-1}(1-u)^a du. \tag{4.3.6}$$

By formula (4.3.6) we have:

$$\int_c^1 u^{r-1}(1-u)^a du = \frac{c^{r-1}(1-c)^{a+1}}{a+1} + \frac{r-1}{a+1}\int_c^1 u^{r-2}(1-u)^{a+1}du. \tag{4.3.7}$$

Then, substituting eq.n.(4.3.7) int eq.n.(4.3.6) we get:

$$\begin{aligned}
\int_c^1 u^r(1-u)^{a-1}du &= \frac{c^r(1-c)^a}{a}+\frac{r}{a}\Big[\frac{c^{r-1}(1-c)^{a+1}}{a+1}+\frac{r-1}{a+1}\int_c^1 u^{r-2}(1-u)^{a+1}du\Big] \\
&= \frac{c^r(1-c)^a}{a}+\frac{r[c^{r-1}(1-c)^{a+1}]}{a(a+1)}+\frac{r(r-1)}{a(a+1)}\int_c^1 u^{r-2}(1-u)^{a+1}du \\
&= \frac{c^r(1-c)^a}{a}+\frac{r[c^{r-1}(1-c)^{a+1}]}{a(a+1)}+\frac{r(r-1)[c^{r-2}(1-c)^{a+2}]}{a(a+1)(a+2)} \\
&+ \ \ldots\ldots\ldots+\frac{r(r-1)\ldots(r-(r-1))}{a(a+1)\ldots(a+r-1)}\int_c^1 u^{r-r}(1-u)^{a+r-1}du \\
&= \frac{c^r(1-c)^a}{a}+\frac{r[c^{r-1}(1-c)^{a+1}]}{a(a+1)}+\frac{r(r-1)[c^{r-2}(1-c)^{a+2}]}{a(a+1)(a+2)} \\
&+ \ \ldots\ldots\ldots+\frac{r(r-1)\ldots(r-(r-1))}{a(a+1)\ldots(a+r-1)}\times\Big[-\frac{(1-u)^{a+r}}{a+r}\Big|_{u=c}^{u=1}\Big]
\end{aligned}$$

$$\begin{aligned}
&= \frac{c^r(1-c)^a}{a}+\frac{r[c^{r-1}(1-c)^{a+1}]}{a(a+1)}+\frac{r(r-1)[c^{r-2}(1-c)^{a+2}]}{a(a+1)(a+2)} \\
&+ \ \ldots\ldots\ldots+\frac{r(r-1)\ldots 3\times 2\times 1(1-c)^{a+r}}{a(a+1)\ldots(a+r)} \\
&= \sum_{j=0}^{r}\frac{r!}{(r-j)!}\frac{c^{(r-j)}(1-c)^{(a+j)}}{\prod_{i=0}^{j}(a+i)}. \qquad (4.3.8)
\end{aligned}$$

Substitute (4.3.8) into (4.3.5) we have:

$$\beta_c(r+1,a)=\frac{r!}{\prod_{j=0}^{r}(a+j)}-\sum_{j=0}^{r}\frac{r!}{(r-j)!}\frac{c^{(r-j)}(1-c)^{(a+j)}}{\prod_{i=0}^{j}(a+i)}.$$

□

In the following proposition, we find the type B $PPWMs$ of a GLD which we use in determining the L-moments of a right-censored distribution.

Proposition 4.3.5. *[23] The type B $PPWMs$ of a $GLD(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ family for singly right censoring are given by*

$$\beta_r^B = \frac{\lambda_1 c^{r+1}}{r+1} + \frac{c^{\lambda_3+r+1}}{\lambda_2(\lambda_3+r+1)} - \frac{1}{\lambda_2}\beta_c(r+1, \lambda_4+1) + \frac{c^{r+1}}{r+1}x(c), \tag{4.3.9}$$

Proof.

$$\begin{aligned}
\beta_r^B &= \int_0^c u^r x(u)du + \frac{1-c^{r+1}}{r+1}x(c) \\
&= \int_0^c u^r \Big[\lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}\Big] du + \frac{1-c^{r+1}}{r+1}x(c) \\
&= \lambda_1 \int_0^c u^r du + \frac{1}{\lambda_2}\int_0^c u^{\lambda_3+r}du - \frac{1}{\lambda_2}\int_0^c (1-u)^{\lambda_4}u^r du + \frac{1-c^{r+1}}{r+1}x(c) \\
&= \lambda_1 \frac{u^{r+1}}{r+1}\Big|_0^c + \frac{1}{\lambda_2} \times \frac{u^{\lambda_3+r+1}}{\lambda_3+r+1}\Big|_0^c - \frac{1}{\lambda_2}\beta_c(r+1, \lambda_4+1) + \frac{1-c^{r+1}}{r+1}x(c) \\
&= \frac{\lambda_1 c^{r+1}}{r+1} + \frac{c^{\lambda_3+r+1}}{\lambda_2(\lambda_3+r+1)} + \frac{1}{\lambda_2}\Big\{\sum_{j=0}^{r} \frac{r!}{(r-j)!}\frac{c^{(r-j)}(1-c)^{(\lambda_4+j+1)}}{\prod_{i=0}^{j}(\lambda_4+i+1)} \\
&- \frac{r!}{\prod_{j=0}^{r}(\lambda_4+j+1)}\Big\} + \frac{1-c^{r+1}}{r+1}x(c), \quad \text{from } eq.n.(4.3.4).
\end{aligned}$$

□

Now, we are about to find the first four L-moments of a right-censored distribution of the GLD. Before doing so, we have to determine the first four B $PPWMs$ of the GLD. Putting $r = 0,\ 1,\ 2,\ 3$ in the expression (4.3.9) and use eq.n.(4.3.4) we get:

$$\begin{aligned}
\beta_0^B &= \lambda_1 c + \frac{c^{\lambda_3+1}}{\lambda_2(\lambda_3+1)} - \frac{1}{\lambda_2}\beta_c(1, \lambda_4+1) + (1-c)x(c) \\
&= \lambda_1 c + \frac{c^{\lambda_3+1}}{\lambda_2(\lambda_3+1)} + \frac{(1-c)^{\lambda_4+1}}{\lambda_2(\lambda_4+1)} + (1-c)x(c) \\
\beta_1^B &= \frac{\lambda_1 c^2}{2} + \frac{c^{\lambda_3+2}}{\lambda_2(\lambda_3+2)} - \frac{1}{\lambda_2}\beta_c(2, \lambda_4+1) + \frac{1-c^2}{2}x(c) \\
&= \frac{\lambda_1 c^2}{2} + \frac{c^{\lambda_3+2}}{\lambda_2(\lambda_3+2)} - \frac{1}{\lambda_2}\Big\{\frac{c(1-c)^{\lambda_4+1}}{\lambda_4+1} + \frac{(1-c)^{\lambda_4+2}-1}{(\lambda_4+1)(\lambda_4+2)}\Big\} \\
&+ \frac{1-c^2}{2}x(c)
\end{aligned}$$

