Extension of compact operators

over non-archimedean fields

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DEDICATION

To ...

My Parents

My Brother

My Sisters

and

My Family

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Abstract

This thesis is a survey of compact linear operators on locally convex spaces over non-Archimedean valued fields.

This thesis is devoted to study the non-Archimedean locally convex spaces X having the following property: For all non-Archimedean locally convex spaces Y and Z with $Y \subset Z$, every compact operator $T : Y \to X$ has an extension to a compact operator $\overline{T} : Z \to X$. The results obtained depend strongly on the spherical completeness of the ground field. On the other hand, the situation here is completely different from its Archimedean counterpart. Our results also lead to some new characterizations of spherically complete fields and of discretely valued fields.

List of symbols

Symbole	Meaning	page
$C(Y,X)$	The set of all compact linear operators from Y into X	1
$\mathbf{K}$	Non-archimedean field	1
CEP	Compact extension property	1
$\mid\ \mid$	Valuation on $\mathbf{K}$	3
$\mathcal{O}_K$	Valuation ring	5
$B_\epsilon(a)$	The ball on a with radius ϵ	5
B_E	The closed unit ball of E	8
$\dim E$	The algebraic dimension of E	9
coF	The absolutely convex hull of F	9
p_A	The $Minkowski\ functional$ of A	11
$\mathcal{L}(X,Y)$	The set of all linear operators from X into Y	12
X^*	$\mathcal{L}(X,\mathbf{K})$	12
$L(X,Y)$	The set of all continuous linear operators from X into Y.	12
$X^{'}$	$L(X,\mathbf{K})$	12
$\prod_{\alpha\in I} E_\alpha$	The product of the family E_α	13
$l^\infty(X)$	The set of all bounded maps from X to $\mathbf{K}$	17
$\mid\mid\ \mid\mid_\infty$	Sup-norm	17
$c_0(X)$	The set of all continuous maps from X to $\mathbf{K}$	17
l^∞	$l^\infty(\mathbf{N})$	17
c_0	$c_0(\mathbf{N})$	17
E/N	The quotient of E modulo N	18
$E_p = E/ker\ p$	The quotient of E modulo $ker\ p$	19
$X_B =< B >$	The vector subspace of X spaned by B	21
$\langle\cdot,\cdot\rangle$	Bilinear form	23
$\langle X,Y\rangle$	Dual pair	24
Λ^*	The $K\ddot{o}the-dual$ of Λ	25
$n(\Lambda,\Lambda^*)$	The normal topology on Λ	26

Contents

Introduction

The theory of compact operator over non-archimedean fields is a diverse area of mathematics and it has many applications in mathematics, physics and engineering.

In this thesis we focus our study on the compact extension property, where we say that X has the compact extension property (CEP for short) if, for every pair of locally convex spaces Y , Z with Y ⊂ Z and every $T \in C(Y, X)$ has an extension $\overline{T} \in C(Z, X)$.

In (1969) J. Lindenstrauss and H.P. Rosenthal studied the CEP for (real and complex) Banach spaces. They prove that a Banach space X has CEP if and only if it is an $\mathcal{L}_\infty$-space, so the question is that: can we prove the CEP with respect to non-archimedean fields.

It turned out that not only was the situation completely different here, but also that the answer was surprisingly simpler.We found, indeed, that when the ground field K is spherically complete, then every non-Archimedean Banach space X over K has the CEP, and if K is not spherically complete, no $X \neq \{0\}$ has this property.

We therefore decided to look at the problem in the much more general frame of non-Archimedean locally convex spaces and called the property CEP (Definition 2.1.1.). Here the situation is more complicated. If K is not spherically complete, still there are no nontrivial locally convex spaces over K with the CEP (Section 2.2.).On the other hand, if K is spherically complete, lots of spaces have the CEP but not all of them (Section 2.4.), and the situation is still different in the special

case when the valuation on K is discrete (Section 2.4.).
The results also lead to some new characterizations of spherically complete fields (Theorem 2.3.37.) and of discretely valued fields (Theorem 2.4.12.). Finally it turns out that the CEP is related to a locally convex version of the notion of weakly injective normed space introduced in [22, Theorem 4.9].

This thesis consist two chapters.

chapter 1, we preset brief summary for valuations, topology and normed spaces. In section 1.3. we present some definitions of compactoid and c-compact sets and we present an important theorem (Theorem 1.4.1.), that we will use in proofs of many theorems. In section 1.5.1. we present the definition of compact operators and some important properties of compactoid and c-compact sets, that we will use in proofs of many theorems. In section 1.6.1. we present the definition of orthogonal and schauder bases. In section 1.7.1. we give some definitions and theorems related to dual pair and sequence spaces.
chapter 2, we have four sections. In section 2.1. we defined the compact extension property and we discussed some basic facts about the compact property. In section 2.2. we studied the CEP over nonspherically complete valued fields. In section 2.3. we studied the CEP over spherically complete valued fields. Finally, we complete this section by giving some applications of the results that introduced in this section to characterize the spherical completeness of the field **K**. In section 2.4. we studied the CEP over discretely valued fields and we complete this section by characterizing discretness of the field **K**.

Chapter 1

Preliminaries

In this chapter we shall give the nescessary facts of valuations, topology, normed spaces, topological vector spaces, linear mappings, orthogonal bases, schauder bases, dual pair and sequence spaces. The purpose of this preliminary chapter is not to establish these results but to clarify terminology, notations, and to give the reader a survey of material that will be used in the later chapters.
For notations, concepts and results not explicitly mentioned we refer the reader to [21], [22] and [25] for more details.

1.1 Valuations

Definition 1.1.1. Let $\mathbf{K}$ be a field. A valuation on $\mathbf{K}$ is a map $|\ |: \mathbf{K} \rightarrow \mathbb{R}$ such that for all $x, y \in \mathbf{K}$, we have :

(i) $|x| \geq 0$,

(ii) $|x| = 0$ if and only if $x = 0$,

(iii) $|x + y| \leq |x| + |y|$,

(iv) $|xy| = |x|\,|y|$.

Remark 1.1.1. Every valuation $|\ |$ define a metric ρ by

$$\rho(x, y) := |x - y| \quad (x, y \in \mathbf{K}),$$

with respect to this metric addition, multiplication and inversion are continuous.

Proof. See [22], p. 1.

□

We call $\mathbf{K}$ complete if it is complete relative to this metric, this mean that every Cauchy sequence in $\mathbf{K}$ converges in $\mathbf{K}$.

Definition 1.1.2. A valuation $|\ |$ is called archimedean if the set $\{|n| : n \in \mathbb{N}\}$ is unbounded , otherwise it is non-archimedean.

Theorem 1.1.1. *Let $|\ |$ be a valuation on $\mathbf{K}$. The following are equivalent.*

(i) The valuation is non-archimedean.

(ii) $|n| \leq 1$ for every $n \in \mathbb{N}$.

(iii) For all $a, b \in \mathbf{K}$, $|a + b| \leq max\{|a|\ ,\ |b|\}$.

(iv) If a, b in $\mathbf{K}$ and $|a| < |b|$, then $|b - a| = |b|$.

Proof. See [22], p. 2.

□

Definition 1.1.3. we define the set $\mathcal{O}_K = \{x \in \mathbf{K} : |x| \leq 1\}$ which is called the " valuation ring ".

1.2 Topology

The metric ρ determined by a non-archimedean valuation on a field $\mathbf{K}$ satisfies the strong triangle inequality $\rho(x,y) \leq max\{\rho(x,z), \rho(z,y)\}$ for all $x, y, z \in \mathbf{K}$. A metric ρ that satisfies this triangle inequality is called an ultrametric.

Remark 1.2.1. (The isosceles triangle principle) A metric ρ on a set X is an ultrametric if and only if for all $x, y, z \in X$, $\rho(x,y) > \rho(y,z)$, then $\rho(x,z) = \rho(x,y)$

Proof. See [20], p. 47.

□

Definition 1.2.1. Let ρ be an ultrametric on X, $a \in X$, $\epsilon > 0$. The ball on a with radius ϵ is the set

$$B_\epsilon(a) := \{x \in X : \rho(x,a) \leq \epsilon\}$$

Remark 1.2.2. Let ρ be an ultrametric on X, then :

(i) If $a, b \in X$, $\epsilon > 0$, and $b \in B_\epsilon(a)$, then $B_\epsilon(a) = B_\epsilon(b)$

(ii) If $a, b \in X$, $\epsilon \geq \delta > 0$, then either $B_\epsilon(a) \cap B_\delta(b) = \phi$ or $B_\epsilon(a) \supset B_\delta(b)$

(iii) Every ball is both closed and open in the topology defined by ρ.

Proof. See [20], p. 48.

□

Proposition 1.2.1. *Let $a_1, a_2, ...$ be a sequence in $\mathbf{K}$.*

(i) If $\lim_{n\to\infty} a_n = a \in \mathbf{K}$ and $a \neq 0$ then $|a_n| = |a|$ for large n.

(ii) $\sum a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$.

Proof. See [20], p. 61.

□

Definition 1.2.2. An ultrametric space $\langle X \ , \ \rho \rangle$ is said to be spherically complete if every shrinking sequence of balls has a nonempty intersection.

Remark 1.2.3. A spherically complete ultrametric space is complete.

Proof. See [22], p. 24.

□

An example of a complete ultrametric space that is not spherically complete is furnished by the set $\mathbb{N}$ under the metric ρ defined by

$$\rho(m, n) = \begin{cases} 0, & \text{if } m = n \\ max\{1 + \frac{1}{n} \ , \ 1 + \frac{1}{m}\} & \text{if } m \neq n \end{cases}$$

Proof. See [22], p. 24.

□

Lemma 1.2.2. *Each of the following conditions on an ultrametric ρ in a set X is equivalent to spherical completeness*

(α) If $\mathfrak{B}$ is a collection of balls such that no two elements of them are disjoint, then $\bigcap \mathfrak{B} \neq \phi$.

(β) Every sequence of balls $B_{\epsilon_1}(a_1) \supset B_{\epsilon_2}((a_2) \supset .....$ for which $\epsilon_1 > \epsilon_2 > .....$ has nonempty intersection .

Proof. See [22], p. 24. □

1.3 Normed Spaces

Definition 1.3.1. A seminorm on a vector space E is a finite real-valued function p defined on E such that for all $x, y \in E$ and $\alpha \in \mathbf{K}$ the following are satisfied:

(i) $p(x) \geq 0$,

(ii) $p(x + y) \leq p(x) + p(y)$,

(iii) $p(\alpha x) = |\alpha| \, p(x)$.

If in addition (ii) is replaced by

(ii') $p(x + y) \leq \max \{p(x), p(y)\}$, then p is called non-archimedean seminorm.

A non-archimedean seminorm is called non-archimedean norm if it satisfies:

(iv) $p(x) = 0$ if and only if $x = 0$

One usually denotes a norm by $|| \; ||$.

Definition 1.3.2. A vector space E is said to be a non-archimedean normed space if for every $x \in E$ there exist a real number $||x||$, called the non-archimedean norm of x, in such a way that satisfies (i), (ii'), (iii) and (iv) in definition (1.3.1). A Banach space E is a complete normed space, this mean that every cauchy sequence in E converges in E with respect the induced metric $(x, y) \mapsto ||x - y||$.

Through this thesis when we say normed space we mean non-archimedean normed space.

By B_E we denotes the closed unit ball of E, i.e.,

$$B_E = \{x \in E : ||x|| \leq 1\}.$$

Lemma 1.3.1. *If* $a_1, ..., a_n$ *are elements of a normed vector space* E *and if* $||a_i|| \neq ||a_j||$ *for some* $i \neq j$, *then*

$$||a_1 + ... + a_n|| = max \; \{||a_i|| \; : \; i = 1, \; 2, ..., \; n\}$$

Proof. See [22], p. 50.

□

1.4 Topological Vector Spaces

Throughout vector spaces are supposed to be over $\mathbf{K}$, where $\mathbf{K}$ is non-archimedean field. The algebraic dimension of a $\mathbf{K}$-vector space E is denoted by $\dim E$.

Definition 1.4.1. A topological vector space (abbreviated T.V.S.) is a vector space E over a scalar field $\mathbf{K}$, with a topology τ defined on E such that the algebraic operations $x+y, \lambda \cdot x$; (where $x, y \in E$ and $\lambda \in \mathbf{K}$) are continuous with respect to the corresponding product topologies. Note that, every normed space is a topological vector space (see [15], p. 4).

Definition 1.4.2. A collection $\mathcal{U}$ of open subsets in E is a local base (for the topology on E) at x in E if for every neighborhood U of x, there exist a set V in $\mathcal{U}$ such that $x \in V \subseteq U$.

Definition 1.4.3. Let A be a subset of a vector space E, then:

(i) A is called convex, if $\alpha x + \beta y + \gamma z \in A$ as soon as $x, y, z \in A$ and $\alpha, \beta, \gamma \in \mathcal{O}_K \subset \mathbf{K}, \quad \alpha + \beta + \gamma = 1$.

(ii) A is called absolutely convex if and only if it is convex and contains zero.

(iii) A is called absorbent, if for each $x \in E$ there is some $\lambda \in \mathbb{R}^+$ such that $x \in \mu A$ for all μ with $|\mu| \geq \lambda$.

Definition 1.4.4. Let F be a subset of E . Let coF be the intersection of all absolutely convex subsets of E that contain F. coF is called the absolutely convex hull of F . If $F = \{a_1, ..., a_n\}$ is a finite set, then

$$co\,\{a_1, ..., a_n\} = \{\lambda_1 a_1 + ... + \lambda_n a_n : \lambda_i \in \mathcal{O}_K\}.$$

Definition 1.4.5. If E is a T.V.S. and A is a subset of E, then:

(i) A is called bounded, if for every nhood U of zero in E there exist a nonzero $\lambda \in \mathbf{K}$ such that $\lambda A \subset U$.

(ii) A is called compactoid in E if for any zero neighborhood $\mathcal{U}$ in E, there is a finite set $F \subset E$ such that $A \subset \mathcal{U} + coF$.

(iii) An non-empty absolutely convex subset A of E is called c-compact if for every collection $\mathcal{C}$ of closed convex subsets of A with the finite intersection property , we have $\bigcap \mathcal{C} \neq \phi$.

Theorem 1.4.1. *Let* $\mathbf{K}$ *be spherically complete , let* $A \subset E$ *be absolutely convex. The following are equivalent.*

(i) A is bounded and c-compact.

(ii) A is compactoid and complete .

Proof. See [21], p. 79.

□

Definition 1.4.6. :

(i) A T.V.S. E is called a locally convex space (abbreviated L.C.S.) if, it is a Hausdorff space and it has a local base $\mathcal{U}$ of zero whose members are convex.

(ii) A L.C.S. is called metrizable if its topology can be defined by a metric.

(iii) A L.C.S. is called quasi-complete if every bounded closed subset of X is complete.

(iv) A *Fréchet* space is a completely metrizable L.C.S. Note that, every Banach space is a *Fréchet* space.

(v) A L.C.S. is called sequentially complete if every cauchy sequence in X converges.

Definition 1.4.7. With every absorbent subset A of a T.V.S. E we associate's *Minkowski functionals* (*gauge*) p_A, which defined by

$$p_A(x) \;=\; \inf\{|\lambda| \;:\; x \;\in\; \lambda A\}$$

The *Minkowski functional* p_A of absolutely convex absorbent subset $A \subset E$ is a seminorm on E, and we have

$$\{x : p_A(x) < 1\} \subseteq A \subseteq \{x : p_A(x) \leq 1\}.$$

By $\mathcal{P}_E$, we denote the set of all continuous seminorms on E.

See [21], p. 11.

Proposition 1.4.2. *Let E be a L.C.S. If p_A is the gauge of the absolutely convex, absorbent set A, then p_A is continuous if and only if A is a nhood of zero.*

Proof. See [14], p. 13.

□

1.5 Linear Mappings

Definition 1.5.1. :

(i) Let X and Y be two vector spaces over the same field $\mathbf{K}$. The mapping T of X into Y is called linear if,

$$T(x+y) = T(x) + T(y), \quad T(\alpha x) = \alpha T(x),$$

for all $x, y \in X$ and all $\alpha \in \mathbf{K}$. A linear mapping is sometimes called a linear operator. By $\mathcal{L}(X,Y)$ we denotes the set of all linear operators from X into Y, if addition and multiplication by scalars can be defined by

$$(T+S)(x) = T(x) + S(x), \quad (\alpha T)(x) = \alpha T(x);$$

so $\mathcal{L}$ becomes a vector space over $\mathbf{K}$. By X^* we denote $\mathcal{L}(X, \mathbf{K})$. When X and Y are both T.V.S.'s, then

(ii) A linear operator $T : X \to Y$ is called continuous, if for each nhood V of zero in Y, there is a nhood U of zero in X such that $T(U) \subseteq V$. We shall denote by $L(X,Y)$ the collection of all continuous linear operators from X into Y. By X' we denote $L(X, \mathbf{K})$.

(iii) A subset T of $L(X,Y)$ is said to be equicontinuous in X if for every neighborhood V in Y, $\bigcap\limits_{f \in T} f^{-1}(V)$ is a neighborhood in X.

(iv) A linear operator $T : X \to Y$ is called open, if $T(A)$ in Y is open for every open set A in X.

(v) A linear operator $T : X \to Y$ is called isomorphism, if T is one-to-one, onto, continuous and has continuous inverse. Tow T.V.S.'s X and Y are said to be isomorphic (we shall write $X \simeq Y$), if there exist an isomorphism T from X onto Y.

(vi) A linear operator $T : X \to Y$ is called compact if there is a zero neighborhood V in X such that $T(V)$ is compactoid in Y. We shall denote by $C(X, Y)$ the collection of all compact linear operators from X onto Y.

(vii) A linear operator $T : X \to Y$ is called finite-dimensional, if $\dim T(X) < \infty$.

Remark 1.5.1. Let E be a finite dimensional normed space, then

$$\dim E = \dim E' = \dim E^*.$$

Proof. (i) see [25], p. 48. □

Definition 1.5.2. Products.

Let $\{E_\alpha : \alpha \in I\}$ be a collection of L.C.S.'s, where I is a linearly ordered set of indices. The product of the family E_α will be written $\prod_{\alpha \in I} E_\alpha$, where

$$\prod_{\alpha \in I} E_\alpha = \left\{ x = x_\alpha : I \to \bigcup_{\alpha \in I} E_\alpha,\ x_\alpha \in E_\alpha \ \forall \alpha \in I \right\}.$$

The projection $\pi_\gamma : \prod_{\alpha \in I} E_\alpha \to E_\gamma$, defined by, $\pi_\gamma(x) = x_\gamma$, is a linear mapping . Since E_α are L.C.S.'s, the space $E = \prod_{\alpha \in I} E_\alpha$ can be made into a convex space as follows:

For all $\gamma \in I$, let $\mathcal{B}_\gamma$ be a base of absolutely convex nhoods in E_γ. Then finite intersection of the sets $\pi_\gamma^{-1}(U_\gamma)$ $(U_\gamma \in \mathcal{B}_\gamma, \gamma \in I)$ form a base of absolutely convex

nhoods in $E = \prod_{\alpha \in I} E_\alpha$ or $U = \prod_{\alpha \in I} U_\alpha$, where $U_\alpha = E_\alpha$ for all but finitely many nhood U_α of zero, form abase of nhoods of zero in E.
If I is a finite set and $E = \prod_{j=1}^{n} E_j$, then the sets $U = \prod_{j=1}^{n} U_j$ form a base of nhoods of zero in E, where U_j is a nhood of zero in E_j.

