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Finite Groups in Stone-Čech Compactification

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"A Thesis Submitted in Partial Fulfillment of The Requirements of The Degree of Master of Mathematics"

2010- 1431

Dedication

To my parents…

To my brother and sisters

To my husband

To all knowledge seekers…

Contents

ACKNOWLEDGEMENTS

First, I'm very grateful to my god for every thing he gave me. Also, I am grateful to my lovely parents for their immolation, subsidization and guidance.

I would like to thank Dr. Ahmed El Mabhouh , my supervisor for his guidance and his helpful suggestions during my work to complete this thesis.

Finally I wish to thank my teachers in the Department of Mathematics in Islamic University for their encouragement and for supporting me in my studying.

Abstract

Let κ be an infinite cardinal. Let $G = \bigoplus_{\alpha<\kappa} G_\alpha$ where G_α is a nontrivial group. Now let βG be the Stone-Čech compactification of G and let $\mathbf{H} = \bigcap_{\alpha<\kappa}\{c\ell_{\beta G}\{x \in G \setminus \{e\} : \min supp(x) \geq \alpha\}$. Then we will show that $\mathbf{H}$ contains no nontrivial finite group. Moreover for the set of natural number we show that every maximal group in the smallest ideal of $(\beta\mathbb{N}, +)$ contains $2^{\mathfrak{c}}$ discrete copies of $(\mathbb{Z}, +)$ the closures of any two of which intersect only at the identity. We also show that the same conclusion applies to copies of the free group on two generators.

Introduction

Compactification theory is of great importance in topology and functional analysis. This is due to the fundamental role of compact spaces in these two branches of mathematics. Many properties of a topological space are a lot easier to deduce if the space is a compact Hausdorff space. Given a space X it is probably not compact and hence it is difficult to handle. Then we must look for a compact Hausdorff space Y, which is very similar to X. We can then use results in Y and translate them back to the original space X. For any topological space X, a compact Hausdorff space Y is called a compactification of X, if there is an embedding ϕ of X into Y such that $\phi(X)$ is dense in Y. In this work we will study a very special compactification, independently introduced during the year 1937 by M. H. Stone and E. Cech. This compactification is called the Stone-Čech compactification of X and is denoted by βX. We take the points of βX to be the ultrafilters on X. In $\beta\mathbb{N}$ there are $2^{\mathfrak{c}}$ minimal right ideals and $2^{\mathfrak{c}}$ minimal left ideals, and consequently $2^{\mathfrak{c}}$ maximal groups in the smallest ideal.

N. Hindman and D. saturass [6] show that every maximal group in the smallest ideal of $(\beta\mathbb{N}, +)$ contains $2^{\mathfrak{c}}$ discrete copies of $(\mathbb{Z}, +)$ the closures of any two of which intersect only at the identity. They also show that the same conclusion applies to copies of the free group on two generators. Also, N. Hindman and J. Pym [8] show that the structural group of $K(\beta\mathbb{N})$ contains as a subgroup the free group on 2^{c} generators. This raised the question of whether any nontrivial finite group exists in $\beta\mathbb{N}$. The question was solved in the negative by Y. Zelenyuk [16]. In fact, it was shown that if X is a countable torsion-free group, then βX contains no nontrivial finite group (see also [2, Section 7.1]) . I. Protasov [2] generalized this result by characterizing the finite groups in βX, where X is an arbitrary countable group. Every finite group in βX has the form $\mathbf{H}p$ for some finite subgroup $\mathbf{H}$ of X and some idempotent p in βX which commutes with all the elements of $\mathbf{H}$. However, nothing has yet been obtained in the uncountable case. Recently, Y. Zelenyuk [17] studied this Result and gave some answers of it when $X = \oplus_\kappa \mathbb{Z}_2$ where X is discrete semigroup and κ an infinite cardinal . Also he show that the smallest ideal of βX is not closed.

This thesis consists of four chapters.

Chapter 1 contains 2 sections, in section 1 we present a brief summary of the notations for abstract algebra that we use and in section 2 we give a basic information for topological space which will be used in the remainder of the thesis.

Chapter 2 is devided into 3 sections, in section 1 we give some remarks on Boolean algebra and the relation between it and stone space. In section 2 we study stone-Čech compactifiction. In section 3 we discuss compactification of discrete space

Chapter 3 also contains 3 sections .In Section 1, we consider the relation between subsemigroups of βG and left invariant topologies on G. In particular, we give a sufficient condition for a subsemigroup of βG to be the ultrafilter semigroup of a regular left invariant topology. In Section 2, we study local homomorphisms of left topological groups. They induce homomorphisms of ultrafilter semigroups. We show that local homomorphisms enjoy a remarkable property of projectivity type. In Section 3, using results of Sections 2 and 3, we show that if κ is infinte cardinal and $G = \oplus G_\alpha$ where G_α is nontrivial group and $\alpha < \kappa$ then $\mathbf{H}$ contains no nontrivial finite group . Finally Chapter 4 contains 2 sections. In Section 1 we show that there are $2^{\mathfrak{c}}$ discrete copies of $\mathbb{Z}$ in each of the maximal groups in the smallest ideal, and that any two of these meet only in the identity. In Section 3 we show that the same results holds for the free group on 2 generators. Since the free group on 2 generators contains copies of $\mathbb{Z}$, the results of Section 2 are a corollary of those of Section 3.

Chapter 1

Preliminaries

In this chapter, we give a basic information which will be used in the remainder of the thesis.

1.1 Algebra

Definition 1.1.1. [4] A *semigroup* is a pair $(S, *)$ where S is a nonempty set and $*$ is a binary associative operation on S.

Formally a *binary operation* on S is a function $* : S \times S \to S$ and the operation is *associative* if and only if $(x * y) * z = x * (y * z)$ for all x,y, and z in S.
we say S is *closed* under $*$ if $x * y \in S$ whenever $x, y \in S$.

Example 1.1.2. *[4] Each of the following is a semigroup.*

(a) *The set of natural numbers* $(\mathbb{N})$ *under addition or multiplication is a semigroup.*

(b) *The set of real numbers* $(\mathbb{R})$ *under addition or multiplication is a semigroup.*

(c) $(S, *)$ *where* S *is a nonempty set and* $x * y = y$ *for all* $x, y \in S$

(d) $(S, *)$ *where* S *is a nonempty set and* $x * y = x$ *for all* $x, y \in S$.

(e) $(\mathbb{N}, \vee)$ *where* $x \vee y = \max\{x, y\}$.

The semigroups of Example 1.1.2 (c) and (d) are called respectively *right zero* and *left zero* semigroups .

Definition 1.1.3. [4] Let S be a semigroup,

(a) S is *commutative* if and only if $xy = yx$ for all $x, y \in S$.

(b) The *center* of S is $\{x \in S : \text{for all } y \in S, xy = yx\}$.

(c) Given $x \in S$, the function $\lambda_x : S \to S$ is defined by $\lambda_x(y) = xy$.

(d) Given $x \in S$, the function $\rho_x : S \to S$ is defined by $\rho_x(y) = yx$.

(e) $L(S) = \{\lambda_x : x \in S\}$.

(f) $R(S) = \{\rho_x : x \in S\}$.

Definition 1.1.4. [4] Let S be a semigroup

(a) An element $x \in S$ is called *left (respectively, right) cancelable* if for all $y, z \in S$, and $xy = xz$, (respectively, $yx = zx$) implies $y = z$.

(b) If each element $x \in S$ is left (respectively, right) cancelable, then S is called *left (respectively, right) cancellative.*

(e) If S is both left cancellative and right cancellative we say S is *cancellative* .

Example 1.1.5. *[18]*

(a) The set of natural number under addition is cancellative semigroup.

(b) A left zero semigroup is right cancellative but not left cancellative.

(c) A right zero semigroup is left cancellative but not right cancellative.

(d) Let S (respectively, $\mathfrak{M}_n$) be the set of all square matrices of order n with real entries (respectively, complex entries). Then under matrix multiplication, S is semigroup. Let A be any element in S. If A is nonsingular, then A is both left cancellable and right cancellable. If A is singular, then A is neither left cancellable nor right cancellable.

Remark 1.1.6. [4] Let S be a semigroup. Then $(L(S), \circ)$ and $(R(S), \circ)$ form semigroups, where $\circ$ is the operation of composite functions.

Definition 1.1.7. [4] Let S be a semigroup. Then

(a) an element $x \in S$ is an idempotent if $xx = x$

(b) the set of all idempotents in S is denoted by $E(S)$; that is, $E(S) = \{x \in S : xx = x\}$,

Example 1.1.8. *(a) The set of idempotents of the semigroup* $(\mathbb{N}, \vee)$ *is* $E(\mathbb{N}) = \mathbb{N}$.

(b) The set of idempotents of the left zero semigroup is $E(S) = S$.

(c) The only idempotent in $(\mathbb{N}, .)$ *is 1*

Definition 1.1.9. A *group* $(G, *)$ is a nonempty set G together with a binary operation $*$ on G such that the following conditions hold:

(a) Associativity: For all $a, b, c \in G$, we have $a * (b * c) = (a * b) * c$.

(b) Identity: G contains an element e (called the *identity*) such that $a * e = e * a = a$ for all $a \in G$.

(c) Inverses: For each $a \in G$ there exusts $b \in G$ (called an *inverse* of a) such that $a * b = b * a = e$. We will denote b by a^{-1}.

The *order* of group G is the number of elements in G.

Definition 1.1.10. [4] Let S be a semigroup,and $T \subseteq S$. Then we say

(a) T is *a subsemigroup* of S if it is a semigroup under the restriction of the operation of S.

(b) T is *a subgroup* of S if S is a group, and T is a group under the restriction of the operation of S.

Given subsets A and B of a semigroup S, by AB we of course mean $\{ab : a \in A$ and $b \in B\}$

Definition 1.1.11. Let G be a group, and T subgroup of G. Then

(a) T is called *a normal* subgroup of G, if $aT = Ta$ for any $a \in G$.

(b) if T is normal in G, then the set $G/T = \{aT : a \in G\}$ is a group under the operations $(aT)(bT) = (ab)T$. This group is called the *quotient group* of G by T.

Definition 1.1.12. Let φ be a function from a set G to a set S. Then

(a) φ is called *one to one or monomorphism,* if for every $a_1, a_2 \in G$, $\varphi(a_1) = \varphi(a_2)$ implies $a_1 = a_2$.

(b) φ is said to be *onto or surjective,* if for every b in S, there is at least one a in G such that $\varphi(a) = b$.

(c) If G and S are groups, then φ is called *a homomorphism* from G to S if it presrves the group operation; that is, $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in A$.

(d) φ is called *an isomorphism* from G to S if it is one to one, onto, and a homorphism.

(e) If φ is a homorphism from G to S, then the *kernal* of φ is the set $\{x \in G : \varphi(x) = e$, where e is the identity of S. The Kernal φ is denoted by $\text{Ker}\varphi$.

Note that $\text{Ker}\varphi$ is a normal subgroup of A.

Theorem 1.1.13. ***First isomorphism Theorem***
Let φ be a group homomorphism from G to S. Then the function $\psi : G/\text{Ker}\varphi \to \varphi(G)$ given by $\psi(g\text{Ker}\varphi) = \varphi(g)$ is an isomorphism.

Definition 1.1.14. [4] Let S be a semigroup and let L, R, I be a nonempty subsets of S. Then

(a) L is *left ideal* of S if $SL \subseteq L$

(b) R is *right ideal* of S if $RS \subseteq R$

(c) I is an *ideal* (sometimes, we say *two sided ideal*) of S if and only if I is both left ideal and right ideal of S.

Any left (respectively, right) ideal L is a subsemigroup of S since $SL \subseteq L$ (respectively, $LS \subseteq L$.)

Example 1.1.15. *[1]*

(a) In the semigroup $(\mathbb{N}, .)$ *the set of all even numbers is an ideal.*

(b) In the multiplicative semigroup $\mathfrak{M}_n$ *of all complex square matrices of order* n*. For a given fixed column, Let* S *be the set of all matrices in which the entries of the fixed column are equal to zero. Then* S *is left ideal but not right if* $n > 1$.

Definition 1.1.16. [4] Let S be a semigroup, L is left ideal of S, and R is right ideal of S. Then

(a) L is *a minimal left (respectively, right) ideal* of S if whenever J is left (respectively, right) ideal of S and $J \subseteq L$, we have $J = L$.

(b) S is *left (respectively, right) simple* if and only if S is a minimal left (respectively, right) ideal of S.

(e) S is *simple* if it is both left simple and right simple .

Note 1.1.17. *If* S *is simple, then the only ideal of* S *is* S *itself.*

Proof. Let L be an ideal of S such that $L \subset S$. Since S is simple then S is minimal ideal of S. so, $L = S$. □

In a semigroup S, an element z is called a *zero element* if $z * s = s * z = z$, for all $s \in S$.

Example 1.1.18. *(a) Semigroups with zero has only one minimal left (right - two sided) ideal of* S *namely the trivial one* $\{0\}$

(b) $(\mathbb{Z}, +)$ *has no minimal ideal .*

(c) *Let* $S = \{a, b, c, d\}$ *where a,b,c , and d are distinct and let* S *has the following multiplicative table. Then* S *is simple but neither left simple nor right simple.*

.	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	d	c
d	c	d	d	c

Clearly S *is semigroup . Also* $\{a, b\}$ *and* $\{c, d\}$ *are right ideals of* S *and* $\{a, c\}$ *and* $\{b, d\}$ *are left deals of* S*.*

Lemma 1.1.19. *[4] Let* S *be a semigroup,*

(a) *suppose* L_1 *and* L_2 *be left ideals of* S*. Then* $L_1 \cap L_2$ *is a left ideal of* S *if and only if* $L_1 \cap L_2 \neq \phi$*.*

(b) *if* $x \in S$*, then* xS *is a right ideal,* Sx *is a left ideal, and* xSx *is an ideal.*

(c) *if* L *is a left ideal of* S *and,* R *is a right ideal of* S*, then* $L \cap R \neq \phi$*.*

Proof. [4]

(a) Since $L_1 \cap L_2$ is a left ideal of S then $L_1 \cap L_2 \neq \phi$. Conversely, let $L_1 \cap L_2 \neq \phi$, since L_1 is a left ideal then $S(L_1 \cap L_2) \subseteq SL_1 \subseteq L_1$ so $(L_1 \cap L_2)$ is a left ideal.

(b) Since $x \in S$ then $xS \neq \phi$. Also $xSS \subseteq xS$ so xS is right ideal of S.
In the same way Sx is a left ideal and xSx is an ideal of S .

(c) Suppose $x \in L$ and $y \in R$. Then we have that $yx \in L$ because L is left ideal and $yx \in R$ because R is right ideal.

□

Lemma 1.1.20. *[4] Let* S *be a semigroup,* L *a left ideal of* S*, and* T *a left ideal of* L*. Then* *(a) for all $t \in T$, Lt is a left ideal of S and $Lt \subseteq T$.*

(b) if L is a minimal left ideal of S, then $T = L$. So minimal left ideals are left simple.

(c) if T is a minimal left ideal of L, then T is a left ideal of S.

Proof. [4]

(a) $S(Lt) = (SL)t \subseteq Lt$ and $Lt \subseteq LT \subseteq T$.

(b) Pick any $t \in T$. By (a), Lt is a left ideal of S and $Lt \subseteq T \subseteq L$, so $Lt = L$. Thus $T = L$.

(c) Pick any $t \in T$. By (a), Lt is a left ideal of S, so Lt is a left ideal of L. Since $Lt \subseteq T$, we have that $Lt = T$. Therefore, $ST = S(Lt) = (SL)t \subseteq Lt = T$.

□

Of course, the right-left switch of the above lemma is hold. That is; if R is a right ideal of S and T is a right ideal of S, moreover if either R is minimal in S or T is minimal in R, then T is right ideal of S.

Lemma 1.1.21. *[4] Let S be a semigroup. If I is an ideal on S, and if L is a minimal left ideal of S, then $L \subseteq I$.*

Proof. Since I is an ideal of S then $S(IL) = (SI)L \subseteq IL$. So IL is a left ideal of S. Also $IL \subseteq L$ because L is a left ideal. But L is minimal, so $IL = L$. Therefore $L = IL \subseteq I$ because I is an ideal. □

Theorem 1.1.22. *[4] Let S be a semigroup and L a minimal left ideal of S, and $T \subseteq S$. Then T is a minimal left ideal of S if and only if there is some $a \in S$ such that $T = La$.*

Proof. Suppose that T is a minimal left ideal of S and pick $a \in T$. Then $SLa \subseteq La$ and $La \subseteq ST \subseteq T$ so La is a left ideal of S contained in T so $La = T$.
Conversely, Let $a \in S$ such that $T = La$. Since $SLa \subseteq La$ then La is a left ideal of S. Assume that B is a left ideal of S and $B \subseteq La$. Let $A = \{s \in L : sa \in B\}$,

then $A \subseteq L$. Since B is non empty then $A \neq \phi$. We claim that A is a left ideal of S, to see this let $s \in A$ and pick $t \in S$. Then $sa \in B$. But B is a left ideal so, $tsa \in B$. since $s \in L$ and L is left ideal then $ts \in L$. Hence $ts \in A$. Since L is minimal and $A \subseteq L$ then $A = L$. Therefore $La \subseteq B$ which implies $La = B$. Since B was arbitrary then $T = La$ is a minimal left ideal. □

Corollary 1.1.23. *Let S be a semigroup and L a minimal left ideal of S. Then*

(a) there is some $a \in S$ such that $L = La$.

(b) if there is a minimal left ideal T of S, then $L = Tb = L(ab)$ for some a and b in S.

Lemma 1.1.24. *[4] Let S be a semigroup and let K be an ideal of S. If K is minimal in $\{J : J$ is an ideal of $S\}$ and I is an ideal of S, then $K \subseteq I$.*

Proof. [4] By Lemma 1.1.15 (c), $K \cap I \neq \phi$. So we have $K \cap I$ is an ideal contained in K. Hence $K \cap I = K$. □

By a bove lemma there is at most one minimal ideal in a semigroup , we called it the *smallest ideal*.

Definition 1.1.25. [4] Let S be a semigroup. If the smallest ideal exists in S, we denote it by $K(S)$.

Theorem 1.1.26. *[4] Let S be a semigroup with a minimal left ideal, then $K(S)$ exists and $K(S) = \bigcup\{L : L$ is a minimal left ideal of $S\}$.*

Proof. [4] Let $I = \bigcup\{L : L$ is a minimal left ideal of $S\}$. By Lemma 1.1.21, if J is any ideal of S, then any minmal left ideal L of S is contained in J. Hence, $I \subseteq J$, so it is suffices to show that I is an ideal of S. We have that $I \neq \phi$ by assumption, so pick $x \in I$ and $s \in S$. Take a minimal left ideal L of S such that $x \in L$. Then $sx \in L \subseteq I$. Also by Theorem 1.1.22, Ls is a minimal left ideal of S so $LS \subseteq I$ while $xs \in Ls$. □

There are many subgroups do not have a smallest ideal. For example $(\mathbb{N}, +)$ and $(\mathbb{N}, .)$

Lemma 1.1.27. *[4] Let S be a semigroup.*

(a) Let L be a left ideal of S. Then L is minimal if and only if $Lx = L$ for every $x \in L$.

(b) Let I be an ideal of S. Then I is the smallest ideal if and only if $IxI = I$ for every $x \in I$.

Proof. (a) If L is minimal and $x \in L$, then Lx is a left ideal of S and $Lx \subseteq L$ so $Lx = L$. Now assume $Lx = L$ for every $x \in L$ and let J be any ideal of S with $J \subseteq L$. Pick $x \in J$. Then $L = Lx \subseteq LJ \subseteq J \subseteq L$, So $J = L$. But J was arbitrary then L is minimal.

(b) Let I be the smallest ideal and $x \in I$, then $IxI \subseteq I$. Since IxI is an ideal of S and I is the smallest ideal so $I \subseteq IxI$.
Conversely, suppose that $IxI = I$ for every $x \in I$ and let J be any ideal of S with $J \subseteq I$. Pick $x \in J$. Then $I = IxI \subseteq IJI \subseteq J$ because J is an ideal so $I \subseteq J$. Thus $J = I$, But J was arbitrary then I is minimal.

□

Theorem 1.1.28. *[4] Let S be a semigroup. If L is a minimal left ideal of S and R is a right ideal of S, then $K(S) = LR$.*

Proof. First we will show that LR is an ideal of S. Since $L \neq \phi$ and $R \neq \phi$ then $LR \neq \phi$. L is a left ideal of S, so $SL \subseteq L$, and $SLR \subseteq LR$. Also, since R is a right ideal of S then $RS \subseteq R$ so, $LRS \subseteq LR$. Hence LR is an ideal of S.
We will use Lemma 1.1.27 to show that $K(S) = LR$ so, let $x \in LR$. Since L is a left ideal then $SL \subseteq L$ which implies $SLRxL \subseteq LRxL$. Since $LRx \subseteq S$ then $LRxL \subseteq SL \subseteq L$ So, $LRxL$ is a left ideal of S which is contained in L .But L is minimal so $LRxL = L$ and hence $LRxLR = LR$. So, from Lemma 1.1.27 part b we get $K(S) = LR$. □

Corollary 1.1.29. *Let S be a semigroup. If L is left ideal of S and R is a minimal a right ideal of S, then $K(S) = LR$.*

Proof. As in a bove theorem LR is an ideal. let $x \in LR$. Since R is a right ideal then $RS \subseteq R$ which implies $RxLRS \subseteq RxLR$. Since $xLR \subseteq S$ then $RxLR \subseteq RS \subseteq R$ So, $RxLR$ is a right ideal of S which is contained in R .But R is minimal so $RxLR = R$ and hence $LRxLR = LR$. So, from Lemma 1.1.27 part b we get $K(S) = LR$. □

Theorem 1.1.30. *[4] Let S be a semigroup and assume there is a minimal left ideal of S which has an idempotent .Then every minimal left ideal has an idempotent.*

Proof. see [4, Theorem 1.56]. □

Theorem 1.1.31. *[4] Let S be a semigroup and assume there is a minimal left ideal of S which has an idempotent. Then there is a minimal right ideal of S which has an idempotent.*

Proof. see [4, Lemma 1.57]. □

Lemma 1.1.32. *[4] Let S be a semigroup and assume there is a minimal left ideal of S which has an idempotent. Then all minimal left ideal of S are isomorphic.*

Proof. see [4, Lemma 1.62]. □

Definition 1.1.33. A group G is called *cyclic* if there exists an element g in G such that $G =< g > = \{g^n | n\ is\ an\ integer\}$ and g is called a *generator* of the group.

If G is a cyclic group of order n then every subgroup of G is cyclic. Moreover, the order of any subgroup of G is a divisor of n and for each positive divisor k of n the group G has exactly one subgroup of order k.

Examples 1.1.34. *(a) The group $\mathbb{Z}$ under addition is infinite cyclic group generated by* 1 *and* -1.

(b) The set $\mathbb{Z}_n$ under addition mod n is finite cyclic group of order n.

Definition 1.1.35. [18] Let $f : X \to Y$ and $g : Y \to X$ be two functions such that the composition $f \circ g : Y \to Y$ is the identity function on Y, then g is a *coretraction* .

In other words g has left inverse function.

1.2 Topology

In this section, we give a basic information for topological space which will be used in the remainder of the thesis. See [14] and [12] for more details and any unfamiliar topological facts encountered in this section.

Definition 1.2.1. A *topological space* is a set X together with τ, a collection of subsets of X, satisfying the following axioms:

(a) ϕ and X are in τ.

(b) The union of any members of τ is also in τ.

(c) The intersection of two sets in τ is also in τ.

Using induction, the intersection of any finite members of τ is also in τ. Usually, a topological space is denoted by (X, τ) or simply by X (if there is no confusion), where τ is a topology on X.
The elements of τ are called *open sets*, and their complements in X are called *closed sets*. A set U may be open, closed, both open and closed, or neither open nor closed. A set that is both closed and open is called a *clopen* set. The intersection of any finite number of open sets is open, but in general the intersection of any number of open sets need not be open. Dually, the union of any finite number of closed sets is closed, but in general the union of any number of closed sets need not be closed.

Examples 1.2.2. *(a) For any set X the collection $\tau = \{\phi, X\}$ forms a topology called the* trivial topology*. The collection $\tau = \mathcal{P}(X)$ the power set of X forms a topology called the* discrete topology*.*

(b) *Let* $X = \{1, 2, 3, 4\}$*. The collection* $\tau = \{\phi, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$ *of six subsets of* X *forms another topology on* X*. item [(c)] Let* $X = \mathbb{Z}$*, the set of integers, and* τ *the collection of finite subsets of the integers together with* $\mathbb{Z}$ *itself, Then* τ *is not a topology, since the union of all singleton sets not including zero is not in* τ*.*

Definition 1.2.3. If (X, τ) is a topological space and $A \subseteq X$, the collection $\tau_{_A} = \{G \cap A : G \in \tau\}$ forms a topology on A, called the *relative topology* on A.