$$
\begin{aligned}
\beta_2^B &= \frac{\lambda_1 c^3}{3} + \frac{c^{\lambda_3+3}}{\lambda_2(\lambda_3+3)} - \frac{1}{\lambda_2}\beta_c(3,\lambda_4+1) + \frac{1-c^3}{3}x(c) \\
&= \frac{\lambda_1 c^3}{3} + \frac{c^{\lambda_3+3}}{\lambda_2(\lambda_3+3)} \\
&+ \frac{1}{\lambda_2}\Big\{\frac{c^2(1-c)^{\lambda_4+1}}{\lambda_4+1} + \frac{2c(1-c)^{\lambda_4+2}}{(\lambda_4+1)(\lambda_4+2)} + \frac{2(1-c)^{\lambda_4+3}-2}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)}\Big\} \\
&+ \frac{1-c^3}{3}x(c)
\end{aligned}
$$

$$
\begin{aligned}
\beta_3^B &= \frac{\lambda_1 c^4}{4} + \frac{c^{\lambda_3+4}}{\lambda_2(\lambda_3+4)} - \frac{1}{\lambda_2}\beta_c(4,\lambda_4+1) + \frac{1-c^4}{4}x(c) \\
&= \frac{\lambda_1 c^4}{4} + \frac{c^{\lambda_3+4}}{\lambda_2(\lambda_3+4)} \\
&+ \frac{1}{\lambda_2}\Big\{\frac{c^3(1-c)^{\lambda_4+1}}{\lambda_4+1} + \frac{3c^2(1-c)^{\lambda_4+2}}{(\lambda_4+1)(\lambda_4+2)} + \frac{6c(1-c)^{\lambda_4+3}}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)} \\
&+ \frac{6((1-c)^{\lambda_4+4}-1)}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)}\Big\} \\
&+ \frac{1-c^4}{4}x(c)\cdot
\end{aligned}
$$

We can write the L-moments of a right-censored distribution from eq.n.(2.2.2) as follows:

$$
L_{r+1}^B = \sum_{m=0}^{r} p_{r,m}^* \beta_m^B, \quad \text{where } p_{r,m}^* = (-1)^{r-m}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} r+m \\ m \end{pmatrix}. \tag{4.3.10}
$$

In particular, the first four L-moments of the right-censored distribution of the GLD that are related to the type B PPWM of the GLD are:

$$
\begin{aligned}
L_1^B &= \beta_0^B \\
L_2^B &= 2\beta_1^B - \beta_0^B \\
L_3^B &= 6\beta_2^B - 6\beta_1^B + \beta_0^B \\
L_4^B &= 30\beta_3^B - 30\beta_2^B + 12\beta_1^B - \beta_0^B.
\end{aligned}
$$

4.3.2 Case 2 - Left Censoring

The B' left censoring results when the observations below a random variable threshold T are censored:

$$\underbrace{X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{m-1:n}}_{n-k \text{ censored}} \leq \underbrace{T \leq X_{m:n} \leq X_{m+2:n} \leq \ldots \leq X_{m:n}}_{k \text{ observed}},$$

where the number of the censored values $(m - 1 = n - k)$ is fixed [28].
For left censoring, type B' PPWMs may be derived by replacing the censored observations with the fixed threshold $x(c)$, below which measurements are unavailable [23].

Example 4.3.2. *In the field of hydrology, left censored data sets arise because river discharges below some measurement threshold are often reported as zero. Such river discharges may have actually been zero or they may have been between zero and the measurement threshold, yet reported as zero [8]. Sometimes it is actually advantageous to intentionally censor (or eliminate) observations in order to better understand the frequency and magnitude of flood and drought events [29].*

Definition 4.3.4. [23] The type B' PPWM of a left-censored distribution is the ordinary PWM of a (complete) distribution with quantile function

$$y^{B'}(u) = \begin{cases} x(c), & 0 < u < c; \\ x(u), & c \leq u < 1. \end{cases} \tag{4.3.11}$$

where $T = x(c)$ is the censoring threshold satisfying $P(X \leq T) = c$.

Note 4.3.6. *[28] The type B' PPWMs, of a left-censored distribution is given by the relation:*

$$\beta_r^{B'} = \frac{c^{r+1}}{r+1} x(c) + \int_c^1 u^r x(u)\, du. \tag{4.3.12}$$

Proof. The type B' PPWMs is obtained from substitution of eq.n.(4.3.11) into eq.n.(4.2.1) leading to

$$\beta_r^{B'} = \int_0^1 x(u) u^r du$$

$$
\begin{aligned}
&= \int_0^1 u^r y^{B'}(u)\, du \\
&= \int_0^c u^r y^{B'}(u)\, du + \int_c^1 u^r y^{B'}(u)\, du \\
&= \int_0^c u^r x(c)\, du + \int_c^1 u^r x(u)\, du \\
&= x(c)\int_0^c u^r\, du + \int_c^1 u^r x(u)\, du \\
&= x(c)\Big[\frac{u^{r+1}}{r+1}\Big]_0^c + \int_c^1 u^r x(u)\, du \\
&= \int_c^1 u^r x(u) du + \frac{c^{r+1}}{r+1}x(c)\cdot
\end{aligned}
$$

□

In the next proposition, we find the type B' $PPWMs$ of a GLD by which we determining the L-moments of a left-censored distribution.