Theorem 1.5.1. *The γ^{th} projection map $\pi_\gamma : \prod_{\alpha \in I} X_\alpha \to X_\gamma$ is continuous and open.*

Proof. See [24], p. 54.

□

Proposition 1.5.2. *Let E be a locally convex space. Then:*

(i) *Closure of bounded set is bounded.*

(ii) *The image by a continuous linear mapping of bounded set is bounded.*

(iii) *If the topology of E is determined by the set Q of seminorms, and $A \subseteq E$, then A is bounded if and only if $p(A)$ is a bounded set of real numbers for each $p \in Q$.*

Proof. See [14], p. 44, p. 45

□

Proposition 1.5.3. *Let E be a normed space . Then:*

(i) *Closure and closed absolutely convex hull of compactoids are compactoids .*

(ii) *Every subset of a compactoid is compactoid .*

(iii) *If X and Y are compactoids, then so are $X \bigcup Y$ and $X + Y$.*

(iv) *IF $X \subset E$ is compactoid and $T \in L(E, F)$, then $T(X)$ is compactoid in F .*

(v) *Every compactoid is bounded .*

(vi) *IF E is finite-dimensional , then the compactoids of E are just the bounded sets .*

Proof. See [22], p. 134.

□

An example of a bounded non-compactoid set is the closed unit ball of any infinite-dimensioanl normed space.
See [23], p. 52.

Proposition 1.5.4. *Let E be a non-archimedean locally convex space, F a subspace of E and B a subset of F.*
Then B is compactoid in F if and only if B is compactoid in E.

Proof. See [4], p. 300.

□

Proposition 1.5.5. *Let $(E_i)_{i \in I}$ be a family of n.a. locally convex spaces and let A_i be a subset of E_i $(i \in I)$. Then $A = \prod_{i \in I} A_i$ is compactoid in $E = \prod_{i \in I} E_i$ if and only if A_i is compactoid in E_i for all $i \in I$.*

Proof. See [4], p. 300.

□

Proposition 1.5.6. *:*

(i) $\mathbf{K}$ *is c-compact .*

(ii) A c-compact set is complete .

(iii) A nonempty closed convex subset of a c-compact set is c-compact.

(iv) Let $\{E_i\}_{i\in I}$ *be a family of Hausdorff locally convex spaces over* $\mathbf{K}$*. Suppose, for each* i *,* C_i *is c-compact in* E_i*. Then* $\prod_{i\in I} C_i$ *is c-compact in* $\prod_{i\in I} E_i$*.*

(v) The image of c-compact set under a continuous linear map is c-compact.

Proof. See [19], p. 2.

□

Remark 1.5.2. Let X and Y be T.V.S.'s.

(i) Every finite dimensional operator $T : X \to Y$ is compact.

(ii) If $T : X \to Y$ is compact and $S : Y \to Z$ is a continuous operator, then $S \circ T$ is a compact operator.

(iii) If $S : X \to Y$ is continuous and $T : Y \to Z$ is a compact operator, then $T \circ S$ is compact operator.

Proof. (i) see [25], p. 246. □

Proof. (ii) Since T is a compact linear operator from X into Y, then there exists a zero neighborhood U in X such that $T(U)$ is compactoid in Y, since S is a continuous linear operator from Y into Z, then (by proposition 1.5.3[iv]),
$S \circ T(U) = S(T(U))$ is compactoid in Z, therefore $S \circ T$ is compact. □

Proof. (iii) Since T is a compact linear operator from Y into Z, then there exists a zero neighborhood U in Y such that $T(U)$ is compactoid in Z, since S is a continuous linear operator from X into Y, then there exists a zero neighborhood V in X such that $S(V) \subset U$, then $T \circ S(V) = T(S(V)) \subset T(U)$, since $T(U)$ is compactoid and $T \circ S(V) \subset T(U)$, then (by proposition 1.5.3[ii]), $T \circ S(U)$ is compactoid in Z, therefore $T \circ S$ is compact. □

Remark 1.5.3. :

(i) Let X be any set. The bounded maps from X to $\mathbf{K}$ form a linear space $l^\infty(X)$, which is a Banach space under the sup-norm $|| \ ||_\infty$ defined by

$$||f||_\infty := \sup\{|f(x)| : x \in X\} \qquad [f \in l^\infty(X)].$$

(ii) $c_0(X) := \{f \in l^\infty(X)$: for every $\epsilon > 0$, there exist only finitely many elements x of X for which $|f(x)| \geq \epsilon\}$ is a closed linear subspace of $l^\infty(X)$.
If $X = \mathbf{N}$, we write l^∞ and c_0 for $l^\infty(\mathbf{N})$ and $c_0(\mathbf{N})$, respectively.

See [22], p. 59.

Definition 1.5.3. Quotient Spaces.

Let N be a subspace of a vector space E. For every $x \in E$, let $\pi(x)$ be the coset of N that contains x; that is,

$$\pi(x) = x + N.$$

These cosets are the elements of a vector space E/N, called the quotient of E modulo N, in which addition and scaler multiplication are defined by

$$\pi(x) + \pi(y) = \pi(x + y), \quad \pi(\alpha x) = \alpha\pi(x) \qquad (*)$$

The zero of E/N is $\pi(0) = 0 + N = N$. By (*), π is a linear mapping of E onto E/N with N as it's kernal (Null space); π is often called the *quotient map* of E onto E/N.

Suppose now that τ is a vector topology on E, i.e., τ makes E into a T.V.S. and that N is a closed subspace of E. Let τ_N be the collection of all sets $G \subseteq E/N$ for which $\pi^{-1}(G) \in \tau$. Then τ_N turns out to be a vector topology on E/N, called the *quotient topology*.

Theorem 1.5.7. *Let N be a closed subspace of a T.V.S. E. Let τ be the topology of E and define τ_N to be quotient topology on E/N as above. Then τ_N is a vector topology on E/N and the quotient map $\pi : E \to E/N$ is linear, continuous and open.*

Proof. See [15], p. 31.

□

Definition 1.5.4. Seminorms and quotients spaces.

Suppose p is a seminorm on a vector space E and

$$ker\ p = \{x \in E : p(x) = 0\},$$

then $ker\ p$ is a subspace of E. Let π be the quotient map of E onto $E_p = E/ker\ p$ and define $\tilde{p}$ on E_p by

$$\tilde{p}(\pi(x)) = p(x).$$

If $\pi(x) = \pi(y)$, then $p(x - y) = 0$, and since

$$|p(x) - p(y)| \leq p(x - y) = 0, \text{ then } p(x) = p(y)$$

it follows that $\tilde{p}(\pi(x)) = \tilde{p}(\pi(y))$. Thus $\tilde{p}$ is well defined on E_p.

Now it is easy to verify that $\tilde{p}$ is a norm on E_p. For, if $x \in ker\ p$, then $p(x) = 0$, so $\tilde{p}(\pi(x)) = p(x) = 0$, thus $\tilde{p}(ker\ p) = 0$; on the other hand, if $\tilde{p}(\pi(x)) = 0$, then $p(x) = 0$, hence, $x \in ker\ p$, so $\pi(x) = ker\ p$. Therefore, $\tilde{p}$ is a norm on E_p. Note that E_p is a metric space defined by

$$d(\pi_p(x), \pi_p(y)) \ = \ \|\pi_p(x) - \pi_p(y)\| \ = \ \|\pi_p(x - y)\| \ = \ p(x - y)$$

Definition 1.5.5. [**Partial-order relation**] Let X be a set . then a partial-order relation on X , denoted by $\prec$ is a relation which is :

(1) Reflexive (i.e $x \prec x$ for each $x \in X$).

(2) Transitive (i.e. $x \prec y$ and $y \prec z$ imply $x \prec z$).

(3) Antisymmetric (i.e. $x \prec y$ and $y \prec x$ imply $x = y$).

A totally ordered set or chain is a partially ordered set such that every two elements x , y of the set , they satisfy $x \prec y$ or $y \prec x$.
An upper bound of a subset W of a partially ordered set M is an element $u \in M$ such that

$$x \prec u \qquad \text{for every } x \in W$$

A maximal element of M is an element $m \in M$ such that

$$m \prec x \qquad \text{for some } x \in M \qquad \text{imply } m = x$$

Lemma 1.5.8. *[* ***Zorn's lemma*** *] Let $M \neq \phi$ be a partially ordered set. Suppose that every chain $C \subset M$ has an upper bound. Then M has at least one maximal element.*

Definition 1.5.6. we say that X contains a copy of Y if X contains a linear subspace X_1 that is linearly homeomorphic to Y.

Definition 1.5.7. Let E be a Banach space. A projection on E is an element P in $L(E,E)$ for which $P^2 = P$.
Let X be a subspace of E, we say that X is complemented in E if there is a projection from E onto X.

Remark 1.5.4. Every complemented subspace of X must be closed.

Proof. see [25], p. 83. □

Definition 1.5.8. A seminorm p on a **K**-vector space E is a polar seminorm if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$.

Definition 1.5.9. Let E be a locally convex space over **K**. E is a strongly polar space if every continuous seminorm on E is polar. E is a polar space if its topology is defined by a family of polar seminorms.

Theorem 1.5.9. *If* **K** *is spherically complete, then each locally convex space is polar.*

Proof. See [17], p. 196.

□

Definition 1.5.10. Let X be a normed space, for any **K**-convex bounded subset B of X, we write $X_B =< B >$ (the vector subspace of X spaned by B ,) and denote by p_B the gauge of B defined on X_B.

Remark 1.5.5. Let X, X_B and p_B as in definition 1.5.10. . Then:

(i) Since B is bounded, then p_B is a norm on X_B.

(ii) The canonical embedding i_B from X_B into X is always continuous.

Proof. See [25], p. (95, 163).

□

Remark 1.5.6. Let E be a vector space;

(i) Let p be a seminorm on E. Then p is continuous on E if and only if it is continuous at the origin. Also, this is equivalent to the boundedness of p.

(ii) If p_1 and p_2 are seminorms with $p_1 \leq p_2$ and p_2 is continuous, then p_1 is continuous.

Proof. (i) see [13], p. 111. □

Proof. (ii) Suppose that p_1 and p_2 are seminorms with $p_1 \leq p_2$ and p_2 is continuous, then by (i) p_2 is bounded, so p_1 is bounded, then by (i) p_1 is continuous. □

Proposition 1.5.10. *Let τ_1, τ_2 be two locally convex topologies on a vector space E. Suppose that every τ_1-continuous seminorm on E is τ_2-continuous, then τ_1 weaker than τ_2.*

Proof. Let U be an absolutely convex zero neighborhood for τ_1. It suffices to see that U is a zeroo neighborhood for τ_2. For that, note that by (Proposition 1.4.2.) the Minkowsky functional p_U associated to U is a τ_1-continuous seminorm on E. Then by assumption p_U is τ_2-continuous. Since by (1.4.7.), U contains the closed unit ball with respect to p_U, we conclude that U is a zero neighborhood for τ_2. □

1.6 Orthogonal and Shauder bases

Definition 1.6.1. A sequuence $x_1, x_2, x_3, ...$ in a locally convex space E is called base if for each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \lambda_n x_n$ with $\lambda_n \in \mathbf{K}$. If the coefficient functionals $f_n : x \in E \mapsto \lambda_n \in \mathbf{K}$ $(n \in N)$ are continuous , then

$x_1, x_2, x_3, ...$ is called Schauder base.

Definition 1.6.2. If the topology of E is defined by a family $\mathcal{P}_E$ of non-archimedean seminorms satisfying the condition

if $x = \sum_{n=1}^{\infty} \lambda_n x_n$, then $p(x) = \max_n p(\lambda_n x_n)$ for all $p \in \mathcal{P}_E$, then $(x_n)_n$ is said to be orthogonal basis in E.

Theorem 1.6.1. *Let (E, τ) be a locally convex space with an orthogonal base $x_1, x_2, x_3, ...$ and coefficient functionals $f_1, f_2, f_3, ... \in E^{'}$. Suppose E is $\sigma(E, E^{'})$-sequentially complete, then $A \subset E$ is compactoid if and only if there exist $y \in E$ such that $A \subset \tilde{y}$, where*

$$\tilde{y} := \{x \in E : |f_n(x)| \leq |f_n(y)| \quad \textit{for all } n \in N\}.$$

Proof. See [5], p. 117.

□

1.7 Dual pair and Sequence spaces

Definition 1.7.1. Let X and Y be two vector spaces over the scaler field $\mathbf{K}$. The scaler-valued function $\langle \cdot, \cdot \rangle : X \times Y \to \mathbf{K}$ defined by $(x, y) \mapsto \langle x, y \rangle$, where $x \in X$ and $y \in Y$ is called bilinear form, (i.e., it is linear in both variables), if $\forall x, x^{'} \in X, \forall y, y^{'} \in Y$ and $\forall \alpha \in \mathbf{K}$ the following are satisfied :

$$\langle x + x^{'}, y \rangle = \langle x, y \rangle + \langle x^{'}, y \rangle, \quad \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$

$$\langle x, y+y'\rangle = \langle x,y\rangle + \langle y,y'\rangle, \quad \langle x,\alpha y\rangle = \alpha\langle x,y\rangle.$$

Definition 1.7.2. Let X and Y be vector spaces over **K**, and suppose that there exist a bilinear functionals ψ on $X \times Y$ satisfying the following separation conditions :-

(S1) If $\psi(x_0, y) = 0$ (for all $y \in Y$), then $x_0 = 0$.

(S2) If $\psi(x, y_0) = 0$ (for all $x \in X$), then $y_0 = 0$.

then (X, Y, ψ) is called a dual pair and ψ is called the canonical bilinear functional of the duality, usually we write

$$\langle x, y\rangle = \psi(x,y) \qquad (\text{ for all } x \in X \text{ and } y \in Y)$$

while the triple (X, Y, ψ) is denoted by $\langle X, Y\rangle$.

Remark 1.7.1. Let (X, ρ) be a locally convex space with the topological dual X', then $\langle X, X'\rangle$ under the natural bilinear form $\langle x, x'\rangle = x'(x)$ (for all $x \in X$ and $x' \in X'$) ia a dual pair.

Proof. See [25], p. 206.

□

Remark 1.7.2. Let $\langle X, X'\rangle$ be as in remark 1.7.1 and suppose that $\langle x, {x'}_1\rangle = \langle x, {x'}_2\rangle$ for all $x \in X$, then ${x'}_1 = {x'}_2$.

Proof. Suppose that $\langle x, {x'}_1\rangle = \langle x, {x'}_2\rangle$ for all $x \in X$, then ${x'}_1(x) = {x'}_2(x)$ for all $x \in X$, so $({x'}_1 - {x'}_2)(x) = 0$ for all $x \in X$, then $\langle x, {x'}_1 - {x'}_2\rangle = 0$ for all $x \in X$, so by $(S2)$ we have $x'_1 - x'_2 = 0$, which imply $x'_1 = x'_2$.

□

Definition 1.7.3. For a given dual pair $\langle X, Y\rangle$, the weak topology on X, denoted by $\sigma(X, Y)$, is defined by the family $\{p_y : y \in Y\}$of seminorms on X, where

$$p_y(x) = |\langle x, y\rangle| \qquad (for\ all\ x \in X)$$

Definition 1.7.4. [**sequence spaces**] Let ω be the collection of all sequences $x = (x_i) = (x_1, x_2, ...)$ where $x_i \in \mathbf{K}\ \ \forall i$. By a sequence space we mean a vector subspace $\Lambda \subseteq \omega$ of infinite sequences. The vector space operations are given by the usual operations on the coordinates.

If Λ is a sequence space, we define

$$\Lambda^* := \{((b_n)_n)\ \in\ \mathbf{K}^N\ :\ \lim_n a_n b_n\ =\ 0\ for\ all\ ((a_n)_n)\ \in \Lambda\}$$

Λ^* is called the $K\ddot{o}the - dual$ of Λ.

We say that Λ is perfect if $\Lambda^{**} = \Lambda$.

Theorem 1.7.1. *A sequence space Λ is perfect if and only if it is sequentially complete .*

Proof. See [11], p. 413.

□

Note : Each $((b_n)_n) \in \Lambda^*$ define a seminorm on Λ by:

$$p_b(a) = \sup_n |a_n b_n|, \quad \text{where } a = ((a_n)_n) \in \Lambda.$$

Proof. Let $x, y \in \Lambda$ and $\alpha \in \mathbf{K}$, then

(1) $p_b(\alpha x) = p_b(\alpha x_1, \alpha x_2, ...) = \sup_n |\alpha x_n b_n| = |\alpha| \sup_n |x_n b_n| = |\alpha| p_b(x)$.

(2) $p_b(x+y) = \sup_n |(x_n+y_n)b_n| = \sup_n |x_n b_n + y_n b_n| \leq \max\{ \sup_n |x_n b_n|, \sup_n |y_n b_n|\} = \max\{p_b(x), p_b(x)\}$.

$\square$

The topology defined by the family of seminorms $\{p_b : b \in \Lambda^*\}$ denoted by $n(\Lambda, \Lambda^*)$ is called the normal topology on Λ.

Note : Every $a = ((a_n)_n) \in \Lambda$ can be written uniquely as $\sum_n a_n e_n$, where for each n, e_n is the sequence with 1 in the nth place and zeros elsewhere.

Note : If Λ is perfect, then the bilinear form defined by

$$\langle a, b\rangle = \sum_n a_n b_n, \quad a = ((a_n)_n) \in \Lambda, \quad ((b_n)_n) \in \Lambda^*.$$

satisfies the separations conditions $(S1)$ and $(S1)$, so (Λ, Λ^*) is a dual pair.
By $\sigma(\Lambda, \Lambda^*)$ we denote the weak topology on Λ associated with this dual pair.

Chapter 2

Extension of compact operators

Compact linear operators are very important in applications. They play a central role in the theory of integral equations and in various problems of mathematical physics.Their theory served as a model for the early work in functional analysis.

In this chapter we study compact operators in non-archimedean locally convex spaces having the following property :
For all non-archimedean locally convex spaces $Z \supset Y$, every compact operator $T : X \to Y$ has an extension to a compact operator $\overline{T} : Z \to Y$. We will study this property on spaces over spherically complete, non spherically complete and discretely valued fields. Section (2.1.) gives us a basic facts about this property which will help us in the others sections. In section (2.2.) we consider this study of that property over non spherically complete valued fields. In section (2.3.) we will focus the light on this property over spherically complete valued fields. In section (2.4.) we focus on this property over discretely valued fields.

2.1 Definition and basic facts

In this section we will define the compact extension property and we will discuss main and basic facts about the compact extension property.
In (Proposition 2.1.3.) we will see that in the compact extension property it is enough to assume that Y and Z are normed spaces.

Definition 2.1.1. We say that X has the compact extension property (CEP for short) if, for every pair of locally convex spaces Y , Z with Y $\subset$ Z and every $T \in C(Y, X)$, there exist $\overline{T} \in C(Z, X)$ that extends T.