The topological space $(A, \tau_{_A})$ is called a subspace of X

Definition 1.2.4. If X is a topological space and $x \in X$. A *neighborhood* of x is a set U which contains an open set V containing x.

Definition 1.2.5. A *base* B for a topological space X with a topology τ is a collection of open sets in τ such that every open set in τ can be written as a union of elements of B. We say that the base generates the topology τ.

Example 1.2.6. *For any* X*, the collection* $\{\{x\} : x \in X\}$ *is a base for the discrete topology on* X*.*

Proposition 1.2.7. *Let* (X, τ) *be a topological space. A family* B *of open subsets of* X *is a basis for* τ *if and only if for any open set* U *and any* $x \in U$*, there is* $A \in B$ *such that* $x \in A \subseteq U$*.*

Definition 1.2.8. Let A be a subset of topological Space X. A point $x \in X$ is called *a limit point* of A if every open set U containing x contains a point of A different from x.

Definition 1.2.9. If X is a topological space and $E \subset X$, the *closure* of E in X is the set

$$\overline{E} = \bigcap \{K \subseteq X \mid K \textit{ is closed and } E \subseteq K\}.$$

Clearly $\overline{E}$ is closed set. In fact it is the smallest closed set containing E. Moreover, A is closed set if and only if $\overline{A} = A$. We denotes A' to be the set of all limit points of X. So we have that $\overline{A} = A \cup A'$.

Definition 1.2.10. Let A be a subset of a topological space X. Then A is said to be *dense* in X if $\overline{A} = X$. Equivalently, A is dense if $A \cap U \neq \phi$ for each $U \in \tau$ and $U \neq \phi$.

For example, the set of rational numbers $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.
In general if $A \subseteq X$ then $x \in \overline{A}$ if and only if for any open set U containing x, we have that $U \cap A \neq \phi$.

Definition 1.2.11. Let (X, τ_1) and (Y, τ_2) be topological spaces and f is a function from X into Y. Then f is said to be *continuous function* if for each $U \in \tau_2$, $f^{-1}(U) \in \tau_1$.

Theorem 1.2.12. *Suppose $Y \subseteq Z$ and $f : X \to Y$. Then f is continuous as a map from X to Y if and only if it is continuous as a map from X to Z.*

Definition 1.2.13. Let X and Y be a topological spaces. A function f from X to Y is called a *homeomorphism* if f is one to one, onto, continuous, and f^{-1} is also continuous. In this case, we say X and Y are homeomorphic.

Example 1.2.14. *The open interval (a, b) in $\mathbb{R}$ is homeomorphic to $(0, 1)$. One homeomorphism being $f(x) = (x - a)/(b - a)$.*

Definition 1.2.15. A topological space X is T_1 space if and only if whenever x and y are distinct points in X. there is two open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$

If X is T_1 space, then each singleton set in X is closed.

Example 1.2.16. *Let X be an infinite set. Define $\tau = \{A \subseteq X : A = \phi$ or $X \setminus A$ is finite$\}$ (called* a cofinite topology*). Then X is T_1 topological space.*

Definition 1.2.17. A topological space X is said to be *Hausdorff* or (T_2 space) if for any pair of distinct points a, b in X there exists open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \phi$.

Let X be an infinite set with cofinite topology then X is Hausdroff space. Moreover, any Hausdroff Space is T_1 but the converse not true .

Definition 1.2.18. Let A be a subset of a topological space X. Then A is said to be *compact* if for every set I and every family of open sets O_α, $\alpha \in I$, such that $A \subseteq \cup_{\alpha\in I} O_\alpha$, there exist a finite subfamily $O_{\alpha 1}, O_{\alpha 2}, ..., O_{\alpha n}$ such that $A \subseteq O_{\alpha 1} \cup O_{\alpha 2} \cup ... \cup O_{\alpha n}$.

Example 1.2.19. *The unit interval [0,1] is compact Hausdroff Space.*

Theorem 1.2.20. *(a) Every closed subset of a compact space is compact.*

(b) A compact subset of Hausdroff space is closed.

Corollary 1.2.21. *A set A in a compact Hausdroff space is closed if and only if it is compact.*

Corollary 1.2.22. *Any finite subset of compact Hausdroff space is closed and compact.*

Definition 1.2.23. A topological space X is said to be *disconnected* if it is the union of two disjoint nonempty open sets. Otherwise, X is said to be *connected.*

A subset A of a topological space X is said to be *connected* if it is connected as a subspace.

Definition 1.2.24. [15] A space X is said to be *totally disconnected* if the clopen sets separate the points of X. That is; for any two distinct points of X, there is a clopen set in X containing one of these two points and not containing the other.

Equivalently, a topological space X is totally disconnected space if the only connected subsets of X are only the singleton sets.

Examples 1.2.25. *[18]*

(1) Every discrete spaces is totally disconnected.

(2) The rational numbers with respect to the relative topology induced by the Euclidean topology of $\mathbb{R}$ is totally disconnected.

(3) The irrational numbers with respect to the relative topology induced by the Euclidean topology of $\mathbb{R}$ is totally disconnected.

Theorem 1.2.26. *A compact Hausdorff space is totally disconnected if and only if the clopen sets form a base for open sets.*

Proof. suppose that X is a compact Hausdorff space which is totally disconnected, and let A be any open set in X. If $A = \phi$ trivial case, so let $A \neq \phi$ and pick $x \in A$. Since X is totally disconnected space, then for all $y \in A^c$ there is a clopen set U_y containing y but not x. Thus $A^c \subseteq \bigcup_{y \in A^c} U_y$ which is an open cover of A^c. Since X is a compact Hausdroff space, and A^c is closed, then A^c is compact. Pick finite subfamily of $\{U_{y_i} : y \in A^c\}$ such that $A^c \subseteq \bigcup_{i=1}^n U_{y_i}$. Since $x \notin U_y$ for all $y \in A^c$, then $x \notin \bigcup_{i=1}^n U_{y_i}$. Thus $x \in (\bigcup_{i=1}^n U_{y_i})^c = \bigcap_{i=1}^n U_{y_i}^c$ where it is clopen supset. Also, $\bigcap_{i=1}^n U_{y_i}^c \subseteq A$. To show this, let $t \in \bigcap_{i=1}^n U_{y_i}^c$, then $t \notin \bigcup_{i=1}^n U_{y_i}$, so $t \notin A^c$ which give $t \in A$.

Conversely, suppose that the clopen sets form a base for open open sets, and let x, y be two distinct points of X. Since X is compact, then $\{x\}$ is a closed set so $\{x\}^c$ is open set containing y, thus from assumption there exists a clopen set containing y and not containing x. So, X is totally disconnected. $\square$

Theorem 1.2.27. *A one to one continuous map from a compact space X onto a Hausdroff space Y is a homeomorphism.*

Definition 1.2.28. (1) A topological space X is called *a regular space* if and only if whenever A is closed in X and $x \notin A$, then there are disjoint open sets U and V with $x \in U$ and $A \subseteq V$. A T_1 regular space is T_3 space.

(2) A topological space X is called *a completely regular space* if for any closed set F and any point x that does not belong to F, there is a continuous function f from X to the real line $\mathbb{R}$ such that $f(x)$ is 0 and $f(y)$ is 1 for every y in F. That is; X is completely regular if x and F can be separated by a continuous function, where F is closed and $x \notin F$.

X is a Tychonoff space, or τ_3 space if and only if it is both completely regular and Hausdorff .

Example 1.2.29. *[18]*

(1) A trivial space is always T_3, and a non-trivial space is regular but not T_3 space.

(2) The real line $\mathbb{R}$ is completely regular space under the Euclidean topology. In fact $\mathbb{R}$ is Tychonoff.

(3) Any compact Hausdorff space is completely regular, and hence a Tychonoff.

Theorem 1.2.30. *The following are equivalent for a topological space X*

(a) X is regular.

(b) if U is open in X and $x \in U$,then there is an open set V containing x such that $\overline{V} \subset U$.

Remark 1.2.31. Any subspace of a completely regular space is completely regular.

Definition 1.2.32. [4]

(a) *A right topological* semigroup is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and for all $x \in S$, $\rho_x : S \to S$ is continuous.

(b) *A left topological* semigroup is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space , and for all $x \in S$, $\lambda_x : S \to S$ is continuous.

(c) *A semitopological semigroup* is a right and left topological semigroup.

(d) *A topological semigroup* is a triple $(S, \cdot, \tau)$ where $(S, \cdot)$ is a semigroup, (S, τ) is a topological space, and $\cdot : S \times S \to S$ is continuous.

(e) *A topological group* is a triple $(S, \cdot, \tau)$ such that $(S, \cdot)$ is a group, (S, τ) is a topological space, $\cdot : S \times S \to S$ is continuous, and $In : S \to S$ is continuous (where $In(x) = x^{-1}$).

Each topological group is a topological semigroup, each topological semigroup is a semitopological semigroup and each semitopological semigroup is both a left and right topological semigroup.
Any semigroup with the discrete topology which is not group provides an example of a topological semigroup which is not a topological group.

Definition 1.2.33. [4] Let S be a right topological semigroup. *The topological center* of S is the set $\Lambda(S) = \{x \in S : \lambda_x$ is continuous $\}$.

Thus, a right topological semigroup S is a semitopological semigroup if and only if $\Lambda(S) = S$. Clearly any topological group is a semitopological semigroup. Note that the center of a right topological semigroup in algebraic sense is contained in its topological center.

Theorem 1.2.34. *[4] Let S be a compact right topological semigroup. Then*

(a) $E(S) \neq \phi$.

(b) Every left ideal of S contains a minimal Left ideal, minimal left ideals are closed, and each minimal left ideal has an idempotent.

(c) S has a smallest ideal $K(S)$ which is the union of all minimal left ideals of S and also the union of all minimal right ideals of S. Each of $\{Se : e \in E(K(S))\}$, $\{eS : e \in E(K(S))\}$, and $\{eSe : e \in E(K(S))\}$ are partitions of $K(S)$.

Theorem 1.2.35. *[4] The intersection of a minimal right ideal and a minimal left ideal is a group, and all these groups are isomorphic.*

Remark 1.2.36. [4] A compact cancellative right topological semigroup is a group.

Definition 1.2.37. A space X is called *extremally disconnected* if the closure of an open set is open or, equivalently, if the closures of disjoint open sets are disjoint.

Every discrete space is extremally disconnected.

Definition 1.2.38. A topological space X is called *zero-dimensional* if it has a base of clopen sets.

Definition 1.2.39. Let Ω be uncountable well ordered set with largest element w_1 with the property that if $\alpha \in \Omega$ with $\alpha < w_1$ then $\{\beta \in \Omega | \beta \leq \alpha \; is \; countable\}$. The elements of Ω are *ordinals* with w_1 being the *first* uncountable ordinal and the set $\Omega_0 = \Omega - \{w_1\}$ is the set of countable ordinals.

Definition 1.2.40. If α and β are ordinals and $\alpha < \beta$, then we say α is *predecessor* of β and β is *successor* of α

β is called an *immediate successor* of α if β is the smallest ordinal larger than α.
Every ordinal α has immediate successor ordinal often denoted by $\alpha + 1$.

Definition 1.2.41. A *limit ordinal* is an ordinal number which have predecessor without immediate predecessor. It is equal to the supremum of all the ordinals below it, but is not zero.

Definition 1.2.42. A family P of nonempty sets is a *partition* of X if

(a) The union of the elements of P is equal to X. (The elements of P are said to cover X.)

(b) The intersection of any two distinct elements of P is empty. (We say the elements of P are pairwise disjoint.)

If X is a space and if each set in P is closed, then we say P is a *closed partition* of X.

Chapter 2

Compactification of discrete Space

In this chapter we are mainly interested in the compactifications of a discrete space. If S is a finite discrete space then S itself is a compactification of S and hence we consider infinite discrete spaces only. Throughout this chapter S denotes an infinite discrete space. In the following we construct several compactifications of S.

2.1 Stone representation Theorem

Definition 2.1.1. [13] A *Boolean algebra* is a non empty set A together with two binary operations $\vee$ and $\wedge$ (on A), a unary operation $'$ and two distinguished elements 0 and 1, satisfying the following axioms: For $p, q, r \in A$,

(1) $0' = 1$ $\qquad$ $1' = 0$

(2) $p \wedge 0 = 0$ $\qquad$ $p \vee 1 = 1$

(3) $p \wedge 1 = p$ $\qquad$ $p \vee 0 = p$

(4) $p \wedge p' = 0$ $\qquad$ $p \vee p' = 1$

(5) $(p')' = p$

(6) $p \wedge p = p$ $\quad\quad$ $p \vee p = p$

(7) $(p \wedge q)' = p' \vee q'$ $\quad\quad$ $(p \vee q)' = p' \wedge q'$

(8) $p \wedge q = q \wedge p$ $\quad\quad$ $p \vee q = q \vee p$

(9) $p \wedge (q \wedge r) = (q \wedge p) \wedge r$ $\quad\quad$ $p \vee (q \vee r) = (q \vee p) \vee r$

(10) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ $\quad\quad$ $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

Examples 2.1.2. *(1) [13]The class of all subsets of a set X is a Boolean algebra under the operations of $\vee$ =union, $\wedge$ =intersection and $'$ = complementation , the 0 element is the empty set and the 1 element is the set X itself.*

(2) A Boolean algebra with only one element is called a trivial Boolean algebra

(3) [18]The simplest non trivial Boolean algebra has only two elements 0, 1 *, and is defined by the rules:*

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

a	0	1
a'	1	0

(4) [18] The set of all subsets of X that are either finite or cofinite is a Boolean algebra under the same operation of part 1.

Definition 2.1.3. [13] *A Boolean subalgebra* of a Boolean algebra A is a subset B of A such that B, together with the 0 and 1 elements is a Boolean algebra under the same operation of A. The algebra A is called a *(Boolean) extension* of B.

Example 2.1.4. *[13] The set of all subsets of X that are either finite or cofinite is a Boolean subalgebra of the set of all subsets of X that are either countable or cocountable of X.*

Every Boolean subalgebra B has the element 1 because if $p \in B$ then $P \vee P' = 1 \in B$. Also $p \wedge P' = 0 \in B$ so, the unit and zero elements in B is the same as the unit and zero in A.

[13]To be a Boolean subalgebra it is not enough to be a subset that is a Boolean algebra in its own right, however natural the Boolean operations may appear. The Boolean operations of a subalgebra, by definition, must be the restrictions of the Boolean operations of the whole algebra.
To illustrate the situation, let Y be a non-empty subset of a set X. Both $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are Boolean algebras in a natural way and clearly every element of $\mathcal{P}(Y)$ is an element of $\mathcal{P}(X)$. Since, however, the unit of $\mathcal{P}(X)$ is X, whereas the unit of $\mathcal{P}(Y)$ is Y, it is not true that $\mathcal{P}(Y)$ is a Boolean subalgebra of $\mathcal{P}(X)$. Another reason why it is not true is, of course, that complementation in $\mathcal{P}(Y)$ is not the restriction of complementation in $\mathcal{P}(X)$.

Definition 2.1.5. [13] Let A be a non-empty collection of subsets of X that is closed under intersection, union, and complement. That is; if P and Q are in A, then so are $P \cap Q$, $P \cup Q$, and P'. Then A is a Boolean algebra which is called a *field of sets.*

Since A contains at least one element, say P then $P' \in A$ it follows that $P \cap P' = \phi \in A$, so A contains ϕ and X. Therefore, in a field of sets, the zero element is ϕ and the 1 element is X.

Examples 2.1.6. *(1) [18] The set of all subsets of X that are either finite or cofinite is a field of sets.*

(2) If X is topological space, then the set $CO(X)$ of all clopen subsets of X is a field of sets.

Definition 2.1.7. [13] *A Boolean homomorphism* is a mapping f from a Boolean algebra B to, a Boolean algebra A such that :

(1) $$f(p \wedge q) = f(p) \wedge f(q)$$

(2) $$f(p \vee q) = f(p) \vee f(q)$$

(3) $$f(p') = (f(p))'$$

whenever p and q are in B.

we shall usually write $f(p)'$ instead of $(f(p))'$.

Definition 2.1.8. [13] The *kernel* of a homomorphism f from a Boolean algebra B to a Boolean algebra A is the set of elements in B that map to 0 in A. In symbols, the kernel of f is defined by

$$Ker(f) = f^{-1}(\{0\}) = \{p \in B : f(p) = 0\}.$$

Note that if $Ker(f) = \{0\}$, then f is one to one.

Example 2.1.9. *[13] consider a field B of subsets of a set X , and let x_0 be an arbitrary point of X. For each set P in B, let*

$$f(P) = \begin{cases} 1, & x_0 \in P \\ 0, & x_0 \notin P \end{cases}$$

To prove that the mapping f is a 2-valued homomorphism on B, we will verify identities (1),(2) and (3).
The definition of f, and the definitions of the Boolean operations in a field of sets and in the Boolean algebra consists only the elements $\{0, 1\}$, justify the following equivalences:

$$\begin{aligned} f(P \cap Q) = 1 \quad & if\ and\ only\ if \quad x_0 \in P \cap Q, \\ & if\ and\ only\ if \quad x_0 \in P\ and\ x_0 \in Q, \\ & if\ and\ only\ if \quad f(P) = 1\ and\ f(Q) = 1, \\ & if\ and\ only\ if \quad f(P) \wedge f(Q) = 1; \end{aligned}$$

So,

$$f(P \cap Q) = f(P) \wedge f(Q).$$

Similarly,

$$\begin{aligned} f(P \cup Q) = 1 \quad & if\ and\ only\ if \quad x_0 \in P \cup Q, \\ & if\ and\ only\ if \quad x_0 \in P\ or\ x_0 \in Q, \\ & if\ and\ only\ if \quad f(P) = 1\ or\ f(Q) = 1, \end{aligned}$$

$$\textit{if and only if} \qquad f(P) \vee f(Q) = 1;$$

So,

$$f(P \cup Q) = f(P) \vee f(Q).$$

Similarly,

$$f(P') = 1 \qquad \textit{if and only if} \qquad x_0 \in P',$$

$$\textit{if and only if} \qquad x_0 \notin P,$$

$$\textit{if and only if} \qquad f(P) = 0,$$

$$\textit{if and only if} \qquad f(P)' = 1;$$

So,

$$f(p') = (f(p))'$$

the kernal of f is

$$Ker(f) = \{p \in B : f(p) = 0\}$$

$$Ker(f) = \{p \in B : x_0 \notin P\}$$

Definition 2.1.10. [13] We define a binary relation $\leq$ in a Boolean algebra by $p \leq q$ or $q \geq p$ if $p \wedge q = p$, or equivalently, $p \vee q = q$. In this case we say that p is *below* q, or q is *above* p .

As a specail case, and for sets, we say $P \leq Q$ if $P \cap Q = P$ $(P \cup Q = Q)$ in the case when $P \subseteq Q$.
The set A of all subelements of p_0, consists of all elements p with $p \leq p_0$,

Definition 2.1.11. [13] A *(Boolean) filter* in a Boolean algebra B is a subset N of B such that

(1) $1 \in N$

(2) if $p \in N$ and $q \in N$, then $p \wedge q \in N$

(3) if $p \in N$ and $q \in B$, then $p \vee q \in N$.

Condition (1) can be replaced by the condition that N be non-empty. Condition (3) can be replaced by (4) if $p \in N$ and $p \leq q$, then $q \in N$

Definition 2.1.12. [13] The filter *generated* by a subset E of a Boolean algebra B is defined to be the intersection of the filters that include E. (There is always one such filter, namely B.) In other words, it is the smallest filter that includes E . A filter N is *principal* if it is generated by a single element p. In this case $N = \{q \in B : p \leq q\}$.

Definition 2.1.13. [13] A *maximal filter*, or an *ultrafilter*-as it is often called-is a proper filter that is not properly included in any other proper filter.

Remark 2.1.14. [19]

(a) Every filter of Boolean algebra A is a subset of some ultrafilter.

(b) If A is a Boolean algebra then any ultrafilter of A consists exactly one of the elements a and a' for each element a of A.

Theorem 2.1.15. *[18] Let B denots a Boolean algebra and F an ultrafiter in it, then for all $a, b \in B$ if $a \vee b \in F$ then either $a \in F$ or $b \in F$.*

Proof. Let B be a Boolean algebra and F a proper filter in it. Suppose to contrary that $a \vee b \in F$, while $a \notin F$ and $b \notin F$. Then by above remark $a' \in F$ and $b' \in F$, and hence $(a' \wedge b') \in F \Rightarrow (a \vee b)' \in F$ contradiction. □

Theorem 2.1.16. *[11] Let U be a Boolean algebra and let $S(U)$ be the set of all ultrafilters on U. For each $x \in U$ put $\lambda(x) = \{p \in S(U) : x \in p\}$. If a topology τ is assigned to $S(U)$ by letting $\{\lambda(x) : x \in U\}$ be an open base for τ then $(S(U), \tau)$ is a compact Hausdorff totally disconnected space. The set $S(U)$, topologized as above, is called the* Stone space *of U.*

Theorem 2.1.17. *[18]**Stone representation Theorem***
Every boolean algebra is isomorphic to the algebra of clopen supset of its stone space

Proof. Let $f : U \to \mathcal{P}(S(U))$ defined by $f(x) = \lambda(x)$. To prove that f is homomorphism, let x and y be elements in U then :

$$\begin{aligned} f(x \vee y) &= \lambda(x \vee y) = \{p \in S(U) : x \vee y \in p\} \\ &= \{p \in S(U) : x \in p \vee y \in p\} \\ &= \{p \in S(U) : x \in p\} \cup \{p \in S(U) : y \in p\} \\ &= \lambda(x) \cup \lambda(y) \\ &= f(x) \cup f(y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1) \end{aligned}$$

The first and last equalities use the definition of f, the second uses the Theorem 2.1.15, the third uses the definition of union. Now,

$$\begin{aligned} f(x \wedge y) &= \lambda(x \wedge y) = \{p \in S(U) : x \wedge y \in p\} \\ &= \{p \in S(U) : x \in p \wedge y \in p\} \\ &= \{p \in S(U) : x \in p\} \cap \{p \in S(U) : y \in p\} \\ &= \lambda(x) \cap \lambda(y) \\ &= f(x) \cap f(y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2) \end{aligned}$$

Similarly,

$$\begin{aligned} f(x') &= \lambda(x') = \{p \in S(U) : x' \in p\} \\ &= \{p \in S(U) : x \notin p\} \\ &= \{p \in S(U) : x \in p\}' \\ &= f(x)' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3) \end{aligned}$$

In order to prove that f is one-to-one, it suffices to show that its $ker(f) = \{0\}$. If $x \neq 0$, then there is a principal ultrafilter p generated by x such that $x \in p$; consequently, the set $\lambda(x) = f(x)$ is not empty. Thus for each nonzero element x in U, $\lambda(x) \neq \phi$ so x can not be in the $ker(f)$.

Next, we will show that the the sets $\{\lambda(x) : x \in U\}$ is the only clopen sets in $S(U)$. First we will show that $\lambda(x)$ is closed for any $x \in U$ by proving that $S(U) \setminus \lambda(x) = \lambda(x')$, so let $p \in S(U) \setminus \lambda(x)$ then $x \notin p$, thus $x' \in p \Rightarrow p \in \lambda(x')$, conversely let $p \in \lambda(x')$ so $x' \in p$ then $x \notin p$ so $p \in S(U) \setminus \lambda(x)$. Hence $\lambda(x)$ is closed.

To show that $\lambda(x)$ is the only clopen sets, suppose that C is any clopen subset of $S(U)$. First from (1), $\lambda(x \vee y) = \lambda(x) \cup \lambda(y)$.

Let

$$B = \{\lambda(x) : x \in U \ \ and \ \ \lambda(x) \subseteq C\}.$$

Since C is open and $\{\lambda(x) : x \in U\}$ is an open base for τ then for all $p \in C$ there is $x_0 \in U$ such that $p \in \lambda(x_0) \subseteq C$ so, $C \subseteq \cup\lambda(x_0)$. Thus B is an open cover of C. Since C is closed, it is compact since τ is compact Hausdroff space. So pick a finite subfamily $\mathcal{F}$ of B such that $C \subseteq \cup_{x\in\mathcal{F}}\lambda(x)$ so, $C = \lambda_{x\in\mathcal{F}}(\vee x)$ from (1). Set $y = \vee x$, $x \in \mathcal{F}$, so $C = \lambda(y)$. Thus if C is clopen subset of $S(U)$ then $C = \lambda(x)$ for some $x \in U$. So, $f(x) = \lambda(x)$, and hence f is onto. So The mapping f is an isomorphism from U onto the Boolean algebra of open-and-closed subsets of $S(U)$. □

It is well known that a Stone space $S(B)$ is compact Hausdorff and totally disconnected and that any topological space is homeomorphic to $S(B)$ for some suitable Boolean algebra B. This result is known as the *Stone-Duality*

Definition 2.1.18. [18] A *complete Boolean algebra* is a Boolean algebra in which every subset has a supremum (least upper bound).

Examples 2.1.19. *[18]*

(1) The class of all subsets of a set X is a complete Boolean algebra.