Proposition 4.3.7. *[23] The type B' $PPWMs$ of a $GLD(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ family for singly left censoring are given by*

$$
\beta_r^{B'} = \frac{\lambda_1(1-c^{r+1})}{r+1} + \frac{(1-c^{\lambda_3+r+1})}{\lambda_2(\lambda_3+r+1)} - \frac{1}{\lambda_2}\{B(r+1,\lambda_4+1) - \beta_c(r+1,\lambda_4+1)\} + \frac{c^{r+1}}{r+1}x(c). \tag{4.3.13}
$$

$$
= \frac{\lambda_1(1-c^{r+1})}{r+1} + \frac{(1-c^{\lambda_3+r+1})}{\lambda_2(\lambda_3+r+1)} - \frac{1}{\lambda_2}\Big\{\sum_{j=0}^{r}\frac{r!}{(r-j)!}\frac{c^{(r-j)}(1-c)^{(\lambda_4+j+1)}}{\prod_{i=0}^{j}(\lambda_4+i+1)}\Big\} + \frac{c^{r+1}}{r+1}x(c).
$$

Proof.

$$
\begin{aligned}
\beta_r^{B'} &= \int_c^1 u^r x(u) du + \frac{c^{r+1}}{r+1}x(c) \\
&= \int_c^1 u^r\Big[\lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}\Big] du + \frac{c^{r+1}}{r+1}x(c) \\
&= \lambda_1\int_c^1 u^r du + \frac{1}{\lambda_2}\int_c^1 u^{\lambda_3+r} du - \frac{1}{\lambda_2}\int_c^1 (1-u)^{\lambda_4} u^r du + \frac{c^{r+1}}{r+1}x(c) \\
&= \lambda_1 \frac{u^{r+1}}{r+1}\Big|_c^1 + \frac{1}{\lambda_2}\times\frac{u^{\lambda_3+r+1}}{\lambda_3+r+1}\Big|_c^1 - \frac{1}{\lambda_2}\Big\{\int_0^1 (1-u)^{\lambda_4}u^r du - \int_0^c (1-u)^{\lambda_4}u^r du\Big\} + \frac{c^{r+1}}{r+1}x(c)
\end{aligned}
$$

$$= \frac{\lambda_1(1-c^{r+1})}{r+1} + \frac{(1-c^{\lambda_3+r+1})}{\lambda_2(\lambda_3+r+1)} - \frac{1}{\lambda_2}\Big\{B(r+1,\lambda_4+1) - \beta_c(r+1,\lambda_4+1)\Big\} + \frac{c^{r+1}}{r+1}x(c)$$

$$= \frac{\lambda_1(1-c^{r+1})}{r+1} + \frac{(1-c^{\lambda_3+r+1})}{\lambda_2(\lambda_3+r+1)} - \frac{1}{\lambda_2}\Big\{\sum_{j=0}^{r}\frac{r!}{(r-j)!}\frac{c^{(r-j)}(1-c)^{(\lambda_4+j+1)}}{\prod_{i=0}^{j}(\lambda_4+i+1)}\Big\}$$

$$+ \frac{c^{r+1}}{r+1}x(c), \quad \text{from eq.n.(4.3.4).}$$

□

Now, we will determine the first four B' $PPWMs$ of the GLD to determine the first four L-moments of a left-censored distribution of the GLD.
Putting $r = 0,\ 1,\ 2,\ 3$ in the expression (4.3.13) we get

$$\beta_0^{B'} = \lambda_1(1-c) + \frac{(1-c^{\lambda_3+1})}{\lambda_2(\lambda_3+1)} - \frac{1}{\lambda_2}\{B(r+1,\lambda_4+1) - \beta_c(1,\lambda_4+1)\}c.x(c)$$

$$\beta_1^{B'} = \frac{\lambda_1(1-c^2)}{2} + \frac{(1-c^{\lambda_3+2})}{\lambda_2(\lambda_3+2)} - \frac{1}{\lambda_2}\{B(2,\lambda_4+1) - \beta_c(2,\lambda_4+1)\} + \frac{c^2}{2}x(c)$$

$$\beta_2^{B'} = \frac{\lambda_1(1-c^3)}{3} + \frac{(1-c^{\lambda_3+3})}{\lambda_2(\lambda_3+3)} - \frac{1}{\lambda_2}\{B(3,\lambda_4+1) - \beta_c(3,\lambda_4+1)\} + \frac{c^3}{3}x(c)$$

$$\beta_3^{B'} = \frac{\lambda_1(1-c^4)}{4} + \frac{(1-c^{\lambda_3+4})}{\lambda_2(\lambda_3+4)} - \frac{1}{\lambda_2}\{B(4,\lambda_4+1) - \beta_c(4,\lambda_4+1)\} + \frac{c^4}{4}x(c)$$

We can write L-moments of a left-censored distribution from eqn.(2.2.2) as :

$$L_{r+1}^{B'} = \sum_{m=0}^{r} p_{r,m}^{*}\beta_m^{B'}, \quad \text{where } p_{r,m}^{*} = (-1)^{r-m}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} r+m \\ m \end{pmatrix}. \qquad (4.3.14)$$

In particular, the first four L-moments of the left-censored distribution of the GLD that are related to the type B' PPWM of the GLD are:

$$L_1^{B'} = \beta_0^{B'}$$

$$L_2^{B'} = 2\beta_1^{B'} - \beta_0^{B'}$$

$$L_3^{B'} = 6\beta_2^{B'} - 6\beta_1^{B'} + \beta_0^{B'}$$

$$L_4^{B'} = 30\beta_3^{B'} - 30\beta_2^{B'} + 12\beta_1^{B'} - \beta_0^{B'}.$$

4.4 L-moments for Censored Distributions Using GLD

In section 4.3, we find the type B PPWMs for GLD. This will be used to find the B PPWMs for Pareto distribution, which is considered to be a special case of GLD and its quantile function will be given soon. The L-moments for Pareto distribution is deduced later.

Definition 4.4.1. [23] The Pareto distribution has the quantile function:

$$x(u) = \alpha[1 - (1-u)^k]/k.$$

Assume that $m_r = \alpha\left[\frac{1-(1-c)^{r+k}}{r+k}\right]$. We will use this assumption to express the L-moments of Pareto distribution.

Now, we are going to calculate the first four L-moments for Pareto distribution.