Lemma 2.1.1. *Let* $T \in C(Y, X)$ *, then* $T_p : Y_p \to X$ *defined by* $T_p(\pi_p(y)) = T(y)$ *is an element in* $C(Y_p, X)$*.*

Proof. Since $T \in C(Y, X)$, then there exist a zero neighborhood $\mathcal{V}$ in Y such that $T(\mathcal{V})$ is compactoid in X , since $\mathcal{V}$ is a zero neighborhood in Y , then $\mathcal{V} + ker\ p = \pi_p(\mathcal{V})$ is a zero neighborhood in Y_p, then $T_p(\pi_p(\mathcal{V})) = T(\mathcal{V})$ is compactoid in X, therefore $T_p \in C(Y_p, X)$.

□

Lemma 2.1.2. *If* $T \in C(Y, X)$ *, then there exist* $p \in \mathcal{P}_Y$ *and* $T_p \in C(Y_p, X)$ *such that* $T = T_p \circ \pi_p$ *.*

Proof. Since $T \in C(Y, X)$, then there exist a zero neighborhood $\mathcal{U}$ in Y such that $T(\mathcal{U})$ is compactoid in X , let $p \in \mathcal{P}_Y$ be the minkowski functional of $\mathcal{U}$, then by the (lemma 2.1.1.) $T_p : Y_p \to X$ defined by $T_p(\pi_p(y)) = T(y)$ is an element in $C(Y_p, X)$ and $T = T_p \circ \pi_p$.

$\square$

Proposition 2.1.3. *If for every pair of normed spaces E , F with $E \subset F$ and for every $T \in C(E,X)$, there exist $\overline{T} \in C(F,X)$ that extends T , then for every pair of locally convex spaces Y , Z with $Y \subset Z$ and every $S \in C(Y,X)$, there exist $\overline{S} \in C(Z,X)$ that extends S. So from now on when using definition 2.1.1., we can restrict our selves to normed spaces Y, Z.*

Proof. Assume that for every pair of normed spaces E , F with $E \subset F$ and for every $T \in C(E,X)$, there exist $\overline{T} \in C(F,X)$ that extends T. ....(*)

Now let $S \in C(Y,X)$, then by (lemma 2.1.2.) , there exist $p \in \mathcal{P}_Y$ and $S_p \in C(Y_p, X)$ such that $S = S_p \circ \pi_p$.

Since Z is a locally convex space, then there is a collection of continuous seminorms $\mathcal{P}_Z$ define the topology on Z, then the restrictions to Y of the continuous seminorms on Z define the topology on Y as a subspace of Z, then we can assume that p is the restriction to $Y, q|Y$ for some $q \in \mathcal{P}_Z$

Define $i : Y_p \to Z_q$ by $\pi_p(y) \mapsto \pi_q(y)$, $y \in Y$,

then i is a linear isometry from Y_p to Z_q. For that we have to prove that i is linear and isometry.

[1] **proof of linearity :** Let x, y be elements in Y and $\lambda \in \mathbf{K}$, then

$$\begin{aligned} i(\pi_p(x) + \lambda\pi_p(y)) &= i(\pi_p(x+\lambda y)) = \pi_q(x+\lambda y)) \\ &= \pi_q(x) + \lambda\pi_q(y) = i(\pi_q(x)) + \lambda i(\pi_q(y)) \end{aligned}$$

[2] proof of isometry :

$$\begin{aligned} d(i(\pi_p(x)), i(\pi_p(y))) &= d(\pi_q(x), \pi_q(y)) = q(x-y) \\ &= p(x-y) = d(\pi_p(x), \pi_p(y)) \end{aligned}$$

Now by (definition 1.5.4.), we have that both Z_q and Y_p are normed spaces. Set $\overline{Y_p} = i(Y_p)$, then $\overline{Y_p}$ is a normed subspace of Z_q.
Define $\bar{i} : Y_p \rightarrow \overline{Y_p}$ to be the restriction of the map i to it's image , then $\bar{i}$ is 1-1 and onto which mean that Y_p is isometric to $\overline{Y_p}$, then $\bar{i}^{-1}$ exist and continuous, so if we define $\overline{S_p} := S_p \circ \bar{i}^{-1}$, then $\overline{S_p} \in C(\overline{Y_p}, X)$ because (by remark 1.5.2. (iii)), composition of compact and continuous operators is compact operator, but $\overline{Y_p}$ is a normed subspace of Z_q, then by (*), there exist $\overline{\overline{S_p}} \in C(Z_p, X)$ that extends $\overline{S_p}$.

Define $\overline{S} = \overline{\overline{S_p}} \circ \pi_q$, then $\overline{S} \in C(Z, X)$ because (by remark 1.5.2. (iii)), composition of compact and continuous operators is compact operator, also $\overline{S}$ is an extension of S because for all $y \in Y$, we have

$$\begin{aligned} \overline{S}(y) &= \overline{\overline{S_P}} \circ \pi_q(y) = \overline{\overline{S_P}}(\pi_q(y)) \\ &= \overline{\overline{S_P}}(i\pi_P(y)) = \overline{\overline{S_P}}(\bar{i}\pi_P(y)) \qquad since\ y \in Y \\ &= \overline{S_P}(\bar{i}\pi_P(y)) = S_P \circ \bar{i}^{-1}(\bar{i}\pi_P(y)) \\ &= S_P \circ \pi_P(y) = S(y) \end{aligned}$$

□

Proposition 2.1.4. *(i) If X has the CEP , then every locally convex space linearly homeomorphic to X has the CEP.*

(ii) If X has the CEP , then every complemented subspace of X has the CEP.

(iii) If $\{X_i\}_{i\in I}$ is a familly of locally convex spaces having the CEP, then $\prod_{i\in I} X_i$ endowed with the product topology has the CEP.

Proof. (i) Suppose that X has the CEP , then for every pair of locally convex spaces Y , Z with Y $\subset$ Z and every $T \in C(Y,X)$ has an extension $\overline{T} \in C(Z,X)$ that extends T.

Now let C be a locally convex space that is linearly homeomorphic to X , then there exist a linear homeomorphism $f : C \to X$, such that both f and f^{-1} are continuous.

Let $S \in C(Y,C)$, then $f \circ S \in C(Y,X)$ because (by remark 1.5.2. (iii)), composition of continuous and compact operators is compact operator, but X has the CEP, then there exist $\overline{f \circ S} \in C(Z,X)$ that extends $f \circ S$. Since f is homeomorphism map, then f^{-1} exist, so we can define $\overline{S} = f^{-1} \circ (\overline{f \circ S})$, then $\overline{S} \in C(Z,C)$ because (by remark 1.5.2. (iii)), composition of continuous and compact operators is compact operator, also $\overline{S}$ is an extension of S because for all $y \in Y$, we have:

$$f^{-1} \circ (\overline{f \circ S})(y) = f^{-1}(\overline{f \circ S}(y)) = f^{-1}(f \circ S(y)) = S(y).$$

□

Proof. (ii) Suppose that C is a complemented subspace of X , then there exist a projection $P \in L(X,X)$ such that $P(X) = C$.

Now , let $T \in C(Y,C)$, then there exist a zero neighborhood $\mathcal{U}$ in Y such that $T(\mathcal{U})$ is compactoid in C. Since C is a subspace of X, then by (Proposition

1.5.4.) $T(\mathcal{U})$ is compactoid in X , so $T \in C(Y, X)$.

Since X has the CEP, then there exists , $\overline{T} \in C(Z, X)$ that extends T, then $P \circ \overline{T} \in C(Z, C)$ because $\overline{T} \in C(Z, X)$ and f is continuous, also $P \circ \overline{T}$ is an extension of T because for all $y \in Y$, we have $P \circ \overline{T}(y) = P(\overline{T}(y)) = P(T(y)) = T(y)$

□

Proof. (iii) Assume that $\{X_i\}_{i \in I}$ is a family of locally convex spaces having the CEP, we want to show that $\prod_{i \in I} X_i$ endowed with the product topology has the CEP, so let $T \in C(Y, \prod_{i \in I} X_i)$, and let $\pi_i : \prod_{i \in I} X_i \to X_i$ be the i^{th} projection map which is continuous, so $\pi_i \circ T \in C(Y, X_i)$ because the composite of operators in which one of them is compact must be compact, but for all $i \in I$, X_i have the CEP, then for all $i \in I$ there exists $\overline{\pi_i \circ T} \in C(Z, X_i)$ that extends $\pi_i \circ T$.

Since $\overline{\pi_i \circ T} \in C(Z, X_i)$, then there is a zero neighborhood $\mathcal{U}$ in Z such that $\overline{\pi_i \circ T}(\mathcal{U})$ is compactoid in X_i, then by (Proposition 1.5.5.), we have $\prod_{i \in I} \overline{\pi_i \circ T}(\mathcal{U})$ is compactoid in $\prod_{i \in I} X_i$.

Define $\overline{T} : Z \to \prod_{i \in I} X_i$ by $\overline{T}(z) = (\overline{\pi_i \circ T}(z))_{i \in I}$, then $\overline{T}$ is compact because $\overline{T}(\mathcal{U}) = (\overline{\pi_i \circ T}(\mathcal{U}))_{i \in I} = \prod_{i \in I} \overline{\pi_i \circ T}(\mathcal{U})$ is compactoid in $\prod_{i \in I} X_i$, also $\overline{T}$ is an extension of T because for all $y \in Y$, we have

$$\overline{T}(y) = (\overline{\pi_i \circ T}(y))_{i \in I} = (\pi_i \circ T(y))_{i \in I} = (\pi_i(T(y)))_{i \in I} = T(y)$$

□

2.2 The nonspherically complete case.

In this section we assume that **K** is not spherically complete. The first goal of this section is to prove that when **K** is not spherically complete, there are no nontrivial locally convex spaces over K with the CEP (Theorem 2.2.9.). The seconde goal is to consider the CEP in the category of polar spaces (we call it p-CEP).Then again for **K** nonspherically complete, there are no nontrivial examples of spaces with the p-CEP (Theorem 2.2.14.).

Theorem 2.2.1. *Let E , F be normed spaces over* **K** *and let $T \in L(E,F)$, then the following are equivalent ,*

(1) T is compactoid.

(2) For each $t \in (0,1)$, there exists a sequence (g_n) in E' and a t-orthogonal sequence (y_n) in F , with $\|g_n\| \leq 1$ and y_n converging to zero such that

$$Tx \;=\; \sum_{1}^{\infty} g_n(x) y_n \qquad (x \in E).$$

(3) For each $t \in (0,1)$, there exists a sequence (g_n) in E' and a t-orthogonal sequence (y_n) in F such that $\|g_n\|\|y_n\|$ tends to zero and

$$Tx \;=\; \sum_{1}^{\infty} g_n(x) y_n \qquad (x \in E).$$

(4) There exists a sequence (h_n) in E' with $\lim \|h_n\| = 0$, such that

$$\|Tx\| \leq \sup_n |h_n(x)| \qquad (x \in E).$$

(5) There exists $S \in C(E, c_0)$ *such that* $\|Tx\| \leq \|Sx\|$ *for all* $x \in E$.

Proof. See [10], p. 335.

□

Theorem 2.2.2. *A locally convex topology on* X *is Hausdorff if and only if for any* $0 \neq x \in X$ *there is a* $p \in \mathcal{P}_X$ *such that* $p(x) \neq 0$.

Proof. See [25], p. 114.

□

Lemma 2.2.3. *Let* X *,* Y *be locally convex spaces with* $X \neq \{0\}$*, then* $C(Y, X) = \{0\}$ *if and only if* $Y^{'} = \{0\}$

Proof. ($\Leftarrow$) Assume that $Y^{'} = \{0\}$. let T be a compact operator from Y to X , take $q \in \mathcal{P}_X$ and let π_q be the quotient map from X onto X_q, then by (theorem 1.5.7.), π_q is continuous, then $\pi_q \circ T \in C(Y, X_q)$ (because composite of a continuous operator and a compact operator is a compact operator). Now by (lemma 2.1.2.), there exist $p \in \mathcal{P}_X$ and $T_p \in C(Y_p, X_q)$ such that $\pi_q \circ T = T_p \circ \pi_p$.

Now we prove that $(Y_p)^{'} = \{0\}$.

Suppose not, then there exists $f \in (Y_p)^{'} \neq \{0\}$ such that $f(\pi_p(y)) \neq 0$ for some $\pi_p(y) \in Y_p$. *i.e.* $f \circ \pi_p(y) \neq 0$ for some $y \in Y$. Since π_p is continuous and linear from Y to Y_p and f is a continuous linear from Y_p into $\mathbf{K}$, then $f \circ \pi_p \in Y^{'}$, but $f \circ \pi_p \neq 0$ which is contradiction with assumption.

Therefore $(Y_p)^{'} = \{0\}$.

Recall that by (definition 1.5.4.) both Y_p, X_q are normed spaces and $T_p \in C(Y_p, X_q)$, so apply (Theorem 2.2.1.[1⇒3]) with $(Y_p)^{'} = \{0\}$, we have $T_p = 0$, so $T_p \circ \pi_p = 0$, which mean that $\pi_q \circ T = 0 \ \ \forall q \in \mathcal{P}_X$, since X is Hausdorff , then by (theorem 2.2.2.) $\exists\, q \in \mathcal{P}_X$ such that $q \neq 0$ which mean $\pi_q \neq 0$, it follows that $T = 0$, therefore $C(Y, X) = \{0\}$.

($\Rightarrow$) suppose that $C(Y, X) = \{0\}$, let $f \in Y'$ be an arbitrary. If $x \neq 0$ is a fixed element of X , define

$$T : Y \to X \qquad \text{by} \qquad T(y) = f(y)x,$$

then T is an operator from X to Y. Since f is bounded, then T is bounded and $dim(T) = 1 < \infty$, then T is finite operator , then by (remark 1.5.2.(i)) $T \in C(Y, X)$, but $C(Y, X) = \{0\}$, then $T = 0$, so $T(y) = f(y)x = 0$ for all $y \in Y$, but $x \neq 0$, then $f(y) = 0$ for all $y \in Y$ which mean that $f = 0$, therefore $Y' = \{0\}$.

□

Theorem 2.2.4. *If T is a continuous linear bijection of a Banach space onto a Banach space , then T is a homeomorphism .*

Proof. See [22], p. 62.

□

Corollary 2.2.5. *The map $T : X \to \prod_{p\in\mathcal{P}_X} X_p$, defined by $T(x) = (\pi_p(x))_{p\in\mathcal{P}_X}$ is a linear homeomorphism from X onto $T(X)$.*

Proof.

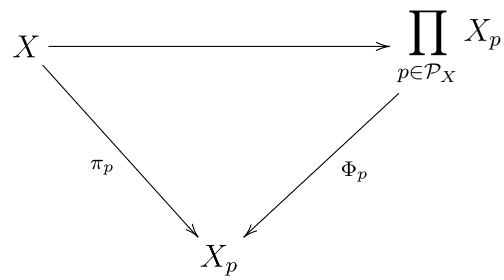

Figure 3.2.1.

Let $T : X \to \prod_{p\in\mathcal{P}_X} X_p$, defined by $T(x) = (\pi_p(x))_{p\in\mathcal{P}_X}$, then $\pi_p = \Phi_p \circ T$, where π_p is the quotient map from X to X_p which is continuous and Φ_p is the p^{th} projection map from $\prod_{p\in\mathcal{P}_X} X_p$ to X_p which is open and continuous. We want to see that :

[**1**] T **is linear** Let x_1, x_2 be elements in X and let $\lambda \in \mathcal{K}$, then

$$\begin{aligned} T(x_1 + \lambda x_2) &= (\pi_p(x_1 + \lambda x_2))_{p\in\mathcal{P}_X} = (\pi_p(x_1) + \lambda(\pi_p(x_2)))_{p\in\mathcal{P}_X} \\ &= (\pi_p(x_1))_{p\in\mathcal{P}_X} + \lambda((\pi_p(x_2)))_{p\in\mathcal{P}_X}. \end{aligned}$$

[**2**] T **is continuous** Since $\pi_p = \Phi_p \circ T$, then $T^{-1} = {\pi_p}^{-1} \circ \Phi_p$.

let U be an open set in $\prod_{p\in\mathcal{P}_X} X_p$, then $T^{-1}(U) = {\pi_p}^{-1} \circ \Phi_p(U) = {\pi_p}^{-1}(\Phi_p(U))$, since Φ_p is the p^{th} projection , then Φ_p is open, so $\Phi_p(U)$ is open, then ${\pi_p}^{-1}(\Phi_p(U))$ is open because π_p is continuous, therefore T is continuous.

[**3**] T **is 1-1** Let $0 \neq x \in \ker T$, then $T(x) = (\pi_p(x))_{p\in\mathcal{P}_X} = (0, 0, ...)$, then $\pi_p(x) = 0$ for all $p \in \mathcal{P}_X$ which mean $p(x) = 0$ for all $p \in \mathcal{P}_X$ which is contradiction with (theorem 2.2.2.), then $\ker T = \{0\}$ and so T is 1-1 .

Then T is a continuous linear bijection of X onto $T(X)$, so by (theorem 2.2.4.),

T is homeomorphic from X onto $T(X)$.

□

Now since $T(X) \subset \prod_{p \in \mathcal{P}_X} X_p$, then we can say that $\prod_{p \in \mathcal{P}_X} X_p$ contains a subspace that is linearly homeomorphic to X.

Recall that [22] by $\breve{X}_p$, we denote the spherical completion of X_p .

Corollary 2.2.6. *If X is T_2 , then $\prod_{p \in \mathcal{P}_X} \breve{X}_p$ contains a subspace which is linearly homeomorphic to X.*

Proof. Since $\breve{X}_p$ contains X_p as a subspace , then $\prod_{p \in \mathcal{P}_X} \breve{X}_p$ contains $\prod_{p \in \mathcal{P}_X} X_p$ as a subspace which by (corollary 2.2.5.) contains a subspace that is linearly homeomorphic to X, then $\prod_{p \in \mathcal{P}_X} \breve{X}_p$ contains a subspace that is linearly homeomorphic to X.

□

Corollary 2.2.7. *If E is spherically complete and $\mathbf{K}$ is not , then $E^{'} = \{0\}$.*

Proof. See [22], p. 100.

□

Lemma 2.2.8. *Suppose that $\mathbf{K}$ is not spherically complete , let X, Y be locally convex spaces over $\mathbf{K}$ with $X \neq \{0\}$, then the following are equivalent :*

(i) For every locally convex space Z containing a copy Y_1 of Y and every $T \in C(Y_1, X)$, there exist a $\overline{T} \in C(Z, X)$ extends T .

(ii) $Y^{'} = \{0\}$.

Proof. $(ii) \Rightarrow (i)$ suppose that $Y^{'} = \{0\}$, then by (lemma 2.2.3.) $C(Y, X) = \{0\}$, since Y_1 is a copy of Y , then $\exists f : Y :\rightarrow Y_1$ such that f is 1-1 , onto and both f, f^{-1} are continuous. Let $T \in C(Y_1, X)$, then $T \circ f \in C(Y, X)$, because composite of a compact operator and a continuous operator is a compact operator, but $C(Y, X) = \{0\}$, then $T \circ f = 0$, but since f is homeomorphism, then f is 1-1 and onto, so $f(x) \neq 0 \; \forall \; x \neq 0$, then $T = 0$, so $C(Y_1, X) = \{0\}$. Then if $T \in C(Y_1, X)$, we have $T = 0$, take $\overline{T} : Z \rightarrow X$ defined by $T(z) = 0$ for all $z \in Z$, then T is finite rank operator and so $T \in C(Z, X)$.