(2) The set of all subsets of X that are either finite or cofinite is incomplete Boolean algebra.

Completion of Boolean algebra [10]

A Boolean subalgebra L contained in a Boolean algebra M is said to be *generates* M if every element of M is supremum of elements of L.

A homomorphism f of Boolean algebras from B to A is said to be *complete* if it preserves any suprema which exist. That is; if a family $\{p_i\}$ of elements in B has

a supremum P, then the family $\{f(p_i)\}$ has $f(p)$ as a supremum.
A complemetion of L is a pair (M, e) where M is a complete Boolean algebra and e is a complete monomorphism of L into M and $e(L)$ generates M .We will usually think of L as a subalgebra of M.
The stone representation Theorem show that every Boolean algebra has a completion since L is isomorphic to the algebra of clopen supset of its stone space.

2.2 Stone-Čech compactification

Recall that by an embedding of a topological space X into a topological space Z, we mean a function $\phi : X \to Z$ which defines a homeomorphism from X onto $\phi[X]$.

Definition 2.2.1. [15] For any topological space X, a compact Hausdorff space Y is called a *compactification* of X, if there is an embedding ϕ of X into Y such that $\phi(X)$ is dense in Y.

We shall identify X with the subspace $\phi(X)$ of Y.

Definition 2.2.2. [15] If $Y \setminus X$ is a singleton set then Y is called *a one-point compactification* of X. If $Y \setminus X$ is a finite (countable or infinite) set, then Y is called *a finite (countable or infinite respectively) compactification* of X.

Example 2.2.3. *Suppose the real line* $\mathbb{R}$ *under the Euclidean topology. Define the homeomorphism* $\varphi : \mathbb{R} \to (0,1)$ *by* $\varphi(x) = \frac{x}{1+|x|}$. *Since* $(0,1)$ *is dense in the compact Hausdorff space* $[0,1]$, *then* $[0,1]$ *is a compactification of* $\mathbb{R}$.

Theorem 2.2.4. *A space* X *has a compactification if and only if it is completely regular.*

Definition 2.2.5. [4] let X be a completely regular topological space. A *Stone-Čech compactification* of X is a pair (ϕ, Z) such that:

(a) Z is a compact Hausdroff space .

(b) ϕ is an embedding of X into Z.

(c) $\phi[X]$ is a dense in Z, and

(d) given any compact space Y and any continuous function $f : X \to Y$ there exists a continuous function $g : Z \to Y$ such that $g \circ \phi = f$.

Proposition 2.2.6. *[4] Let X be a completely regular topological space and (ϕ, Z) and (τ, W) be Stone-Čech compactification of X. Then there is a homeomorphism $\gamma : Z \to W$ such that $\gamma \circ \phi = \tau$*

Proof. Since (τ, W) is a Stone-Čech compactification of X then τ, is an embedding of X into W, hence $\tau : X \to \tau(X)$ is continuous. By Theorem 1.2.12 $\tau : X \to W$ is continuous. But W is compact and (ϕ, Z) is a Stone–Čech compactification of X then there exists a continuous function $\gamma : Z \to W$ such that $\gamma \circ \phi = \tau$. □

The a bove remark can be viewed as saying:" The Stone-Čech compactification of X is unique up to homoemorphism"
Throughout this section S will be denote an infinite discrete space. Recall that the power set $\mathcal{P}(S)$ of all subsets of S together with the usual set operations is a Boolean algebra. As a special case a filter $\mathcal{U}$ on a set S is a nonempty collection of nonempty subsets S with the following properties:

(a) if F_1, $F_2 \in \mathcal{U}$ then $F_1 \cap F_2 \in \mathcal{U}$,

(b) if $F \in \mathcal{U}$ and $F \subseteq E$, then $E \in \mathcal{U}$.

A proper filter which is maximal among the class of proper filters is called an ultrafilter of B.

Remark 2.2.7. Let S be a infinite discrete space. We will denote $S(S)$ (the set of all ultrafilters of S) by βS and for any $A \subseteq S$ we will denote $\lambda(A)$ by $\hat{A}$, so $\hat{A} = \{U \in \beta S | A \in U\}$ and $\{\hat{A} \mid A \subseteq S\}$ forms a base for a topology on βS.

Theorem 2.2.8. *[4] Let S be a discrete space, $a \in S$, and $e : S \to \beta S$ defined by $e[a] = \{A \subseteq S : a \in A, \}$ then $(e, \beta S)$ is a Stone–Čech compactification of S.*

Remark 2.2.9. For each $a \in S$, $e(a)$ is the principal ultrafilter corresponding to a. For any $A \subseteq S$, $e(A) = \bigcup e[a]$ for all $a \in A$. The principal ultrafilters are being identified with the points of S, so $S \subseteq \beta S$ and we denote $S^* = \beta(S) \setminus S$.

Note that from Theorem 2.2.8 we conclude that S is dense in βS and given any compact space Y and any function $f : S \to Y$ there exists a continuous function $g : \beta S \to Y$ such that $g_{|S} = f$.

Theorem 2.2.10. *[4] Let S be a infinite discrete space and let A, $B \subseteq S$.*

(a) $\widehat{(A \cup B)} = \widehat{A} \cup \widehat{B}$.

(b) $\widehat{(A \cap B)} = \widehat{A} \cap \widehat{B}$.

(c) $\widehat{(S \setminus A)} = \beta S \setminus \widehat{A}$.

Definition 2.2.11. [4] Let D be a discrete space, let Y be a compact space, and let $f : D \to Y$. Then $\widehat{f}$ is the continuous function from βD to Y such that $\widehat{f}_{|D} = f$.

In the following theorem we will show that for any discrete semigroup S, there is a natural extension of the operation $(\cdot)$ of S to βS making βS a compact right topological semigroup with S contained in its topological center.

Theorem 2.2.12. *[4] Let S be a discrete space and let $\cdot$ be a binary operation defined on S. Then there is a unique extension binary operation $* : \beta S \times \beta S \to \beta S$ satisfying the following three conditions:*

(a) for every $s, t \in S$, $s * t = s \cdot t$,

(b) for each $q \in \beta S$ the function $\rho_q : \beta S \to \beta S$ is continuous where $\rho_q(p) = p*q$,

(c) for each $s \in S$, the function $\lambda_s : \beta S \to \beta S$ is continuous, where $\lambda_s(q) = s*q$.

We will denote the operation on βS by the same symbol as that used for the operation on S.

Definition 2.2.13. [4] Let S be a discrete space, Y a topological space, $p \in \beta S$, and $y \in Y$. If $A \in p$ and $f : A \to Y$, then we shall write $lim_{a \to p} f(a) = y$ if and only if for every neighborhood V of y, there is neighborhood U of p in βS such that $f[A \cap U] \subseteq V$.

The statements in the following proposition follow immediately from the fact that λ_s is continuous for every $s \in S$ and ρ_q is continuous for every $q \in \beta S$.

Proposition 2.2.14. *[4] Let $\cdot$ be a binary operation on a discrete space S.*

(a) If $s \in S$ and $q \in \beta S$, then $s \cdot q = lim_{t \to q} s \cdot t$

(b) If $p, q \in \beta S$, then $p \cdot q = lim_{s \to p}(lim_{t \to q} s \cdot t)$

Where s, t denote elements of S

Theorem 2.2.15. *[4] Let $(S, .)$ be a semigroup .Then the extended operation on βS is associative.*

Proof. Let $p, q, r \in \beta$. From Proposition 2.2.14 we consider $\lim_{a \to p} \lim_{b \to q} \lim_{c \to r} (a.b).c$, where a, b and c denote elements of S .We have:

$$\begin{aligned}
\lim_{a\to p}\lim_{b\to q}\lim_{c\to r}(a.b).c &= \lim_{a\to p}\lim_{b\to q}(a.b).r && (\textit{because } \lambda_{a.b} \textit{ is continuous}) \\
&= \lim_{a\to p}(a.q).r && (\textit{because } \rho_r \circ \lambda_a \textit{ is continuous}) \\
&= (p.q).r && (\textit{because } \rho_r \circ \rho_q \textit{ is continuous}).
\end{aligned}$$

Also:

$$\begin{aligned}
\lim_{a\to p}\lim_{b\to q}\lim_{c\to r} a.(b.c) &= \lim_{a\to p}\lim_{b\to q} a.(b.r) && (\textit{because } \lambda_a \circ \lambda_b \textit{ is continuous}) \\
&= \lim_{a\to p} a.(q.r) && (\textit{because } \lambda_a \circ \rho_r \textit{ is continuous}). \\
&= p.(q.r) && (\textit{because } \rho_{q.r} \textit{ is continuous}).
\end{aligned}$$

Since S is associative then $(a.b).c = a.(b.c)$ so $(p.q).r = p.(q.r)$. □

As a consequence of Theorems 2.2.8, 2.2.12 and 2.2.15 we see that βS is a compact right topological semigroup.

Remark 2.2.16. Let $(S, .)$ be a semigroup .If $p, q \in S^*$ then $p.q \in S^*$

Proof. Since $p, q \in S^*$ then $p = lim_{t \to p} t$ and $q = lim_{s \to q} s$ where s, t elements in S so

$$\begin{aligned}
&\lim_{t\to p}\lim_{s\to q} ts = \lim_{t\to p}(\lim_{s\to q} ts) = \lim_{t\to p} tq && (\textit{because } \lambda_t \textit{ is continuous}) \\
&= \lim_{t\to p}\rho_q(t) = \rho_q(\lim_{t\to p} t) = \rho_q(p) = pq && (\textit{because } \rho_q \textit{ is continuous})
\end{aligned}$$

□

Definition 2.2.17. Let $(S, .)$ be a semigroup, let $A \subseteq S$ and let $s \in S$. Then we define

(a) $s^{-1}A = \{y \in S : s \cdot y \in A\}$.

(a) $As^{-1} = \{y \in S : y \cdot s \in A\}$.

Theorem 2.2.18. *[4] Let $(S, .)$ be a semigroup, let $A \subseteq S$. Then*

(a) For any $s \in S$ and $q \in \beta S$, $A \in s \cdot q$ if and only if $s^{-1}A \in q$.

(b) For any p, $q \in \beta S$, $A \in p.q$ if and only if $\{s \in S : s^{-1}A \in q\} \in p$.

Definition 2.2.19. [4] Let S be any set and let $\mathcal{U}$ be a filter on S .A family $\mathcal{A}$ is *a filter base* for $\mathcal{U}$ if and only if $\mathcal{A} \subseteq \mathcal{U}$ and for each $B \in \mathcal{U}$ there is some $A \in \mathcal{A}$ such that $A \subseteq B$.

Remark 2.2.20. [17] Let $(S, .)$ be a semigroup and let p and $q \in \beta S$ then the ultrafilter pq has a base of subsets $\bigcup\{xB_x : x \in A\}$, where $A \in p$ and $B_x \in q$.

Proof. Let $\mathcal{A}$ be a family of subsets $\bigcup\{xB_x : x \in A\}$. First we will prove that $\mathcal{A} \subseteq pq$, so let $H \in \mathcal{A}$ then $H = \bigcup\{xB_x : x \in A\}$ for some $A \in p$ and $B_x \in q$. Since $A \in p$ then $A \neq \phi$, so let $x_0 \in A$ then $\{x_0 B_{x_0} : x_0 \in A\} \subseteq H$ so, $B_{x_0} \subseteq x_0^{-1}H$. Since $B_{x_0} \in q$ and q is an ultrafilter, then $x_0^{-1}H \in q$. which implies that $x_0 \in \{t \in S : t^{-1}H \in q\}$. Hence $A \subseteq \{t \in S : t^{-1}H \in q\} \in p$ since $A \in p$. From Theorem 2.2.18 $H \in pq$.
Seconed we will prove that for any $B \in pq$ there is some $H \in \mathcal{A}$ such that $H \subseteq B$. Let $B \in pq$ then from Theorem 2.2.18, $\{t \in S : t^{-1}B \in q\} \in p$. Let $A = \{t \in S : t^{-1}B \in q\}$, then $A \in p$, and for all $X \in A$ we have $x^{-1}B \in q$. Put $B_x = x^{-1}B$ and let $H = \bigcup\{xB_x : x \in A\}$. Clearly $H \subseteq B$. □

2.3 Compactifications of Discrete Spaces

Remark 2.3.1. [15]

Let S be infinite discrete space and let B be a Boolean subalgebra of $\mathcal{P}(S)$ then a proper filter U of B is maximal if and only if B $\setminus U$ is closed under finite unions.

Proof. Let U be an ultrafiter and let $\{A_1, A_2, ..., A_n\}$ be subsets of $\mathrm{B} \setminus U$ then $A_i^c \in U$ for all $i = 1, 2, .., n$, so by definition of filter, $\cap_{i=1}^{n} A_i^c \in U$. Since U is an ultrafilter, $(\cap_{i=1}^{n} A_i^c)^c \notin U$. Hence $(\cap_{i=1}^{n} A_i^c)^c = \cup_{i=1}^{n} A_i \in \mathrm{B} \setminus U$ .Conversely , suppose that $\mathrm{B} \setminus U$ is closed under finite unions and assume by contrary that U is not maximal then there exist a ultrafilter D of B such that $U \subset D$ then choose $A \in D \setminus U$ so $A^c \notin D$ which implies $A^c \notin U$. Since $\mathrm{B} \setminus U$ is closed under finite unions then $S = A \cup A^c \in \mathrm{B} \setminus U$. Since U is filter then $U \neq \phi$. choose $A \in U$, then $A \subseteq S$. From definition of filter $S \in U$ contradiction. □

Let B_0 be the Boolean subalgebra of all subset of S that are either finite or cofinite.

Theorem 2.3.2. *[3] Let S be an infinite discrete space. Then $\beta\mathrm{B}_0$ is the one-point compactification of S and $\beta\mathcal{P}(S)$ is the Stone-Čech compactification of S.*

In the following we shall prove that $\beta\mathrm{B}$ is a compactification of S, for any Boolean subalgebra B of $\mathcal{P}(S)$ containing B_0.

Theorem 2.3.3. *[15] Let S be an infinite discrete pace and* B *a Boolean subalgebra of $\mathcal{P}(S)$ containing* B_0. *Then the Stone space $\beta\mathrm{B}$ is a compactification of S in which S is open.*

Proof. It is well known that $\beta\mathrm{B}$ is a compact Hausdorff space. Now define a map $\alpha : S \to \beta\mathrm{B}$ by $\alpha(d) = \{A \in \mathrm{B} | d \in A\}$. Since B containing B_0 then $\{d\} \in \mathrm{B}$ for all $d \in S$. First we will show $\alpha(d)$ is an ultrafilter, since $\{d\} \in \alpha(d)$ then $\alpha(d) \neq \phi$. Moreover if A_1, $A_2 \in \alpha(d)$ then $d \in A_1 \cap A_2$, so $A_1 \cap A_2 \in \alpha(d)$. Now let $A_1 \in \alpha(d)$ so for any set A_2 such that $A_1 \subseteq A_2$ we have $d \in A_1 \subseteq A_2$ which implies that $\alpha(d)$ is filter. To prove that it is maximal let $\{A_1, ..., A_n\}$ be in $\mathrm{B} \setminus \alpha(d)$ so $d \notin A_i$ for all $i = 1, ..., n$. Hence $d \notin \cup_{i=1}^{n} A_i$. Therefore $\mathrm{B} \setminus \alpha(d)$ is closed under finite union and so $\alpha(d)$ is maximal.
Also, for any $d \in S$, $\alpha(d) = \widehat{\{d\}}$, since any ultrafilter of B containing $\{d\}$ must be equal $\alpha(d)$, To see this let U be an ultrafilter containing $\{d\}$ and let $A \in \alpha(d)$ then $d \in A$ so $\{d\} \subseteq A$ which impliese that $A \in U$ from the definition of ultrafilter. Hence $\alpha(d) \subseteq U$. Since $\alpha(d)$ is ultrafilter, $\alpha(d) = U$.

To prove α is one-one, let $d_1, d_2 \in S$ such that $\alpha(d_1) = \alpha(d_2)$, since B containing B_0 then $\{d_1\}$ is in B and so $\{d_1\} \in \alpha(d_1) = \alpha(d_2)$ so $d_2 \in \{d_1\}$ which implies $d_1 = d_2$.

Since S is discrete, we have $\alpha : S \to \beta$B is a continuous map. Now we shall prove that α is an embedding of S into βB. For any $d \in S$, since $\{\alpha(d)\} = \alpha(S) \cap \widehat{\{d\}}$ and hence each singleton set is open in $\alpha(S)$. This proves that $\alpha(S)$ is discrete and hence α is an embedding of S into βB. Next, if $\widehat{A}$ is a non empty basic open set in βB, then $A \neq \phi$ and if $a \in A$, then $\alpha(a) \in \alpha(S) \cap \widehat{A}$ and so $\alpha(S) \cap \widehat{A} \neq \phi$. Therefore $\alpha(S)$ is dense in βB. Thus βB is a compactification of S. Also since $\{\alpha(d)\} = \widehat{\{d\}}$, This proves that $\alpha(S)$ is open in βB. □

Remark 2.3.4. [15] If A and B are clopen subsets of topological space X and D is a dense subset of X then:

(a) $A \cap D = \phi \Leftrightarrow A = \phi$

(b) $A \cap D \subseteq B \cap D \Leftrightarrow A \subseteq B$.

(c) $A \cap D = B \cap D \Leftrightarrow A = B$.

Proof. (a) By contrapositive, since A is open and D is a dense subset then $A \neq \phi$ if and only if $A \cap D \neq \phi$.

(b) Suppose that $A \cap D \subseteq B \cap D$ and let $x \in A$.
Case(1): If $x \in D$ then

$$x \in A \cap D \subseteq B \cap D \Rightarrow x \in B$$

Case(2): If $x \notin D$. Suppose to contrary that $x \notin B$ then $x \in B^c$ which is open set. so $A \cap B^c$ is open set containing x. Since D is a dense subset of X then $(A \cap B^c) \setminus \{x\} \cap D \neq \phi$. Pick $y \in A \cap B^c \cap D \subseteq B \cap D \cap B^c = \phi$ contradiction. Hence $x \in B$.
Conversely if $A \subseteq B$ then $A \cap D \subseteq B \cap D$.

(c) By using part (b) we have

$$A \cap D = B \cap D \Leftrightarrow A \cap D \subseteq B \cap D \textit{ and } B \cap D \subseteq A \cap D$$

$$\Leftrightarrow A \subseteq B \ and \ B \subseteq A \Leftrightarrow A = B.$$

□

Now we have the following, which is a converse of Theorem 2.3.3, in the sense that any totally disconnected compactification of S must necessarily be (homeomorphic to) the Spectrum of a Boolean subalgebra of $\mathcal{P}(S)$ containing B_0.

Theorem 2.3.5. *[15] A compactification of an infinite discrete space S is totally disconnected if and only if it is homeomorphic to βB for some Boolean subalgebra B of $\mathcal{P}(S)$ containing* B_0.

Proof. Let Y be a compactification of S. Suppose that Y is totally disconnected. Consider $\mathrm{B} = \{A \cap S \mid A$ is a clopen subset of Y $\}$. It is not difficult to show that B is a Boolean subalgebra of $\mathcal{P}(S)$. Since S is discrete then for any $d \in S$, $\{d\}$ is open in S. But Y is totally disconnected hence there is a clopen subset A of Y such that $d = A \cap S$. This implies that $\{d\} \in \mathrm{B}$ for all $d \in S$ and hence all finite subsets of S must be in B, to prove this let A be finite subset of S then $A = \{d_1, ..., d_n\}$ where $d_i \in S$ for all $i = 1, ..., n$. Since $\{d\} \in \mathrm{B}$ for all $d \in S$ then for each $\{d_i\} \in A$ there exist clopen subset A_i such that $\{d_i\} = A_i \cap S$. Hence

$$A = \cup_{i=1}^{n}\{d_i\} = \cup_{i=1}^{n}(A_i \cap S) = (\cup_{i=1}^{n} A_i) \cap S \in B.$$

Also if $U \in S$, and $S \setminus U$ is finite, then $S \setminus U \in B$ so $S \setminus U = A \cap S$ for some clopen subset A of Y. Hence $U = (S \setminus U)^c \cap S = (A \cap S)^c \cap S = (A^c \cup S^c) \cap S = A^c \cap S$. Since A^c is clopen subset of Y, $U \in B$. So that B contains B_0. Now we shall prove that $Y \cong \beta\mathrm{B}$. Define $f : Y \to \beta\mathrm{B}$ by

$$f(y) = \{A \cap S | A \, is \ clopen \ in \ Y \ and \, y \in A\}.$$

Firstly, we will show that $f(y)$ is an ultrafilter of B. First $f(y) \neq \phi$ since $S = Y \cap S \in f(y)$. Now let A and $B \in f(y)$ then $A = A_1 \cap S$ and $\mathrm{B} = \mathrm{B}_1 \cap S$ where A_1 and B_1 are clopen subsets of Y and $y \in A_1 \cap B_1$. Hence

$$A \cap B = (A_1 \cap S) \cap (\mathrm{B}_1 \cap S) = (A_1 \cap \mathrm{B}_1) \cap S \in f(y).$$

If $A \in f(y)$, then $A = A_1 \cap S$ where A_1 is clopen subset of Y and $y \in A_1$. If $A \subseteq C$ for some $C \in$ B, then $A_1 \cap S \subseteq C = C_1 \cap S$ where C_1 is clopen subset of Y, so from Remark 2.3.4 $y \in A_1 \subseteq C_1$ so $C \in f(y)$. Therefore $f(y)$ is a filter. To show that it is maximal, Let $\{A_1, A_2, ..., A_n\} \in$ B $\setminus f(y)$. Since $A_i \in$ B for all $i = 1, ..., n$. then $A_i = C_i \cap S$ where C_i is clopen subset of Y. Moreover since $A_i \notin f(y)$, then $y \notin C_i$. Thus $y \notin \cup_{i=1}^{n} A_i$. Suppose by contrary that $\cup_{i=1}^{n} A_i \in f(y)$, then $\cup_{i=1}^{n} A_i = C \cap S$ for some clopen subset C of Y and $y \in C$. But

$$\cup_{i=1}^{n} A_i = \cup_{i=1}^{n} (C_i \cap S) = (\cup_{i=1}^{n} C_i) \cap S.$$

Hence $(\cup_{i=1}^{n} C_i) \cap S = C \cap S$. Sinse $(\cup_{i=1}^{n} C_i)$ and C are clopen subsets of Y and S is dense, then from Remark 2.3.4 $(\cup_{i=1}^{n} C_i) = C$ which is contradiction sinse $y \in C$ and $y \notin \cup_{i=1}^{n} C_i$. Therefore B $\setminus f(y)$ is closed under finite union so it is maximal.

Secondly, we will show that f is an injection. Let y_1 and $y_2 \in Y$ such that $f(y_1) = f(y_2)$. Since Y is totally disconnected, if $y_1 \neq y_2$, then there is a clopen subset A of Y such that A contains one of y_1 or y_2 and not contains the other, say $y_1 \in A$. Then $A \cap D \in f(y_1) = f(y_2)$ so $y_2 \in A$ which is contradiction. Thus $y_1 = y_2$.

Thiredly, to prove that f is onto, let U be any ultrafilter of B. Then U satisfies the finite intersection property and so is $\{A \mid A \text{ is clopen in } Y \text{ and } A \cap S \in U\}$. By the compactness of Y, there exists $y \in Y$ such that, for any clopen set A in Y such that $A \cap S \in U$ we have $y \in A$, otherwise suppose that for all $y \in Y$ there exists clopen set A_y in Y such that $A_y \cap S \in U$ and $y \notin A_y$ so $Y = \bigcup (A_y)^c$. But Y is compact so $Y = \bigcup_{i=1}^{n} (A_{y_i})^c$ where $\{(A_{y_1})^c, (A_{y_2})^c, ..., (A_{y_n})^c\}$ is finite subfamily from $\{(A_y)^c : y \in Y\}$ and $A_{y_i} \cap S \in U$. Thus $\phi = \cap_{i=1}^{n} A_{y_i} = \cap_{i=1}^{n} (A_{y_i} \cap S)$ contradiction since U satisfies the finite intersection property .

From this, it follows that $U \subseteq f(y)$ and hence $U = f(y)$ because U is an ultrafilter. Therefore f is a surjection too.

Finally, for any $A \cap S \in B$, where A is clopen in Y.

$$f^{-1}(\widehat{A \cap S}) = \{y \in Y | f(y) \in \widehat{A \cap S}\}$$

$$= \{y \in Y | A \cap S \in f(y)\}$$

$$= \{y \in Y | y \in A\} = A$$

This implies that f is continuous and hence f is a homeomorphism (since both Y and βB are compact Hausdorff spaces).
The converse follows from Theorem 2.3.3

$\square$

Definition 2.3.6. [13] The *(direct) product* of two Boolean algebras B and C is the algebra

$$A = BC = \{(p, q) \ : p \in B \ \ and \ \ q \in C\}$$

with the operations :

(1) $(p, q) \wedge (r, s) = (p \wedge r, q \wedge s)$.