First, we calculate β_0^B for Pareto distribution:

$$\begin{aligned}
\beta_0^B &= \lambda_1 c + \frac{c^{\lambda_3+1}}{\lambda_2(\lambda_3+1)} + \frac{(1-c)^{\lambda_4+1}}{\lambda_2(\lambda_4+1)} + (1-c)x(c) \\
&= \frac{c}{\frac{k}{\alpha}} + \frac{(1-c)^{k+1}}{\frac{k}{\alpha}} + \frac{\alpha}{k(k+1)} - \frac{1}{\frac{k}{\alpha}(k+1)} + (1-c)[1-(1-c)^k]\frac{\alpha}{k} \\
&= \frac{\alpha}{k}\Big[c + \frac{(1-c)^{k+1}}{k+1} - \frac{1}{k+1} + (1-c) - (1-c)^{k+1}\Big] \\
&= \frac{\alpha}{k}\Big[\frac{(1-c)^{k+1}-1}{k+1} + 1 - (1-c)^{k+1}\Big] \\
&= \frac{\alpha}{k}\Big[1-(1-c)^{k+1}\Big]\Big(\frac{-1}{k+1} + 1\Big) \\
&= \frac{\alpha}{k}\Big[1-(1-c)^{k+1}\Big]\Big(\frac{-1+k+1}{k+1}\Big) \\
&= \frac{\alpha}{k}\Big[1-(1-c)^{k+1}\Big]\Big(\frac{k}{k+1}\Big) \\
&= \alpha\Big[\frac{1-(1-c)^{k+1}}{k+1}\Big] = \alpha m_1.
\end{aligned}$$

Thus, the 1^{st} L-moment for Pareto distribution is given by:

$$L_B^1 = \beta_0^B = \alpha m_1.$$

Second, we calculate β_1^B for Pareto distribution:

$$\begin{aligned}
\beta_1^B &= \frac{\lambda_1 c^2}{2} + \frac{c^{\lambda_3+2}}{\lambda_2(\lambda_3+2)} - \frac{1}{\lambda_2}\Big\{\frac{c(1-c)^{\lambda_4+1}}{\lambda_4+1} + \frac{(1-c)^{\lambda_4+2}-1}{(\lambda_4+1)(\lambda_4+2)}\Big\} \\
&+ \frac{1-c^2}{2}x(c) \\
&= \frac{c^2}{2(\frac{k}{\alpha})} + \frac{1}{k/\alpha}\Big\{\frac{c(1-c)^{k+1}}{k+1} + \frac{(1-c)^{k+2}-1}{(k+1)(k+2)}\Big\} + \frac{1-c^2}{2}\Big[1-(1-c)^k\Big]\frac{\alpha}{k} \\
&= \frac{\alpha}{k}\Big[\frac{c^2}{2} + \frac{c(1-c)^{k+1}}{k+1} + \frac{(1-c)^{k+2}-1}{(k+1)(k+2)} + \frac{1-c^2}{2} - \frac{1-c^2}{2}(1-c)^k\Big] \\
&= \frac{\alpha}{k}\Big[\frac{c^2}{2} + \frac{c(1-c)^{k+1}}{k+1} + \frac{(1-c)^{k+2}-1}{(k+1)(k+2)} + \frac{1}{2} - \frac{c^2}{2} - \frac{1+c}{2}(1-c)^{k+1}\Big] \\
&= \frac{\alpha}{k}\Big[\frac{c(1-c)^{k+1}}{k+1} + \frac{(1-c)^{k+2}-1}{(k+1)(k+2)} + \frac{1}{2} - \frac{1+c}{2}(1-c)^{k+1}\Big] \\
&= \frac{\alpha}{k}\Big[\frac{c(1-c)^{k+1}}{k+1} + \frac{(1-c)^{k+2}}{(k+1)(k+2)} - \frac{1}{(k+1)(k+2)} + \frac{1}{2} - \frac{1+c}{2}(1-c)^{k+1}\Big] \\
&= \frac{\alpha}{k}\Big[\frac{2(k+2)c(1-c)^{k+1} + 2(1-c)^{k+2} - 2 + (k+1)(k+2)}{2(k+1)(k+2)} \\
&+ \frac{-(k+1)(k+2)(1+c)(1-c)^{k+1}}{2(k+1)(k+2)}\Big] \\
&= \frac{\alpha}{k}\Big[\frac{2(k+2)c(1-c)^{k+1} + 2(1-c)^{k+2} - 2 + k^2 + 3k + 2}{2(k+1)(k+2)} \\
&+ \frac{-(k+1)(k+2)(1+c)(1-c)^{k+1}}{2(k+1)(k+2)}\Big]
\end{aligned}$$

$$= \frac{\alpha}{k}\Big[\frac{(1-c)^{k+1}\big[2(k+2)c-(k+1)(k+2)(1+c)\big]+2(1-c)^{k+2}+k^2+3k}{2(k+1)(k+2)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(1-c)^{k+1}\big[2(k+2)c-(k+1)(k+2)-(k+1)(k+2)c\big]+2(1-c)^{k+2}+k^2+3k}{2(k+1)(k+2)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(1-c)^{k+1}\big[c\big\{2(k+2)-(k+1)(k+2)\big\}-(k+1)(k+2)\big]+2(1-c)^{k+2}+k^2+3k}{2(k+1)(k+2)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(1-c)^{k+1}\big[c(k+2)\big\{2-(k+1)\big\}-(k+1)(k+2)\big]+2(1-c)^{k+2}+k^2+3k}{2(k+1)(k+2)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(k+2)(1-c)^{k+1}\big[c(-k+1)-(k+1)\big]+2(1-c)^{k+2}+k^2+3k}{2(k+1)(k+2)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(k+2)(1-c)^{k+1}\big[c(-k+1)-(-k+1)+(-k+1)-(k+1)\big]}{2(k+1)(k+2)}$$
$$+ \frac{2(1-c)^{k+2}+k^2+3k}{2(k+1)(k+2)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(k+2)(1-c)^{k+1}\big[(-k+1)(c-1)-2k\big]+2(1-c)^{k+2}+k^2+3k}{2(k+1)(k+2)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{-(k+2)(-k+1)(1-c)^{k+2}-2k(k+2)(1-c)^{k+1}+2(1-c)^{k+2}+k^2+3k}{2(k+1)(k+2)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(1-c)^{k+2}\big[-(k+2)(-k+1)+2\big]-2k(k+2)(1-c)^{k+1}+k^2+3k}{2(k+1)(k+2)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(1-c)^{k+2}(k^2+k)-2k(k+2)(1-c)^{k+1}+k^2+3k}{2(k+1)(k+2)}\Big].$$

$$= \frac{\alpha}{k}\Big[\frac{(1-c)^{k+2}(k^2+k)-(k^2+k)+(k^2+k)}{2(k+1)(k+2)}$$

$$+ \frac{-2k(k+2)(1-c)^{k+1}+2k(k+2)-2k(k+2)+k^2+3k}{2(k+1)(k+2)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(k^2+k)[(1-c)^{k+2}-1]-2k(k+2)[(1-c)^{k+1}-1]}{2(k+1)(k+2)}$$

$$+ \frac{k^2+k-2k^2-4k+k^2+3k}{2(k+1)(k+2)}\Big] + \alpha\Big[-\frac{1-(1-c)^{k+2}}{2(k+2)} + \frac{1-(1-c)^{k+1}}{(k+1)}\Big].$$