$(i) \Rightarrow (ii)$ since Y is Hausdorff , then by (corollary 2.2.6.) Y is linearly homeomorphic to a subspace Y_1 of the locally convex space $Z := \prod_{p \in P_Y} \breve{Y}_p$. Now since $\breve{X}_p$ is spherically complete and **K** is not , then by (corollary 2.2.7.) $(\breve{X}_p)^{'} = \{0\}$ for all $p \in \mathcal{P}_Y$ and so $Z^{'} = \{0\}$, then by (Lemma 2.2.3.), we have $C(Z, X) = \{0\}$, so if we apply (i) then we have $C(Y_1, X) = \{0\}$ and again apply (Lemma 2.2.3.) we have $Y^{'} = \{0\}$.

□

Theorem 2.2.9. *If* **K** *is not spherically complete , then no locally convex space* $X \neq \{0\}$ *over* **K** *has the CEP .*

Proof. Assume that $\mathbf{K}$ is not spherically complete and $X \neq \{0\}$ is a locally convex space over $\mathbf{K}$ has the " CEP " , apply (Lemma 2.2.8.) for $Y = \mathbf{K}$, then $\mathbf{K}' = \{0\}$ which is contradiction because the identity mapping from $\mathbf{K}$ onto $\mathbf{K}$ is an element in $\mathbf{K}'$. □

One can weaken the " CEP " by making smaller the category of locally convex spaces in which it is defined , considering only normed or Banach spaces . The new goal would be to find out whether of these weaker conditions we obtain nontrivially spaces satisfying them.

We now consider the " CEP " in the category of polar spaces (we call it p-CEP). then again there are no nontrivial examples of spaces with the p-CEP.

Definition 2.2.1. :

(1) We say that X is injective if for every pair of locally convex spaces Y, Z with $Y \subset Z$ and every $T \in L(Y, X)$,there exist $\overline{T} \in L(Z, X)$ that extends T with $||T|| = ||\overline{T}||$.

(2) Let $0 < t \leq 1$. X is t-injective if for every pair of locally convex spaces Y, Z with $Y \subset Z$ and every $T \in L(Y, X)$,has an extension $\overline{T} \in L(Z, X)$ such that $t||\overline{T}|| \leq ||T||$.

(3) We say that X is weakly injective if for every pair of locally convex spaces Y, Z with $Y \subset Z$ and every $T \in L(Y, X)$,there exist $\overline{T} \in L(Z, X)$ that extends T.

Theorem 2.2.10. *The following conditions on* $\mathbf{K}$ *are equivalent .*

(i) $\mathbf{K}$ *is spherically complete .*

(ii) *There exist a* $g \in (l^{\infty})'$ *such that*

$$g(x) = \sum_{n=1}^{\infty} x_n \qquad for\ every\ x \in C_0.$$

(iii) $\mathbf{K}$ *is injective, i.e. if* D *is any linear subspace of any normed vector space* E*, then every* $f \in D'$ *has an extension* $g \in E'$ *with* $||g|| = ||f||$.

(iv) $\mathbf{K}$ *is weakly injective, i.e. if* D *is any linear subspace of any normed vector space* E*, then every* $f \in D'$ *has an extension* $g \in E'$.

Proof. See [22], p. 108.

□

Corollary 2.2.11. *Let* $x \in X\backslash\{0\}$ *, define* $[x] = \{\lambda x \ : \lambda \in \mathbf{K}\}$ *. If for every* $S \in L(c_0, [x])$ *has an extension* $\overline{S} \in L(l^{\infty}, [x])$*, then* $\mathbf{K}$ *is spherically complete .*

Proof. Assume that every $S \in L(c_0, [x])$ has an extension $\overline{S} \in L(l^{\infty}, [x])$, let $f : c_0 \to \mathbf{K}$, be defined by $f(x) = \sum_{n=1}^{\infty} x_n$ *for every* $x \in c_0$, then f is continuous linear isometry. For that let x, y be elements in c_0, then

$$f(x+y) = \sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n = f(x) + f(y).$$

$$|f(x)| = |\sum_{n=1}^{\infty} x_n| \leq \sup_n |x_n| = ||x||,$$

then f is a bounded linear map and so it is a continuous linear map from c_0 into $\mathbf{K}$.

Let $\psi : [x] \to \mathbf{K}$, defined by $\psi(\lambda x) = \lambda$, then ψ is linear, 1-1, onto and continuous. For that let $\lambda x, \mu x$ be elements is $[x]$, then

[1] ψ **is linear :** $\psi(\lambda x + \mu x) = \lambda + \mu = \psi(\lambda x) + \psi(\mu x)$

[2] ψ **is 1-1 :** Suppose that $\lambda x \neq \mu x$, then $\lambda \neq \mu$, so $\psi(\lambda x) \neq \psi(\mu x)$

[3] ψ **is onto :** Let $\lambda \in \mathbf{K}$, then $\lambda x \in [x]$ and $\psi(\lambda x) = \lambda$

[4] ψ **is continuous :** $|\psi(\lambda x)| = |\lambda| = (1/||x||)||\lambda x||$, since $x \in X\backslash\{0\}$, then $||x|| \neq 0$ and constant, then ψ is bounded and so is continuous, then ψ^{-1} exist and continuous, so $\psi^{-1} \circ f \in L(C_0, [x])$ because composite of two continuous maps is continuous, then by assumption $\psi^{-1} \circ f$ has an extension $\overline{\psi^{-1} \circ f} \in L(l^{\infty}, [x])$.
Define $g = \psi \circ (\overline{\psi^{-1} \circ f})$, then $g \in (l^{\infty})'$.

Now if $x \in c_0$, then we have:

$$g(x) = \psi \circ (\overline{\psi^{-1} \circ f})(x) = \psi((\overline{\psi^{-1} \circ f})(x)) = \psi(\psi^{-1} \circ f(x)) = f(x),$$

so $g(x) = \sum_{n=1}^{\infty} x_n$ for every $x \in c_0$. and so by (theorem 2.2.10) $\mathbf{K}$ is spherically complete.

□

Proposition 2.2.12. *Let D be a finite dimensional subspace of a hausdorff polar space E, then each $f \in D'$ can be extended to an $\overline{f} \in E'$.*

Proof. See [17], p. 204.

□

Definition 2.2.2. Let us say that a linear subspace D of E has the weak extension property ($W.E.P$ for short) if every $f \in D'$ has an extension $\overline{f} \in E'$.

Lemma 2.2.13. *If D is a one-dimensional subspace of E, then the following properties are equivalent :*

(i) D has the W.E.P.

(ii) D is a complemented subspace.

(iii) There exist $f \in E^{'}$ such that $f(D) \neq \{0\}$

Proof. Let D be a one–dimensional subspace of E, then there exist $a \in E$ such that $D = [a] = \{\lambda a\ : \lambda \in \mathbf{K}\}$, define $\psi : D \to \mathbf{K}$, defined by $\psi(\lambda a) = \lambda$, then ψ is linear, 1-1, onto and continuous, then ψ^{-1} exist and continuous.

($i \Rightarrow ii$) Let $i : D \to D$ be the identity map on D which is continuous, then $\psi \circ i \in D^{'}$, then by (i) there is $\overline{\psi \circ i} \in E^{'}$ that extends $\psi \circ i$. Let $P = \psi^{-1} \circ (\overline{\psi \circ i})$, then we have:

(1) $P \in L(E, D)$, because composite of two continuous maps is continuous.

(2) For all $x \in E$, we have

$$\begin{aligned} P^2(x) &= (\psi^{-1} \circ (\overline{\psi \circ i}))^2(x) = (\psi^{-1} \circ (\overline{\psi \circ i}))(\psi^{-1} \circ (\overline{\psi \circ i})(x)) \\ &= (\psi^{-1} \circ (\psi \circ i))(\psi^{-1} \circ (\overline{\psi \circ i})(x)) = (\psi^{-1} \circ (\overline{\psi \circ i})(x)) = P(x) \end{aligned}$$

from (1), (2) and (Definition 1.5.7.), we have that P is a projection from E onto D, then D is complemented subspace in E.

($ii \Rightarrow iii$) Suppose that D is complemented subspace in E, then there is a projection from E onto D, then $\psi \circ P \in E'$ and

$$\psi \circ P(D) = \psi(D) = \mathbf{K} \neq \{0\}.$$

$(iii \Rightarrow i)$ Suppose that there exist $f \in E'$ such that $f(D) \neq \{0\}$. For every $g \in D'$, define $\overline{g} : E \to \mathbf{K}$ by $\overline{g}(x) = (g(a)/f(a))f(x)$, since $(g(a)/f(a))$ is constant and $f \in E'$, then $\overline{g}(x) = (g(a)/f(a))f(x) \in E'$, also $\overline{g}$ is an extension of g because for all $y = \lambda a \in D$, we have

$$\begin{aligned} \overline{g}(y) &= (g(a)/f(a))f(y) = (\lambda g(a)/\lambda f(a))f(y) \\ &= (g(\lambda a)/f(\lambda a))f(y) = (g(y)/f(y))f(y) = g(y) \end{aligned}$$

□

Theorem 2.2.14. *If there exist a polar space $X \neq \{0\}$ over $\mathbf{K}$ with the p-CEP, then $\mathbf{K}$ is spherically complete .*

Proof. Let $X \neq \{0\}$ be a polar space over $\mathbf{K}$ with the " p-CEP " , take $x \in X \backslash \{0\}$. Define $[x] = \{\lambda x \ : \lambda \in \mathbf{K}\}$, then $[x]$ is a one-dimensional linear subspace of X .

To prove that $\mathbf{K}$ is spherically complete we will use (corollary 2.2.11.), so we have to prove that every $S \in L(c_0, [x])$ has an extension $\overline{S} \in L(l^\infty, [x])$.

So let $S \in L(c_0, [x])$, if $i : [x] \to X$ is the canonical inclusion from $[x]$ into X , (In fact $S \in C(c_0, [x])$ because $S(c_0) \subset [x]$ which is one-dimensional linear subspace of X, then S is an operator of finite rank and so is compact operator), then $T = i \circ S$ is a compact linear operator from c_0 into X because composite of continuous and compact operators is compact operator,.

Now by assumption T has an extension $\overline{T} \in C(l^{\infty}, X)$.

Since X is polar , and $[x]$ is a finite dimensional subspace of X , then by (proposition 2.2.12), every $f \in [x]'$ can be extends to an $\overline{f} \in X'$, but this is the definition of W.E.P and since $[x]$ is one-dimensional subspace of X , and $[x]$ has the W.E.P. ,then by (lemma 2.2.13), $[x]$ is complemented subspace of X, then by definition of complemented subspace , there exist a continuous linear projection $P : X \to [x]$, then $\overline{S} = P \circ \overline{T}$ is compact because $\overline{T} \in C(l^{\infty}, X)$ and P is continuous, also composition of continuous and compact operators is compact operator, and so $\overline{T}$ is continuous, also we have that $\overline{S}$ is an extension of S, because if $x \in c_0$, we have:

$\overline{S}(x) = (P \circ \overline{T})(x) = p(\overline{T}(x)) = p(T(x)) = p(i \circ S(x)) = (p \circ i)(S(x)) = S(x)$.

Therefore, **K** is spherically complete.

□

2.3 The spherically complete case

In this section we assume that $\mathbf{K}$ is spherically complete. We have the following main results.

(1) Every metrizable locally convex space X over $\mathbf{K}$ (and so every normed space) has the CEP (Theorem 2.3.6.).

(2) Every weakly sequentially complete locally convex space X over $\mathbf{K}$ with an orthogonal basis has the CEP (Theorem 2.3.7.).

(3) Every weakly injective locally convex space X over $\mathbf{K}$ has the CEP (Theorem 2.3.28.).

(4) We finish this section by giving some applications of the results we study through this section to characterize spherical completeness of the field $\mathbf{K}$ (Theorem 2.3.34.).

Lemma 2.3.1. *Let D be a subspace of E , then there exists an element $a \in E \backslash D$ such that $E = D + [a]$.*

Proof. Define $D_a = \{x + \alpha a \ : x \in D \ and \ \alpha \in \mathbf{K}\} \ = D + [a]$

Let $M = \{D + [b] : b \in E \backslash D\}$ and order M by $D + [b] \prec D + [c]$ if and only if $D + [b] \subset D + [c]$, then we have that $\prec$ is a partial order because :

(1) $D + [b] \subset D + [b]$ for all $b \in E \backslash D$.

(2) If $D + [a] \subset D + [b]$ and $D + [b] \subset D + [c]$, then $D + [a] \subset D + [c]$.

(3) Suppose that $D+[a] \subset D+[b]$ and $D+[b] \subset D+[a]$, then $D+[a] = D+[b]$.

Let $C = \{D + [a_\alpha]\}$ with a running through some index set I , be a chain subset of M, then for any $\alpha, \beta \in I$ we have $D + [a_\alpha] \subset D + [a_\beta]$ or $D + [a_\beta] \subset D + [a_\alpha]$

Now if $D + [a_\alpha] \subset D + [a_\beta]$, then

$$[a_\alpha] \subset [a_\beta] ....(*)$$

Let $H = \bigcup_{\alpha \in I} D + [a_\alpha]$ then by (*) we have $H = D + [a_\delta]$ for some $\delta \in I$, then H is an upper bound of C , and so by Zorn's lemma M has a maximal element say it $D + [a]$, we want to see that $E = D + [a]$. Suppose not , then by the first part of our proof we can extends $D + [a]$ to $(D + [a]) + [b] = D + [a + b]$ for some $b \in E \backslash D + [a]$ since D is a subspace , then $a + b \in E \backslash D$, then $D + [a + b] \in M$ which is contradiction with the maximality of $D + [a]$, therefore $E = D + [a]$ □

Lemma 2.3.2. *Let E , F be normed vector spaces , F spherically complete , Let D be a linear subspace of E, $S \in L(D, F)$, Let $\mathcal{U}$ be a nonempty subset of $L(E, F)$ and for each $U \in \mathcal{U}$, let $\epsilon_U > 0$ be so that*

$$\|U - V\| \leq \ max \ \{\epsilon_U, \epsilon_V\} \qquad (U, V \in \mathcal{U})$$

$$\|Sx - Ux\| \leq \epsilon_U \|x\| \qquad (U \in \mathcal{U}, x \in D)$$

then S has an extension $\overline{S} \in L(E, F)$ such that

$$\|\overline{S} - U\| \leq \epsilon_U \qquad (U \in \mathcal{U})$$

Proof. By (lemma 2.3.1) we can find an element $a \in E \backslash D$ such that $E = D + [a]$. For $x \in D$ *and* $U \in \mathcal{U}$ let :

$$B_{x_U} := B_{\epsilon_U \|x+a\|}(Ux + Ua - Sx)$$

For all $x, y \ inD$ and $U, V \in \mathcal{U}$, with $\epsilon_U \leq \epsilon_V$, we have

$$
\begin{aligned}
\|(Ux+Ua-Sx)-(Vy+Va-Sy\| &= \|(U-V)(y+a)-[S(x-y)-U(x-y)]\| \\
&\le max\{\|(U-V)(y+a)\|, \|S(x-y)-U(x-y)\|\} \\
&\le max\{\|U-V\|\|y+a\|, \|S(x-y)-U(x-y)\|\} \\
&\le max\{\{\epsilon_U, \epsilon_V\}\|y+a\|, \epsilon_U\|x-y\|\} \\
&= max\{\epsilon_V\|y+a\|, \epsilon_U\|x-y\|\} \\
&= max\{\epsilon_V\|y+a\|, \epsilon_U\|x+a-a-y\|\} \\
&\le max\{\epsilon_V\|y+a\|, \epsilon_U\|x+a\|, \epsilon_U\|y+a\|\} \\
&\le max\{\epsilon_V\|y+a\|, \epsilon_U\|x+a\|\}
\end{aligned}
$$

It follows that $B_{x_U} \bigcap B_{y_V} \neq \phi$. thus any two balls have non empty intersection, so by spherically completeness of F and by (Lemma 1.2.2.[α]), there is a $z_0 \in F$ that lies in every B_{x_U} , then for all $x \in D$ and $U \in \mathcal{U}$

$$\|Sx + z_0 - U(x+a)\| \le \epsilon_U \|x+a\|$$

then for all $x \in D$, $\lambda \in \mathbf{K}$ *and* $U \in \mathcal{U}$, we have

$$\|(Sx + \lambda z_0) - U(x+\lambda a)\| \le \epsilon_U \|x+\lambda a\|$$

Define $\overline{S} : x + \lambda a \mapsto S(x) + \lambda z_0$.

Then $\overline{S}$ is an extension of S because if $x \in D$, then $x = x + 0a$, so $\overline{S}(x) = \overline{S}(x+0a) = S(x) + 0z_0 = S(x)$ and we have : for $||x + \lambda a|| \neq 0$,

$$\frac{\|(\overline{S}-U)(x+\lambda a)\|}{\|x+\lambda a\|} \leq \frac{\epsilon_U\|x+\lambda a\|}{\|x+\lambda a\|} = \epsilon_U$$

so ϵ_U is an upper bound of the set

$$\{\frac{\|(\overline{S}-U)(x+\lambda a)\|}{\|x+\lambda a\|} \mid \overline{S}-U \in L(E,F)\ ,\ x+\lambda a \in E\setminus\{0\}\}$$

then

$$\begin{aligned}\|\overline{S}-U\| &= \sup\{\frac{\|(\overline{S}-U)(x+\lambda a)\|}{\|x+\lambda a\|} \mid \overline{S}-U \in L(E,F)\ ,\ x+\lambda a \in E\setminus\{0\}\} \\ &\leq \epsilon_U.\end{aligned}$$

□

Theorem 2.3.3. *Let E , F be normed vector spaces , D a linear subspace of E , if F is spherically complete, then every $S \in L(D,F)$ has an extension $\overline{S} \in L(E,F)$ such that $\|\overline{S}\| = \|S\|$*

Proof. Let $S \in L(D,F)$, let $\mathcal{U} = \{0\}$ and $\epsilon_U = \|S\|$, then by (lemma 2.3.2) we have S has an extension $\overline{S} \in L(E,F)$ such that $\|\overline{S}-U\| \leq \epsilon_U = \|S\|$, then

$$\|\overline{S}\| = \|\overline{S}-0\| \leq \|S\|$$

but for ever we have $\|S\| \leq \|\overline{S}\|$,

therefore, $\|S\| = \|\overline{S}\|$. □

Corollary 2.3.4. *If* **K** *is spherically complete, then for every normed space E, let D be a linear subspace of E, then every $f \in D'$ has an extension $g \in E'$ with $\|f\| = \|g\|$.*

Proof. Apply (theorem 2.3.3) with $F = \mathbf{K}$.

□

Lemma 2.3.5. *Suppose X is metrizable, then for* $\mathbf{K}$*-convex bounded subset A of X there exists a* $\mathbf{K}$*-convex bounded subset* $B \supset A$ *such that on A the topologies induced by X and* X_B *coincide.*

Proof. See [25], p. 170.