(2) $(p, q) \vee (r, s) = (p \vee r, q \vee s)$.

(3) $(p, q)' = (p', q')$.

where $p \wedge r$ and $p \vee r$ are the operation of p and r in B, while $q \wedge s$ and $q \vee s$ are the operation of q and s in C. Also, p' and q' are the complements of p and q in B and C respectively.

The product A is a Boolean algebra with zero $(0, 0)$ and unit $(1, 1)$.
In the case when B and C are fields of subsets of disjoint sets Y and Z respectively, their product A represents itself naturally as a field of subsets of the union $X = Y \cup Z$. Every subset S of X can be written in one and only one way as a union $S = P \cup Q$ of a subset P of Y and a subset Q of Z. Indeed,

$$P = S \cap Y \qquad and \qquad Q = S \cap Z.$$

Furthermore, if S_1 and S_2 are subsets of X, say

$$S_1 = P_1 \cup Q_1 \qquad and \qquad S_2 = P_2 \cup Q_2,$$

then

(1) $S_1 \cap S_2 = (P_1 \cap P_2) \cup (Q_1 \cap Q)$.

(2) $S_1 \cup S_2 = (P_1 \cup P_2) \cap (Q_1 \cup Q)$.

(3) $S'_1 = P'_1 \cup Q'_1$.

The representation f of the product A as a field of subsets of X maps each pair (P, Q) in A to the union $P \cup Q$. Since every subset of X can be written in only one way as such a union, the mapping f is one-to-one. Almost everything that has been said can be generalized.

Definition 2.3.7. [13] The *(direct) product* of a family $\{A_i\}_{i \in I}$ of Boolean algebras is the algebra

$$A = \prod A_i,$$

The universe of the product consists of the functions p with domain I such that $p(i)$ - or p_i as we shall usually write is an element of A_i for each index i. The operation of p and q in A are the functions $p \wedge q$ and $p \vee q$ on I defined by

$$(p \wedge q)_i = p_i \wedge q_i \qquad \text{and} \qquad (p \vee q)_i = p_i \vee q_i,$$

while the complement of p is the function p' on I defined by

$$(p')_i = p'_i.$$

The right sides of these equations are computed in the Boolean algebra A_i for each i. Under these operations, the product A is a Boolean algebra with zero and unit of the product are the functions 0 and 1 on I defined by

$$0 = 0_i \ \text{and} \ 1 = 1_i,$$

where the elements on the right sides of these equations are the zero and unit of A_i for each i. The algebras A_i are the factors of the product A.

If each member of a family $\{A_i\}$ of Boolean algebras is a field of subsets of a set X_i, and if the sets X_i are mutually disjoint, then the product $A = \prod A_i$ is naturally represented as a field of subsets of the union $X = \bigcup X_i$ via the mapping f that assigns to each element P in A the subset $\bigcup Pi$ of X. (Recall that P is a function on I, and P_i is a subset of X_i for each i.) Since every subset of X can be written in only one way as such a union, the mapping f is one-to-one.

Theorem 2.3.8. *[15] Let Y be a compactification of an infinite discrete space S and $Y \setminus S$ finite. Then the following hold.*

(1) Y is totally disconnected.

(2) There exist pair-wise disjoint infinite sets $S_1, S_2, ..., S_n$ of S such that $S = \cup_{i=1}^{n} S_i$ and Y is the topological union of the one-point compactifications of $S_i's$.

(3) Let $B_i = \{X \subseteq S_i \mid X$ or $S_i \setminus X$ is finite$\}$. Then $B_1 \times B_2 \times ... \times B_n$ is isomorphic to the Boolean algebra of all clopen subsets of Y and hence Y is homeomorphic to $\beta(B_1 \times B_2 \times ... \times B_n)$.

Proof. (1) Let $Y\setminus S = \{x_1, x_2, ..., x_n\}$. Since Y is Hausdorff, we can find open sets $A_1, A_2, ..., A_n$ in Y such that $x_i \in A_i$ and $A_i \cap A_j = \phi$ for all $i \neq j$....(1). Since $Y \setminus S$ is finite and Y is Hausdroff, then $Y \setminus S$ is closed in Y, and hence S is open in Y. Also, since S is discrete, $\{d\}$ is open in S and hence in Y, for all $d \in S$. The class of all singleton sets $\{d\}$, $d \in S$, together with the A_is forms an open cover for Y. Since Y is compact, there exists a finite subset E of S such that $E \cup A_1 \cup A_2 \cup ... \cup A_n = Y$.....(2),

Each $A_i \setminus E$ is open in Y because both of A_i and E^c are open in Y. Then from (1)and (2) $\{E, A_1 \setminus E, A_2 \setminus E, ..., A_n \setminus E\}$ is a class of open sets in Y which are pairwise disjoint and cover Y. Therefore, each $A_i \setminus E = (\cup_{j\neq i} A_j \setminus E)^c = \cap_{j\neq i}(A_j \setminus E)^c$. So it is closed also. Since each $\{d\}$ is clopen in Y, it follows that the points of Y are separated by clopen subsets of Y. Thus Y is totally disconnected.

(2) There exist pairwise disjoint clopen sets $Y_1, Y_2, ..., Y_n$ in Y such that $Y = Y_1 \cup Y_2 \cup, , \cup Y_n$ and $x_i \in Y_i$ for each i. (For example, we can take $Y_1 = E \cup A_1$ and $Y_i = A_i \setminus E$ for $i > 1$). Put $S_i = Y_i \cap S$ .Then $S_i \neq \phi$. (since S is dense and Y_i is a nonempty open set in Y). Also, each S_i is infinite; otherwise $S_i \cup (Y \setminus Y_i)$ is a closed set containing S and hence $Y = S_i \cup (Y \setminus Y_i)$ because S is dense and Y is smallest closed set containing S. Further since $S_i \subset S$ for all i, then $\cup_{i=1}^{n} S_i \subseteq S$. Conversely, let $x \in S$ then $x \in Y$ because $S \subseteq Y$,

so $x \in Y_i$ for some i. Hence $x \in S_i$ so, $S \subseteq \cup_{i=1}^{n} S_i$. Therefore $S = \cup_{i=1}^{n} S_i$. Now we will prove that $Y_i \setminus S_i = \{x_i\}$. Clearly $x_i \in Y_i \setminus S_i$, suppose that $y \in Y_i \setminus S_i$ and $y \neq x_i$, Since $y \notin S_i$ then $y \notin S$ so $y = x_j$ and $j \neq i$ so $y \in Y_j$, but $y \in Y_i$ so $Y_i \cap Y_j \neq \phi$ contradiction. Since Y_i is closed in Y, Y_i is compact and hence Y_i is the one-point compactification of S_i. Thus Y is the topological union of the one-point compactifications of S_i's.

(3) It is well-known from Theorem 2.3.5 that Y_i is homeomorphic to the Stone space βB_i, where

$$B_i = \{X \subseteq S_i | X \text{ or } S_i \setminus X \text{ is finite}\}.$$

Further $Y = \cup_{i=1}^{n} Y_i \cong \cup_{i=1}^{n} \beta B_i$. To see this let $x \in Y$. Then $x \in Y_i$ for some unique i. Since Y_i is homeomorphic to the Stone space βB_i, there exists a homoeomorphism $g_i : Y_i \to \beta B_i$. Define $F : Y \to \cup_{i=1}^{n} \beta(B_i)$ given by $F(x) = g_i(x)$. F is homoeomorphism because g_i does. Also $\cup_{i=1}^{n} \beta(B_i) \cong \beta(B_1 \times ... \times B_2)$. Hence $B_1 \times ... \times B_n$ is isomorphic to the Boolean algebra of all clopen subsets of Y (by the Stone duality).

□

So, we proved that any finite compactification of S is totally disconnected. we do not know whether finiteness can be dropped in this. However, there are examples of infinite compactifications, other than the Stone-Čech compactification, which are totally disconnected. Consider the following example:

Example 2.3.9. *[15] Let $S = \cup_{n=1}^{\infty} S_n$, where each S_n is infinite and $S_n \cap S_m = \phi$ for all $n \neq m$. Let*

$$B = \{A \subseteq S| \text{ for each } n, \text{ either } S_n \cap A \text{ or } S_n \setminus A \text{ is finite}\}.$$

Then B is a Boolean subalgebra of $\mathcal{P}(S)$ containing B_0 because for all $A \in B_0$ either A is finite or $S \setminus A$ is finite, but in so $\cup_{n=1}^{\infty} S_n \cap A$ is finite or $\cup_{i=1}^{\infty} S_n \setminus A$ is finite. Thus for each n $S_n \cap A$ or $S_n \setminus A$ is finite so $A \in B$.
By Theorem 2.3.5, βB is a compactification of S which is totally disconnected. It can be easily seen that B is an incomplete Boolean algebra and hence B can not be

isomorphic to $\mathcal{P}(S)$ so that, by the Stone duality, $\beta(B)$ can not be homeomorphic to $\beta(\mathcal{P}(S))$. This says that $\beta(B)$ is not the Stone-Čech compactification of S. Further, for each positive integer n, let

$$\mathcal{U}_n = \{A \in B | S_n \cap \quad \textit{is infinite}\}.$$

First we will prove that $\mathcal{U}_n$ is an ultrafilter of B.

(1) *Since $S_n \in \mathcal{U}_n$ so $\mathcal{U}_n \neq \phi$. Also $\phi \notin \mathcal{U}_n$ because $S_n \cap \phi = \phi$ is finite.*

(2) *Let A_1 and $A_2 \in \mathcal{U}_n$ then $S_n \cap A_1$ and $S_n \cap A_2$ are infinite. Since $A_1 \in B$ and $A_2 \in B$ then $A_1 \cap A_2 \in B$ because B is Boolean subalgebra of $\mathcal{P}(S)$. Since $S_n \cap A_1$ and $S_n \cap A_2$ are infinite. Then from definition of B we have $S_n \setminus A_1$ and $S_n \setminus A_2$ are finite. So, $S_n \setminus A_1 \cap A_2$ is finite. But S_n is infinite so $S_n \cap A_1 \cap A_2$ is infinite. Hence $A_1 \cap A_2 \in \mathcal{U}_n$.*

(3) *If $A \in \mathcal{U}_n$ and $A \subseteq B$ then $S_n \cap A \subseteq S_n \cap B$ is infinite .*

(4) *To show that $\mathcal{U}_n$ is maximal, it is enough to show that $B \setminus \mathcal{U}_n$ is closed under finite unions. So let $\{A_1, A_2, ..., A_m\} \in B \setminus \mathcal{U}_n$. Suppose by contrary that $\cup_{i=1}^m A_i \in \mathcal{U}_n$, then $\cup_{i=1}^m A_i \cap S_n$ is infinite. Since $A_i \in B$ for all i, then either $S_n \cap A$ or $S_n \setminus A$ is finite. If for all $i = 1, ..., m$, $A_i \cap S_n$ is finite, then $\cup_{i=1}^m A_i \cap S_n$ is finite contradiction. So, there is $j = 1, ..., m$ such that $A_j \cap S_n$ is infinite which implies $A_j \in \mathcal{U}_n$ contradiction, Therefore $\mathcal{U}_n$ is an ultrafilter and $\mathcal{U}_n \in \beta B$.*

Let $f : S \to \beta B$ be the usual embedding defined by

$$f(d) = \{A \in B | d \in A\}.$$

Then as in the proof of Theorem 2.3.3, $f(d)$ is an ultrafilter. Also, $\mathcal{U}_n \neq f(d)$ for all $d \in S$ and for all $n \in \mathbb{N}$, because $S \setminus \{d\} \in \mathcal{U}_n$ but $S \setminus \{d\} \notin f(d)$. Moreover, for each n we have $S_n \in \mathcal{U}_n$. Since $S_n \cap S_m = \phi$ for all $n \neq m$, $\mathcal{U}_n \neq \mathcal{U}_m$. Thus $\beta B \setminus f(S)$ contains infinitely many points. So βB is an infinite compactification of S.

Finally we will show that each, $A \in B$ can be uniquely expressed as $A = \cup_{n=1}^\infty A_n$

where A_n is in the Boolean algebra $B_n = \{E \subseteq S_n|$ either E or $S_n \setminus E$ is finite $\}$ and that B is isomorphic to the product algebra $\prod_{n=1}^{\infty} B_n$, we will prove this in three steps:

(1) *Step (1): Suppose that $A \in B$ and let $A_n = S_n \cap A$. Clearly $\cup_{n=1}^{\infty} A_n \subseteq A$. Conversely; let $x \in A$. Since $A \subseteq S$, then $x \in S \Rightarrow x \in S_n$ for some n. Thus $x \in A_n$ and hence $x \in \cup_{n=1}^{\infty} A_n$. To show the uniqueness suppose that $A = \cup_{n=1}^{\infty} H_n$ where H_n is in the Boolean algebra B_n. Since $A_n = S_n \cap A$ then $A_n = S_n \cap (\cup_{n=1}^{\infty} H_n). = \cup_{n=1}^{\infty}(S_n \cap H_n)$. So, $S_n \cap H_n \subseteq A_n$. But $H_n \in B_n$. So, $H_n \subseteq S_n$ which implies $S_n \cap H_n = H_n$. Hence $H_n \subseteq A_n$. Suppose that $H_n \subset A_n$. Then there is $x \in A_n$ and $x \notin H_n$. So, $x \in H_m$ for some m. But $H_m \subseteq A_m$. Then $x \in A_m$ which implies that $A_n \cap A_m \neq \phi$. Hence $S_n \cap S_m \neq \phi$, which is a contradiction.*

(2) *Step (2): Since $A \in B$ then $S_n \cap A$ or $S_n \setminus A$ is finite. So A_n or $S_n \setminus A_n$ is finite. Since $A_n \subseteq S_n$, we have that $A_n \subseteq B_n$.*

(3) *Step (3): Since $S_n \cap S_m \neq \phi$ then $A_n \cap A_m \neq \phi$. Define $f : \prod_{n=1}^{\infty} B_n \rightarrow B$ by $f(p) = \bigcup_{n=1}^{\infty} A_n$ where $p_n = A_n$ for all n. Now we will show that f is an isomorphism.*

    (1) *Since every subset of B can be written in only one way as such a union, then the mapping f is one-to-one.*
    (2) *For all $A \in B$, $A = \cup_{n=1}^{\infty} A_n$, where $A_n \subseteq B_n$ for each n. Take $P \in \prod_{n=1}^{\infty} B_n$ such that $p_n = A_n$ for all n, then $f(P) = A$ Hence f is onto.*
    (3) *Let P and $Q \in \prod_{n=1}^{\infty} B_n$ then $(P \wedge Q)_n = P_n \cap Q_n)$. Thus $f(P \wedge Q) = \bigcup_{n=1}^{\infty}(P \wedge Q)_n = \bigcup_{n=1}^{\infty} P_n \cap Q_n = \bigcup_{n=1}^{\infty} P_n \cap \bigcup_{n=1}^{\infty} Q_n = f(P) \cap f(Q)$. Similarly,$f(P \vee Q) = \bigcup_{n=1}^{\infty}(P \vee Q)_n = \bigcup_{n=1}^{\infty} P_n \cup Q_n = \bigcup_{n=1}^{\infty} P_n \cup \bigcup_{n=1}^{\infty} Q_n = f(P) \cup f(Q)$. Also,*
    Since $(P')_i = P'_i$ then $f(P') = \bigcup_{n=1}^{\infty}(P')_n = \bigcup_{n=1}^{\infty} P'_n = (\bigcup_{n=1}^{\infty} P_n)^c = f(p)^c$.
    By the Stone duality, βB is homeomorphic to the direct sum of the spaces βB_n, $n \in \mathbb{N}$.

Chapter 3

Finite groups in Stone-Čech compactification

3.1 Ultrafilter semigroups and topologies

In this section all topologies are assumed to satisfy the T_1 separation axiom.

Definition 3.1.1. [14] A filter p on a topological space X is said to be *convergent* to x if and only if for each neighborhood U of x, $U \in p$.

Definition 3.1.2. [4] Let G be a group. A topology τ on G is *left invariant* if for every $U \in \tau$ and $a \in G$, $aU \in \tau$. Equivalently, τ is left invariant if for every $a \in G$, the left shift $\lambda_a : G \to G$ is continuous in τ.

Thus, left invariant topologies on G are those that make G into a left topological group.
Note that a left invariant topology is completely determined by the neighborhood filter at the identity.

Definition 3.1.3. [17] Let G be a group and let τ be a left invariant topology on G. *The ultrafilter semigroup* of τ is defined as a closed subsemigroup of G, by

$$Ult(\tau) = \{p \in G^* : p \text{ converges to the identity } \ e \in G \ \ in \ \ T\}.$$

Definition 3.1.4. [18] For an infinite family of groups G_α for $\alpha \in I$, the direct sum $\bigoplus G_\alpha$ consists of the elements (a_α) such that a_α is the identity element of G_α for all but finitely many α.

For illustrated example let $X = \bigoplus_{i=1}^{\infty} \mathbb{R}$ then the element $(1, 0, 0, 0, ...) \in X$ but the element $(1, 1, 1, 1, ...) \notin X$

Definition 3.1.5. Let κ be an infinite cardinal. Suppose for every ordinal $\alpha < \kappa$, we have non-trivial group G_α. Set $G = \bigoplus_{\alpha<\kappa} G_\alpha$, then

(a) [7] if e is the identity of G, then for each $x \in G$, we define $supp(x) = \{\alpha < \kappa : x_\alpha \neq e\}$.

(b) [17] we define a set

$$\mathbf{H} = \bigcap_{\alpha<\kappa} \{c\ell_{\beta G}\{x \in G \setminus \{e\} : \min supp(x) \geq \alpha\}\}.$$

(c) [17] let τ_0 denote the group topology on G with a neighborhood base of $e \in G$ consisting of subgroups

$$H_\alpha = \{x \in G : supp(x) \cap \alpha = \phi\}$$

Where $\alpha < \kappa$.

Remark 3.1.6. For every $x \in G = \bigoplus_{\alpha<\kappa} G_\alpha$, $supp(x)$ is finite.

Note that the sets H_α contains the identity and all elements such that $minsupp(x) \geq \alpha$.

Lemma 3.1.7. *[17] Let κ be an infinite cardinal. For every ordinal $\alpha < \kappa$, let G_α be a nontrivial group, let $G = \bigoplus_{\alpha<\kappa} G_\alpha$, and let*

$$\mathbf{H} = \bigcap_{\alpha<\kappa} \{c\ell_{\beta G}\{x \in G \setminus \{e\} : \min supp(x) \geq \alpha\}\}.$$

then $\mathbf{H} = Ult(\mathcal{T}_0)$

Proof. Let $S_\alpha = \{x \in G \setminus \{e\} : \min supp(x) \geq \alpha\}$ then $\mathbf{H} = \bigcap_{\alpha<\kappa} c\ell_{\beta G} S_\alpha$. Note that $H_\alpha = S_\alpha \cup \{e\}$. Let $p \in Ult(\mathcal{T}_0)$ then p converge to e. Pick $\alpha < \kappa$, so $H_\alpha \in p$. Suppose by contrary that $p \notin c\ell_{\beta G} S_\alpha$, then there is $A \in p$ such that $\widehat{A} \cap S_\alpha = \phi$. Since $A \cap H_\alpha \neq \phi$ then the only element in this intersection is $\{e\}$. Thus $\{e\} \in p$ which implies that p is principal ultrafilter generated by e contradiction since $p \in G^*$.

Conversely let $p \in \mathbf{H}$ and pick H_α neighborhood of e. Suppose by contrary that $H_\alpha \notin p$ then $H_\alpha^c \in p$. Since $p \in c\ell_{\beta G} S_\alpha$ then $\widehat{H_\alpha^c} \cap S_\alpha \neq \phi$ which is a contradiction. □

Lemma 3.1.8. *[17] Let G be a group, then the ultrafilter semigroup $Ult(\tau)$ is a closed subsemigroup of G^*.*

Proof. Let $S = Ult(\tau)$. To see that S is closed, let $p \in G^* \setminus S$. Then p does not converge to the identity, so there is a neighborhood U of e such that $U \notin p$. Put $C = G \setminus U$, then $C \in p$ so, $p \in \widehat{C}$. Moreover, $\widehat{C} \cap S = \phi$, otherwise if there exist $q \in \widehat{C} \cap S$ then $C \in q$ and q converges to e. Thus $U \in q$ and $C \in q$ which implies $C \cap U \neq \phi$ which is a contradiction.

To see that S is a semigroup, let $p, q \in S$. From Remark 2.2.16, $pq \in G^*$, and so it suffices to prove that pq converges to e. Let U be an open neighborhood of e. For every $x \in U$, put $V_x = x^{-1}U$. Then $U = \bigcup_{x\in U} xV_x$. To prove this, let $x \in U$ then $x = ex \in eV_e$ so $U \subseteq \bigcup_{x\in U} xV_x$. On the other hand, for each $x \in U$ we have $V_x = x^{-1}U = \{t \in G : xt \in U\}$ then $xV_x = \{xt \in G : xt \in U\} \subseteq U$, so $\bigcup_{x\in U} xV_x \subseteq U$. Since $U \in p$ and $V_x \in q$, $U = \bigcup_{x\in U} xV_x \in pq$. Hence, $U \in pq$. □

Lemma 3.1.9. *[17] Let G be a group, then for every nonempty open subset U in (G, τ), we have $\widehat{U}.Ult(\tau) \subseteq \widehat{U}$*

Proof. Let $p \in \widehat{U}$ and $q \in Ult(\tau)$, so $U \in p$ and q converge to e. As in the proof of Lemma 3.1.8, for every $x \in U$, $U = \bigcup_{x\in U} xV_x$. Hence, $U \in pq$ and $pq \in \widehat{U}$ □

Lemma 3.1.10. *[17] If $Ult(\tau)$ has only one minimal right ideal, then τ is extremally disconnected.*

Proof. Let $S = Ult(\tau)$. Suppose to the contrary that τ is not extremally disconnected. Then there are two disjoint open subsets U and V such that

$$c\ell(U) \cap c\ell(V) \neq \phi.$$

Also $\widehat{U} \cap S$ and $\widehat{V} \cap S$ are disjoint, since if $p \in (\widehat{U} \cap S) \cap (\widehat{V} \cap S)$ then $U \in p$ and $V \in p$, so $U \cap V \neq \phi$. Now from Lemma 3.1.9, $(\widehat{U} \cap S)S \subseteq \widehat{U} \cap S$ and $(\widehat{V} \cap S)S \subseteq \widehat{V} \cap S$. Thus $\widehat{U} \cap S$ and $\widehat{V} \cap S$ are two disjoint right ideals of S. By duality of Theorem 1.1.26 we have $K(S) = R$ where R is the minimal right ideal. so $K(S) \subseteq \widehat{U} \cap S$ and $K(S) \subseteq \widehat{V} \cap S$ which contradicts that $\widehat{U} \cap S$ and $\widehat{V} \cap S$ are two disjoint right ideals. □

Definition 3.1.11. [17] A subsemigroup S of a semigroup T is *left saturated* in τ if for every $x \in T \setminus S$, $xS \cap S = \phi$

Lemma 3.1.12. *[17] Let τ be a regular left invariant topology on a group G, and $S = Ult(\tau)$. Then $S^1 = S \cup \{e\}$ is left saturated in βG.*

Proof. Let $p \in \beta G \setminus S^1$. Since S^1 is closed and $p \notin S$, then there is a neighborhood U of e in τ with $U \notin p$. Since τ is regular then from Theorem 1.2.30 one may suppose that U is closed. Let $C = G \setminus U$, then $p \in \widehat{C}$, C is open, and $\widehat{C} \cap S^1 = \phi$, otherwise if $q \in \widehat{C} \cap S^1$ then q converge to e so $U \in q$ which contradicts that $q \in \widehat{C}$. By Lemma 3.1.9, $pS^1 \subseteq \widehat{C}$, and so $pS^1 \cap S^1 = \phi$ □

Definition 3.1.13. [17] Given a filter $\mathcal{F}$ on G. Then we define

$$\widehat{\mathcal{F}} = \{p \in \beta G : \mathcal{F} \subseteq p\}.$$

Since each filter is contained in ultrafilter then there ultrafilter p such that $\mathcal{F} \subseteq p \in \widehat{\mathcal{F}}$ then $\widehat{\mathcal{F}} \neq \phi$.

Theorem 3.1.14. *[4] Let X be a discrete space :*

(a) If $\mathcal{F}$ is filter on X, then $\widehat{\mathcal{F}}$ is a closed subset of βX.

(b) If $S \subseteq \beta X$ and $\mathcal{F} = \cap S$, then $\mathcal{F}$ is a filter on X and $\widehat{\mathcal{F}} = c\ell S$.

Proof. (a) Let $p \in \beta X \setminus \widehat{\mathcal{F}}$, then $p \neq \mathcal{F}$. Pick $B \in \mathcal{F} \setminus p$. Then $X \setminus B \in p$ and so, $p \in \widehat{X \setminus B}$. Thus $\widehat{X \setminus B}$ is a neighborhood of p. Now if there is $q \in \widehat{\mathcal{F}} \cap \widehat{X \setminus B}$, then $B \in \mathcal{F} \subseteq q$. Hence B and $X \setminus B$ will be in q which is a contradiction.