Thus, the 2^{nd} L-moment for Pareto distribution is given by:

$$\begin{aligned} L_2^B &= 2\beta_1^B - \beta_0^B \\ &= \alpha\Big[-\frac{1-(1-c)^{k+2}}{(k+2)} + \frac{2[1-(1-c)^{k+1}]}{(k+1)} - \frac{1-(1-c)^{k+1}}{(k+1)}\Big] \\ &= \alpha\Big[\frac{1-(1-c)^{k+1}}{(k+1)} - \frac{1-(1-c)^{k+2}}{(k+2)}\Big] \\ &= \alpha(m_1 - m_2)\cdot \end{aligned}$$

Third, we calculate β_2^B for Pareto distribution:

$$\begin{aligned} \beta_2^B &= \frac{\lambda_1 c^3}{3} + \frac{c^{\lambda_3+3}}{\lambda_2(\lambda_3+3)} + \frac{1}{\lambda_2}\Big\{\frac{c^2(1-c)^{\lambda_4+1}}{\lambda_4+1} + \frac{2c(1-c)^{\lambda_4+2}}{(\lambda_4+1)(\lambda_4+2)} \\ &+ \frac{2(1-c)^{\lambda_4+3}-2}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)}\Big\} + \frac{1-c^3}{3}x(c) \\ &= \frac{c^3}{3(\frac{k}{\alpha})} + \frac{1}{k/\alpha}\Big\{\frac{c^2(1-c)^{k+1}}{k+1} + \frac{2c(1-c)^{k+2}}{(k+1)(k+2)} + \frac{2(1-c)^{k+3}-2}{(k+1)(k+2)(k+3)}\Big\} \\ &+ \frac{1-c^3}{3}\Big[1-(1-c)^k\Big]\frac{\alpha}{k} \\ &= \frac{\alpha}{k}\Big[\frac{c^3}{3} + \frac{c^2(1-c)^{k+1}}{k+1} + \frac{2c(1-c)^{k+2}}{(k+1)(k+2)} + \frac{2(1-c)^{k+3}-2}{(k+1)(k+2)(k+3)} \\ &+ \frac{1-c^3}{3} - \frac{1-c^3}{3}(1-c)^k\Big] \end{aligned}$$

$$= \frac{\alpha}{k}\Big[\frac{1}{3} + \frac{\{(1-c)^2 - 2(1-c) + 1\}(1-c)^{k+1}}{k+1} + \frac{2\{-(1-c)+1\}(1-c)^{k+2}}{(k+1)(k+2)}$$

$$+ \quad \frac{2(1-c)^{k+3} - 2}{(k+1)(k+2)(k+3)} - \frac{\{(1-c)^3 - 3(1-c)^2 + 3(1-c)\}}{3}(1-c)^k\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{1}{3} + \frac{(1-c)^{k+3} - 2(1-c)^{k+2} + (1-c)^{k+1}}{k+1} + \frac{-2(1-c)^{k+3} + 2(1-c)^{k+2}}{(k+1)(k+2)}$$

$$+ \quad \frac{2(1-c)^{k+3} - 2}{(k+1)(k+2)(k+3)} + \frac{-(1-c)^{k+3} + 3(1-c)^{k+2} - 3(1-c)^{k+1}}{3}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(k+1)(k+2)(k+3)}{3(k+1)(k+2)(k+3)} + \frac{3(k+2)(k+3)(1-c)^{k+3} - 6(k+2)(k+3)(1-c)^{k+2}}{3(k+1)(k+2)(k+3)}$$

$$+ \quad \frac{3(k+2)(k+3)(1-c)^{k+1}}{3(k+1)(k+2)(k+3)}$$

$$+ \quad \frac{-6(k+3)(1-c)^{k+3} + 6(k+3)(1-c)^{k+2}}{3(k+1)(k+2)(k+3)} + \frac{6(1-c)^{k+3} - 6}{3(k+1)(k+2)(k+3)}$$

$$+ \quad \frac{-(k+1)(k+2)(k+3)(1-c)^{k+3} + 3(k+1)(k+2)(k+3)(1-c)^{k+2}}{3(k+1)(k+2)(k+3)}$$

$$+ \quad \frac{-3(k+1)(k+2)(k+3)(1-c)^{k+1}}{3(k+1)(k+2)(k+3)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(k+1)(k+2)(k+3) - 6}{3(k+1)(k+2)(k+3)} + \frac{(1-c)^{k+1}\big[3(k+2)(k+3) - 3(k+1)(k+2)(k+3)\big]}{3(k+1)(k+2)(k+3)}$$

$$+ \quad \frac{(1-c)^{k+2}\big[-6(k+2)(k+3) + 6(k+3) + 3(k+1)(k+2)(k+3)\big]}{3(k+1)(k+2)(k+3)}$$

$$+ \quad \frac{(1-c)^{k+3}\big[3(k+2)(k+3) - 6(k+3) + 6 - (k+1)(k+2)(k+3)\big]}{3(k+1)(k+2)(k+3)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{k^3+6k^2+11k}{3(k+1)(k+2)(k+3)} + \frac{3(k+2)(k+3)(1-c)^{k+1}(1-k-1)}{3(k+1)(k+2)(k+3)}$$

$$+ \frac{3(k+3)(1-c)^{k+2}\big[-2(k+2)+2+(k+1)(k+2)\big]}{3(k+1)(k+2)(k+3)}$$

$$+ \frac{(1-c)^{k+3}\big[3k^2+15k+18-6k-18+6-k^3-6k^2-11k-6\big]}{3(k+1)(k+2)(k+3)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{k^3+6k^2+11k}{3(k+1)(k+2)(k+3)} + \frac{-3k(k+2)(k+3)(1-c)^{k+1}}{3(k+1)(k+2)(k+3)}$$

$$+ \frac{3(k+3)(1-c)^{k+2}(-2k-4+2+k^2+3k+2)}{3(k+1)(k+2)(k+3)} + \frac{(1-c)^{k+3}(-k^3-3k^2-2k)}{3(k+1)(k+2)(k+3)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{k^3+6k^2+11k}{3(k+1)(k+2)(k+3)} + \frac{-3k(k+2)(k+3)(1-c)^{k+1}}{3(k+1)(k+2)(k+3)}$$