□

Theorem 2.3.6. *Every metrizable locally convex space over* $\mathbf{K}$ *has the CEP.*

Proof. First assume that X is a normed space, then by (definition 1.4.1.), X is locally convex space. We will use (proposition 2.1.3.), so let Y, Z be normed spaces with $Y \subset Z$ and $T \in C(Y, X)$, then by (theorem 2.2.1[1 $\Rightarrow$ 3]), there exists a sequence (f_n) in Y' and a t-orthogomnal sequence (x_n) in X , $(t \in (0, 1))$ such that:

$$\|f_n\|\|x_n\| \text{ tends to zero and}$$

$$Ty \ = \ \sum_{n=1}^{\infty} f_n(y)x_n \qquad (y \in Y).$$

Since $\mathbf{K}$ is spherically complete, then by (corollary 2.3.4.) every $f_n \in Y'$ has an extension $g_n \in Z'$ with $\|g_n\| = \|f_n\|$

Define

$$\overline{T} \ : Z \to X \text{ by } \overline{T}(z) \ = \ \sum_{n=1}^{\infty} g_n(z)x_n \qquad (z \in Z)..................(1),$$

and we have

$$\|g_n\|\|x_n\| = \|f_n\|\|x_n\| \text{ tends to zero } ....(2).$$

Then $\overline{T}$ is an extension of T because for every $y \in Y$, we have

$$\overline{T}(y) \;=\; \sum_{n=1}^{\infty} g_n(y)x_n \;=\; \sum_{n=1}^{\infty} f_n(y)x_n \;=\; T(y).$$

Also we have $\overline{T} \in C(Z, X)$ which follows from (1), (2) and applying (theorem 2.2.1[$3 \Rightarrow 1$]).

Now suppose that X is metrizable locally convex space, and let Y , Z and $T \in C(Y, X)$ be as above, then the image of B_Y is compactoid in X, where B_Y is the closed unit ball in Y. Since $\mathbf{K}$ is spherically complete and $T(B_Y)$ is compactoid in X, then by (theorem 1.4.1.), $T(B_Y)$ is bounded in X, then by (lemma 2.3.5.), there exist an absolutely convex bounded subset B of X such that $T(B_Y) \subset B$ and the topologies induced by X and X_B on $T(B_Y)$ coincide, then $T(B_Y)$ is compactoid in X_B , then

$$T_B \;:\; Y \rightarrow X_B \quad , \quad y \in Y \;\mapsto\; T(y) \in X_B$$

is a compact linear operator from Y into X_B.

Since the normed space X_B has the CEP, we can derive the existence of $\overline{T_B} \in C(Z, X_B)$ that extends T_B. Set $\overline{T} \;:=\; i_B \circ \overline{T_B}$, where i_B is the continuous canonical inclusion from X_B into X which is continuous.

Then we have that $\overline{T}$ is an extension of T because for any $y \in Y$, we have:

$$\overline{T}(y) \;=\; i_B \circ \overline{T_B}(y) \;=\; i_B(\overline{T_B}(y)) \;=\; i_B(T_B(y)) \;=\; i_B(T(y)) \;=\; T(y),$$

also $\overline{T} \in C(Z,X)$ because composition of continuous and compact operators is compact.

□

Theorem 2.3.7. *Every (weakly) sequentially complete locally convex space X over $\mathbf{K}$ with an "orthogonal" basis has the CEP.*

Proof. Let $(x_n)_n$ be an orthogonal basis for X with associated coefficient functionals $f_n \in X'$. Let Y, Z be normed spaces with $Y \subset Z$ and $T \in C(Y,X)$. then $T(B_Y)$ is compactoid in X, so by (theorem 1.6.1.), there exist an $a \in X$ such that:

$$T(B_Y) \subset \tilde{a} = \{x \in X : |f_n(x)| \leq |f_n(a)| \forall\ n \in N\}.$$

For each $n \in N$, let $g_n = f_n \circ T$, then $g_n \in Y'$.

Now for all $y \in B_Y$, we have:

$g_n(y) = f_n \circ T(y) = f_n(T(y)) = f_n(x_y)$ for some $x_y \in \tilde{a}$, then

$|g_n(y)| = |f_n(x_y)| \leq |f_n(a)|$ for all $n \in N$, so if $\pi \in \mathbf{K}$ with $|\pi| > 1$, we have $|g_n(y)| < |\pi||f_n(a)|$ for all $y \in B_Y$, then

$$\|g_n\| = \sup_{\|y\| \leq 1} |g_n(y)| = \sup_{y \in B_Y} |g_n(y)| < |\pi||f_n(a)| = |f_n(\pi a)|,$$

also for all $y \in B_Y$, we have $T(y) \in X$ and $(x_n)_n$ is an orthogonal basis for X with coefficient functionals $f_n \in X'$, then $T(y)$ can be written uniquely as

$$T(y) \;=\; \sum_{n=1}^{\infty} f_n(T(y))(x_n) = \sum_{n=1}^{\infty}(f_n \circ T)(y)(x_n) = \sum_{n=1}^{\infty} g_n(y)(x_n)$$

Since $\mathbf{K}$ is spherically complete, then by (corollary 2.3.5.), every $g_n \in Y^{'}$ has an extension $h_n \in Z^{'}$ with $\|h_n\| = \|g_n\|$, then
$\overline{T} : Z \to X$ defined by $\overline{T}(z) = \sum_{n=1}^{\infty} h_n(z)(x_n)$ is an extension of T .

Now we have

$$\begin{aligned} \overline{T}(B_Z) &\subset \{x \in X : |f_n(x)| \leq |\pi||f_n(a)| \; for \; all \; n\} \\ &= \{x \in X : |f_n(x)| \leq |f_n(\pi a)| \; for \; all \; n\} \; = \; \tilde{\pi a} \end{aligned}$$

because if $m \in \overline{T}(B_Z)$, then $m = \sum_{n=1}^{\infty} h_n(z)(x_n)$ for some $z \in B_Z$, so

$f_n(m) = f_n(\sum_{n=1}^{\infty} h_n(z)(x_n)) = \sum_{n=1}^{\infty} h_n(z) f_n(x_n) = \sum_{n=1}^{\infty} h_n(z)$, then

$|f_n(m)| = |\sum_{n=1}^{\infty} h_n(z)| \leq \sup_n |h_n(z)| \leq \|h_n\| = \|g_n\| < \pi |f_n(a)|$,

then $m \in \tilde{\pi a}$, so $T(B_Z) \subset \tilde{\pi a}$ and $\pi a \in X$, then by (theorem 1.6.1.), we have $\overline{T}(B_Z)$ is compactoid in X , therefore $\overline{T}$ is compact.

□

Lemma 2.3.8. *Every cauchy sequence in X is contained in a bounded and closed subset .*

Proof. Let $B = \{x_n : n \in \mathbf{N}\}$, where $(x_n)_n$ is a cauchy sequence in X, then $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbf{N}$ s.t. $m, n \geq N(\epsilon)$ implies

$$|q(x_n) - q(x_m)| \; \leq q(x_n - x_m) \; < \epsilon$$

for any continuous seminorm which mean that $(q(x_n))_n$ is a cauchy sequence in $\mathbb{R}$ and so is bounded. Then by (proposition 1.5.2.) B is bounded and it's closure

$\overline{B}$ is bounded and closed .

□

Proposition 2.3.9. *If X is quasi-complete, then we have every cauchy sequence in X is convergent .*

Proof. Let $B = \{x_n : n \in \mathbf{N}\}$, where $(x_n)_n$ is a cauchy sequence in X, then by (Lemma 2.3.8.) $\overline{B}$ is bounded and closed, then by quasi-completeness of X, we have $\overline{B}$ is complete, then (x_n) has a limit in $\overline{B}$ and so in X, therefore every cauchy sequence is convergent i.e X is sequentially complete .

□

Lemma 2.3.10. *If (x_i) is a schauder basis for X with corresponding coefficient functionals (f_i), then (f_i) is a $\sigma(X', X)$ schauder basis for X' .*

Proof. Since (x_i) is a schauder basis for X with corresponding coefficient functionals (f_i), then every $x \in X$ can be written uniquely as:

$$x = \sum_{i=1}^{\infty} \lambda_i x_i = \sum_{i=1}^{\infty} f_i(x) x_i$$

Now if $a \in X'$, then for all $x \in X$, we have:

$$\begin{aligned} \langle x, a \rangle &= a(x) = a(\sum_{i=1}^{\infty} f_i(x) x_i) \\ &= \sum_{i=1}^{\infty} f_i(x) a(x_i) = \sum_{i=1}^{\infty} f_i(x) \langle x_i, a \rangle \\ &= \langle x, \sum_{i=1}^{\infty} \langle x_i, a \rangle f_i \rangle \qquad \forall x \in X. \end{aligned}$$

So by (Remark 1.7.2.) we have $a = \sum_{i=1}^{\infty} \langle x_i, a\rangle f_i$.

Now we want to show that this representation is unique, so let

$$\begin{aligned} a &= \sum_{i=1}^{\infty} \langle x_i, a\rangle f_i = \sum_{i=1}^{\infty} \langle y_i, a\rangle f_i \quad \forall a \in X' \\ &\Rightarrow \sum_{i=1}^{\infty} \langle x_i - y_i, a\rangle f_i = 0 \quad \forall a \in X' \\ &\Rightarrow \langle x_i - y_i, a\rangle = 0 \quad \forall a \in X' \ and \ \forall i \in N \\ &\Rightarrow x_i - y_i = 0 \ and\ therefore\ x_i = y_i \ \forall i \in N \end{aligned}$$

Therefore the representation is unique .

Now we want to show that the coefficient functionals

$\phi_i : X' \rightarrow \mathbf{K}$, defined by $\phi_i(a) = \langle x_i, a\rangle$ are $\sigma(X', X)$ continuous,

but $|\phi_i(a)| = |\langle x_i, a\rangle| = |a(x_i)| = p_a(x_i)$ which by (definition 1.7.3.) is $\sigma(X', X)$-continuous seminorm , then ϕ_i is $\sigma(X', X)$-continuous $\forall i \in N$.

□

Definition 2.3.1. Let (x_i) be a schauder basis for X with corresponding coefficient functionals (f_i). Define the following sequence spaces Λ and Δ.

$$\Lambda := \{(\alpha_i) : \sum_{i=1}^{\infty} \alpha_i x_i \ converges\ inX\}$$

$$\Delta := \{(\lambda_i) : \sum_{i=1}^{\infty} \lambda_i f_i \ converges\ in\ X_\sigma{}'\}$$

where $X_\sigma{}'$ denotes the space X' with the $\sigma(X', X)$ topology.

Proposition 2.3.11. *:*

(i) There are algebraic isomorphism $\Lambda \cong X$ *and* $\Delta \cong X'$.

(ii) $\Delta \subset \Lambda^*$.

(iii) If ${X_\sigma}'$ *is sequentially complete, then* $\Delta = \Lambda^*$.

(iv) If X_σ *and* ${X_\sigma}'$ *are sequentially complete, then* Λ *is a perfect sequence space.*

Proof. (i) Since (x_i) is a schauder basis for X with corresponding coefficient functionals (f_i), then every $x \in X$ can be written uniquely as:
$x = \sum_{i=1}^{\infty} \alpha_i x_i$, since $x \in X$ then $\sum_{i=1}^{\infty} \alpha_i x_i$ is converges in X, so $(\alpha_i) \in \Lambda$.

Define $f : X \to \Lambda$ by $f(x) = (\alpha_i)$, we want to see that f is $1-1$, onto and linear.
Since x can be written uniquely as $x = \sum_{i=1}^{\infty} \alpha_i x_i$, then f is $1-1$.
Now let $(\alpha_i) \in \Lambda$ then $\alpha_i \in \mathbf{K}$ for all $i \in N$, then $\sum_{i=1}^{\infty} \alpha_i x_i$ converges and belongs to X and $f(\sum_{i=1}^{\infty} \alpha_i x_i) = (\alpha_i)$, then f is onto.
Also for all $x_1, x_2 \in X$ and $\beta_1, \beta_2 \in \mathbf{K}$, we have:

$$\begin{aligned} f(\beta_1 x_1 + \beta_2 x_2) &= f(\beta_1 \sum_{i=1}^{\infty} {\alpha_i}^1 x_i + \beta_2 \sum_{i=1}^{\infty} {\alpha_i}^2 x_i) \\ &= f(\sum_{i=1}^{\infty} (\beta_1 {\alpha_i}^1 + \beta_2 {\alpha_i}^2) x_i) \\ &= (\beta_1 {\alpha_i}^1 + \beta_2 {\alpha_i}^2) \\ &= \beta_1 f(x_1) + \beta_2 f(x_2). \end{aligned}$$

then f is $1-1$, onto and linear, therefore $X \cong \Lambda$.

Now since (x_i) is a schauder basis for X, then by (Lemma 2.3.10.) (f_i) is $\sigma(X', X)$ schauder basis for X' and by the same way we have $X' \cong \Delta$.

$\square$

Proof. (ii) Let $(\lambda_i) \in \Delta$, we want to show that $(\lambda_i) \in \Lambda^*$

$$i.e. \quad \lim_i \lambda_i \alpha_i = 0 \qquad \forall (\alpha_i) \in \Lambda \text{ , } so$$

Let $(\alpha_i) \in \Lambda$, if $a \in X'$ with $a = \sum_{i=1}^{\infty} \lambda_i f_i$, then $\lambda_i = \langle x_i, a \rangle$ for $i = 1, 2, 3, ...$

If $x = \sum_{i=1}^{\infty} \alpha_i x_i$, then

$$\langle x, a \rangle = \langle \sum_{i=1}^{\infty} \alpha_i x_i, a \rangle = \sum_{i=1}^{\infty} \alpha_i \langle x_i, a \rangle = \sum_{i=1}^{\infty} \alpha_i \lambda_i$$

so $\sum_{i=1}^{\infty} \alpha_i \lambda_i$ is a converges series in $\mathbf{K}$, then by (proposition 1.2.1.[ii]) $\lim_i \lambda_i \alpha_i = 0$. Therefore

$$(\lambda_i) \in \Lambda^* \qquad i.e. \ \Delta \subset \Lambda^*.$$

$\square$

Proof. (iii) Suppose that ${X_\sigma}'$ is sequentially complete and let $(\beta_i) \in \Lambda^*$, we want to see that $(\beta_i) \in \Delta$.

If $x \in X$, then $x = \sum_{i=1}^{\infty} \alpha_i x_i$ with $(\alpha_i) \in \Lambda$ and

$$\begin{aligned} \langle x, \beta_i f_i \rangle &= \langle \sum_{j=1}^{\infty} \alpha_j x_j, \beta_i f_i \rangle \\ &= \beta_i f_i (\sum_{j=1}^{\infty} \alpha_j x_j) \\ &= \beta_i (f_i (\sum_{j=1}^{\infty} \alpha_j x_j)) \\ &= \beta_i \alpha_i, \end{aligned}$$

then $\langle x, \lim_i \beta_i f_i \rangle = \lim_i \beta_i \alpha_i = 0$ (because $(\alpha_i) \in \Lambda$ and $(\beta_i) \in \Lambda^*$) for all $x \in X$, then by (Definition 1.7.2.(S2)), we have that $\lim_i \beta_i f_i = 0$ in $X_\sigma{}'$, so (by proposition 1.2.1.[ii]), $\sum_{i=1}^{\infty} \beta_i f_i$ converges in $X_\sigma{}'$ which mean that $(\beta_i) \in \Delta$ *i.e.* $\Lambda^* \subset \Delta$, but by (ii) we have $\Delta \subset \Lambda^*$, therefore $\Delta = \Lambda^*$.

□

Proof. (iv) Suppose that X_σ and $X_\sigma{}'$ are sequentially complete, since $\Lambda \cong X$ and $X' \cong \Delta = \Lambda^*$, so we can assume that $\Lambda \cong X$ and $X' = \Lambda^*$, then (X, X') and (Λ, Λ^*) corresponds isomorphically. Since X_σ is weakly sequentially complete, then Λ_σ is sequentially complete. So by (Theorem 1.7.1.) Λ is perfect space.

□

Definition 2.3.2. If (E, E') is a dual pair, any convex topology on E which the dual is E' is called a topology of the dual pair (E, E').

Proposition 2.3.12. *If (E, E') is a dual pair and A is a convex subset of E, then the closure $\overline{A}$ is the same for every topology of the dual pair (E, E').*

Proof. See [14], p. 34.

□

Corollary 2.3.13. *If $(E, E^{'})$ is a dual pair, then the closed absolutely convex sets are the same for all topologies of the dual pair $(E, E^{'})$.*

Proof. If τ is any topology of the dual pair $(E, E^{'})$, the τ-closed absolutely convex are certainly $\sigma(E, E^{'})$-closed absolutely convex.
conversely let A be a $\sigma(E, E^{'})$-closed absolutely convex set. then by (Proposition 2.3.12.), $\overline{A}$ is the same for every topology of the dual pair $(E, E^{'})$, but A is closed, then $A = \overline{A}$ is closed absolutely convex set with respect to τ.

□

Proposition 2.3.14. *Let E be a locally* **K***-convex space with dual $E^{'}$. The* **K** *c-compact subsets of E are the same in every topology of the dual pair $(E, E^{'})$.*

Proof. Let τ_1, τ_2 be any topologies of the dual pair $(E, E^{'})$, A is a non-empty c-compact subset with respect to τ_1 if and only if for every collection $\mathcal{C}$ of τ_1-closed convex subsets of A with the finite intersection property, we have $\bigcap \mathcal{C} \neq \emptyset$ if and only if by (Corollary 2.3.13.), for every collection $\mathcal{C}$ of τ_2-closed convex subsets of A with the finite intersection property, we have $\bigcap \mathcal{C} \neq \emptyset$ if and only if A is c-compact subset with respect to τ_2.

□

Proposition 2.3.15. *Let X be a locally convex space with a schauder basis, If X is quasi-complete and $(X^{'}, \sigma(X^{'}, X))$ is sequentially complete, then X has the CEP.*

Proof. Since X is *quasi-complete*, then by (proposition 2.3.9.) X is sequentially complete. Since both X and $(X', \sigma(X', X))$ are sequentially complete, then by (proposition 2.3.11.[iv]), X can be algebraically identified with a perfect sequence space Λ while $X' = \Lambda^*$.

Now if τ_Λ denotes the topology on Λ associated with the original topology on X, then

$$\sigma(\Lambda, \Lambda^*) \;\; \textit{weaker than} \;\; \tau_\Lambda \;\; \textit{weaker than} \;\; n(\Lambda, \Lambda^*) \;\; \ldots\ldots\ldots\ldots\ldots\ldots \;\; (*)$$

proof of (*)

For the first inequality is true always, we want to prove the second inequality.