(b) $\mathcal{F}$ is anintersection of filters, so $\mathcal{F}$ is a filter. Furthermore, for each $p \in S$ we have $\mathcal{F} = \cap S \subseteq p$. So, we have that $S \subseteq \widehat{\mathcal{F}}$, and by (a) $c\ell S \subseteq \widehat{\mathcal{F}}$. To see that $\widehat{\mathcal{F}} \subseteq c\ell S$, let $p \in \widehat{\mathcal{F}}$ and let $B \in p$. Suppose to contrary $\widehat{B} \cap S = \phi$. Then for each $q \in S$, $q \in \widehat{X \setminus B}$ so $X \setminus B \in q$ and then $X \setminus B \in \mathcal{F} \subseteq p$ which is a contradiction.

□

From Theorem 3.1.14 we conclude that every nonempty closed subset of βG can be represented in such a form.

Remark 3.1.15. [17]Given a filter $\mathcal{F}$ on G, then $\widehat{\mathcal{F}} = \bigcap\{\widehat{A} : A \in \mathcal{F}\}$.

Proof. Let $p \in \bigcap\{\widehat{A} : A \in \mathcal{F}\}$. Then $p \in \widehat{A}$ for all $A \in \mathcal{F}$ which implies that $A \in p$ for all $A \in \mathcal{F}$. Hence $\mathcal{F} \subseteq p$, and $p \in \mathcal{F}$. Now let $p \in \beta(G)$ such that $\mathcal{F} \subseteq p$, then for all $A \in \mathcal{F}$ we have $A \in p$. So $p \in \widehat{A}$ for all $A \in \mathcal{F}$ which implies that $p \in \bigcap\{\widehat{A} : A \in \mathcal{F}\}$. □

Proposition 3.1.16. *[17] Let S be a closed subsemigroup of a group G. Suppose that $S^1 = S \cup \{e\}$ is left saturated in βG and that S has a finite left ideal. Then there is a regular left invariant topology τ on G with $Ult(\tau) = S$.*

Proof. Since S^1 is a closed subsemigroup of G, then from Theorem 3.1.14 we can find a filter $\mathcal{F}$ on G such that $\widehat{\mathcal{F}} = S^1$ .We first show that :

(a) $\bigcap \mathcal{F} = \{e\}$, and

(b) For every $U \in \mathcal{F}$, there is a $V \in \mathcal{F}$ such that for all $x \in V$, $x^{-1}U \in \mathcal{F}$.

To prove (a), since $\{e\} \subseteq S^1 = \widehat{\mathcal{F}}$, then $\{e\} \subseteq \bigcap \widehat{A}$ for all $A \in \mathcal{F}$, so $\{e\} \subseteq \widehat{A}$ for all $A \in \mathcal{F}$, which implies $A \in e$ for all $A \in \mathcal{F}$. So, $\bigcap A = \{e\}$ for all $A \in \mathcal{F}$ and then $\bigcap \mathcal{F} = \{e\}$.

To prove (b), let $U \in \mathcal{F}$, L a finite left ideal of S and let $C = G \setminus U$. Since $C \cap U = \phi$, then $\widehat{C} \cap \widehat{U} = \phi$, otherwise if there is $p \in \widehat{C} \cap \widehat{U}$ then $C \in p$ and $U \in p$ so, $C \cap U \neq \phi$. Also, since $\widehat{\mathcal{F}} \subseteq \widehat{U}$ then $\widehat{C} \cap \widehat{\mathcal{F}} = \phi$. Thus $\widehat{C} \cap S_1 = \phi$. Since S^1 is left saturated in βG, $\widehat{C} \cdot S^1 \cap S^1 = \phi$ ....(1)
Since L is an ideal of S, then $L \subseteq S \subseteq S^1$ so, for every $q \in L$ we have $\widehat{C} \cdot q \subseteq \widehat{C} \cdot S^1$...(2)
So, from (1) and (2) we conclude that $\widehat{C}q \cap S^1 = \phi$. For every $q \in L$ we can choose $W_q \in \mathcal{F}$ such that $\widehat{C} \cdot q \bigcap \widehat{W_q} = \phi$. To show this suppose by contrary that $\widehat{C}q \cap \widehat{W} \neq \phi$ for all $W \in \mathcal{F}$ then $\widehat{C}q \cap \mathcal{F} \neq \phi$ hence, $\widehat{C}q \cap S^1 \neq \phi$ which is a contradiction. Since $W_q \in \mathcal{F}$, $\mathcal{F} \in \widehat{W_q}$, and $\widehat{\mathcal{F}} \subseteq \widehat{W_q}$. Put $W = \bigcap_{q \in L} W_q$. Then $W \in \mathcal{F}$, as L is finite, and $\mathcal{F}$ is filter. Moreover,

$$(\widehat{C} \cdot L) \cap \widehat{W} = \phi \text{..(3)}$$

Next, since L is a left ideal of S, it follows that $S^1.L \subseteq L$. Since $\mathcal{F}$ is filter and $W_q, U \in \mathcal{F}$ then $U \cap W_q \neq \phi$. Now for every $q \in L$, choose $V_q = U \cap W_q \in \mathcal{F}$. Since $V_q \subseteq U$ then $V_q \cap C = \phi$,thus $\widehat{V_q} \cap \widehat{C} = \phi$ so from (3) $\widehat{V_q}.q \subseteq \widehat{W}$. Put $V = \bigcap_{q \in L} V_q$. Then $V \in \mathcal{F}$ and

$$\widehat{V} \cdot L \subseteq \widehat{W} \text{...............................(4)}$$

We claim that for all $x \in V$, $x^{-1}U \in \mathcal{F}$. Suppose to contrary that for some $x \in V$, $x^{-1}U \notin \mathcal{F}$ that is $(x^{-1}U)^c \in \mathcal{F}$ thus $\mathcal{F} \in \widehat{(x^{-1}U)^c}$. But $\mathcal{F} \in \widehat{U}$ so $\widehat{(x^{-1}U)^c} \cap \widehat{U} \neq \phi$ so, there is $p = \mathcal{F} \in \widehat{U}$ and $xp \notin \widehat{U}$ that is $xp \in \widehat{C}$ and $p \in S$. Take any $q \in L$. Then, from (3),

$$xpq = xp \cdot q \in \widehat{C} \cdot L \subseteq \beta G \setminus \widehat{W},$$

and form (4),

$$xpq = x \cdot pq = \in \widehat{V}.L \subseteq \widehat{W},$$

which is a contradiction.

Condition (b) can be restated in the following stronger form:

(c) for every $U \in \mathcal{F}$, there is a $V \in \mathcal{F}$ such that $V \subseteq U$ and for all $x \in V$, $x^{-1}V \in \mathcal{F}$.

To see this, for every $U \in \mathcal{F}$, let

$$U^\circ = \{x \in U : x^{-1}U \in \mathcal{F}\}.$$

By (b) there is a $V \in \mathcal{F}$ such that for all $x \in V$, $x^{-1}U \in F$. Since $\mathcal{F}$ is filter then $V \cap U \neq \phi$ so $V \subseteq U^\circ \in \mathcal{F}$. We claim that $(U^\circ)^0 = U^\circ$

Clearly from definition of $(U^\circ)^0$ that $(U^\circ)^0 \subseteq U^\circ$. Let $x \in U^\circ$ and let $V = x^{-1}U$. Then from definition of U°, $V \in F$. Also $xV \subseteq U$ because for all $t \in V$ we have $t \in x^{-1}U$ and then $xt \in U$. Now we will show that $xV^\circ \subseteq U^0$. Let $y \in V^\circ$ and let $W = y^{-1}V$. Then from definition of V°, $W \in F$. Also $yW \subseteq V$ because for all $t \in W$ we have $t \in y^{-1}V$ and then $yt \in V$. Consequently,

$$xyW \subseteq xV \subseteq U.$$

Hence $W \subseteq (xy)^{-1}U \in \mathcal{F}$. Since $y \in V = x^{-1}U$ then $xy \in U$. Therefore $xy \in U^\circ$ Since $xV^\circ \subseteq U^0$. then $V^\circ \subseteq x^{-1}U^\circ \in \mathcal{F}$ so, $x \in (U^\circ)^0$. Now let $V = U^\circ$ to get the result.

It follows from (a) - (c) that there is a left invariant topology τ on G in which $\mathcal{F}$ is the neighborhood filter of e, and so $Ult(\tau) = S$. We now show that τ is regular. Assume by contrary that τ is not regular. Then there is a neighborhood U, of e such that for every neighborhood V of e,

$$c\ell(V) \setminus U \neq \phi$$

For every open neighborhood V of e, choose $x_v \in c\ell(V) \setminus U$. Since $x_v \in c\ell(V)$, there is an ultrafilter on G containing V and converging to x_v. Consequently, there is a filter $p \to x_v$. Since $x_v \notin U$ then $x_v \neq e$. But $p \to x_v$ so p not in S.

Since $p \to x_v$ so, $x_v^{-1}p \to x_v^{-1}x_v = e$. Let $p_v = x_v^{-1}p$ then $p_v \in S$. Since $V \in p$ then $x_v^{-1}V \in x_v^{-1}p = p_v$ so, $V \in x_v p_v$.

Take any $q \in L$, since $q \in L$ then $q \in S = Ult(\tau)$. Also since $V \in x_v p_v$ then $x_v p_v \in \widehat{V}$. By Lemma 3.1.9, $\widehat{V}Ult(\tau) \subseteq \widehat{V}$ which impliese $x_v p_v q \in \widehat{V}$ and,

$$V \in x_v p_v q,$$

Since L is a left ideal of S then $p_v q \in L$. But L is finite, then $p_v q = q_i$ for some $q_i \in L$. Thus $V \in x_v q_i$,
Therefore, we obtained that there is an ultrafilter p not in S and ultrafilter q_i in S such that $pq_i \in S$ which impliese that S^1 is not left saturated in βG, contradiction. $\square$

3.2 Local homomorphisms and projectivity

Definition 3.2.1. [17] Let G be a group, τ a left invariant topology on G, and X be open neighborhood of e in τ. A mapping $f : X \to S$, where S is a semigroup, is called *a local homomorphism* if for every $x \in X \setminus \{e\}$, there is a neighborhood U_x of e such that $f(xy) = f(x)f(y)$ for all $y \in U_x \setminus \{e\}$.

Lemma 3.2.2. *[17] Let G be a group, τ a left invariant topology on G, and X an open neighborhood of e in τ. Let $f : X \to T$ be a local homomorphism into a compact right topological semigroup T such that $f(X) \subseteq \Lambda(T)$, $\widehat{f} : \widehat{X} \to T$ be the continuous extension of f, and let $f^* = \widehat{f}|_{Ult(\tau)}$. Then $f^* : Ult(\tau) \to T$ is a homomorphism. Furthermore, if for every neighborhood U of e, $f(U \setminus \{e\})$ is dense in T, then f^* is onto.*

Proof. [17] Since for any neighborhood U of e, $U \in q$, then from a bove definition we have if $q \in Ult(\tau)$, then $f(xy) = f(x)f(y)$ for all $x \in X$ and $y \in U$ for all $U \in q$. Also if $f(x) \in \Lambda(T)$ then $\lambda_{f(x)}$ is continuous so $\lim_{y \to q} \lambda_{f(x)}(f(y)) = \lambda_{f(x)} \lim_{y \to q} f(y) = \lambda_{f(x)} \widehat{f}(q)$
To show that $f^* : Ult(\tau) \to T$ is a homomorphism, let $p, q \in Ult(\tau)$, Then

$$\begin{aligned}
\widehat{f}(pq) &= \widehat{f}(\lim_{x \to p} \lim_{y \to q}(xy)) \text{ and } x, y \in X \\
&= \lim_{x \to p} \lim_{y \to q} f(xy) \text{ (because } \widehat{f} \text{ is continuous extension of } f) \\
&= \lim_{x \to p} \lim_{y \to q} f(x)f(y) \text{ (because } f \text{ is local homomorphism)} \\
&= = \lim_{x \to p} f(x)\widehat{f}(q) \quad \text{(because } f(x) \in \Lambda(T)) \\
&= \widehat{f}(p)\widehat{f}(q)
\end{aligned}$$

To check that f^* is onto, let $t \in T$. Since for every neighborhood U of

$e \in X$, $f(U \setminus \{e\})$ is dense in T, then for every neighborhood V of t in T, $V \cap f(U \setminus \{e\}) \neq \phi$, thus there exists an $x \in U \setminus \{e\}$ such that $f(x) \in V$. Let $A_u \subseteq U \setminus \{e\}$ such that $f(A_u) \subseteq V$, then $A_u \neq \phi$. Let p be the ultrafilter containing A_u, then from definition of A_u, p contains any neighborhood U of e. So, $p \to e$ that is $p \in Ult(\tau)$. Also we can see $\lim_{x \to p} f(x) = t$ since for every neighborhood V of t in T, there is a neighborhood $f(A_u)$ of p such that $f(A_u \cap X) = f(A_u) \subseteq V$. Thus from Definition 2.2.13, we get the result.

It follows that there exists $p \in Ult(T)$ such that $\widehat{f}(p) = \widehat{f}(lim_{x \to p} x) = \lim_{x \to p} f(x) = t$. □

Example 3.2.3. *[17]Let κ be an infinite cardinal. For every ordinal $\alpha < \kappa$, let G_α be a nontrivial group. Let $G = \bigoplus \mathbb{G}_\alpha$ and*

$$D = \{x \in G : |supp(x)| = 1\}.$$

Suppose $f_0 : D \to S$ is a mapping from D into a semigroup S. Then for all $x \in G$ there exist $n \in \mathbb{N}$ such that $x = x_1 \cdot x_2 ... \cdot x_n$ and $x_i \in D$, for all $i = 1, ..., n$. Extend f_0 to the mapping $f : G \to S$ by

$$f(x_1 \cdot ... \cdot x_n) = f_0(x_1) \cdot ... \cdot f_0(x_n),$$

where $x_1, ..., x_n \in D$ are such that if $supp(x_1) = \{\alpha_1\}$, ..., $supp(x_n) = \{\alpha_n\}$, then $\alpha_1 < ... < \alpha_n$ The value $f(e)$ does not matter, and we can consider $f(e) = f_0(e)$. Let $x, y \in G \setminus \{e\}$ such that $\max supp(x) < \min supp(y)$. If $x = x_1 \cdot x_2 ... \cdot x_n$, $y = y_1 \cdot y_2 ... \cdot y_m$, $x_i, y_j \in D$ and if $supp(x) = \{\alpha_i, i = 1, ..n\}$ and $supp(y) = \{\beta_j, j = 1, ..n\}$, then $\alpha_n < \beta_1$. Hence $f(xy) = f_0(x_1) \cdot ... \cdot f_0(x_n) f_0(y_1) \cdot ... \cdot f_0(y_n) = f(x)f(y)$. Let τ_0 be the topology on G and for any $x \in G \setminus \{e\}$, take $n = \max supp(x)$, and let $U_x = H_{n+1} = \{x \in G : supp(x) \cap n + 1 = \phi\})$. Thus form definition of τ_0, U_x is a neighborhood of e. Also for all $y \in U_x \setminus \{e\}$ we have $\max supp(x) < \min supp(y)$. Hence, $f : (G, \tau_0) \to S$ is a local homomorphism ....(1)

Now let λ be a cardinal such that $|G_\alpha| \geq \lambda$ for all $\alpha < \kappa$ and let $\mu = \max\{\kappa, \lambda\}$. Let T be any compact right topological semigroup containing a dense subset A such that $|A| \leq \mu$ and $A \subseteq \Lambda(T)$. For any $\alpha < \mu$ define $\varphi_\alpha : G_\alpha \to D$ by $\varphi_\alpha(x) = y$ where $y_\alpha = x$ and $y_\beta = e$ for all $\beta \neq \alpha$. Since $|G_\alpha| \geq \lambda$ then for $\gamma < \mu$ and

$x_\gamma \in G_\alpha$, define $D_\gamma = \{\varphi_\alpha(x_\gamma)$ for all $\alpha < \mu\}$. From definition of D_γ we conclude that $D_\gamma \subseteq D$ and then $\bigcup D_\gamma \subseteq D$. Now let $x \in D$ then so, $supp(x) = \{\alpha\}$. Thus $x_\alpha \neq e_\alpha$. Suppose that $x_\alpha = y_\beta$ for some $y_\beta \in G_\alpha$, then $\varphi_\alpha(y_\beta) = x$. Hence $x \in D_\beta$. Therefore $D = \bigcup D_\gamma$ for all $\gamma < \mu$. Let $\beta < \gamma \leq \mu$, and suppose by contrary that $D_\gamma \cap D_\beta \neq \phi$ for $\gamma \neq \beta$. Pick w in this intersection, then $w = \varphi_{\alpha_1}(x_\gamma)$ and $w = \varphi_{\alpha_2}(x_\beta)$ where x_γ and x_β lie in G_{α_1} and G_{α_2} repectivaly. Thus $w_{\alpha_1} = x_\gamma$ and $w_{\alpha_2} = x_\beta$. Since $w \in D$ then $\alpha_1 = \alpha_2$ and consequently $x_\gamma = x_\beta$; that is, $\beta = \gamma$ which is a contradiction. Thereore $\{D_\gamma : \gamma < \mu\}$ make a partition of D. Now for every $\gamma < \mu$ define U_γ such that :

(a) $\phi \notin U_\gamma$ and $D_\gamma \in U_\gamma$,

(b) for any $\alpha < \kappa$, $H_\alpha \in U_\gamma$,

(c) if $A \subseteq G$ such that $H_\alpha \subseteq A$ then $A \in U_\gamma$,

(d) if $A \subseteq G$ such that $D_\gamma \subseteq A$ then $A \in U_\gamma$,

Notic that U_γ is an ultrafilter, to see this it is suffices to prove that $H_\alpha \cap D_\gamma \neq \phi$ for any $\alpha < \kappa$. So, $\varphi_\alpha(x_\gamma) = y$ where $y_\alpha = x_\gamma \neq e_\gamma$. So, $\varphi_\alpha(x_\gamma) \in H_\alpha$. Since $H_\alpha \in U_\gamma$. for all $\alpha < \kappa$ then U_γ converge to the identity in τ_0. Hence $\widehat{D_\gamma} \cap \mathcal{H} \neq \phi$. Now, choose any surjection $g : \mu \to A$. For any $x \in D$, $x \in D_\gamma$ for some $\gamma < \mu$ so define $f_0 : D \to A$ by

$$f_0(x) = g(\gamma) \quad \textit{if } x \in D_\gamma$$

Let $f : G \to T$ be defined by $f(x_1 \cdot ... \cdot x_n) = f_0(x_1) \cdot ... \cdot f_0(x_n) = g(\gamma_1) \cdot ... \cdot g(\gamma_n)$ where $x_1, ..., x_n \in D$ and $x_i \in D_{\gamma_i}$. So from (1), f is local homomorphism. Then $f^* : Ult(\tau_0) \to T$ is a surjective homomorphism.

Definition 3.2.4. [17]Let κ be an infinite cardinal. For every ordinal $\alpha < \kappa$, let G_α be a nontrivial group. Let $G = \bigoplus \mathbb{G}_\alpha$, τ any nondiscrete zero-dimensional left invariant topology on G such that $\tau_0 \subseteq \tau$, and X an open neighborhood of $e \in G$ in τ. An *independent system* in X is a mapping M such that:

(a) $D = dom(M)$ is a subset of $X \setminus \{e\}$;

(b) for every $x \in D$, $M(x)$ is a clopen subset of $X \setminus \{e\}$ with $x \in M(x)$;

(c) $M(x) \cap M(y) = \phi$ for all distinct $x, y \in D$.

If M is an independent system, then an $M-$product is a product of the form $x_0 x_1 ... x_n$,such that for each $i \leq n$, $x_i \in D$ and $x_i \cdot ... \cdot x_n \in M(x_i)$.

Lemma 3.2.5. *[17]A decomposition into an* M-product *is unique.*

Proof. Let $x_0 \cdot ... \cdot x_n$ and $y_0 \cdot ... \cdot y_m$ be M-products, and let $x_0 \cdot ... \cdot x_n = y_0 \cdot ... \cdot y_m$ We prove that $n = m$ and $x_i = y_i$ for all $i \leq n$. We proceed by induction on $\min\{n, m\}$.

Suppose that $\min\{n, m\} = 0$. Now from definition of independent system $x_0 \in M(x_0)$, with out loss of generality, let $n = 0$ then

$$x_0 = y_0 \cdot ... \cdot y_m \in M(y_0)$$

Since $\{M(x) : x \in D\}$ is pairwise disjoint, it follows that $x_0 = y_0$,
Moreover, $m = 0$ Indeed,otherwise $y_1 \cdot ... \cdot y_m = e$, but $y_1 \cdot ... \cdot y_m \in M(y_1)$, so $e \in M(y_1)$ which contradict that $M(y_1)$ is a clopen subset of $X \setminus \{e\}$.
Let $\min\{n, m\} = K > 0$ and suppose that the statment is true for all $m < K$. Again from definition of M-products we find that

$$x_0 \cdot ... \cdot x_n \in M(x_0) \quad \textit{and} \quad x_0 \cdot ... \cdot x_n = y_0 \cdot ... \cdot y_m \in M(y_0)$$

,so $x_0 = y_0$ But then $x_1 \cdot ... \cdot x_n = y_1 \cdot ... \cdot y_m$. So if we apply the inductive assumption we get the result. □

Proposition 3.2.6. *[17] Let κ be an infinite cardinal. For every ordinal $\alpha < \kappa$, let G_α be a nontrivial group. Let $G = \bigoplus \mathbb{G}_\alpha$, τ any nondiscrete zero-dimensional left invariant topology on G such that $\tau_0 \subseteq \tau$, and X an open neighborhood of $e \in G$ in τ. Then for every homomorphism $g : R \to Q$of a semigroup R onto a semigroup Q and for every local homomorphism $f : X \to Q$, there is a local homomorphism $h : X \to R$ such that $f = g \circ h$.*

Proof. [17] For every $x \in X \setminus \{e\}$, since f is local homomorphism, then we can choose a neighborhood U_x of $e \in X$ such that $f(xy) = f(x)f(y)$ for all $y \in U_x \setminus \{e\}$..................$(*)$,
Let $\nu(x) = \max\ supp(x)$, and let

$$F_x = \{y \in X \setminus \{e\} : \nu(y) < \nu(x) \text{ and } y(\beta) = x(\beta) \text{ for all } \beta \in supp(y)\}.$$

Note that F_x is finite. To show this, suppose by contrary that F_x is infinite. Then for all $y \in F_x$, $y(\beta) = x(\beta)$ for all $\beta \in supp(y)$, so there is infinite many terms X_β such that $X_\beta \neq e_\beta$ (i.e $supp(x)$ is infinite) contradiction. Also for every $\alpha < \kappa$, let

$$X_\alpha = \{x \in X : \max\ supp(x) = \alpha\}.$$

Define inductively an increasing sequence $(M_\alpha)_{\alpha<\kappa}$ of independent systems in X by the following conditions:

(a) $M_0 = \phi$;

(b) $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ if α is a limit ordinal;

(c) $D_{\alpha+1} = D_\alpha \cup (X_\alpha \setminus \bigcup_{x \in D_\alpha} M_\alpha(x))$, and

(d) for every $x \in D_{\alpha+1} \setminus D_\alpha$,

$$M_{\alpha+1}(x) = (x(U_x \cap H_{\nu(x)+1})) \setminus \bigcup_{y \in F_x \cap D\alpha} M_\alpha(y).$$