$$+ \frac{3k(k+1)(k+3)(1-c)^{k+2}}{3(k+1)(k+2)(k+3)} + \frac{-k(k+1)(k+2)(1-c)^{k+3}}{3(k+1)(k+2)(k+3)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{k^3+6k^2+11k}{3(k+1)(k+2)(k+3)}$$

$$+ \frac{-3k(k+2)(k+3)(1-c)^{k+1}+3k(k+2)(k+3)-3k(k+2)(k+3)}{3(k+1)(k+2)(k+3)}$$

$$+ \frac{3k(k+1)(k+3)(1-c)^{k+2}-3k(k+1)(k+3)+3k(k+1)(k+3)}{3(k+1)(k+2)(k+3)}$$

$$+ \frac{-k(k+1)(k+2)(1-c)^{k+3}+k(k+1)(k+2)-k(k+1)(k+2)}{3(k+1)(k+2)(k+3)}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{k^3+6k^2+11k}{3(k+1)(k+2)(k+3)}$$

$$+ \frac{3k(k+2)(k+3)[1-(1-c)^{k+1}]-3k(k+2)(k+3)}{3(k+1)(k+2)(k+3)}$$

$$
\begin{aligned}
&+ \quad \frac{-3k(k+1)(k+3)[1-(1-c)^{k+2}] + 3k(k+1)(k+3)}{3(k+1)(k+2)(k+3)} \\
&+ \quad \frac{k(k+1)(k+2)[1-(1-c)^{k+3}] - k(k+1)(k+2)}{3(k+1)(k+2)(k+3)}\Big] \\
&= \quad \frac{\alpha}{k}\Big[\frac{k^3+6k^2+11k-3k(k+2)(k+3)+3k(k+1)(k+3)-k(k+1)(k+2)}{3(k+1)(k+2)(k+3)} \\
&+ \quad \frac{k[1-(1-c)^{k+1}]}{(k+1)} + \frac{-k[1-(1-c)^{k+2}]}{(k+2)} + \frac{k[1-(1-c)^{k+3}]}{3(k+3)}\Big] \\
&= \quad \alpha\Big[\frac{1-(1-c)^{k+1}}{(k+1)} - \frac{1-(1-c)^{k+2}}{(k+2)} + \frac{1-(1-c)^{k+3}}{3(k+3)}\Big].
\end{aligned}
$$

Thus, the 3^{ed} L-moment for Pareto distribution is given by:

$$
\begin{aligned}
L_3^B &= 6\beta_2^B - 6\beta_1^B + \beta_0^B \\
&= \alpha\Big[6\frac{1-(1-c)^{k+1}}{(k+1)} - 6\frac{1-(1-c)^{k+2}}{(k+2)} + 2\frac{1-(1-c)^{k+3}}{(k+3)} + 3\frac{1-(1-c)^{k+2}}{(k+2)} \\
&- 6\frac{1-(1-c)^{k+1}}{(k+1)} + \frac{1-(1-c)^{k+1}}{(k+1)}\Big] \\
&= \alpha\Big[\frac{1-(1-c)^{k+1}}{(k+1)} - 3\frac{1-(1-c)^{k+2}}{(k+2)} + \frac{1-(1-c)^{k+3}}{(k+3)}\Big] \\
&= \alpha(m_1 - 3m_2 + 2m_3).
\end{aligned}
$$

Finally, we calculate β_3^B for Pareto distribution:

$$\begin{aligned}
\beta_3^B &= \frac{\lambda_1 c^4}{4} + \frac{c^{\lambda_3+4}}{\lambda_2(\lambda_3+4)} \\
&+ \frac{1}{\lambda_2}\Big\{\frac{c^3(1-c)^{\lambda_4+1}}{\lambda_4+1} + \frac{3c^2(1-c)^{\lambda_4+2}}{(\lambda_4+1)(\lambda_4+2)} + \frac{6c(1-c)^{\lambda_4+3}}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)} \\
&+ \frac{6((1-c)^{\lambda_4+4}-1)}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)}\Big\} + \frac{1-c^4}{4}x(c) \\
&= \frac{c^4}{4(\frac{k}{\alpha})} + \frac{1}{k/\alpha}\Big\{\frac{c^3(1-c)^{k+1}}{k+1} + \frac{3c^2(1-c)^{k+2}}{(k+1)(k+2)} + \frac{6c(1-c)^{k+3}}{(k+1)(k+2)(k+3)} \\
&+ \frac{6(1-c)^{k+4}-6}{(k+1)(k+2)(k+3)(k+4)}\Big\} + \frac{1-c^4}{4}\Big[1-(1-c)^k\Big]\frac{\alpha}{k} \\
&= \frac{\alpha}{k}\Big[\frac{c^4}{4} + \frac{c^3(1-c)^{k+1}}{k+1} + \frac{3c^2(1-c)^{k+2}}{(k+1)(k+2)} + \frac{6c(1-c)^{k+3}}{(k+1)(k+2)(k+3)} \\
&+ \frac{6(1-c)^{k+4}-6}{(k+1)(k+2)(k+3)(k+4)} + \frac{1-c^4}{4}\big[1-(1-c)^k\big]\Big] \\
&= \frac{\alpha}{k}\Big[\frac{c^4}{4} + \frac{c^3(1-c)^{k+1}}{k+1} + \frac{3c^2(1-c)^{k+2}}{(k+1)(k+2)} + \frac{6c(1-c)^{k+3}}{(k+1)(k+2)(k+3)} \\
&+ \frac{6(1-c)^{k+4}-6}{(k+1)(k+2)(k+3)(k+4)} + \frac{1-c^4}{4} - \frac{1-c^4}{4}(1-c)^k\Big] \\
&= \frac{\alpha}{k}\Big[\frac{1}{4} + \frac{c^3(1-c)^{k+1}}{k+1} + \frac{3c^2(1-c)^{k+2}}{(k+1)(k+2)} + \frac{6c(1-c)^{k+3}}{(k+1)(k+2)(k+3)} \\
&+ \frac{6(1-c)^{k+4}-6}{(k+1)(k+2)(k+3)(k+4)} - \frac{1-c^4}{4}(1-c)^k\Big]
\end{aligned}$$