Now since **K** is spherically complete, then by (theorem 1.5.9.), we have both X and Λ are polar spaces, so let p be a polar τ_Λ-continuous seminorm on Λ and let $b = (b_n)_n \in \Lambda^*$ such that $|(b_n)_n| \leq p$

Since every $x = (x_n)_n \in \Lambda$ can be written uniquely as $x = \sum_n x_n e_n$ such that for each $n \in N$, e_n is the sequence with 1 in the n^{th} place and zero elsewhere, then

$$(b_n)_n (x_n)_n = (b_n x_n)_n = \sum_n (b_n x_n) e_n$$

then

$$\begin{aligned} |(b_n)_n (x_n)_n| &= |\sum_n (b_n x_n) e_n| \leq \sup_n |(b_n x_n) e_n| \\ &= \sup_n |(b_n x_n)||e_n| = \sup_n |(b_n x_n)| \\ &= p_b(x). \end{aligned}$$

then $|(b_n)_n(x)| \leq p_b(x)$, as soon as $|(b_n)_n(x)| \leq p(x)$. So by polarity of p, we have $p \leq p_b$, but p_b is is $n(\Lambda, \Lambda^*)$-continuous seminorm,then by (Remark 1.5.6.(ii)), p is $n(\Lambda,\Lambda^*)$-continuous seminorm, then by (Proposition 1.5.10.) τ_Λ weaker than $n(\Lambda,\Lambda^*)$.

Now it is enough to prove that if Y is a normed space over $\mathbf{K}$, and $T : Y \to \Lambda$ is a linear operator, then T is τ_Λ-compact if and only if T is $n(\Lambda,\Lambda^*)$-compact.
For: suppose that T is $n(\Lambda,\Lambda^*)$-compact and let B_Y be the closed unit ball in Y, then $T(B_Y)$ is $n(\Lambda,\Lambda^*)$-compactoid in Λ w.r.t. $n(\Lambda,\Lambda^*)$-topology.
Now let U be a τ_Λ-zero neighborhood in Λ, since $\tau_\Lambda \leq n(\Lambda,\Lambda^*)$, then U is a $n(\Lambda,\Lambda^*)$-zero neighborhood in Λ, since $T(B_Y)$ is a $n(\Lambda,\Lambda^*)$-compactoid in Λ and U is $n(\Lambda,\Lambda^*)$-zero neighborhood in Λ, then there exist a finite set F in Λ such that $T(B_Y) \subset U + coF$, so T is τ_Λ-compact.

To prove the converse, assume that T is τ_Λ-compact and let B_Y be the closed unit ball in Y, then $T(B_Y)$ is τ_Λ-compactoid and so by (Proposition 1.5.3.[i]) $\overline{T(B_Y)}^{\tau_\Lambda}$ is compactoid and by τ_Λ-*quasi-completeness* of Λ, $\overline{T(B_Y)}^{\tau_\Lambda}$ is complete.
Since $\mathbf{K}$ is spherically complete, $\overline{T(B_Y)}^{\tau_\Lambda}$ is compactoid and complete , then by (theorem 1.4.1.) $\overline{T(B_Y)}^{\tau_\Lambda}$ is c-compact.

Since τ_Λ and $n(\Lambda,\Lambda^*)$ have the same topological dual, then by (Proposition 2.3.14.), $\overline{T(B_Y)}^{\tau_\Lambda}$ is c-compact in $(\Lambda, n(\Lambda,\Lambda^*))$ and bounded, so by (theorem 1.4.1.), $\overline{T(B_Y)}^{\tau_\Lambda}$ is compactoid in $(\Lambda, n(\Lambda,\Lambda^*))$.
Since $\tau_\Lambda \leq n(\Lambda,\Lambda^*)$, then $\overline{T(B_Y)}^{n(\Lambda,\Lambda^*)} \subset \overline{T(B_Y)}^{\tau_\Lambda}$ and $\overline{T(B_Y)}^{\tau_\Lambda}$ is compactoid, then by (proposition 1.5.3.(ii)), $\overline{T(B_Y)}^{n(\Lambda,\Lambda^*)}$ is compactoid in $(\Lambda, n(\Lambda,\Lambda^*))$, since $T(B_Y) \subset \overline{T(B_Y)}^{n(\Lambda,\Lambda^*)}$, then by (proposition 1.5.3.[ii]), $T(B_Y)$ is compactoid in

$(\Lambda, n(\Lambda, \Lambda^*))$, therefore T is $n(\Lambda, \Lambda^*)$-compact.

For its enough : Since both X and $(X', \sigma(X', X))$ are sequentially complete, then by (proposition 2.3.11.[iii]), Λ is perfect sequence space (by theorem 1.7.1.), if and only if Λ is sequentially complete. So Λ is sequentially complete and $\{e_i\}_{i \in I}$ is an orthogonal basis for Λ, then (by theorem 2.3.7.), $(\Lambda, n(\Lambda, \Lambda^*))$ has the CEP, then (Λ, τ_Λ) has the CEP because by (*) τ_Λ is weaker than $n(\Lambda, \Lambda^*)$. Since τ_Λ denotes the topology on Λ associated with the original topology on X, then X has the CEP.

□

Lemma 2.3.16. *Let $T \in L(Y, X)$, then $T_p : Y_p \rightarrow X$ defined by $T_p(\pi_p(y)) = T(y)$ is an element in $L(Y_p, X)$.*

Proof. Since $T \in L(Y, X)$, then there exists a zero neighborhood $\mathcal{V}$ in Y such that $T(\mathcal{V})$ is bounded in X , since $\mathcal{V}$ is a zero neighborhood in Y , then $\pi_p(\mathcal{V}) = \mathcal{V} + ker\ p$ is a zero neighborhood in Y_p , then $T_p(\pi_p(\mathcal{V})) = T(\mathcal{V})$ is bounded in X, therefore $T_p \in L(Y_p, X)$.

□

Lemma 2.3.17. *If $T \in L(Y, X)$, then there exist $p \in \mathcal{P}_Y$ and $T_p \in L(Y_p, X)$ such that $T = T_p \circ \pi_p$.*

Proof. Since $T \in L(Y, X)$, then there exists a zero neighborhood $\mathcal{U}$ in Y such that $T(\mathcal{U})$ is bounded in X , let $p \in \mathcal{P}_Y$ be the minkowski functional of $\mathcal{U}$ then by (Lemma 2.3.16.) $T_p : Y_p \to X$ defined by $T_p(\pi_p(y)) = T(y)$ is an element in $L(Y_p, X)$ and $T = T_p \circ \pi_p$.

□

The proof of the next proposition is similar to the proof of proposition 2.1.3. page (27).

Proposition 2.3.18. *Let X be a normed space. If, for every pair of normed spaces E , F with $E \subset F$ and every $T \in L(E, X)$, there exist $\overline{T} \in L(F, X)$ that extends T , then for every pair of locally convex spaces Y , Z with $Y \subset Z$ and every $S \in L(Y, X)$, there exists $\overline{S} \in L(Z, X)$ that extends S .*

Proof. Assume that for every pair of normed spaces E , F with $E \subset F$ and for every $T \in L(E, X)$, there exists $\overline{T} \in L(F, X)$ that extends T................(*)

Now let $S \in L(Y, X)$, then by (Lemma 2.3.17.) , there exist $p \in \mathcal{P}_Y$ and $S_p \in L(Y_p, X)$ such that $S = S_p \circ \pi_p$.

Since Z is a locally convex space, then there is a collection of continuous seminorms $\mathcal{P}_Z$ define the topology on Z, then the restrictions to Y of the continuous seminorms on Z define the topology on Y as a subspace of Z, then we can assume that p is the restriction to $Y, q|_Y$ for some $q \in \mathcal{P}_Z$

Define $i : Y_p \to Z_q$ by $\pi_p(y) \mapsto \pi_q(y)$, $y \in Y$,

then i is a linear isometry from Y_p to Z_q. For that we have to prove that i is linear and isometry.

[1] proof of linearity : Let x, y be elements in Y and $\lambda \in \mathbf{K}$, then

$$\begin{aligned} i(\pi_p(x) + \lambda\pi_p(y)) &= i(\pi_p(x + \lambda y)) = \pi_q(x + \lambda y)) \\ &= \pi_q(x) + \lambda\pi_q(y) = i(\pi_q(x)) + \lambda i(\pi_q(y)) \end{aligned}$$

[2] proof of isometry :

$$\begin{aligned} d(i(\pi_p(x)), i(\pi_p(y))) &= d(\pi_q(x), \pi_q(y)) = q(x - y) \\ &= p(x - y) = d(\pi_p(x), \pi_p(y)) \end{aligned}$$

Now, by (definition 1.5.4.), we have that both Z_q and Y_p are normed spaces. Set $\overline{Y_p} = i(Y_p)$, then $\overline{Y_p}$ is a normed subspace of Z_q.
Define $\bar{i} : Y_p \to \overline{Y_p}$ to be the restriction of the map i to it's image , then $\bar{i}$ is 1-1 and onto which mean that Y_p is isometric to $\overline{Y_p}$, then $\bar{i}^{-1}$ exist and continuous, so if we define $\overline{S_p} := S_p \circ \bar{i}^{-1}$, then $\overline{S_p} \in L(\overline{Y_p}, X)$ because composition of continuous operators is continuous operator, but $\overline{Y_p}$ is a normed subspace of Z_q, then by (*), there exist $\overline{\overline{S_p}} \in L(Z_p, X)$ that extends $\overline{S_p}$.

Define $\overline{S} = \overline{\overline{S_p}} \circ \pi_q$, then $\overline{S} \in L(Z, X)$ because composition of continuous operators is a continuous operator, also $\overline{S}$ is an extension of S because for all $y \in Y$, we have

$$\begin{aligned} \overline{S}(y) &= \overline{\overline{S_P}} \circ \pi_q(y) = \overline{\overline{S_P}}(\pi_q(y)) \\ &= \overline{\overline{S_P}}(i\pi_P(y)) = \overline{\overline{S_P}}(\bar{i}\pi_P(y)) \qquad \textit{since } y \in Y \\ &= \overline{S_P}(\bar{i}\pi_P(y)) = S_P \circ \bar{i}^{-1}(\bar{i}\pi_P(y)) \\ &= S_P \circ \pi_P(y) = S(y) \end{aligned}$$

□

Theorem 2.3.19. *The following conditions on a normed vector space F are equivalent .*

(1) F is weakly injective .

(2) F is linearly homeomorphic to a spherically complete Banach space .

(3) There exists a t $\in (0,1]$ such that F is $t-injective$.

(4) If E is a normed vector space that contains F, then F is complemented in E.

Proof. See [22], p. 106.

□

Lemma 2.3.20. *If $\{X_i\}_{i\in I}$ is a collection of locally convex spaces, then $X = \prod_{i\in I} X_i$ endowed with the product topology is weakly injective if and only if X_i is weakly injective for all $i \in I$.*

Proof. [$\Leftarrow$] Suppose that $\{X_i\}_{i\in I}$ is a family of locally convex spaces such that X_i is weakly injective for all $i \in I$, let Y , Z be locally convex spaces such that $Y \subset Z$ and let $T \in L(Y, \prod_{i\in I} X_i)$.
Let $\pi_i : \prod_{i\in I} X_i \rightarrow X_i$ be the i^{th} projection map which is continuous and open for all $i \in I$, then $\pi_i \circ T : Y \rightarrow X_i$ is continuous because composition of tow continuous maps is continuous . Since X_i is weakly injective for all $i \in I$, then by

assumption there exist $\overline{\pi_i \circ T} \in L(Z, X_i)$ that extend $\pi_i \circ T$, then ${\pi_i}^{-1} \circ \overline{\pi_i \circ T} \in L(Z, \prod_{i \in I} X_i)$, also ${\pi_i}^{-1} \circ \overline{\pi_i \circ T}$ is extension of T because for all $y \in Y$, we have,

$$ {\pi_i}^{-1} \circ \overline{\pi_i \circ T}(y) = {\pi_i}^{-1}(\overline{\pi_i \circ T}(y)) = {\pi_i}^{-1}(\pi_i \circ T(y)) = T(y). $$

[⇒] Suppose that $X = \prod_{i \in I} X_i$ is weakly injective, let Y, Z be locally convex spaces with $Y \subset Z$ and let $T_i \in L(Y, X_i)$.

Let $T : Y \to \prod_{i \in I} X_i$ defined by $y \to (T_i(y))_{i \in I}$, then by definition of product topology $T \in L(Y, \prod_{i \in I} X_i)$.

Since $\prod_{i \in I} X_i$ is weakly injective, then there exists $\overline{T} \in L(Z, \prod_{i \in I} X_i)$ that extends T.

Let $\overline{T_i} = \pi_i \circ \overline{T}$, where π_i is the i^{th} projection map which is continuous, then $\overline{T_i} \in L(Z, X_i)$, also $\overline{T_i}$ is an extension of T_i because for all $y \in Y$, we have:

$$ \overline{T_i}(y) = \pi_i \circ \overline{T}(y) = \pi_i(\overline{T}(y)) = \pi_i(T(y)) = \pi_i(T_i(y))_{i \in I} = T_i(y) $$

□

Proposition 2.3.21. *X is weakly injective if and only if for every locally convex space Y that contains a copy X_1 of X, X_1 is complemented in Y . In particular, every weakly injective space is complete.*

Proof. [⇐] Suppose that for every locally convex space Y that contains a copy X_1 of X, X_1 is complemented in Y.

Since X is Hausdorff, then by (Corollary 2.2.6.) X is linearly homeomorphic to a subspace X_1 of the locally convex space $Y := \prod_{p \in P_X} \breve{X}_p$, where foe each $p \in \mathcal{P}_X$,

$\breve{X}_p$ is the spherical completion of X_p, then by assumption X_1 is complemented in Y .

Since $\breve{X}_p$ is a spherically complete Banach space, then by (Theorem 2.3.19.[1] $\Leftrightarrow$ [2]),

$\breve{X}_p$ is weakly injective for all $p \in \mathcal{P}_X$............ (1)

By (1) and (Lemma 2.3.20.), we have $Y := \prod_{p \in P_X} \breve{X}_p$ is also weakly injective .

Since X_1 is complemented in Y, then by (Theorem 2.3.19.[4] $\Leftrightarrow$ [1]) X_1 is weakly injective.

Since X_1 is a copy of X , then X_1 is linearly homeomorphic to X, and so there exist a $g : X_1 \to X$ such that g is 1-1 , onto and both g and g^{-1} are continuous. Let E, F be normed spaces such that $E \subset F$ and let $T \in L(E, X)$, then

$g^{-1} \circ T \in L(E, X_1)$ because composition of two continuous maps is continuous map, but X_1 is weakly injective , then there exist $\overline{g^{-1} \circ T} \in L(F, X_1)$ that extends $g^{-1} \circ T$, then $g \circ \overline{g^{-1} \circ T} \in L(F, X)$ is extension of T, therefore X is weakly injective.

Conversely, suppose that X is weakly injective and let Y be a locally convex space that contains a copy X_1 of X. Since X is weakly injective, then as the first side proof of this proposition, we have X_1 is weakly injective.

Since $I_{dn} \in L(X_1, X_1)$ and X_1 is weakly injective, then there exist $\overline{I_{dn}} \in L(Y, X_1)$ that extends I_{dn}, then $\overline{I_{dn}}$ is a projection from Y onto X_1 because :

(1) $\overline{I_{dn}}$ is continuous .

(2) $(\overline{I_{dn}})^2(y) = \overline{I_{dn}}(\overline{I_{dn}}(y)) = \overline{I_{dn}}(y)$

From (1) and (2) and (Definition 1.5.7.), we have that $\overline{I_{dn}}$ is a projection from Y onto X_1 , therefore X_1 is complemented in Y .

Then if X is weakly injective, then we can obtain that X is complemented in its completion, hence by (Remark 1.5.4.) it is closed in its completion, then X is complete which prove " In Particular "

□

Corollary 2.3.22. *If* $\mathbf{K}$ *is spherically complete, the canonical map* $J_X : X \to (X^{'})^*$ *is injective.*

Proof. Since $\mathbf{K}$ is spherically complete, then by (theorem 1.4.1.) X is polar space and X is Hausdorff.

Now let $x \in X, x \neq 0$, then there exist a polar continuous seminorm p with $p(x) \neq 0$, since X is polar, then we have

$$p = \sup\{|f| : f \in X^*, |f| \leq p\},$$

so there exist an $f \in X^*$ such that $J_X(x) = f(x) \neq 0$, therefore J_X is injective.

□

Theorem 2.3.23. *If* $\langle X, X^{'} \rangle$ *is a dual pair, then the completion of* $(X, \sigma(X, X^{'}))$ *is* $(X^{'*}, \sigma(X^{'*}, X^{'}))$*, and it is topologically isomorphic to* $\mathbf{K}^d$*, where* d *is the algebraic dimension of* $X^{'}$*.*

Proof. See [11], p. 248.

□

Lemma 2.3.24. *If* $f_0, f_1, ..., f_n$ *are linear forms on the vector space* E*, then either* f_0 *is a linear combination of* $f_1, f_2, ..., f_n$ *or there is an element* a *of* E *with* $f_0(a) = 1$ *and* $f_1(a) = f_2(a) = ... = f_n(a) = 0$*.*

Proof. See [14], p. 33.

□

Proposition 2.3.25. *If $\langle X, X^{'}\rangle$ is a dual pair, then the dual of E under $\sigma(E, E^{'})$ is $E^{'}$.*

Proof. By definition of weak topology, we have that $E^{'} \subset (E, \sigma(E, E^{'}))^{'}$. We want to prove the converse,

so let f be any $\sigma(E, E^{'})$-continuous linear form, then $|f(x)| \leq \alpha < 1$ on some neighborhood of the form

$$U = \{x : \sup_{1\leq i\leq n} |\langle x, {x_i}^{'}\rangle| \leq 1\} \qquad ({x_i}^{'} \in E^{'}).$$

By (Lemma 2.3.24.) either f is a linear combination of ${x_1}^{'}, {x_2}^{'}, ..., {x_n}^{'}$ or there is some $a \in E$ with $f(a) = 1$ and $\langle a, {x_i}^{'}\rangle = 0$ for $1 \leq i \leq n$, but the last imply $a \in U$ and $1 = |f(a)| > \alpha$ which is contradiction with $|f(x)| \leq \alpha$, then $f = \sum_{i=1}^{n} \lambda_i {x_i}^{'}$, then f is an element in $E^{'}$, so $(E, \sigma(E, E^{'}))^{'} \subset E^{'}$, therefore $E^{'} = (E, \sigma(E, E^{'}))^{'}$

□

Definition 2.3.3. Let E be a polar Hausdorff locally convex space over $\mathbf{K}$ with topology τ_0 , $E^{'} = (E, \tau_0)^{'}$.

A polar topology ν on E is τ_0-compatible if $(E, \nu)^{'} = E^{'}$.

Corollary 2.3.26. *If $\mathbf{K}$ is spherically complete, then $\sigma(E, E^{'})$ is τ_0-compatible.*

Proof. Since $\mathbf{K}$ is spherically complete, then by (Theorem 1.5.9.), (E, τ_0) is polar Hausdorff locally convex space, so by (Proposition 2.3.25.) the dual of E under

$\sigma(E,E^{'})$ is $E^{'} = (E,\tau_0)^{'}$, therefore $\sigma(E,E^{'})$ is τ_0-compatible.

□

Theorem 2.3.27. *All τ_0-compatible topologies on E have the same bounded sets.*

Proof. See [14], p. 67.