Recall that for every $\alpha < \kappa$, $H_\alpha = \{x \in G : supp(x) \cap \alpha = \phi\}$.
For $n = 0$, $M_0 = \phi$ is an independent system. Suppose For all $\alpha < \kappa$, M_α is an independent system. Now, for $n = \alpha + 1$ It follows from (c) that for every $x \in D_{\alpha+1} \setminus D_\alpha$, $x \in (X_\alpha \setminus \bigcup_{y \in D_\alpha} M_\alpha(y))$ so, $x \notin \bigcup_{y \in D_\alpha} M_\alpha(y)$. Since M_α is an independent systems, then for every $x \in D_\alpha$, $M_\alpha(x)$ is a clopen subset of $X \setminus \{e\}$ with $x \in M(x)$. Then $M_\alpha(y)$ is a clopen subset of $X \setminus \{e\}$ for all $y \in F_x \cap D_\alpha$. Since $F_x \cap D_\alpha$ is finite, so $\bigcup_{y \in F_x \cap D\alpha} M_\alpha(y)$ is closed which implies $\bigcap_{y \in F_x \cap D\alpha} M^c_\alpha(y)$ is open ....(1) .
Since $H_{\nu(x)+1} \in \tau_0 \subseteq \tau$, $U_x \cap H_{\nu(x)+1} \in \tau$. But τ is a left invariant topology on G

so $x(U_x \cap H_{\nu(x)+1})$ is open in τ ....(2).
From (1) and (2), $M_{\alpha+1}(x)$ is open. Also from part (d) $M^c_{\alpha+1}(x) = \bigcup_{y\in F_x \cap D\alpha} M_\alpha(y)$ which is open so $M_{\alpha+1}(x)$ is closed. Therefore $M_{\alpha+1}(x)$ is a clopen neighborhood of $x \in X \setminus \{e\}$. To see that

$$M_{\alpha+1}(x) \cap M_{\alpha+1}(y) = \phi.$$

whenever $x, y \in D_{\alpha+1}$ and $x \neq y$, suppose that $x \in D_{\alpha+1} \setminus D_\alpha$ then from part (c) $x \in X_\alpha \setminus \bigcup_{x\in D_\alpha} M_\alpha(x)$. So, $\upsilon(x) = \max\ supp(x) = \alpha$. If $y \in F_x$, then $\upsilon(y) < \upsilon(x)$ i.e max $supp(y) < \alpha$. Thus $y \notin X_\alpha$ which implies from part (c) that $y \in D_\alpha$, and so from part (d) $M_{\alpha+1}(x) \bigcap M_{\alpha+1}(y) = \phi$. If $y \notin F_x$, then $\upsilon(y) \geq \upsilon(y)$ or there is a $\beta \in supp(y)$ with $y(\beta) \neq x(\beta)$, so we have two cases:
Case (1): If $\nu(y) \geq \nu(x)$, suppose that $y(\beta) = x(\beta)$ for all $\beta \in supp(y)$, then $\nu(y) \leq \nu(x)$. Hence $\nu(y) = \nu(x)$. Since $x \neq y$, there is $\gamma \in supp(x)$ such that $y_\gamma = e_\gamma$. Suppose to contrary $xH_{\nu(x)+1} \cap yH_{\nu(y)+1} \neq \phi$. Then there is $r \in H_{\nu(x)+1}$, and $m \in H_{\nu(y)+1}$ such that $xr = ym$. Since $\gamma < \nu(y) = \nu(x)$, then $r_\gamma = m_\gamma = e_\gamma$. Thus $x_\gamma = x_\gamma r_\gamma = (xr)_\gamma = (ym)_\gamma = y_\gamma m_\gamma = y_\gamma = e_\gamma$, which is a contradiction.
Case (2): If there is a $\beta \in supp(y)$ with $y(\beta) \neq x(\beta)$. Suppose to contrary that $xH_{\nu(x)+1} \cap yH_{\nu(y)+1} \neq \phi$, then there is $r \in H_{\nu(x)+1}$, and $m \in H_{\nu(y)+1}$ such that $xr = ym$. Since $y \in D_{\alpha+1}$, then from part (c), $\nu(y) \leq \nu(x)$, so $r_\beta = m_\beta = e_\beta$, where $\beta < \nu(y) \leq \nu(x)$. Thus $x_\beta = x_\beta r_\beta = (xr)_\beta = (ym)_\beta = y_\beta m_\beta = y_\beta = e_\beta$, which is a contradiction. Therefore in two cases, $xH_{\nu(x)+1} \cap yH_{\nu(y)+1} = \phi$, and again by part (d), $M_{\alpha+1}(x) \cap M_{\alpha+1}(y) = \phi$.
Thus, $(M_\alpha)_{\alpha<\kappa}$ is indeed an increasing sequence of independent systems in X. Put $M = \bigcup_{\alpha<\kappa} M_\alpha$. then M is an independent system in X. To show this, let $D = Dom(M) = \bigcup_{i=1}^{\infty} D(M_\alpha)$

(1) Since each $D(M_\alpha)$ is subset of $X \setminus \{e\}$ so Dom (M) is subset of $X \setminus \{e\}$

(2) Let $x \in D$ then $x \in D(M_\alpha)$ for some $\alpha < \kappa$. Since $(M_\alpha)_{\alpha<\kappa}$ is an increasing sequance then $M(x) = M_\alpha(x)$ so $M(x)$ is a clopen subset of $X \setminus \{e\}$ with $x \in M(x)$.

(3) Let x, y in D then there is α,β such that $x \in D(M_\alpha)$ and $y \in D(M_\beta)$. Since

$(M_\alpha)_{\alpha<\kappa}$ is an increasing sequance then there exist $\gamma = \max\{\alpha, \beta\}$ such that x, $y \in M_\gamma$ .But M_γ is an independent system then $M(x) \bigcap M(y)$= $M_\gamma(x) \bigcap M_\gamma(y) = \phi$

Now, We claim that:

(1) every $X \setminus \{e\}$ can be decomposed into an M-product, and

(2) $f(x_0 \cdot ... \cdot x_n) = f(x_0) \cdot ... \cdot f(x_n)$ whenever $x_0...x_n$ is an M-product.

To see (a), let $x \in X \setminus \{e\}$, then $x \in X_\alpha$ for some α, so fix $\alpha < \kappa$. We show that every $x \in X_\alpha$ can be decomposed into an $M_{\alpha+1}-$product. We proceed by induction on $|supp(x)|$.

Let $|supp(x)| = 1$. Then $x \in D_{\alpha+1}$. Indeed, otherwise from (c) $x \in M_\alpha(y)$ for some $y \in D_\alpha$, then from part (d) $x = yz$ for some $z \in H_{\upsilon(y)+1}$ and $\upsilon(y) < \beta$ where $\beta = \min supp(z)$, and so $|supp(x)| > 1$ because at least x_β and $x_{\upsilon(y)}$ in $supp(x)$ which is a contradiction. Since $x \in D_{\alpha+1}$, $x = x$ is an $M_{\alpha+1}-$product. Now suppose that $|supp(x)| > 1$ and the statement holds for all $z \in X_\alpha$ with

$$|supp(z)| < |supp(x)|.$$

If $x \in D_{\alpha+1}$, then $x = x$ is an $M_{\alpha+1}-$product. Otherwise from (c) $x \in M_\alpha(y)$ for some $y \in D_\alpha$. Then from part (d) $x = yz$ for some $z \in H_{\upsilon(y)+1}$, and $\upsilon(y) < \beta$ where $\beta = \min\ supp(z)$. Since $x \in X_\alpha$, then $\max\ supp(x) = \alpha$. Since $\upsilon(y) < \beta$, then $supp(y) \bigcap supp(z) = \emptyset$, so $x_\alpha = (zy)_\alpha = z_\alpha$ then $z_\alpha \neq e$. Moreover, if there exists $\gamma > \alpha$ such that $z_\gamma \neq e$, then $\max\ supp(x) \neq \alpha$ which ic contradiction. Therefore $\max\ supp(z) = \alpha$ which implies $z \in X_\alpha$.

Since $\upsilon(y) < \beta$, then $supp(y) \bigcap supp(z) = \emptyset$. Since $x = yz$ then $|supp(x)| = |supp(y)| + |supp(z)|$. Thus $|supp(z)| < |supp(x)|$.

By the inductive hypothesis, z can be decomposed into an $M_{\alpha+1}-$product $z = z_1 \cdot ... \cdot z_n$. Then $x = yz_1 \cdot ... \cdot z_n$. Since M_α is an increasing sequance and $x \in M_\alpha(y)$ then $x \in M_{\alpha+1}(y)$. Hence x is an $M_{\alpha+1}-$product. Now for all $x \in M$, $x \in M_\alpha$ for some $\alpha < \kappa$ so x is $M_{\alpha+1}$-product then x is M-product

To prove (b),first we will prove that $M(x_0) \subseteq x_0 U_{x_0}$, Let $h \in M(x_0)$ so there is

$h \in M_\alpha(x_0)$ for some $\alpha < \kappa$ then from part (d), $h \in x_0 U_{x_0}$.
Let $x_0 \cdot ... \cdot x_n$ be an M-product, then $x_0 \cdot ... \cdot x_n \in M(x_0)$. But $M(x_0) \subseteq x_0 U_{x_0}$ then $x_0 \cdot ... \cdot x_n \in x_0 U_{x_0}$ so, $x_1 \cdot ... \cdot x_n \in U_{x_0}$. Then from $(*)$

$$f(x_0 \cdot ... \cdot x_n) = f(x_0) f(x_1 \cdot ... \cdot x_n).$$

In the same way $M(x_1) \subseteq x_1 U_{x_1}$, so $x_1 \cdot ... \cdot x_n \in x_1 U_{x_1} \Rightarrow x_2 \cdot ... \cdot x_n \in U_{x_1}$. Then from $(*)$

$$f(x_1 \cdot ... \cdot x_n) = f(x_1) f(x_2 \cdot ... \cdot x_n).$$

Thus

$$f(x_0 \cdot ... \cdot x_n) = f(x_0) f(x_1 \cdot ... \cdot x_n) = f(x_0) f(x_1) f(x_2 \cdot ... \cdot x_n).$$

Continue in this process we have

$$f(x_0 \cdot ... \cdot x_n) = f(x_0) f(x_1) f(x_2) \cdot ... \cdot f(x_n).$$

We now construct $h : X \to R$.
Since $D = dom(M)$ is subset of $X \setminus \{e\}$ then for every $x \in D$, choose $h(x) = g^{-1}(f(x))$. Since each element in $X \setminus \{e\}$ can be decomposed into an M-product then we can extend h over X by

$$h(x_0 \cdot ... \cdot x_n) = h(x_0) \cdot ... \cdot h(x_n),$$

where $x_0 \cdot ... \cdot x_n$ is an $M-$product. Now for every $x \in D$ we have $(g \circ h)(x) = g(h(x)) = g(g^{-1}(f(x))) = f(x)$ so $(g \circ h)|_D = f|_D$.
Since g is a homomorphism, then

$$\begin{aligned} gh(x_0 \cdot ... \cdot x_n) &= g(h(x_0) \cdot ... \cdot h(x_n)) \\ &= gh(x_0) \cdot ... \cdot gh(x_n) \\ &= f(x_0) \cdot ... \cdot f(x_n) \\ &= f(x_0 \cdot ... \cdot x_n), \end{aligned}$$

and so $f = g \circ h$.
To see that h is a local homomorphism, let $x \in X \setminus \{e\}$ be given and let $x_0 \cdot ... \cdot x_n$

be the decomposition of x into an $M-$product. For each $i \leq n$, one has $x_i \cdot ... \cdot x_n \in M(x_i)$. Define

$$V_x = \bigcap_{i \leq n} (x_i \cdot ... \cdot x_n)^{-1} M(x_i)$$

Since G is a group and $(x_i \cdot ... \cdot x_n) \in G$, then $(x_i \cdot ... \cdot x_n)^{-1} \in G$. Sinse $x_i \cdot ... \cdot x_n \in M(x_i)$ for all $i \leq n$, then $e = (x_i \cdot ... \cdot x_n)^{-1}(x_i \cdot ... \cdot x_n) \in (x_i \cdot ... \cdot x_n)^{-1} M(x_i)$ for all $i \leq n$ so,$e \in V_x$.

Since $M(x_i)$ is open and τ is a left invariant topology on G then $(x_i \cdot ... \cdot x_n)^{-1} M(x_i)$ is open for all $i \leq n$ Since the finite intersection of open sets is open then V_x is open .Therefore V_x is neighborhood of e.

Let $y \in V_x \setminus \{e\}$, and $y = y_0 \cdot ... \cdot y_m$ the decomposition of y into an $M-$product. Then for each $i \leq n$ $y = (x_i \cdot ... \cdot x_n)^{-1} t$ where $t \in M(x_i)$. Thus for all $i \leq n$, we have $(x_i \cdot ... \cdot x_n) y = (x_i \cdot ... \cdot x_n)(x_i \cdot ... \cdot x_n)^{-1} t = et = t \in M(x_i)$. If $i = 0$ it follows that $x_0 \cdot ... \cdot x_n y_0 \cdot ... \cdot y_m$ is an M-product and we see that

$$h(xy) = x_0 \cdot ... \cdot x_n y_0 \cdot ... \cdot y_m$$

$$= h(x_0) \cdot ... \cdot h(x_n) h(y_0) \cdot ... \cdot h(y_m)$$

$$= h(x)h(y).$$

□

3.3 Finite groups in Stone-Čech compactification

Theorem 3.3.1. *[17] Let κ be an infinite cardinal. For every ordinal $\alpha < \kappa$, let G_α be a nontrivial group, $G = \bigoplus_{\alpha<\kappa} G_\alpha$, let*

$$\mathbf{H} = \bigcap_{\alpha<\kappa} \{c\ell_{\beta G}\{x \in G \setminus \{e\} : \min supp(x) \geq \alpha\}\}.$$

Then $\mathbf{H}$ contains no nontrivial finite group.

Proof. Assume, on the contrary, that there is a nontrivial finite group Q in $\mathbf{H}$. If Q is not cyclic then we can find cyclic subgroup G in $\mathbf{H}$ generated by any element in Q, so we will consider Q as a cyclic group. Let u be the identity of Q. Consider the subset

$$S = \{x \in G^* : xQ = Q\}.$$

In G^* clearly Q is a left ideal of S, because for all $x \in S, xQ = Q$ so $SQ = Q$. To show that it is minimal, let $t \in Q$ then $St \subseteq Q$. Now let $h \in Q$. Since Q is a group, then $t^{-1} \in Q$. Thus $h = ht^{-1}t \in St$, since $ht^{-1} \in S$. Since t was arbitrary then from Lemma 1.1.27 it is minimal.

Also S can be defined as

$$S = \{x \in G^* : xu \in Q\}.$$

To prove this let $x \in G^*$ such that $xu \in Q$, and let $t \in xQ$, then $t = xh, h \in Q$. Since u is the identity and Q is abelain then $t = xhu = xuh \in Q$, so $xQ \subseteq Q$. Since Q is group, then it is minimal and hence $xQ = Q$.

Conversely let $x \in G^*$ such that $xQ = Q$ then $xu \in Q$.

We will prove that S is a closed subsemigroup in G^*, and S^1 is left saturated in βG.

To check that S is a subsemigroup, let $x, y \in S$. Then $(xy)Q = x(yQ) = xQ = Q$. Since $x, y \in G^*$, then $xy \in G^*$, and so, $xy \in S$.

To see that S is closed, let $\sigma = \rho_{u|G^*}$. Since the function $\rho_u : \beta G \to \beta G$ is defined by $\rho_u(x) = xu$, then for all $x \in G^*$, $\sigma(x) = xu$. Hence, $\sigma^{-1}(Q) = \{x \in G^* : xu \in Q\} = S$. Since τ is T_1 topology, then any finite set is closed which implies Q is closed. Since σ is continuous function, and Q is closed of G^*, then $S = \sigma^{-1}(Q)$ is closed.

To see that S^1 is left saturated, Suppose to contrary S^1 is not left saturated, then $(\beta G)S^1 \cap S^1 \neq \phi$ so, there is $x \in \beta G$ such that $xy = z$ for some $y, z \in S^1$. Thus $xyQ = zQ$. Since $zy \in S^1$, then $yQ = Q$ and $zQ = Q$ so, $xQ = xyQ = zQ = Q$. Suppose to contrary that $x \in G \setminus \{e\}$. Since $Q \subseteq \mathbf{H} = Ult(\tau_0)$, then for all ultrafilter $p \in Q$ we have $P \to e$. Since λ_x is continuous, for all $x \in G \setminus \{e\}$, then $xp \longrightarrow x$. But $xQ = Q$, then $xp \longrightarrow e$ for all $p \in Q$. contradiction. Hence $x \notin G \setminus \{e\}$. Hence $x \in S^1$ which is a contradiction.

By Proposition 3.1.16, there is a regular left invariant topology τ on G with

$Ult(\tau) = S$.

Since Q is a minimal left ideal, there is only one minimal right ideal of S. To prove this suppose by contrary there exist two minimal right ideals R_1 and R_2 of S such that $R_1 \neq R_2$. Since Q is a minimal left ideal, then by Theorem 1.1.28 $K(S) = QR_1 = QR_2$.

Let $a \in R_1$, be arbitrary and let $q \in Q$ then $qa \in QR_1$ so, $qa = q'b$ for some $q' \in Q$, and $b \in R_2$. Since Q is group, then q^{-1} exist. Thus, $a = q^{-1}q'b \in QR_2$. Since a was arbitrary then $R_1 \subseteq QR_2 = K(S)$. But $K(S)$ is the smallest ideal of S then $R_1 = K(S)$. ....(1)

In the same way we can prove that $R_2 \subseteq QR_1 = K(S)$. Thus $R_2 = K(S)$ because $K(S)$ is the smallest ideal. ....(2)

From (1) and (2), we have $R_1 = R_2$ which is a contradiction.

Since S have only one minimal right ideal Then by Lemma 3.1.10, τ is extremally disconnected. Being regular extremally disconnected, τ is zero-dimensional.

Next, for every $p \in Q$, put

$$Sp = \{x \in S : xu = p\}.$$

Claim: $\{S_p : p \in Q\}$ is a closed partition of S

(a) To show tat $\bigcup S_p = S$, let $x \in S$, then $xu \in Q$. So, $xu = p$ for some $p \in Q$. Thus, $x \in S_p$ so $S \subseteq \bigcup S_p$. Since $S_p \subseteq S$ for all $p \in Q$, then $\bigcup S_p \subseteq S$.

(b) If $S_p \cap S_q \neq \phi$ for some $p, q \in Q$, then there is $x \in S_p \cap S_q$ so $xu = p$ and $xu = q$ so, $\rho_u(x) = p$ and $\rho_u(x) = q$ so $p = q$ which implies that $S_p = S_q$.

(c) To show that S_p is closed, let $\sigma = \rho_{u|G^*}$, then $S_p = \sigma^{-1}\{p\}$. Since τ is T_1 topology then any finite set is closed so $\{p\}$ is closed. Since σ is continuous function, and p is closed of G^*, then S_p is closed.

Now for all $p \in Q$, we have $pu = p$, because u is the identity of Q. Thus, $p \in S_p$ for all $p \in Q$. Since S_p is closed, then from Theorem 3.1.14 we can find F_p the filter on G with $\widehat{F_p} = S_p$. For every $p \in Q$, $S_p \cap S_q = \phi$. If $p \neq q$ so, $\widehat{F_p} \cap \widehat{F_q} = \phi$. Thus there is $V_p \in F_p$, and $V_q \in F_q$, such that $V_p \cap V_q = \phi$. Now we will show that for each $p \in Q$, there is a $W_p \in F_p$ where

$$W_p S_q \subseteq \widehat{V_{pq}}$$

for all $q \in Q$.

Let $\tau = \rho_u|_{\beta G \setminus \{e\}}$. Then $\tau^{-1}(pq) = \{x \in S : rho_u(x) = pq\}$=$\{x \in S : xu = pq\}$ $= S_{pq}$. Since $V_{pq} \in F_{pq} \Rightarrow \widehat{F_{pq}} \subseteq \widehat{V_{pq}}$ so, $\tau^{-1}(pq) = S_{pq} = \widehat{F_{pq}} \subseteq \widehat{V_{pq}}$. Since $\rho_u(pq) = pqu$, then $pq \in S_{pq}$. But S_{pq} is closed so, there is $C_{pq} \in pq$ such that $pq \in \widehat{C_{pq}} \subseteq S_{pq}$.

Now

$$\tau^{-1}(\widehat{C_{pq}}) = \{x \in \beta G \setminus \{e\} : xu \in \widehat{C_{pq}}\}$$

$$= \{x \in \beta G \setminus \{e\} : xu \in S_{pq}\}$$

$$= \{x \in \beta G \setminus \{e\} : xu \in \tau^{-1}(pq)\}$$

$$= \{x \in \beta G \setminus \{e\} : \tau(xu) = pq\}$$

$$= \{x \in \beta G \setminus \{e\} : xuu = pq\}$$

$$= \{x \in \beta G \setminus \{e\} : xu = pq\} = \tau^{-1}(pq) \subseteq \widehat{V_{pq}}.$$

Next, we will prove that there is a $W_p \in F_p$ such that $W_p q \subseteq \widehat{C_{pq}}$ for all $q \in Q$. Let $q \in Q$. Since $S_p u = \{xu : x \in S_p\} = \{p\}$, then $\widehat{F_p} q = S_p q = S_p u q = pq$. Since $F_p \in \widehat{F_p}$, then $F_p q = pq$ so, $C_{pq} \in F_p q$, hence from Theorem 2.2.18 there is $W_q = \{x \in S : x^{-1} C_{pq} \in q\} \in F_p$. To show that $W_q q \subseteq \widehat{C_{pq}}$ let $x \in W_q$ then $x^{-1} C_{pq} \in q \Rightarrow C_{pq} \in xq \Rightarrow xq \in \widehat{C_{pq}}$. thus $W_q q \subseteq \widehat{C_{pq}}$. Let $W_p = \bigcap_{q \in Q} W_q$, since Q is finite, then $W_p \in F_p$. Also, $W_p q \subseteq W_q q \subseteq \widehat{C_{pq}}$. for all $q \in Q$.

Since $S_q u = \{xu : x \in S_q\} = \{q\}$, then

$$W_p S_q u = W_p q \subseteq \widehat{C_{pq}},$$

Since $W_p S_q u = \{xu : x \in W_p S_q\} \subseteq \widehat{C_{pq}}$, then $W_p S_q \subseteq \{x \in \beta G \ \{e\} : xu \in \widehat{C_{pq}}\} \subseteq \widehat{V_{pq}}$.

Since $W_p \in F_p$, and $V_p \in F_p$, where F_p is a filter then $W_p \bigcap V_p \in F_p$, and $W_p \bigcap V_p \neq \phi$. Since $W_p S_q \subseteq \widehat{V_{pq}}$, then $(W_p \bigcap V_p) S_q \subseteq \widehat{V_{pq}}$. So we can choose the subsets W_p such that

$$W_p = W_p \bigcap V_p \subseteq V_p$$

and

$$X = \bigcup_{p \in Q} W_p \cup \{e\}$$

is open in τ. Then define $f : X \to Q$ by

$$f(x) = p \quad if \quad x \in W_p,$$

since $W_p \subseteq V_p$, and $V_p \cap V_q = \phi$. If $p \neq q$, then $W_p \cap W_q = \phi$ for all $p \neq q$ and hence f is well defined.

To show that f is a local homomorphism let $x \in X \setminus \{e\}$, then $x \in W_p$ for some $p \in Q$. For each $q \in Q$, choose $U_{x,q} \in F_q$ such that

$$U_{x,q} \subseteq W_q \quad and \quad xU_{x,q} \subseteq V_{pq}.$$

To show this, let $x \in W_p$ Since $W_p S_q \subseteq \widehat{V_{pq}}$, then $W_p \widehat{F_q} \subseteq \widehat{V_{pq}}$. But $x \in W_p$ and, $F_q \in \widehat{F_q}$ then $xF_q \in W_p \widehat{F_q} \subseteq \widehat{V_{pq}}$. So, $xF_q \in \widehat{V_{pq}}$; that is, $V_{pq} \in xF_q$. Thus from Theorem 2.2.18 $x^{-1}V_{pq} \in F_q$. Let $M_{x,q} = x^{-1}V_{pq}$, then $M_{x,q} \in F_q$. Also, $xM_{x,q} = V_{pq}$. Since $M_{x,q} \in F_q$, and $W_q \in F_q$, then $M_{x,q} \cap W_q \neq \phi$. Let $U_{x,q} = M_{x,q} \cap W_q$, then $U_{x,q} \subseteq W_q$. Moreover, $xU_{x,q} \subseteq xM_{x,q} = V_{pq}$.

Since X is open and $e \in X$ then there exist a neighborhood U_x of e such that $U \subseteq X = \bigcup_{p \in Q} W_p \cup \{e\}$. Since τ is a left invariant topology then $xU_x \in \tau$. So, xU_x is a a neighborhood of x and $xU_x \subseteq X$. Since $U_{x,q} \subseteq W_q$, then we can choose a neighborhood U_x of $e \in X$ such that

$$U_x \subseteq \bigcup_{q \in Q} U_{x,q} \cup \{e\} \quad and \quad xU_x \subseteq X$$

Now, let $y \in U_x \setminus \{e\}$. Then $y \in U_{x,q}$ for some $q \in Q$. Since $xU_{x,q} \in Vpq$, one has $xy \in Vpq$. But then, since $V_p \cap V_q = \phi$, $W_p \cap W_q = \phi$ for all $p \neq q$. Since $xU_x \subseteq X$, and $xy \in Vpq$ it follows that, $xy \in W_{pq}$. Hence

$$f(xy) = pq = f(x)f(y).$$

Let $f^* : Ult(\tau) \to Q$. Now for each $q \in Q$, $f^*(q) = f^*(Lim_{x \to q} x)$ $= lim_{x \to q} f(x) =$ $lim_{x \to q}(q) = q$ so f^* is a coretraction.

On the other hand, let R be a cyclic group of order $|Q|^2$, and let $g : R \to Q$ be a surjective homomorphism. By Proposition 3.2.6, there is a local homomorphism $h : X \to R$ such that $f = g \circ h$. Since $h : X \to R$, then $h^* : Ult(\tau) \to R$ so, $g \circ h^* : Ult(\tau) \to Q$.