$$= \frac{\alpha}{k}\Big[\frac{1}{4} + \frac{\{-(1-c)^3 + 3(1-c)^2 - 3(1-c) + 1\}(1-c)^{k+1}}{k+1}$$

$$+ \frac{3\{(1-c)^2 - 2(1-c) + 1\}(1-c)^{k+2}}{(k+1)(k+2)}$$

$$+ \frac{6\{-(1-c) + 1\}(1-c)^{k+3}}{(k+1)(k+2)(k+3)} + \frac{6(1-c)^{k+4} - 6}{(k+1)(k+2)(k+3)(k+4)}$$

$$- \frac{-(1-c)^4 + 4(1-c)^3 - 6(1-c)^2 + 4(1-c)}{4}(1-c)^k\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{1}{4} + \frac{-(1-c)^{k+4} + 3(1-c)^{k+3} - 3(1-c)^{k+2} + (1-c)^{k+1}}{k+1}$$

$$+ \frac{3(1-c)^{k+4} - 6(1-c)^{k+3} + 3(1-c)^{k+2}}{(k+1)(k+2)}$$

$$+ \frac{-6(1-c)^{k+4} + 6(1-c)^{k+3}}{(k+1)(k+2)(k+3)} + \frac{6(1-c)^{k+4} - 6}{(k+1)(k+2)(k+3)(k+4)}$$

$$+ \frac{(1-c)^{k+4} - 4(1-c)^{k+3} + 6(1-c)^{k+2} - 4(1-c)^{k+1}}{4}\Big]$$

$$= \frac{\alpha}{k}\Big[\frac{(k+1)(k+2)(k+3)(k+4)}{4(k+1)(k+2)(k+3)(k+4)}$$

$$+ \frac{4(k+2)(k+3)(k+4)\{-(1-c)^{k+4} + 3(1-c)^{k+3} - 3(1-c)^{k+2} + (1-c)^{k+1}\}}{4(k+1)(k+2)(k+3)(k+4)}$$

$$+ \frac{4(k+3)(k+4)\{3(1-c)^{k+4} - 6(1-c)^{k+3} + 3(1-c)^{k+2}\}}{4(k+1)(k+2)(k+3)(k+4)}$$

$$+ \frac{-24(k+4)(1-c)^{k+4} + 24(k+4)(1-c)^{k+3}}{4(k+1)(k+2)(k+3)(k+4)} + \frac{24(1-c)^{k+4} - 24\}}{4(k+1)(k+2)(k+3)(k+4)}$$

$$+ \frac{(k+1)(k+2)(k+3)(k+4)\{(1-c)^{k+4} - 4(1-c)^{k+3} + 6(1-c)^{k+2} - 4(1-c)^{k+1}\}}{4(k+1)(k+2)(k+3)(k+4)}\Big]$$

$$
\begin{aligned}
&= \frac{\alpha}{k}\Big[\frac{(k+1)(k+2)(k+3)(k+4)-24}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{(1-c)^{k+1}\big[4(k+2)(k+3)(k+4)-4(k+1)(k+2)(k+3)(k+4)\big]}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{(1-c)^{k+2}\big[-12(k+2)(k+3)(k+4)+12(k+3)(k+4)}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{6(k+1)(k+2)(k+3)(k+4)\big]}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{(1-c)^{k+3}\big[12(k+2)(k+3)(k+4)-24(k+3)(k+4)+24(k+4)}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{-4(k+1)(k+2)(k+3)(k+4)\big]}{4(k+1)(k+2)(k+3)(k+4)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(1-c)^{k+4}\big[-4(k+2)(k+3)(k+4)+12(k+3)(k+4)-24(k+4)+24}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{(k+1)(k+2)(k+3)(k+4)\big]}{4(k+1)(k+2)(k+3)(k+4)}\Big] \\
&= \frac{\alpha}{k}\Big[\frac{(k+1)(k+2)(k+3)(k+4)-24}{4(k+1)(k+2)(k+3)(k+4)} + \frac{-4k(k+2)(k+3)(k+4)(1-c)^{k+1}}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{6k(k+1)(k+3)(k+4)(1-c)^{k+2}}{4(k+1)(k+2)(k+3)(k+4)} + \frac{-4k(k+1)(k+2)(k+4)(1-c)^{k+3}}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{k(k+1)(k+2)(k+3)(1-c)^{k+4}}{4(k+1)(k+2)(k+3)(k+4)}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha}{k}\Big[\frac{k^4+10k^3+35k^2+50k}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{4k(k+2)(k+3)(k+4)\big[1-(1-c)^{k+1}\big]-4k(k+2)(k+3)(k+4)}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{-6k(k+1)(k+3)(k+4)\big[1-(1-c)^{k+2}\big]+6k(k+1)(k+3)(k+4)}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{4k(k+1)(k+2)(k+4)\big[1-(1-c)^{k+3}\big]-4k(k+1)(k+2)(k+4)}{4(k+1)(k+2)(k+3)(k+4)} \\
&+ \frac{-k(k+1)(k+2)(k+3)\big[1-(1-c)^{k+4}\big]+k(k+1)(k+2)(k+3)}{4(k+1)(k+2)(k+3)(k+4)}\Big] \\
&= \alpha\Big[\frac{k^4+10k^3+35k^2+50k-4k(k+2)(k+3)(k+4)+6k(k+1)(k+3)(k+4)}{4k(k+1)(k+2)(k+3)(k+4)k(k+1)(k+2)(k+3)} \\
&+ \frac{-4k(k+1)(k+2)(k+4)+k(k+1)(k+2)(k+3)}{4k(k+1)(k+2)(k+3)(k+4)k(k+1)(k+2)(k+3)} \\
&+ \frac{1-(1-c)^{k+1}}{(k+1)} - \frac{3\big[1-(1-c)^{k+2}\big]}{2(k+2)} + \frac{1-(1-c)^{k+3}}{(k+3)} - \frac{1-(1-c)^{k+4}}{4(k+4)}\Big] \\
&= \alpha\Big[\frac{1-(1-c)^{k+1}}{(k+1)} - \frac{3\big[1-(1-c)^{k+2}\big]}{2(k+2)} + \frac{1-(1-c)^{k+3}}{(k+3)} - \frac{1-(1-c)^{k+4}}{4(k+4)}\Big].
\end{aligned}
$$