□

Proposition 2.3.28. *$(X,\sigma(X,X^{'}))$ is complete if and only if $X = (X^{'})^*$. In particular if X is a normed space, then $(X,\sigma(X,X^{'}))$ is weakly injective if and only if $\dim X < \infty$.*

Proof. Since **K** is spherically complete, then by (Corollary 2.3.20.), the canonical map $J_X : X \to (X^{'})^*$ is injective . In this case we can assume that X is a subspace of $J_X(X)$.
Now apply (Theorem 2.3.23.), we have the completion of $(X,\sigma(X,X^{'}))$ is $((X^{'})^*,\sigma((X^{'})^*,X^{'})$ which mean that X is complete if and only if $X = (X^{'})^*$.
Suppose that $dimX = n < \infty$, then $(X,\sigma(X,X^{'}))$ is linearly homeomorphic to $\mathbf{K}^n$, since **K** is spherically complete, then by (theorem 2.2.10. [1 $\Leftrightarrow$ 4]), **K** is weakly injective if and only if by (Lemma 2.3.20.) $\mathbf{K}^n$ is weakly injective, then $(X,\sigma(X,X^{'}))$ is weakly injective which is the proof of "if" of " In Particular ".
For the "only if", note that if $(X,\sigma(X',X))$ is weakly injective then (by proposition 2.3.21), $(X,\sigma(X',X))$ is complete, so (by theorem 2.3.23.) $\dim X < \infty$.

□

Corollary 2.3.29. *Let G be an infinite dimensional weakly injective normed space over* **K** *(e.g., take $G = l^\infty$). Then, $X := (G,\sigma(G,G^{'}))$ is a nonweakly* *injective locally convex space satisfying that for every pair of normed spaces E, F with $E \subset F$, every $T \in L(E, X)$ has an extension $\overline{T} \in L(F, X)$.*

Proof. Since $\dim G = \infty$, then by (Proposition 2.3.28.) $X := (G, \sigma(G, G'))$ is not weakly injective .

For the rest of proof :

Since **K** is spherically complete, then by (Corollary 2.3.26.) $\sigma(E, E')$ is τ_0-compatible, then by (Theorem 2.3.27.) G and $X := (G, \sigma(G, G'))$ have the same bounded subsets.

This mean that $L(E, X) = L(E, G)$ and $L(F, X) = L(F, G)$ for all normed spaces E, F with $E \subset F$.

Now let $T \in L(E, X) = L(E, G)$, but G is weakly injective, then T has an extension $\overline{T} \in L(F, G) = L(F, X)$ □

Theorem 2.3.30. *Every weakly injective locally convex space X over* **K** *has the CEP.*

Proof. Since X is Hausdorff, then by (corollary 2.2.5.), X is linearly homeomorphic to a subspace X_1 of the product space $E := \prod_{p \in \mathcal{P}_X} X_p$. But for each $p \in \mathcal{P}_X$, we have X_p is a metrizable locally convex space over **K** and **K** is spherically complete, then by (theorem 2.3.6.) X_p has the CEP for every $p \in \mathcal{P}_X$, so by (proposition 2.1.4. [iii]) $E := \prod_{p \in \mathcal{P}_X} X_p$ has the CEP. Now since X is weakly injective and E is a locally convex space contains a copy X_1 of X, then by (proposition 2.3.19.) X_1 is complemented subspace in E, then by (proposition 2.1.4. [i]) X_1 has the CEP, therefore X has the CEP.

□

Definition 2.3.4. A (non-archimedean) seminorm p on a vector space E is called of finite type if $E_p := E/kerp$ is finite dimensional. A locally convex space E is called of finite type if each continuous seminorm is of finite type.

A subset X of a locally convex space E is a local compactoid in E if for every zero neighborhood U in E there exists a finite-dimensional subspace D of E such that $X \subset U + D$.

Theorem 2.3.31. *If $T : X \to Y$ is a linear map, then*

(i) $\ker T := \{x \in X : Tx = 0\}$ *is a subspace of X.*

(ii) $imT := \{Tx : x \in X\}$ *is a subspace of Y.*

(iii) $X/\ker T$ *is isomporphic to* imT *by the linear*

$$\tilde{T} : X/\ker T \to imT; \qquad [x] \mapsto Tx$$

Proof. (iii)

(1) We need to check that $\tilde{T}$ is well-defined.

$$\begin{aligned} [x] = [x^{'}] &\Rightarrow x \sim x^{'} \Rightarrow x - x^{'} \in \ker T \\ &\Rightarrow T(x - x^{'}) = 0 \Rightarrow Tx = Tx^{'} \end{aligned}$$

so $\tilde{T}$ is well defined.

(2) $\tilde{T}$ is liner: Let $[x], [y] \in X/\ker T, \quad \lambda, \mu \in \mathbf{K}$, then

$$\tilde{T}(\lambda[x] + \mu[y]) = \tilde{T}[\lambda x + \mu y] = T(\lambda x + \mu y) = \lambda Tx + \mu Ty = \lambda\tilde{T}[x] + \mu\tilde{T}[y].$$

(3) $\tilde{T}$ is one-to-one,

$$\begin{aligned}\tilde{T}[x] = \tilde{T}[y] &\Rightarrow Tx = Ty \Rightarrow T(x-y) = 0\\ &\Rightarrow x - y \in \ker T \Rightarrow x \sim y\\ &\Rightarrow [x] = [y]\end{aligned}$$

(4) $\tilde{T}$ is onto : Let $y \in imT$, then there exist $x \in X$ such that $T(x) = y$, then $\tilde{T}[x] = y$.

□

Definition 2.3.5. A closed subspace M of the normed space X is complemented in X if there exists a closed subspace N such that $M \oplus N = X$, i.e., $M + N = X$ and $M \bigcap N = \emptyset$.

Remark 2.3.1. :

(1) M is complemented in X if M is finite dimensional.

(2) M is complemented in X if M has finite codimension, i.e., $\dim X/M < \infty$.

Proof. See [25], p. 83. □

Remark 2.3.2. Let X ba an infinite-dimensional normed space and let $V \subset X$ be a weakly open set containing the origin, then V contains a closed subspace of finite codimension in X.

Proof. Since V is a weakly open set containing the origin, then there exist an $\epsilon > 0$ and $f_1, ..., f_n \in X'$ such that $\bigcap_{i=1}^{n}\{x \in X : |f_i(x)| \leq \epsilon\} \subset V$. Define

$$\Phi : X \to \mathbf{K}^n \ ; \quad x \mapsto (f_1(x), ..., f_n(x)),$$

then Φ is well defined because if $x = y$, then $f_i(x) = f_i(y)$ for every $i = 1, ..., n$, then $(f_1(x), ..., f_n(x)) = (f_1 y), ..., f_n(y))$, so $\Phi(x) = \Phi(y)$, also Φ is linear because for all x, y in X and $\lambda \in \mathbf{K}$, we have

$$\Phi(x+\lambda y) = (f_1(x+\lambda y), ..., f_n(x+\lambda y)) = (f_1(x), ..., f_n(x))+\lambda(f_1(y), ..., f_n(y))$$
$$= \Phi(x) + \lambda\Phi(y), \text{ and}$$
$$\ker\Phi = \{x \in X : \Phi(x) = (f_1(x), ..., f_n(x)) = 0\} = \bigcap_{i=1}^{n} \ker f_i$$
$$\subset \bigcap_{i=1}^{n} \{x \in X : |f_i(x)| \leq \epsilon\} \subset V,$$

which is closed because $\ker f_i$ is closed for every $i = 1, ..., n$. By (Theorem 2.3.31.[iii]) $X/\ker\Phi \cong im\Phi \subset \mathbf{K}^n$, then $\ker\Phi$ has finite codimension.

□

Definition 2.3.6. We say that a seminorm p on a vector space E is of bounded range if there is $C \in \mathbb{R}^+$ such that $p(x) \leq C$ for all $x \in E$.

Remark 2.3.3. Let E be a locally convex space, if p is a seminorm on E of bounded range, then p is identically zero.

Proof. : Suppose $p(x) \neq 0$ for some $x \in E$. Then $x \neq 0$, so we can multiply $p(x)$ with $|a|$ where a in K. We can make $|a|$ as large as we wish, contradicting boundedness of range.

□

Theorem 2.3.32. *For a locally convex space $E = (E, \tau)$ the following are equivalent.*

(i) E is of finite type.

(ii) τ *is a weak topology.*

(iii) E *is a local compactoid in* E.

Proof. ($i \Rightarrow ii$) We want to prove that τ is equal the weak topology $\sigma = \sigma(E, E^{'})$. obviously, σ is weaker than τ. Conversely, let p be a τ-continuous seminorm on E, since E is of finite type, then E_p is finite dimensional, so by (Remark 1.5.1.) $\dim E_p = \dim E_p^*$, so let $\{g_1, ..., g_n\} \subset E_p^*$ be a basis for E_p^*. Define

$h : E_p \longrightarrow [0, \infty)$; $h(y) = \max |g_i(y)|$ for all $y \in E_p$, then h is a norm on E_p because of the following :

(1) $h(y) = max|g_i(y)| \geq 0$ for every $y \in E_p$.

(2) $h(\lambda y) = \max |g_i(\lambda y)| = |\lambda| \max |g_i(y)| = |\lambda| h(y)$.

(3) $h(x+y) = \max |g_i(x+y)| = \max |g_i(x)+g_i(y)| \leq \max\{\max |g_i(x)|, \max |g_i(y)\}$
$= \max\{h(x), h(y)\}$.

(4) Suppose that $h(x) = \max |g_i(x)| = 0$ for some $x \in E_p$, then $|g_i(x)| = 0$ for every $1 \leq i \leq n$, so $g_i(x) = 0$ for every $1 \leq i \leq n$. Since $\{g_1, ..., g_n\} \subset E_p^*$ is a basis for E_p^*, then $f(x) = 0$ for every $f \in E_p^*$, then $x = 0$.

Therefore h is a norm on E_p, also $\tilde{p}$ (see Definition 1.5.4.) defined by $\tilde{p}(\pi(x)) = p(x)$, is a norm on E_p, where π is the natural map from E into E_p, since E_p is a finite-dimensional space, then both h and $\tilde{p}$ are equivalent ....(*)

Define $f_i : E \rightarrow \mathbf{K}$; $f_i(x) = g_i(\pi(x))$ for every $x \in E$, then $\{f_1, ..., f_n\} \subset E^*$.

Define $q : E :\longrightarrow [0, \infty)$; $(x) \mapsto \max |f_i(x)|$.

Since $f_i \in E^*$, then $|f_i|$ is a seminorm, so $q = \max |f_i(x)|$ is a seminorm on E. Moreover q is equivalent to p. For that $q(x) = \max |f_i(x)| = \max |g_i(\pi(x))|$ which by (*) is equivalent norm to $\tilde{p}(\pi(x)) = p(x)$, then q is τ-continuous seminorm on E, but for every $1 \leq i \leq n \ \ |f_i| \leq q$, so $f_1, ..., f_n$ are τ-continuous, hence σ-continuous, then $q = max|f_n(x)|$ is σ-continuous seminorm , then p is σ-continuous seminorm. It follows that τ is weaker than σ.

$(ii \Rightarrow iii)$ Let U be a zero neighborhood in (E, τ). By (ii) there exist an $\epsilon > 0$ and $f_1, ..., f_n \in E^{'}$ such that $\bigcap_{i=1}^{n}\{x \in E : |f_i(x)| \leq \epsilon\} \subset U$, so by (Remark 2.3.2.) U contains $H := \bigcap_{i=1}^{n} \ker f_i$, a space of finite-codimension. Let D be a complement of H. Then by (Remark 2.3.1.) D is finite-dimensional and $E \subset H + D \subset U + D$.

$(iii \Rightarrow i)$ Let p be a τ-continuous seminorm on E, let D be a finite dimensional subspace of E such that $E \subset \{x \in E : p(x) < 1\} + D$, define

$$q : E \longrightarrow [0, \infty) \ ; \quad x \mapsto \inf_{d \in D} p(x - d),$$

then q is a seminorm. For that, since D is subspace, then $\lambda D = D$ for all $\lambda \neq 0$, so :

$$\begin{aligned} q(\lambda x) &= \inf_{d \in D} p(\lambda x - d) = \inf_{\lambda d^{'} \in D} p(\lambda x - \lambda d^{'}) \\ &= |\lambda| \inf_{d^{'} \in D} p(x - d^{'}) = |\lambda| q(x) \end{aligned}$$

$$\begin{aligned} q(x + y) &= \inf_{d \in D} p(x + y - d) = \inf_{d_1, d_2 \in D} p(x - d_1 + y - d_2) \\ &\leq \max\{\inf_{d_1 \in D} p(x - d_1), \inf_{d_2 \in D} p(x - d_2) = \max\{q(x), q(y)\}, \end{aligned}$$

also $q \leq p$, then q is smaller than p, since p is τ-continuous, then by (Remark 1.5.6.(ii)) q is τ-continuous.

Also q is of bounded range. For that, since $E \subset \{x \in E : p(x) < 1\} + D$, then any element y in E can be written as $y = x + d$, where $x \in \{x \in E : p(x) < 1\}$ and $d \in D$, so $q(y) = q(x + d) \leq \max\{q(x), q(d)\}$, but for every $d \in D$ we have $0 \leq q(d) = \inf_{h \in D} p(d - h) \leq p(d - d) = 0$, so $q(d) = 0$, then $q(y) \leq \max\{q(x), q(d)\} = q(x) = \inf_{d' \in D} p(x - d^{'})$, since $0 \in D$, then $p(x - 0) = p(x) < 1$, so $q(y) = \inf_{d' \in D} p(x - d^{'}) < 1$, then q is of bounded range, hence by (Remark 2.3.3.), q is identically zero.

Then $dist(x, D) = q(x) = \inf_{d \in D} p(x - d) = 0$ for all $x \in E$, then x in the closure of D for all $x \in E$, so E in the closure of D with respect to the topology induced by p. Since $\ker p$ is closed, then $D + \ker p = \bigcup_{d \in D} d + \ker p$ is closed that contains D, so $E \subset D + \ker p$, so that $E_p = E / \ker p$ is finite-dimensional.

□

Corollary 2.3.33. *Let E be a locally convex space. The weakly bounded and the weakly compactoid subsets of E coincide.*

Proof. Let X be a weakly bounded subset of E, let p be a weakly continuous seminorm, let $E_p = E/kerp$ with norm $\overline{p}$ and let $\pi_p : E \to E_p$ be the natural quotient map.

We have to find a finite set $G \subset E$ such that $X \subset \{x \in E : p(x) \leq 1\} + coG$.

Since π_p is continuous, then by (proposition 1.5.2.[ii]), The set $\pi_p(X)$ is bounded in the normed space E_p, which by (Theorem 2.3.32.) is finite-dimensional. Then (by proposition Proposition 1.5.3.[iv]), $\pi_p(X)$ is a compactoid set in E_p, so there is a finite set $G \subset E$ such that:

$$\begin{aligned} \pi_p(X) &\subset \{z \in E_p : \overline{p}(z) \le 1\} + co(\pi_p(G)) \\ &= \pi_p(\{x \in E : p(x) \le 1\}) + \pi_p(co(G)). \end{aligned}$$

Hence

$$\begin{aligned} X &\subset \{x \in E : p(x) \le 1\} + co(G) + Kerp \\ &\subset \{x \in E : p(x) \le 1\} + co(G). \end{aligned}$$

□

Proposition 2.3.34. *Let $X \neq \{0\}$ be a normed space over $\mathbf{K}$. Then $X_\sigma = (X, \sigma(X, X'))$ has the CEP if and only if X is weakly injective.*

Proof. [$\Leftarrow$] (1) Since $\mathbf{K}$ is spherically complete, then by (Theorem 2.3.27.) X and X_σ have the same bounded sets.

This mean that $L(E, X_\sigma) = L(E, X)$ and $L(F, X_\sigma) = L(F, X)$ for every locally convex spaces E, F.

So if $T \in L(E, X_\sigma)$, then $T \in L(E, X)$, but X is weakly injective, then T has an extension $\overline{T} \in L(F, X)$, but $L(F, X_\sigma) = L(F, X)$, so $\overline{T} \in L(F, X_\sigma)$, which means that X_σ is weakly injective (by proposition 2.3.28.) if and only if $\dim X < \infty$.

(2) Since X is finite dimensional, then by (proposition 1.5.3.[vi]), the compactoids of X are just the bounded sets.

Now let E, F be normed spaces with $E \subset F$ and let $S \in C(E, X_\sigma)$, then $S \in L(E, X_\sigma)$ but X_σ is weakly injective, so S has an extension $\overline{S} \in L(F, X_\sigma)$, then there exists a zero neighborhood U in F such that $\overline{S}(U)$ is bounded in X_σ, then by (1) $\overline{S}(U)$ is bounded in X, so by (2) $\overline{S}(U)$ is compactoid in X, but the topology on X_σ is weaker than the topology on X, so $\overline{S}(U)$ is compactoid in X_σ, then $\overline{S} \in C(F, X_\sigma)$, therefore X_σ have the CEP.

[⇒] Suppose X with the weak topology has the CEP. Let us see that X is weakly injective.

By (Proposition 2.3.18.), it suffices to see that X is weakly injective in the category of normed spaces, but it is true because of the following:

(1) The weakly bounded and the bounded subsets of X coincide (Theorem 2.3.27.).

(2) The weakly bounded and the weakly compactoid subsets of X coincide (Corollary 2.3.33.).

(3) Applying (1) and (2), we have that $L(Y, X) = L(Y, X_\sigma)$ for every normed space Y, i.e. X is weakly injective, then X_σ has the CEP.

□

Remark 2.3.4. Let $(h_n)_n \in \ell^\infty(E^{'})$, where $\ell^\infty(Y^{'})$ is the set of all bounded sequences in $Y^{'}$. Define

$$T : E \to \ell^\infty : x \mapsto (h_n(x))_n,$$

then T is a continuous linear map.

Proof. Since $(h_n)_n \in \ell^\infty(E^{'})$, then $(h_n(x))$ in $\mathbf{K}$, so $(h_n(x))_n$ in ℓ^∞

[1] T **is well defined :** Let x, y $\in E$ and suppose that $Tx \neq Ty$, then $(h_n(x))_n \neq (h_n(y))_n$, then there is some $j \in \mathbb{N}$ such that $h_j(x) \neq h_j(y)$, so $x \neq y$.

[**2**] T **is linear :** Let x,y be elements in E and $\lambda \in \mathbf{K}$, then

$$T(x+\lambda y) = (h_n(x+\lambda y))_n = (h_n(x)+\lambda h_n(y))_n = (h_n(x))_n+\lambda(h_n(y))_n = Tx+\lambda Ty$$

[**3**] T **is continuous :** $||T(x)|| = ||(h_n(x))_n|| = \sup_n |h_n(x)| \leq C||x||$ for some $C \in \mathbb{R}^+$, then T is bounded and so T is continuous.

□

Lemma 2.3.35. *Let Y be a normed space. The map*

$$\psi : L(Y, \ell^\infty) \to \ell^\infty(Y^{'}) \ : \ T \mapsto (T^{'}(f_{e_n}))_n$$

is a linear isometry from $L(Y, \ell^\infty)$ onto $\ell^\infty(Y^{'})$, where $\ell^\infty(Y^{'})$ is the set of all bounded sequences in $Y^{'}$ and $T^{'}$ is the algebraic adjoint of T defined from $(\ell^\infty)^{'}$ into $Y^{'}$ by $\langle x, T^{'}y^{'}\rangle = \langle Tx, y^{'}\rangle \qquad (x \in Y, y^{'} \in Y^{'})$, and
$f_{e_n} : \ell^\infty \to \mathbf{K}$ defined by $f_{e_n}(y) = y_n \qquad (y \in \ell^\infty)$.