Moreover, since $f = g \circ h$, then $g \circ h^*(q) = g(h^*(q)) = g(h^*(Lim_{x \to q} x) = g(lim_{x \to q} h(x)) = lim_{x \to q} g \circ h = lim_{x \to q} f(x) = f^*(q)$. It follows that $f^* = g \circ h^*$.

Since g is surjective homomorphism then by first isomorphism Theorem $R/Ker(g) = Q$ so $|R/Ker(g)| = |Q|$. But $|R/Ker(g)| = \frac{|R|}{|Ker(g)|} = \frac{|Q|^2}{|Ker(g)|}$, hence $|Ker(g)| = |Q|$.

Since f^* is a coretraction, g is a coretraction as well. However, this is false because R has only one subgroup of order $|Q|$ and it is the kernel of g. □

Corollary 3.3.2. *[17] Let κ be an infinite cardinal, and $G = \bigoplus_\kappa \mathbb{Z}_2$. Then*

$$\mathbb{H}_\kappa = \bigcap_{\alpha < \kappa} \{c\ell_{\beta G}\{x \in G \setminus \{0\} : \min supp(x) \geq \alpha\}.$$

contains no nontrivial finite group.

Chapter 4

Discrete Groups in $\beta\mathbb{N}$

4.1 Copies of $\mathbb{Z}$

In this section we will show that there are $2^{\mathfrak{c}}$ discrete copies of $\mathbb{Z}$ in each of the maximal groups in the smallest ideal of $(\beta\mathbb{N}, +)$, and that any two of these meet only on the identity. Recall that $\mathfrak{c}$ is cardinal number of $\mathbb{R}$.

Definition 4.1.1. *The discrete copy* of $\mathbb{Z}$ is a countably infinite discrete space homomorphic to $\mathbb{Z}$.

Recall that we take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, the principal ultrafilters being identified with the points of $\mathbb{N}$.[5] Given $A \subseteq \mathbb{N}$, $\widehat{A} = c\ell A = \{p \in \beta\mathbb{N} : A \in p\}$. The set $\{\widehat{A} : A \subseteq N\}$ is a basis for the open sets (as well as a basis for the closed sets) of $\beta\mathbb{N}$.

Remark 4.1.2. A fundamental topological property of $\beta\mathbb{N}$ which we shall need, is that every neighborhood $U \subseteq \beta\mathbb{N}$ of an ultrafilter $p \in \beta\mathbb{N}$ satisfies $U \cap \mathbb{N} \in p$.

We write $\mathbb{N}^*$ for $\beta\mathbb{N} \setminus \mathbb{N}$ and denote the set of finite nonempty subsets of a set X by $\mathcal{P}_f(X)$. Also, $\omega = \mathbb{N} \cup \{0\}$.

Definition 4.1.3. [4] Let p be an idempotent in $\beta\mathbb{N}$, then we define the set $H(p) = \bigcup\{G : G \text{ is a subgroup of } \beta\mathbb{N} \text{ and } p \in G\}$

Theorem 4.1.4. *[4] Let p be an idempotent in $\beta\mathbb{N}$. Then $H(p)$ is the largest subgroup of $\beta\mathbb{N}$ with p is an identity.*

Theorem 4.1.5. *[4] Let S be a semigroup. If there is a minimal left ideal of S which has an idempotent p then the following are equivalent.*

(a) $p \in K(S)$ where $k(S)$ is the smallest ideal of S.

(b) $pSp = H(p)$.

If we apply Theorems 4.1.4 and 4.1.5 on $\beta\mathbb{N}$ we conclude that if q is an idempotent of $\beta\mathbb{N}$ and $q \in K(\beta\mathbb{N})$, then $q + \beta\mathbb{N} + q = H(q)$ with identity q.
[4]In $\beta\mathbb{N}$ there are $2^{\mathfrak{c}}$ minimal right ideals and $2^{\mathfrak{c}}$ minimal left ideals, and consequently $2^{\mathfrak{c}}$ maximal groups in the smallest ideal.
Recall that if p and q in $\beta\mathbb{N}$ and $A \subseteq \mathbb{N}$, $A \in p+q$ if and only if $\{x \in \mathbb{N} : -x+A \in q\} \in p$, where $-x + A = \{y \in \mathbb{N} : x + y \in A\}$.

Definition 4.1.6. [6] Given $x \in \mathbb{N}$, we define *supp*(x) is the $H \in \mathcal{P}_f(\omega)$. such that $x = \sum_{t\in H} 2^t$ and this representation is unique.

Definition 4.1.7. [6] $\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}}$

Theorem 4.1.8. *[4] The set $\mathbb{H}$ is compact subsemigroup in $(\beta\mathbb{N}, +)$.*

Lemma 4.1.9. *[4] Let q be an idempotent in $(\beta\mathbb{N}, +)$. Then for every $n \in \mathbb{N}$, $n\mathbb{N} \in q$*

From lemma 4.1.9 we can show that all idempotents of $(\beta\mathbb{N}, +)$ lie in $\mathbb{H}$.
A topological space is said to be $\sigma-$*compact* if it is the union of countably many compact subspaces.

Theorem 4.1.10. *[4] Let S be a discrete space and let A and B be $\sigma-$compact subsets of βS. If $A \cap c\ell B = c\ell A \cap B = \phi$, then $c\ell A \cap c\ell B = \phi$*

Remark 4.1.11. If $y \in 2^s\mathbb{N}$ for some $s \in \mathbb{N}$, then $\min supp(y) \geq s$.

Proof. Let $s \in \mathbb{N}$, and let $y \in 2^s\mathbb{N}$, then $y = 2^s m$ for some $m \in \mathbb{N}$. So, there is $H \in \mathcal{P}_f(\omega)$ such that $m = \sum_{t\in H} 2^t$. Thus $y = 2^s \sum_{t\in H} 2^t = \sum_{t\in H} 2^{t+s}$. Since $t \geq 0$ for all $t \in H$ then $2^{t+s} \geq 2^s$. Hence $\min supp(y) \geq s$. □

Theorem 4.1.12. *[6] Let A and B be infinite disjoint subsets of $\mathbb{N}$. Let $q = q + q \in \beta K(\beta\mathbb{N})$, let $u \in \mathbb{N}^* \cap c\ell\{2^n : n \in A\}$, and let $v \in \mathbb{N}^* \cap c\ell\{2^n : n \in B\}$. Let φ and ψ be the homomorphisms from $\mathbb{Z}$ into the group $q + \beta\mathbb{N} + q$ such that $\varphi(1) = q + u + q$and $\psi(1) = q + v + q$. Then $c\ell\{\varphi(n) : n \in \mathbb{Z} \setminus \{0\}\} \cap c\ell\{\psi(n) : n \in \mathbb{Z} \setminus \{0\}\} = \phi$.*
If $q \notin c\ell\{\varphi(n) : n \in \mathbb{Z} \setminus \{0\}\}$, then $\{\varphi(n) : n \in \mathbb{Z} \setminus \{0\}\}$ is a discrete copy of $\mathbb{Z}$.
If $q \notin c\ell\{\psi(n) : n \in \mathbb{Z} \setminus \{0\}\}$, then $\{\psi(n) : n \in \mathbb{Z} \setminus \{0\}\}$ is a discrete copy of $\mathbb{Z}$.

Proof. First we will show that $\psi(1) \in \mathbb{H}$. Since q is an idempotent then $q \in \mathbb{H}$. Since H is semigroup then it is enough to show that $v \in \mathbb{H}$. Since $v \in c\ell\{2^n : n \in B\}$, then there is sequence $X_m \in \{2^n : n \in B\}$ such that $\lim_{m\to\infty} 2^m = v$. where $m \in B$. Let $k \in \mathbb{N}$ be fixed. for all $m \geq k$, $2^k \mid 2^m$ that is there is $r_m \in \mathbb{N}$ such that $2^k r_m = 2^m$. Hence $v = \lim_{m\to\infty} 2^m = \lim_{m\geq k} 2^m = \lim_{m\geq k} 2^k r_m$. Therefore $v \in \overline{2^k\mathbb{N}}$. Since k was arbitrary then $v \in \mathbb{H}$. So, $\psi(1) \in \mathbb{H}$ and consequently $\psi(n) \in \mathbb{H}$ for all $n \in \mathbb{Z}$ that is $\psi(\mathbb{Z}) \subseteq \mathbb{H}$. In the same way we can show that $\varphi(\mathbb{Z}) \subseteq \mathbb{H}$.
Now We show that for any $m \in \mathbb{Z}$ and any $n \in \mathbb{N}$,

$$\{x \in \mathbb{N} : |supp(x) \cap A| \equiv 0 \ (mod\, n)\} \in \psi(m) \ \ and$$

$$\{x \in \mathbb{N} : |supp(x) \cap B| \equiv 0 \ (mod\, n)\} \in \varphi(m).$$

It suffices to establish the first statement. Let $n \in \mathbb{N}$, and let $C = \{x \in \mathbb{N} : |supp(x) \cap A| \equiv 0(mod\, n)\}$. We show first that $C \in q$. For each $i \in \{0, 1, 2, ..., n-1\}$ Let $A_i = \{x \in \mathbb{N} : |supp(x) \cap A| \equiv i(mod\, n)\}$ then the collection of sets $\{A_0, ..., A_{n-1}\}$ is partition of $\mathbb{N}$. To show this let $x \in \mathbb{N}$. Since the set $\{0, 1, 2, ..., n-1\}$ is complete residue system modulo n then $|supp(x) \cap A| \equiv i(mod\, n)$ fore some $i \in \{0, 1, 2, ..., n-1\}$, hence $x \in A_i$. So, $\mathbb{N} \subseteq \bigcup_{i=0}^{n-1} A_i$. Since each $A_i \subseteq \mathbb{N}$, then $\bigcup_{i=0}^{n-1} A_i \subseteq \mathbb{N}$. Therefore $\bigcup_{i=0}^{n-1} A_i = \mathbb{N}$.
Suppose by contrary there is $x \in A_i \cap A_j$ for some $0 \leq i < j \leq n-1$ then $|supp(x) \cap A| \equiv i(mod\, n)$ and $|supp(x) \cap A| \equiv j(mod\, n)$ which implies that $i \equiv j(mod\, n)$ which is a contradiction.
Now we will prove that one of $A_i \in q$. Suppose by contrary it is not true then

$A_i^c \in q$ for all i. Thus $\bigcap_{i=0}^{n-1} A_i^c \in q$. But $\mathbb{N} = \bigcup_{i=0}^{n-1} A_i$, then $\phi = \mathbb{N}^c = \bigcap_{i=0}^{n-1} A_i^c \in q$, contradiction. Therefore we can choose $i \in \{0, 1, 2, ..., n-1\}$ such that $D = \{x \in \mathbb{N} : |supp(x) \cap A| \equiv i \,(mod\, n)\} \in q$. Since $q = q + q$, then $D \in q + q$. So, $\{x \in \mathbb{N} : -x + D \in q\} \in q$. Hence, $\{x \in \mathbb{N} : -x + D \in q\} \cap D \neq \phi$ so, we can pick $x \in D$ such that $-x + D \in q$. Let $t = \max supp(x)$. From Lemma 4.1.9, since q is an idempotent, then $2^{t+1}\mathbb{N} \in q$. Since q is an ultrafilter, then $(-x + D) \cap D \cap 2^{t+1}\mathbb{N} \neq \phi$. Pick $y \in (-x + D) \cap D \cap 2^{t+1}\mathbb{N}$.
we will show that $|supp(x + y)| =|supp(x)| + |supp(y)|$. Since $y \in 2^{t+1}\mathbb{N}$, and $t = \max supp(x)$, then from Remark 4.1.11 $|supp(x)| \cap |supp(y)| = \phi$. Let H_1,H_2 and $H_3 \in \mathcal{P}_f(w)$ such that $x = \sum_{t\in H_1} 2^t$,$y = \sum_{s\in H_2} 2^s$, and $x + y = \sum_{r\in H_3} 2^r$. Since H_1 and H_2 are finite then $x + y = \sum_{t\in H_1} 2^t + \sum_{s\in H_2} 2^s = \sum_{k\in H_1\cup H_2} 2^k$. From definition of $supp(x + y)$, we have $\sum_{r\in H_3} 2^r = \sum_{k\in H_1\cup H_2} 2^k$. Then $H_3 = H_1 \cup H_2$. Since $H_1 \cap H_2 = \phi$, then $|H_3|= |H_1| + |H_2|$; that is, $|supp(x + y)| =|supp(x)| + |supp(y)|$. Hence $|supp(x + y) \cap A| =|supp(x) \cap A| + |supp(y) \cap A|$. Since $y \in (-x + D)$, then $y + x \in D$ so, $i \equiv |supp(x + y) \cap A|= |supp(x) \cap A| + |supp(y) \cap A| = i + i \ (mod\ n)$ so, $i = 0$, that is; $D = C$.
Now we show by induction on $m \in \omega$ that $C \in \psi(m)$. Since q is the identity of the group $q + \beta\mathbb{N} + q$, then $\psi(0) = q$. Hence $C \in \psi(0)$. Assume that $m \in \omega$ and $C \in \psi(m)$. Now

$$\psi(m) = \psi(1) + ... + \psi(1) \qquad m \ times$$

$$= q + v + q + q + v + q + ... + q + v + q \quad m \ times$$

$$= q + v + q + v + q + ... + q + v + q \quad since \ q + q = q$$

So,

$$\psi(m + 1) = \psi(1) + ... + \psi(1) \qquad m + 1 \ times$$

$$= q + v + q + q + v + q + ... + q + v + q + q + v + q \qquad m + 1 \ times$$

$$= q + v + q + v + q + ... + q + v + q + v + q \qquad since \ q + q = q$$

$$= \psi(m) + v + q.$$

Hence we will show that $C \in \psi(m)+v+q$. First we prove $C \subseteq \{x \in \mathbb{N} : -x+C \in v + q\}$, so, let $x \in C$ and let $t = \max supp(x)$. We claim that for $s > t$ and $s \in B$

we have $C \cap 2^{s+1}\mathbb{N} \subseteq -2^s + (-x + C)$. So, let $y \in C \cap 2^{s+1}\mathbb{N}$. Since $s > t$, then $|supp(x+y) \cap A| = |supp(x) \cap A| + |supp(y) \cap A|$. Since A and B are disjoint, then $s \notin A$. Since $supp(2^s) = \{s\}$, then $supp(2^s) \cap A = \phi$ so, $|supp(x + 2^s + y) \cap A| = |supp(x + y) \cap A|$. Since x an $y \in C$, then $|supp(x) \cap A| \equiv |supp(y) \cap A| \equiv 0$. Hence $|supp(x + 2^s + y) \cap A| = |supp(x) \cap A| + |supp(y) \cap A| \equiv 0 \ mod(\ n)$. Thus $x + 2^s + y \in C$. that is $y \in -2^s + (-x + C)$.

Since q is an idempotent then by lemma 4.1.9, $2^{s+1}\mathbb{N} \in q$ so, $C \cap 2^{s+1}\mathbb{N} \in q$, and then $-2^s + (-x + C) \in q$. Thus $\{2^s : s \in B \text{ and } s > t\} \subseteq \{z \in \mathbb{N} : -z + (-x + C) \in q\}$. Since $\{2^s : s \in Bands > t\} \in v$, and v is an ultrafilter, then $\{z \in \mathbb{N} : -z + (-x + C) \in q\} \in v$. Therefore $-x + C \in v + q$. which implies $C \subseteq \{x \in \mathbb{N} : -x + C \in v + q\}$. By induction hypotheses $C \in \psi(m)$, and $\psi(m)$ is an ultrafilter, so $\{x \in \mathbb{N} : -x + C \in v + q\} \in \psi(m)$, that is; $C \in \psi(m) + v + q = \psi(m + 1)$.

To complete this portion of the proof, we let $m \in \mathbb{N}$ and show that $C \in \psi(-m)$. Since the sets $\{A_0, ..., A_{n-1}\}$ is partition of $\mathbb{N}$, and $\psi(-m)$ is an ultrafilter, then we can pick $i \in \{0, 1, ..., n-1\}$ such that $D = \{x \in \mathbb{N} : |supp(x) \cap A| \equiv i\ (mod\, n)\} \in \psi(-m)$. Since $\psi(0) = q$, then $C \in q = \psi(0) = \psi(m + (-m)) = \psi(m) + \psi(-m)$ so, $\{x \in \mathbb{N} : -x + C \in \psi(m)\} \in \psi(-m)$. Hence $D \cap \{x \in \mathbb{N} : -x + C \in \psi(m)\} \neq \phi$. So, pick $x \in D$ such that $-x + C \in \psi(m)$. Let $t = \max supp(x)$. Since $\psi(m) \in \mathbb{H}$ and $(-x + C) \cap C \in \psi(m)$ so, $c\ell((-x + C) \cap C) \cap 2^{t+1}\mathbb{N} \neq \phi$. Thus there is a principal ultrafilter y in $2^{t+1}\mathbb{N}$ such that $(-x + C) \cap C \in y$, that is; $y \in (-x + C) \cap C \cap 2^{t+1}\mathbb{N}$. Since $t = \max supp(x)$ and $y \in 2^{t+1}\mathbb{N}$, then $|supp(x + y)| = |supp(x)| + |supp(y)|$. Since $y \in -x + C$, then $x + y \in C$. Hence $0 \equiv |supp(x + y) \cap A| = |supp(x) \cap A| + |supp(y) \cap A| \equiv i + 0 \ mod(\ n)$, so $i = 0$. That is $D = C$.

By a nearly identical proof, one can also establish that for any $m \in \mathbb{Z}$, and any $n \in \mathbb{N}$

$$\{x \in \mathbb{N} : |supp(x) \cap A| \equiv m\ (mod\, n)\} \in \varphi(m) \ \ and$$

$$\{x \in \mathbb{N} : |supp(x) \cap B| \equiv m\ (mod\, n)\} \in \psi(m).$$

Notice in particular that this shows that if $k \neq m$ in $\mathbb{Z}$, then $\varphi(m) \neq \varphi(k)$

because we can find $n \in \mathbb{N}$ such that k an m are not congruence mod n. So, $U_m = \{x \in \mathbb{N} : |supp(x) \cap A| \equiv m \; mod(\; n) \;\} \in \varphi(m)$ and $U_m \notin \varphi(k)$. In the same way we can show that $\psi(m) \neq \psi(k)$ if $m \neq K$.

Now suppose that $c\ell\{\varphi(n) : n \in \mathbb{Z} \setminus \{0\}\} \cap c\ell\{\psi(n) : n \in \mathbb{Z} \setminus \{0\}\} \neq \phi$. Then by Theorem 4.1.10, either $\{\varphi(n) : n \in \mathbb{Z} \setminus \{0\}\} \cap c\ell\{\psi(n) : n \in \mathbb{Z} \setminus \{0\}\} \neq \phi$ or $\{\psi(n) : n \in \mathbb{Z} \setminus \{0\}\} \cap c\ell\{\varphi(n) : n \in \mathbb{Z} \setminus \{0\}\} \neq \phi$. Assume without loss of generality, that the former holds and pick $m \in \mathbb{Z} \setminus \{0\}$ such that $\varphi(m) \in c\ell\{\psi(n) : n \in \mathbb{Z} \setminus \{0\}\}$. Since $U = \{x \in \mathbb{N} : |supp(x) \cap A| \equiv m \; (mod \; |m| + 1)\} \in \varphi(m)$, then $c\ell U$ is a neighborhood of $\varphi(m)$. We claim that $c\ell U \cap \{\psi(n) : n \in \mathbb{Z}\} = \phi$. Suppose by contrary it is not true and pick $n \in \mathbb{Z}$ such that $\psi(n) \in c\ell U$. Then $U \in \psi(n)$, and $S = \{x \in \mathbb{N} : |supp(x) \cap A| \equiv 0 \; (mod \; |m| + 1)\} \in \psi(n)$. Since $\psi(n)$ is an ultrafilter, then $U \cap S \neq \phi$. So, there exist $x \in \mathbb{N}$ such that $|supp(x) \cap A| \equiv 0 \equiv m \; (mod \; |m|+1)\}$ contradiction. So, $c\ell U \cap \{\psi(n) : n \in \mathbb{Z}\} = \phi$. then $\varphi(m) \notin c\ell\{\psi(n) : n \in \mathbb{Z} \setminus \{0\}\}$. Contradiction.

To complete the proof, we may assume that $q \notin c\ell\{\varphi(n) : n \in \mathbb{Z} \setminus \{0\}\}$. Pick a neighborhood U of q which misses $\{\varphi(n) : n \in \mathbb{Z} \setminus \{0\}\}$. Given $n \in \mathbb{Z} \setminus \{0\}$, since $q = \varphi(0) = \varphi(n + (-n)) = \varphi(n) + \varphi(-n)$ then U is a neighborhood of $\varphi(n) + \varphi(-n)$. Let $\widehat{A}$ be a basic neighborhood of $\varphi(n)+\varphi(-n)$, such that $\varphi(n)+\varphi(-n) \in \widehat{A} \subseteq U$, then $A \in \varphi(n) + \varphi(-n)$; that is, $B = \{x \in \mathbb{N} : -x + A \in \varphi(-n)\} \in \varphi(n)$. We claim that $\widehat{B} + \varphi(-n) \subseteq \widehat{A}$. Let $P + \varphi(-n) \in \widehat{B} + \varphi(-n)$ where $p \in \widehat{B}$, then $B = \{x \in \mathbb{N} : -x + A \in \varphi(-n)\} \in p$ which implies that $A \in p + \varphi(-n)$. Hence $\widehat{B} + \varphi(-n) \subseteq U$. We claim that $\widehat{B} \cap \{\varphi(m) : m \in \mathbb{Z} \setminus \{n\}\} = \phi$, otherwise if there is $m \in \mathbb{Z} \setminus \{n\}$ such that $\varphi(m) \in \widehat{B}$, then $\varphi(m) + \varphi(-n) \subseteq \widehat{B} + \varphi(-n) \subseteq U$. Thus $\varphi(m - n) \in U$. which is a contradiction. Hence $\widehat{B} \cap \{\varphi(m) : m \in \mathbb{Z}\} = \{\varphi(n)\}$ so $\{\varphi(n)\}$ is open in the subspace generated by $\{\varphi(n) : n \in \mathbb{Z}\}$. So, it is discrete copy of $\mathbb{Z}$. □

Corollary 4.1.13. *[6] Let $q = q + q \in K(\beta\mathbb{N})$. For each $p \in \mathbb{N}^* \cap \overline{\{2^n : n \in \mathbb{N}\}}$ let φ_p be the homomorphism from $\mathbb{Z}$ to the group $q+\beta\mathbb{N}+q$ for which $\varphi_p(1) = q+p+q$. If $p \neq r \in \mathbb{N}^* \cap \overline{\{2^n : n \in \mathbb{N}\}}$, then $c\ell\{\varphi_p(n) : n \in \mathbb{Z} \setminus \{0\}\} \cap c\ell\{\varphi_r(n) : n \in \mathbb{Z} \setminus \{0\}\} = \phi$ and for all but at most one $p \in \mathbb{N}^*$, $\{\varphi_p(n) : n \in \mathbb{Z}$ is a discrete copy of $\mathbb{Z}$.*

Proof. let $p \neq r \in \mathbb{N}^* \cap \overline{\{2^n : n \in \mathbb{N}\}}$. Since $p \neq r$, then there is disjoint subsets A and B of $\mathbb{N}$ such that $A \in p$ and $B \in r$. Since $p \in \overline{\{2^n : n \in \mathbb{N}\}}$, then $\overline{\{2^n : n \in \mathbb{N}\}} \cap \mathbb{N} \in p$, hence, $\{2^n : n \in A\} = \overline{\{2^n : n \in \mathbb{N}\}} \cap \mathbb{N} \cap A \in p$. In the same way we can show that $\{2^n : n \in B\} = \overline{\{2^n : n \in \mathbb{N}\}} \cap \mathbb{N} \cap B \in r$.
Now if we applied the Theorem 4.1.12 on the homomorphisms $\varphi_p(n)$ and $\varphi_r(n)$ we get $c\ell\{\varphi_p(n) : n \in \mathbb{Z} \setminus \{0\}\} \cap c\ell\{\varphi_r(n) : n \in \mathbb{Z} \setminus \{0\}\} = \phi$.
Suppose by contrary there is two element $p, r \in \mathbb{N}^*$, and $p \neq r$ such that $q \in c\ell\{\varphi_p(n) : n \in \mathbb{Z} \setminus \{0\}\}$, and $q \in c\ell\{\varphi_r(n) : n \in \mathbb{Z} \setminus \{0\}\}$, then $c\ell\{\varphi_p(n) : n \in \mathbb{Z} \setminus \{0\}\} \cap c\ell\{\varphi_r(n) : n \in \mathbb{Z} \setminus \{0\}\} \neq \phi$ which is a contradiction. So, there is at most one $p \in \mathbb{N}^*$, such that $p \in c\ell\{\varphi_r(n) : n \in \mathbb{Z} \setminus \{0\}\}$. Hence from Theorem 4.1.12 we conclude that for all but at most one $p \in \mathbb{N}^*$, $\{\varphi_p(n) : n \in \mathbb{Z}\}$ is a discrete copy of $\mathbb{Z}$. □

4.2 Discrete free groups and semigroups in $\mathbb{N}^*$

Recall that we say a group G is generated by a subset S, if every element of the group can be expressed as the combination (under the group operation) of finitely many elements of the subset S and their inverses.