Thus, the 4^{th} L-moment for Pareto distribution is given by:

$$
\begin{aligned}
L_4^B &= 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0 \\
&= \alpha\Big[20\frac{1-(1-c)^{k+1}}{(k+1)} - 30\frac{1-(1-c)^{k+2}}{(k+2)} + 20\frac{1-(1-c)^{k+3}}{(k+3)} - 5\frac{1-(1-c)^{k+4}}{4(k+4)} \\
&- 30\frac{1-(1-c)^{k+1}}{(k+1)} + 30\frac{1-(1-c)^{k+2}}{(k+2)} - 10\frac{1-(1-c)^{k+3}}{(k+3)} - 6\frac{1-(1-c)^{k+2}}{(k+2)} \\
&+ 12\frac{1-(1-c)^{k+1}}{(k+1)} - \frac{1-(1-c)^{k+1}}{(k+1)}\Big]
\end{aligned}
$$

$$= \alpha\Big[\frac{1-(1-c)^{k+1}}{(k+1)} - 6\frac{1-(1-c)^{k+2}}{(k+2)} + 10\frac{1-(1-c)^{k+3}}{(k+3)} - 5\frac{1-(1-c)^{k+4}}{4(k+4)}\Big]$$

$$= \alpha\Big[m_1 - 6m_2 + 10m_3 - 5m_4\Big].$$

In Table 4.2 the numerical values of the first four L-moments of Pareto distribution for different censoring values are compared with the values obtained by the corresponding GLD approximation [23].

Table 4.2: :[23] L-moment of Pareto distribution for censoring fraction c

Distribution	c	method	L_1^B	L_2^B	L_3^B	L_4^B
	0.99	direct	0.2437	0.1326	0.0533	0.0283
		gld	0.2437	0.1326	0.0533	0.0283
	0.9	direct	0.2104	0.1010	0.0250	0.0043
		gld	0.2104	0.1010	0.0250	0.0043
	0.8	direct	0.1810	0.0760	0.0074	-0.0050
		gld	0.1810	0.0760	0.0074	-0.0050
Pareto	0.7	direct	0.1546	0.0562	-0.0026	-0.0064
$\alpha = 1/5$,		gld	0.1546	0.0562	-0.0026	-0.0064
$k = -1/5$						
	0.6	direct	0.1299	0.0401	-0.0075	-0.0043
		gld	0.1299	0.0401	-0.0075	-0.0043
	0.5	direct	0.1064	0.02772	-0.0089	-0.0013
		gld	0.1064	0.0272	-0.0089	-0.0013

Table (4.2) strongly recommends the use of GLD for modeling univariate continuous distributions using their PWMs and L-moments even for censored observations [23].

4.5 Fitting of the Distributions to Censored Data Using GLD

In this section, we estimate type B PPWMs for right censoring distribution and type B' PPWMs for left censoring distribution. Using this estimations, we can estimate the parameters of the right and the left censoring GLD.

PPWM Estimators for Right Censoring

Type B PPWM is computed from the completed sample, where the $n-m$ censored values in

$$\underbrace{X_{1:n} \leq X_{2:n} \leq ... \leq X_{m:n}}_{m-1 \text{ observed}} \leq \underbrace{T \leq X_{m+1:n} \leq X_{m+2:n} \leq ... \leq X_{n-1:n} \leq X_{n:n}}_{n-m \text{ censored}}$$

are replaced by the censoring threshold T[28].

Definition 4.5.1. [28] Let $\underbrace{x_{1:n} \leq x_{2:n} \leq ... \leq x_{m:n}}_{m-1 \text{ observed}} \leq \underbrace{T \leq x_{m+1:n} \leq x_{m+2:n} \leq ... \leq x_{n-1:n} \leq x_{n:n}}_{n-m \text{ censored}}$ be an order sample, and define the sample type B PPWM for right censoring distribution, b_r^B as:

$$b_r^B = \frac{1}{n}\Big\{ \sum_{j=1}^{m} \frac{(j-1)(j-2)...(j-r)}{(n-1)(n-2)...(n-r)}\, x_{j:n} + \Big(\sum_{j=m+1}^{n} \frac{(j-1)(j-2)...(j-r)}{(n-1)(n-2)...(n-r)} \Big) T \Big\}. \quad (4.5.1)$$

Note 4.5.1. *[28] The samples type B PPWMs, b_r^B are unbiased estimators of the type B PPWM β_r^B for $r = 1, 2, 3, \ldots$*

To estimate the parameters of the right censored GLD in the case of type one single censoring, we can equate the sample and population PPWMs. As for estimation usually B type PPWMs are preferred by comparing the first four theoretical and sample moments obtained from expressions (4.3.2) and (4.5.1), we can obtain the appropriate values of the parameters $\lambda_1, \lambda_2, \lambda_3$ and λ_4 [20].

PPWM Estimators for Left Censoring

Type B' PPWM is computed from the completed sample, where the $n-k$ censored values in

$$\underbrace{X_{1:n} \leq X_{2:n} \leq ... \leq X_{m-1:n}}_{n-k \text{ censored}} \leq \underbrace{T \leq X_{m:n} \leq X_{m+2:n} \leq ... \leq X_{n:n}}_{k \text{ observed}}, \quad {\scriptstyle (m-1=n-k)}$$

are replaced by the censoring threshold T[28].

Definition 4.5.2. [28] Let $\underbrace{x_{1:n} \leq x_{2:n} \leq ... \leq x_{m-1:n}}_{n-k \text{ censored}} \leq \underbrace{T \leq x_{m:n} \leq x_{m+2:n} \leq ... \leq x_{n:n}}_{k \text{ observed}}$, $(m-1 = n-k)$ ba an order sample and define the sample type B' PPWM for lift censoring distribution, $b_r^{B'}$ as:

$$b_r^{B'} = \frac{1}{n}\Big\{ \sum_{j=1}^{n-k} \frac{(j-1)(j-2)...(j-r)}{(n-1)(n-2)...(n-r)} T + \sum_{j=n-k+1}^{n} \frac{(j-1)(j-2)...(j-r)}{(n-1)(n-2)...(n-r)} \, x_{j:n} \Big\}. \quad (4.5.2)$$

Note 4.5.2. *[23] The sampls type $b_r^{B'}$ are unbiased estimators of the type B', $\beta_r^{B'}$ for $r = 1, 2, 3, \dots$*

In eq.n.(4.5.2), $k = n - m + 1$. In the case of type B' censoring T is to be replaced by $X_{m:n}$ in the above expressions. So, by comparing the First 4 theoretical and sample PPWMs using expressions (4.3.13) and (4.5.2), we can fit a GLD for a left censored data [23].

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