Proof. We want to show that ψ is a linear isometry from $L(Y, \ell^\infty)$ onto $\ell^\infty(Y^{'})$.
For that we have to show that :

[**1**] ψ **is well defined :** Let T, S be elements in $L(Y, \ell^\infty)$ and suppose that $\psi(T) \neq \psi(S)$, then $(T^{'}(f_{e_n})_n) \neq (S^{'}(f_{e_n})_n)$, then there is some $j \in \mathbb{N}$ such that $T^{'}(f_{e_j}) \neq S^{'}(f_{e_j})$, then $T^{'} \neq S^{'}$, then there is some $f \in (\ell^\infty)^{'}$ such that $T^{'}(f) \neq S^{'}(f)$ then $f \circ T \neq f \circ S$, so $T \neq S$.

[**2**] ψ **is linear :** Let T, S be elements in $L(Y, \ell^\infty)$ and $\lambda \in \mathbf{K}$, then
$\psi(T + \lambda S) = ((T + \lambda S)^{'}(f_{e_n}))_n = (f_{e_n} \circ (T + \lambda S))_n = ((f_{e_n} \circ T) + \lambda(f_{e_n} \circ S))_n =$
$(f_{e_n} \circ T)_n + \lambda(f_{e_n} \circ S)_n = (T^{'}(f_{e_n})_n) + \lambda(S^{'}(f_{e_n})_n) = \psi(T) + \lambda\psi(S)$

[3] ψ **is onto :** Let $(h_n)_n \in \ell^\infty(E^{'})$. Define

$$T : E \to \ell^\infty : x \mapsto (h_n(x))_n,$$

then by (Remark 2.3.4.), T is continuous linear from E into ℓ^∞ and

$$\psi(T) = (T^{'}(f_{e_n})_n = (f_{e_n} \circ T)_n =^{(*)} (h_n)_n.$$

For (*) $T(x) = (h_n(x))_n$, then $f_{e_n}(T(x)) = f_{e_n}((h_n(x))_n) = h_n(x)$ for all $x \in Y$, then $f_{e_n} \circ T(x) = h_n(x)$ for all $x \in Y$, so $f_{e_n} \circ T = h_n$, so ψ is onto.

[4] ψ **is isometry :** Let $T \in L(Y, \ell^\infty)$, then

$$||T|| = \sup_{x\in E\ ,\ ||x||\le 1} ||T(x)|| = \sup_{x\in E\ ,\ ||x||\le 1} ||([T^{'}(f_{e_n})](x))|| = \sup_{x\in E\ ,\ ||x||\le 1} ||(f_{e_n}(T(x))_n||$$

$$= \sup_n ||f_{e_n} \circ T|| = \sup_n ||T^{'}(f_{e_n})|| = ||(T^{'}(f_{e_n}))_n|| = ||\psi(T)||,$$

then ψ is isometry.

$\square$

Corollary 2.3.36. *ℓ^∞ is weakly injective if and only if for every pair of normed spaces Y, Z with $Y \subset Z$ and every bounded sequence $(h_n)_n$ in $Y^{'}$, there exists a bounded sequence $(g_n)_n$ in $Z^{'}$ such that $g_n|_Y = h_n$ for all n.*

Proof. [$\Rightarrow$] Suppose that ℓ^∞ is weakly injective, let Y, Z be normed spaces with $Y \subset Z$ and let $(h_n)_n$ be a bounded sequence in $Y^{'}$.

Define $T : Y \to \ell^\infty$ by $T(y) = (h_n(y))_n$, then by (Remark 2.3.4.) $T \in L(Y, \ell^\infty)$, also by (Lemma 2.3.35.) we have $\psi(T) = (T^{'}(f_{e_n}))_n$ which by (*) in (Lemma

2.3.35.) equal $(h_n)_n$.

Since ℓ^∞ is weakly injective, then there exists $K \in L(Z, \ell^\infty)$ that extends T, since ψ is a linear isometry from $L(Y, \ell^\infty)$ into $\ell^\infty(Y^{'})$, then $\psi(K) = (K^{'}(f_{e_n}))_n$ is a bounded sequence in $Z^{'}$.

For all $n \in N$, let $g_n = K^{'}(f_{e_n})$, then g_n is an extension of h_n, because for all $y \in Y$, we have:

$$g_n(y) = K^{'}(f_{e_n})(y) = f_{e_n}(K(y)) = f_{e_n}(T(y)) = T^{'}(f_{e_n})(y) = h_n(y).$$

[⇐] Suppose that for every pair of normed spaces Y, Z with $Y \subset Z$ and for every bounded sequence $(h_n)_n$ in $Y^{'}$, there exists a bounded sequence $(g_n)_n$ in $Z^{'}$ such that $g_n|_Y = h_n$ for all n.

Let $T \in L(Y, \ell^\infty)$, since ψ is a linear isometry from $L(Y, \ell^\infty)$ into $\ell^\infty(Y^{'})$, then $\psi(T) = (T^{'}(f_{e_n}))_n$ is a bounded sequence in $Y^{'}$, let $h_n = T^{'}(f_{e_n})$ for all $n \in N$, then (h_n) is a bounded sequence in $Y^{'}$, so by assumption there exists bounded sequence $(g_n)_n$ in $Z^{'}$ such that $g_n|_Y = h_n$ for all n. Define $\overline{T} : Z \to \ell^\infty$ by $\overline{T}(z) = (g_n(z))_n$ for all $z \in Z$, then $\overline{T} \in L(Z, \ell^\infty)$, also $\overline{T}$ is an extension of T because for all $y \in Y$, we have:

$$\overline{T}(y) = (g_n(y))_n = (h_n(y))_n = (T^{'}(f_{e_n})(y))_n = T(y)$$

□

Theorem 2.3.37. *For a non-archimedean valued field* **K**, *the following properties are equivalent.*

(1) $\mathbf{K}$ *is spherically complete.*

(2) For every pair of normed spaces Y, Z *with* $Y \subset Z$ *and every sequence* $(f_n)_n$ *in* Y' *with* $\lim_n ||f_n|| = 0$*, there exists a sequence* $(g_n)_n$ *in* Z' *with* $\lim_n ||g_n|| = 0$*, and* $g_n|_Y = f_n$ *for all* n*.*

(3) For every pair of normed spaces Y, Z *with* $Y \subset Z$ *and every compactoid sequence* $(f_n)_n$ *in* Y'*, there exists a compactoid sequence* $(g_n)_n$ *in* Z' *such that* $g_n|_Y = f_n$ *for all* n*.*

(4) For every pair of normed spaces Y, Z *with* $Y \subset Z$ *and every bounded sequence* $(f_n)_n$ *in* Y'*, there exists a bounded sequence* $(g_n)_n$ *in* Z' *such that* $g_n|_Y = f_n$ *for all* n*.*

Proof. :

[$1 \Leftrightarrow 2$] (By Theorem 2.2.10.[$(i) \Leftrightarrow (iii)$]) $\mathbf{K}$ is spherically complete if and only if $\mathbf{K}$ is injective.

[$2 \Rightarrow 3$] (By Theorem 2.2.10.[$(iii) \Leftrightarrow (iv)$]) $\mathbf{K}$ is injective if and only if $\mathbf{K}$ is weakly injective, then by (Theorem 2.3.30.), $\mathbf{K}$ has the CEP.

[$3 \Rightarrow 1$] Assume that (3) holds and suppose that $\mathbf{K}$ is not spherically complete, then by (Theorem 2.2.9.) " If K is not spherically complete , then no locally convex space $X \neq \{0\}$ over $\mathbf{K}$ has the CEP and so $\mathbf{K}$ has not the CEP, which is contradiction with assumption.

[$1 \Leftrightarrow 4$] $\mathbf{K}$ is spherically complete (By Theorem 2.2.10.[$(i) \Leftrightarrow (iv)$]) if and only if $\mathbf{K}$ is weakly injective, (by lemma 2.3.20.) if and only if ℓ^∞ is weakly injective, (by corollary 2.3.36.) if and only if for every pair of normed spaces Y, Z with $Y \subset Z$ and every bounded sequence $(f_n)_n$ in Y', there exists a bounded sequence $(g_n)_n$ in Z' such that $g_n|_Y = f_n$ for all n.

□

2.4 The case when the valuation of K is discrete.

Definition 2.4.1. The valuation of $\mathbf{K}$ is discrete if 1 is not an accumulation point of the value group $|\mathbf{K}^{\times}|$, otherwise, the valuation is dense.

Proposition 2.4.1. *If the metric on a complete ultrametric space X is discrete, then X is spherically complete.*

Proof. see [21], p. 52.

□

Corollary 2.4.2. *If the valuation of $\mathbf{K}$ is discrete, then $\mathbf{K}$ is spherically complete.*

Proof. Since $\mathbf{K}$ is complete ultrametric space with the metric defined by $\rho(x, y) := |x - y|$ for all $(x, y \in \mathbf{K})$, then by (Proposition 2.4.1.) $\mathbf{K}$ is spherically complete.

□

Remark 2.4.1. c_0 is a perfect sequence space and so by (Theorem 1.7.1.), c_0 is sequentially complete.

Remark 2.4.2. c_0 is spherically complete if and only if the valuation of $\mathbf{K}$ is discrete.

Proof. see [22], p. 97.

□

Corollary 2.4.3. *If the valuation of $\mathbf{K}$ is discrete, then for every $t \in (0, 1]$, every Banach space is t-injective.*

Proof. see [22], p. 104.

□

Corollary 2.4.4. *Let the valuation of* $\mathbf{K}$ *is discrete, then every Banach space is weakly injective.*

Proof. If we take $t = 1$ in (Corollary 2.4.4.), then every Banach space is 1-injective hence every Banach space is injective and so is weakly injective.

□

Corollary 2.4.5. c_0 *is weakly injective if and only if the valuation of* $\mathbf{K}$ *is discrete.*

Proof. The valuation of $\mathbf{K}$ is discrete if and only if (by Remark 2.4.2.),the Banach space c_0 is spherically complete if and only if by (Theorem 2.3.19.[1$\Leftrightarrow$ 2]), c_0 is weakly injective.

□

Corollary 2.4.6. *If the valuation of* $\mathbf{K}$ *is dense, then* c_0 *is not complemented in* ℓ^∞*.*

Proof. see [22], p. 181.

□

Corollary 2.4.7. *If the valuation of* $\mathbf{K}$ *is dense, then* c_0 *is not weakly injective.*

Proof. Suppose that the the valuation of $\mathbf{K}$ is dense, then by (Corollary 2.4.6.), c_0 is not complemented in ℓ^∞, then ℓ^∞ is a locally convex space contains a copy of

c_0 which is not complemented in ℓ^∞, so by (Proposition 2.3.21.), c_0 is not weakly injective.

$\square$

Example 2.4.1. ***(of locally convex spaces without the CEP).***

Suppose that the valuation of $\mathbf{K}$ is dense. It follows from (Corollary 2.4.7.) that c_0 is not weakly injective. So, by using Proposition 2.3.34, we obtain that the CEP is not satisfied for $X := (c_0, \sigma(c_0, \ell^\infty))$.

Lemma 2.4.8. *A linear map $T : X \to c_0$ is continuous if and only if T can be written as*

$$T(x) = (a_n(x))_n,$$

where $(a_n)_n$ is an equicontinuous $\sigma(X', X)$-null sequence in X'.

Proof. see [25], p. 71.

$\square$

Corollary 2.4.9. *c_0 is weakly injective if and only if for every pair of normed spaces Y, Z with $Y \subset Z$ and every bounded sequence $(f_n)_n$ in Y' with $\lim_n f_n = 0$ in $\sigma(Y, Y')$, then there exists a bounded sequence $(g_n)_n$ in Z' with $\lim_n g_n = 0$ in $\sigma(Z, Z')$ and $g_n|_Y = f_n$ for all $n \in N$.*

Proof. [$\Rightarrow$] Suppose that c_0 is weakly injective, let Y, Z be normed spaces with $Y \subset Z$ and let $(f_n)_n$ be a bounded sequence in Y' with $\lim_n f_n = 0$ in $\sigma(Y, Y')$.

Define $T : Y \to c_0$ by

$$T(y) = (f_n(y))_n$$

for all $y \in Y$, then by (Remark 2.3.4.) $T \in L(Y, c_0)$.

Since c_0 is weakly injective, then there exist $\overline{T} \in L(Z, c_0)$ that extends T, since $\overline{T} \in L(Z, c_0)$, then by (Lemma 2.4.8.) $\overline{T}$ can be written as $\overline{T}(z) = (g_n(z))_n$, where $(g_n)_n$ is an equicontinuous $\sigma(X', X)$-null sequence in X', then g_n is an extension of f_n, because for all $y \in Y$, we have

$$(g_n(y))_n = \overline{T}(y) = T(y) = (f_n(y))_n$$

if and only if $g_n(y) = f_n(y)$ for all $y \in Y$ if and only if $g_n|_Y = f_n$ for all $n \in N$.

[$\Leftarrow$] Assume that for every pair of normed spaces Y, Z with $Y \subset Z$ and every bounded sequence $(f_n)_n$ in Y' with $\lim_n f_n = 0$ in $\sigma(Y, Y')$, then there exist a bounded sequence $(g_n)_n$ in Z' with $\lim_n g_n = 0$ in $\sigma(Z, Z')$ and $g_n|_Y = f_n$ for all $n \in N$.

Let $T \in L(Y, c_0)$, then by (Lemma 2.4.8.) T can be written as

$$T(y) = (a_n(y))_n,$$

where $(a_n)_n$ is an equicontinuous sequence in Y' converging pointwise to zero in $\sigma(Y, Y')$, then $(a_n)_n$ is a bounded sequence in Y' with $\lim_n a_n = 0$ in $\sigma(Y, Y')$, then by assumption, there exists a bounded sequence $(b_n)_n$ in Z' with $\lim_n b_n = 0$ in $\sigma(Z, Z')$ and $b_n|_Y = a_n$ for all $n \in N$, define $\overline{T} : Z \to c_0$ by $\overline{T}(z) = (b_n(z))_n$, then $\overline{T} \in L(Z, c_0)$, also $\overline{T}$ is an extension of T because for all $y \in Y$, we have

$$\overline{T}(y) = (b_n(y))_n = (a_n(y))_n = T(y),$$

therefore c_0 is weakly injective.

$\square$

Definition 2.4.2. A locally convex space X is locally complete if and only if for every absolutely convex closed bounded set A in X, the normed space X_A is complete.

Lemma 2.4.10. *Every sequentially complete is locally complete.*

Proof. Suppose that X is sequentially complete and let A be an absolutely convex closed bounded set in X, let $(x_n)_n$ be a cauchy sequence in X_A, then by sequently completeness of X, $(x_n)_n$ converges in X, since A is closed, then X_A is closed, then $(x_n)_n$ converges in X_A, so X_A is complete, hence X is locally complete.

$\square$

Theorem 2.4.11. *Every locally complete, hence every (weakly) sequentially complete, locally convex space over a discretely valued field* $\mathbf{K}$ *has the CEP.*

Proof. Let Y, Z be normed spaces with $Y \subset Z$ and let $T \in C(Y, X)$, then $T(Y) \subset X$ but $T(B_Y)$ is an absolutely convex zero neighborhood in X, so $T(B_Y)$ absorbs X, then $T(Y) \subset span[(T(B_Y))]$, since $T(B_Y) \subset \overline{T(B_Y)}$, then $span[T(B_Y)] \subset span[\overline{T(B_Y)}]$, take $A = \overline{T(B_Y)}$, then

$$T(Y) \subset span[\overline{T(B_Y)}] = X_A$$

Since $\overline{T(B_Y)}$ is an absolutely convex closed and bounded, then locally completeness of X implies that X_A is a Banch space , since the valuation of $\mathbf{K}$ is discrete, then by (Corollary 2.4.4.), X_A is weakly injective, then the continuous linear operator

$$S : Y \to X_A \quad x \mapsto T(x)$$

can be extended to a continuous linear operator $\overline{S} : Z \to X_A$.

Recall that the inclusion $i_A : X_A \to X$ is compact. For that, since T is compact, then $T(B_Y)$ is compactoid, then by (Proposition 1.5.3. (i)), $A = \overline{T(B_Y)}$ is compactoid. Then by (Proposition 1.5.3. (ii)), the open unit ball of X_A which is contained in A is also compactoid, then the inclusion is continuous by (Remark 1.5.5.(ii)) and maps this open unit ball ito compactoid, hence i_A is compact.

Set $\overline{T} = i_A \circ \overline{S}$, then $\overline{T} \in C(Z, X)$ because composite of compact and continuous is compact, also $\overline{T}$ is an extension of T because for all $y \in Y$, we have

$$\overline{T}(y) = i_A \circ \overline{S}(y) = i_A(\overline{S}(y)) = i_A(S(y)) = i_A(T(y)) = T(y).$$

□

Theorem 2.4.12. *For a non-archimedean valued field* $\mathbf{K}$, *the following are equivalent.*

(1) The valuation of $\mathbf{K}$ *is discrete.*

(2) Every locally complete space over $\mathbf{K}$ *has the CEP.*

(3) Every (weakly) sequentially complete space over $\mathbf{K}$ *has the CEP.*

(4) $(c_0, \sigma(c_0, \ell^\infty))$ *has the CEP.*

(5) For every pair of normed spaces Y, Z *with* $Y \subset Z$ *and every bounded sequence* $(f_n)_n$ *in* Y' *with* $\lim_n f_n = 0$ *in* $\sigma(Y, Y')$, *there exists a bounded sequence*

$(g_n)_n$ in $Z^{'}$ with $\lim_n g_n = 0$ in $\sigma(Z, Z^{'})$ and $g_n|_Y = f_n$ for all $n \in N$.

Proof. :

[1] $\Rightarrow$ [2] $\Rightarrow$ [3] follows from (Theorem 2.4.11.).

[3 $\Rightarrow$ 4] Since c_0 is a perfect sequence space then by (Theorem 1.7.1.), c_0 is sequentially complete, then by [3] $(c_0, \sigma(c_0, \ell^\infty))$ has the CEP

[4 $\Rightarrow$ 1] Assume that $(c_0, \sigma(c_0, \ell^\infty))$ has the CEP and suppose that the valuation of **K** is discrete, then by (Example 2.4.1.), $X = (c_0, \sigma(c_0, \ell^\infty))$ has not the CEP, which is contradiction with assumption.

[1 $\Leftrightarrow$ 5] The valuation of **K** is discrete if and only if (by Corollary 2.4.5.), c_0 is weakly injective if and only if (by Corollary 2.4.9.) for every bounded sequence $(f_n)_n$ in $Y^{'}$ with $\lim_n f_n = 0$ in $\sigma(Y, Y^{'})$, there exist a bounded sequence $(g_n)_n$ in $Z^{'}$ with $\lim_n g_n = 0$ in $\sigma(Z, Z^{'})$ and $g_n|_Y = f_n$ for all $n \in N$.

□

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