Definition 4.2.1. [18] A group F is called *free* if there is a subset S of F such that any element of F can be written in one and only one way as a product of finitely many elements of S and their inverses.

Construction[18] The free group F with free generating set S can be constructed as follows. S is a set of symbols and we suppose for every s in S, there is a corresponding "inverse" symbol, s^{-1}, in a set S^{-1}. Let $T = S \cup S^{-1}$, and define *a word* in S to be any written product of elements of T. The empty word is the word with no symbols at all. For example, if $S = \{a, b, c\}$, then $T = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$ and $abc^{-1}ca^{-1}c$ is a word in S. If an element of S lies immediately next to its inverse, the word may be simplified by omitting the s, s^{-1} pair:
$abc^{-1}ca^{-1}c \longrightarrow aba^{-1}c$.

A word that cannot be simplified further is called *reduced* The free group F is defined to be the group of all reduced words in S. To illustrate the operation group on F we give an example.
$(ab^{-3}a^{-2}b^2).(b^{-2}a^3b^{-4}c) = (ab^{-3}ab^{-4}c)$.

Theorem 4.2.2. *[4] Let $(S, .)$ be a semigroup, $\mathcal{A} \subseteq \mathcal{P}(S)$ have the finite intersection property . Let $(T, .)$ be a compact right topological semigroup and $\varphi : S \to T$ satisfy $\varphi(S) \subseteq \Lambda(T)$. Assume that there is some $A \in \mathcal{A}$ such that for each $x \in A$, there exist $B \in \mathcal{A}$ for which $\varphi(x.y) = \varphi(x).\varphi(y)$ for every $y \in B$. Then for all $p, q \in \bigcap_{A \in \mathcal{A}} \widehat{A}$, $\widehat{\varphi}(p.q) = \widehat{\varphi}(p).\widehat{\varphi}(q)$.*

Lemma 4.2.3. *[4] Let A be a set and let F be the free group generated by A, let G be an arbitrary group and let $\varphi : A \to G$ be any mapping . There is a unique homomorphism $\widehat{\varphi} : F \to G$ for which $\widehat{\varphi}(g) = \varphi(g)$ for every $g \in A$.*

Theorem 4.2.4. *[4] Let A be a set, and let F be the free group generated by A. Then F can be embedded in a compact topological group. This means that there is a compact topological group C and a one to one homomorphism $\varphi : F \to C$.*

Corollary 4.2.5. *[6] There is a compact topological group C which contains a free group F on the distinct generators $\{a_1, a_2, a_3, a_4\}$.*

Proof. Let $A = \{a_1, a_2, a_3, a_4\}$. Then by above theorem, here is a compact topological group C and a one to one homomorphism $\varphi : F \to C$. □

Lemma 4.2.6. *[6] Let C and F be as in Corollary* 4.2.5. *Let A_1, A_2, A_3, and A_4 be pairwise disjoint infinite subsets of $\mathbb{N}$ and let $q \in K(\beta\mathbb{N})$. For $i \in \{1, 2, 3, 4\}$ pick*

$$u_i \in \mathbb{N}^* \cap c\ell\{2^n : n \in A_i\}$$

and let $r_i = q + u_i + q$. Let G be the subgroup of $q + \beta\mathbb{N} + q$ generated by $\{r_1, r_2, r_3, r_4\}$. There is a continuous homomorphism $\sigma : \{0\} \cup \mathbb{H} \to C$ such that $\sigma_{|G}$ is an isomorphism onto F and $\sigma(r_i) = a_i$ for each $i \in \{1, 2, 3, 4\}$.

Proof. Denote the identity of C by e. Since each $n \in \omega$, can be expressed uniquely as $n = \sum_{i \in H} 2^i$, where $H \in \mathcal{P}_f(\omega)$ then define $f : \omega \to C$ as follows. *For $n \in \omega$,*

$$f(2^n) = \begin{cases} a_i, & n \in A_i \\ \text{e}, & n \notin \bigcup_{i=1}^4 A_i. \end{cases}$$

Given $L \in \mathcal{P}_f(\omega)$, $f(\sum_{n\in L} 2^n) = \prod_{n\in L} f(2^n)$ where the product is taken in increasing order of indices. And $f(0) = e$. Let $\widetilde{f} : \beta\omega \to C$ be the continuous extension of f, and let σ be the restriction of $\widetilde{f}$ to $\{0\} \cup \mathbb{H}$. Now we will show that σ is a homomorphism. Since C is a compact topological group then $\Lambda(C) = C$. Take the collection $\mathcal{A} = \{2^n\omega : n \in \mathbb{N}\}$. Let $F = \mathcal{P}_f(\omega)$, and let $S = \cap_{k\in F} 2^k w$. We claim that $S \neq \phi$. Let $t = \max F$ then for all $k \in F$, $k \geq t \Rightarrow 2^k \mid 2^t$. So, $2^t = 2^k m$ for some $m \in \mathbb{N}$. Thus $2^t \in S$. Since F was arbitrary, then $\mathcal{A}$ has a finite intersection property.

Let $A = \{2\omega\}$, let $x \in A$ and let $t = \max supp(x)$ then for all $y \in B = \{2^t\omega\}$, $\min supp(y) \geq 2^{t+1}$. Thus $supp(x) \cap supp(y) = \phi$ which implies $f(x + y) = f(x)f(y)$. Therefore By Theorem 4.2.2 applied to the collection $\mathcal{A} = \{2^n\omega : n \in \mathbb{N}\}$, σ is a homomorphism.

To see that $\sigma[G] = F$, it suffices to let $i \in \{1, 2, 3, 4\}$, and show that $\sigma(r_i) = a_i$. Since $u_i \in c\ell\{2^n : n \in A_i\}$, then there is a sequence (x_n) in $\{2^n : n \in A_i\}$ such that x_n converge to u_i. Since f is constantly equal to a_i on $\{2^n : n \in A_i\}$, and σ is continuous, we have that $\sigma(u_i) = lim\sigma(x_n) = a_i$. Since $q + q = q$, then $\sigma(q).\sigma(q) = \sigma(q)$. Since C is group then $\sigma(q) = e$. Since $r_i = q + u_i + q$. then $\sigma(r_i) = ea_ie = a_i$.

From lemma 4.2.3, we can pick a homomorphism $h : F \to G$, such that $h(a_i) = r_i$ for each $i \in \{1, 2, 3, 4\}$. Then $h[F] = G$. Therefore $\sigma \circ h : F \to F$ and $\sigma \circ h(a_i) = \sigma(h(a_i)) = \sigma(r_i) = a_i$, for each $i \in \{1, 2, 3, 4\}$, so $\sigma \circ h$ is the identity on F so σ is injective. □

Lemma 4.2.7. *[4]* $\mathbb{N}$ *is the center of* $(\beta\mathbb{N}, +)$ *and* $(\beta\mathbb{N}, .)$

Theorem 4.2.8. *[6] Let* A_1, A_2, A_3, A_4, C, F, G, u_1, u_2, u_3, u_4, r_1, r_2, r_3, r_4, *and* σ *be as in Lemma* (4.2.6). *Let* G_1 *be the subgroup of* G *generated by* $\{r_1, r_2\}$ *and let* G_2 *be the subgroup of* G *generated by* $\{r_3, r_4\}$. *Then* $c\ell(G_1 \setminus \{q\}) \cap c\ell(G_2 \setminus \{q\}) = \phi$. *If* $i \in \{1, 2\}$ *and* $q \notin c\ell(G_i \setminus \{q\})$, *then* G_i *is a discrete copy of* F.

Proof. Suppose that $c\ell(G_1 \ \{q\}) \cap c\ell(G_2 \ \{q\}) \neq \phi$. By Theorem 4.1.10, either $(G_1 \setminus \{q\}) \cap c\ell(G_2 \setminus \{q\}) \neq \phi$. or $c\ell(G_1 \setminus \{q\}) \cap (G_2 \setminus \{q\}) \neq \phi$. Assume without loss of generality that $(G_1 \setminus \{q\}) \cap c\ell(G_2 \setminus \{q\}) \neq \phi$, and pick w in this intersection. We shall show that $w = q$. Let s_1 and s_2 denote the inverses of r_1 and r_2 in G_1. Since $w \in (G_1 \setminus \{q\})$, then ω can be written as a linear combination of the elements of $\{r_1, r_2, s_1, s_2\}$. Pick $m \in \mathbb{N}$ and $p_1, p_2, ..., p_m \in \{r_1, r_2, s_1, s_2\}$ such that $w = p_1 + p_2 + ... + p_m$.

Define $\theta : \mathbb{N} \to \omega$ by $\theta(n) = \Sigma\{2^t : t \in supp(n) \cap (A_1 \cup A_2)\}$, and let $\widehat{\theta} : \beta\mathbb{N} \to \beta\omega$ be its continuous extension. Take the collection $\mathcal{A} = \{2^n\omega : n \in \mathbb{N}\}$. then As in proof of lemma 4.2.6 $\mathcal{A}$ has a finite intersection property. Now, let $A = \{2\omega\}$, let $x \in A$ and let $t = \max supp(x)$ then for all $y \in B = \{2^t\omega\}$, $\min supp(y) \geq 2^{t+1}$. Thus $supp(x) \cap supp(y) = \phi$ so, $|supp(x + y)| = |supp(x)| + |supp(y)|$. Hence $|supp(x + y) \cap (A_1 \cup A_2)| = |supp(x) \cap (A_1 \cup A_2)| + |supp(y) \cap (A_1 \cup A_2)|$. If $x = \sum_{s \in H_1} 2^s$ and $y = \sum_{k \in H_2} 2^k$ for some H_1, $H_2 \in \mathcal{P}_f(\omega)$, then $x + y = \sum_{s+k \in H_1 \cup H_2} 2^{s+k}$. Since $supp(x) \cap supp(y) = \phi$, then $f(x+y) = \Sigma\{2^{s+k} : s+k \in supp(x + y) \cap (A_1 \cup A_2)\} = \Sigma\{2^s : s \in supp(x) \cap (A_1 \cup A_2)\}+ \Sigma\{2^k : k \in supp(y) \cap (A_1 \cup A_2)\}$; that is, $f(x+y) = f(x) + f(y)$. If we consider $\theta : \mathbb{N} \to \beta\omega$, then $\beta\omega$ is a compact right topological semigroup and $\theta(\mathbb{N}) \subseteq \omega = \Lambda(\beta\omega)$. By Theorem 4.2.2 $\widehat{\theta}_{|\mathbb{H}}$ is a homomorphism.

Also, $\widehat{\theta}(\mathbb{H}) \subseteq \mathbb{H} \cup \{0\}$. To show this let $p \in \mathbb{H}$. Then $p \in \overline{2^n\mathbb{N}}$ for all $n \in \mathbb{N}$. Fix $k \in \mathbb{N}$, since $p \in \overline{2^k\mathbb{N}}$, then there is a sequence $x_s = 2^k s \in 2^k\mathbb{N}$ such that x_s converge to p. Now we have two cases:

Case (1): If $k \notin A_1 \cup A_2$, then $\theta(2^k) = 0$. Since $\widehat{\theta}$ is continuous, then $\widehat{\theta}(p) = \lim \widehat{\theta}(2^k)\widehat{\theta}(s) = \lim \theta(2^k)\theta(s) = 0$.

Case (2): If $k \in A_1 \cup A_2$, then $\theta(2^k) = 2^k$. So, $\widehat{\theta}(p) = \lim \widehat{\theta}(2^k)\widehat{\theta}(s) = \lim \theta(2^k)\theta(s) = \lim 2^k\theta(s)$. If $\theta(s) = 0$ for all but finitely number of x_s, then $\lim \theta(s) = 0$ which give $\widehat{\theta}(p) = 0$. If there is infinite number of x_s such that $\theta(s) \neq 0$ then we can find subsequence y_{m_s} of x_s such that $\theta(2^k m) \neq 0$ for all $2^k m \in y_{m_s}$. Hence $\widehat{\theta}(p) = \lim 2^k\theta(m)$. Moreover, $y_{m_s} \in 2^k\mathbb{N}$ so, $\widehat{\theta}(p) \in \overline{2^k\mathbb{N}}$. Since k was arbitrary, then $\widehat{\theta}(p) \in \overline{2^n\mathbb{N}}$ for all $n \in \mathbb{N}$; that is, $\widehat{\theta}(p) \in \mathbb{H}$.

For $i \in \{1, 2\}$, and $n \in A_i$, we have $\theta(2^n) = 2^n$. So, θ is the identity on

$\{2^n : n \in A_i\}$. Since $u_i \in c\ell\{2^n : n \in A_i\}$, then $\widehat{\theta}(u_i) = u_i$ and since $q + q = q$, then $\sigma(\widehat{\theta}(q)) = \sigma(\widehat{\theta}(q)) + \sigma(\widehat{\theta}(q))$. Since C is a group, then $\sigma(\widehat{\theta}(q)) = e$. Thus for $i \in \{1, 2\}$,

$$\begin{aligned}\sigma(\widehat{\theta}(r_i)) &= \sigma(\widehat{\theta}(q + u_i + q)) \\ &= \sigma(\widehat{\theta}(q)) + \sigma(\widehat{\theta}(u_i)) + \sigma(\widehat{\theta}(q)) \\ &= e\sigma(u_i)e \\ &= \sigma(q)\sigma(u_i)\sigma(q) \\ &= \sigma(q + u_i + q) \\ &= \sigma(r_i).\end{aligned}$$

Since $e = \sigma(s_i)\sigma(r_i)$, and $e = \sigma(\widehat{\theta}(r_i))\sigma(\widehat{\theta}(r_i)^{-1}) = \sigma(r_i)\sigma(\widehat{\theta}(r_i)^{-1})$ and C is a group, then $\sigma(\widehat{\theta}(s_i) = \sigma(\widehat{\theta}(r_i^{-1})) = \sigma(\widehat{\theta}(r_i)^{-1}) = \sigma(s_i)$.
Next we note that for $i \in \{3, 4\}$, and $n \in A_i$, $\theta(2^n) = 0$, because $n \notin (A_1 \cup A_2)$. Since $u_i \in c\ell\{2^n : n \in A_i\}$, then $\widehat{\theta}(u_i) = 0$ and since $q + q = q$, then $\widehat{\theta}(q) = \widehat{\theta}(q) + \widehat{\theta}(q)$. Thus for $i \in \{3, 4\}$,

$$\begin{aligned}\sigma(\widehat{\theta}(r_i)) &= \sigma(\widehat{\theta}(q + u_i + q)) \\ &= \sigma(\widehat{\theta}(q) + \widehat{\theta}(u_i) + \widehat{\theta}(q)) \\ &= \sigma(\widehat{\theta}(q) + 0 + \widehat{\theta}(q)) \\ &= \sigma(\widehat{\theta}(q)) = e.\end{aligned}$$

Also, $\sigma(\widehat{\theta}(s_i) = \sigma(\widehat{\theta}(r_i)^{-1}) = e$. Thus we have $\sigma \circ \widehat{\theta}[G_2] = \{e\}$. Since $w \in c\ell(G_2 \setminus \{q\})$, then $\sigma(\widehat{\theta}(w)) = e$; that is, $\sigma(\widehat{\theta}(p_1+p_2+...+p_m)) = e$. But $w \in (G_1\setminus\{q\})$, then $\sigma(\widehat{\theta}(p_1+p_2+...+p_m)) = \sigma(p_1+p_2+...+p_m)$ because $p_1, p_2, ..., p_m \in \{r_1, r_2, s_1, s_2\}$. Hence $\sigma(p_1 + p_2 + ... + p_m) = e$. from proof of lemma 4.2.6 we have $\sigma(q) = e$ and σ is an isomorphism on G so, $w = p_1 + p_2 + ... + p_m = q$ which is a contradiction.
Now to complete the proof let $i \in \{1, 2\}$ and $q \notin c\ell(G_i\setminus\{q\})$. Pick a neighborhood U of q which misses $G_i \setminus \{q\}$. From lemma (4.2.3) we can find a homomorphism $h : F \longrightarrow G_i$. Pick $x \in F$. Then $h(e) = h(xx^{-1}) = h(x) + h(x^{-1}) = q$. Let $\widehat{A}$ be a basic neighborhood of $h(x) + h(x^{-1})$ such that $h(x) + h(x^{-1}) \subseteq \widehat{A} \subseteq U$,

then $A \in h(x) + h(x^{-1})$. So, $B = \{x \in G : -x + A \in h(x^{-1})\} \in h(x)$. We claim that $\widehat{B} + h(x^{-1}) \subseteq \widehat{A}$ So, let $p + h(x^{-1}) \in \widehat{B} + h(x^{-1})$. Since $B = \{x \in G : -x + A \in h(x^{-1})\} \in p$, then $A \in p + h(x^{-1})$. Thus $p + h(x^{-1}) \in \widehat{A} \subseteq U$. We claim that $\widehat{B} \cap \{h(y) : y \in F \setminus \{x\}\} = \phi$. Suppose by contrary it is not true and pick $y \in F \setminus \{x\}$ such that $h(y) \in \widehat{B}$ then $h(y) + h(x^{-1}) = h(yx^{-1}) \in \widehat{A} \subseteq U$, contradiction. Hence $\widehat{B} \cap \{h(y) : y \in F\} = \{h(x)\}$. then G_i is a discrete copy of F. □

Corollary 4.2.9. *[6] Let $q = q + q \in K(\beta\mathbb{N})$. There exist $2^{\mathfrak{c}}$ copies of the free group on 2 generators in $(q + \beta\mathbb{N} + q) \cap \mathbb{H}$. The intersection of the closures of any two of these is $\{q\}$.*

Proof. Partition $\mathbb{N}^* \cap \{2^n : n \in \mathbb{N}\}$ into two element subsets $H_\alpha = \{x_\alpha, y_\alpha\}$ for $\alpha < 2^c$. For each $\alpha < 2^c$, let G_α be the subgroup of $q + \beta\mathbb{N} + q$ generated by $q + x_\alpha + q$, and $q + y_\alpha + q$. If $\alpha < \beta < 2^c$, then $x_\alpha \neq y_\alpha \neq x_\beta \neq y_\beta$. Pick disjoint subsets A_1, A_2, A_3, A_4 of $\mathbb{N}$ such that $A_1 \subseteq x_\alpha$, $A_2 \subseteq y_\alpha$, $A_3 \subseteq x_\beta$, and $A_4 \subseteq y_\beta$. Since $\{2^n : n \in \mathbb{N}\} \in x_\alpha$, then $\{2^n : n \in A_1\} = \{2^n : n \in \mathbb{N}\} \cap A_1 \cap \mathbb{N} \in x_\alpha$. In the same way we can show that $\{2^n : n \in A_2\}$, $\{2^n : n \in A_3\}$, $\{2^n : n \in A_4\}$ are members of y_α, x_β, y_β respectively. By Theorem 4.2.8 $c\ell(G_\alpha\ \{q\}) \cap c\ell(G_\beta\ \{q\}) = \phi$. Hence there is at most one $\alpha < 2^{\mathfrak{c}}$ for which there is some $\delta \neq \alpha$ with $c\ell(G_\alpha\ \{q\}) \cap c\ell(G_\delta\ \{q\}) \neq \phi$. □

Definition 4.2.10. [4] A semigroup S is *weakly left cancellative* if and only if for all $u, v \in S$, $\{x \in S : xu = v\}$ is finite.

Of course a left cancellative semigroup is weakly left cancellative . On the other hand the semigroup $(\mathbb{N}, \vee)$ is weakly left cancellative.

Definition 4.2.11. [14] A subset of topological space is G_δ if and only if it is countable intersection of open sets.

Theorem 4.2.12. *[4] Let S be an infinite right cancellative and weakly left cancellative semigroup. Then every G_δ subset of S^* which contains an idempotent contains a copy of $\mathbb{H}$.*

Corollary 4.2.13. *[6] Let S be an infinite discrete right cancellative, and weakly left cancellative semigroup. Let U be a G_δ subset of $\beta S \setminus S$ which contains an idempotent. There is a set $D \subseteq U$ of idempotents such that $|D| = 2^{\mathfrak{c}}$, and for each $q \in D$, there exist $2^{\mathfrak{c}}$ copies of the free group on 2 generators in $q\beta Sq$. The intersection of the closures of any two of these is $\{q\}$.*

Proof. By Theorem 4.2.12, U contains a copy of $\mathbb{H}$. Since $\mathbb{H}$ contains all idempotents of $\beta\mathbb{N}$, then There is a set $D \subseteq U$ of idempotents such that $|D| = 2^{\mathfrak{c}}$. Now for each idempotent $q \in D$ apply Corollary 4.2.9 to get the result. □

Note that one can take $U = \beta S \backslash S$ in Corollary 4.2.13, so any right cancellative and weakly left cancellative semigroup S has these discrete copies of the free group on 2 generators in $\beta S \setminus S$.

Definition 4.2.14. [9] Let X be an infinite set. A set $\mathcal{A}$ is a set of almost disjoint subsets of X if and only if $A \subseteq \mathcal{P}(X)$, for each $A \in \mathcal{A}$, $|A| = |X|$, and for $A \neq B$ in $\mathcal{A}$, $A \cap B$ is finite .

Note 4.2.15. *[9] There is a set $\mathcal{A}$ of $\mathfrak{c}$ almost disjoint subsets of $\mathbb{N}$. Probably the simplest example of a set of c almost disjoint subsets of a countably infinite set can be obtained as follows: For each $\alpha \in R$, choose an increasing sequence $\langle x_{\alpha,n} \rangle_{n=0}^{\infty}$ in $\mathbb{Q}$ which converges to α. Then $\{\{x_{\alpha,n} : n \in \omega\} : \alpha \in \mathbb{R}$ is a set of almost disjoint subsets of $\mathbb{Q}$.*

Definition 4.2.16. [4] A subset D of a topological space X is *strongly discrete*, if there is an indexed family $\langle U_x \rangle_{x \in D}$ such that for each $x \in D$, U_x is a neighborhood of x, and $U_x \cap U_y = \phi$ when $x \neq y$.

Theorem 4.2.17. *[6] There is a strongly discrete copy of the free semigroup with identity on $\mathfrak{c}$ generators in $\mathbb{H}$ (which is therefore discrete in $\mathbb{N}^*$.)*

Proof. Pick an indexed family $\langle A_\alpha \rangle_{\alpha < \mathfrak{c}}$ of almost disjoint subsets of $2\mathbb{N}+1$. (That can be done from note 4.2.15. For each $\alpha < \mathfrak{c}$, pick $p_\alpha \in \mathbb{N}^* \cap c\ell\{2^n : n \in A_\alpha\}$. Since $\mathbb{H} \cap c\ell\{x \in \mathbb{N} : supp(x) \subseteq 2\mathbb{N}\}$ is compact subsemigroup in $\beta\mathbb{N}$, then from Theorem 1.2.34 part (a), it has an idempotent. Pick $q = q + q \in c\ell\{x \in \mathbb{N} :$

$supp(x) \subseteq 2\mathbb{N}\}$. For $\alpha < \mathfrak{c}$, let $r_\alpha = q + p_\alpha + q$, and let $S = \{q\} \cup \{r_\alpha : \alpha < \mathfrak{c}\}$. Since q is an idempotent, then $q \in \mathbb{H}$. Since $p_\alpha \in c\ell\{2^n : n \in B\}$, then there is sequence $X_m \in \{2^n : n \in A_\alpha\}$ such that $\lim_{m\to\infty} 2^m = p_\alpha$, where $m \in A_\alpha$. Let $k \in \mathbb{N}$ be fixed for all $m \geq k$, $2^k \mid 2^m$. That is; there is $r_m \in \mathbb{N}$ such that $2^k r_m = 2^m$. Hence $p_\alpha = \lim_{m\to\infty} 2^m = \lim_{m\geq k} 2^m = \lim_{m\geq k} 2^k r_m$. Therefore $p_\alpha \in \overline{2^k\mathbb{N}}$. Since k was arbitrary, then $p_\alpha \in \mathbb{H}$. Since $\mathbb{H}$ is semigroup, then $r_\alpha \in \mathbb{H}$. So, $S \subseteq \mathbb{H}$.

For each finite sequence $\langle \alpha_1, \alpha_2, ..., \alpha_k \rangle$ in $\mathfrak{c}$, let $B_{(\alpha_1,\alpha_2,...,\alpha_k)} = \{x \in \mathbb{N} : supp(x) \cap (2\mathbb{N}+1) = \{n_1, n_2, ..., n_k\}$, where $n_1 < n_2 < ... < n_k$ and each $n_i \in A_{\alpha_i}\}$. Now we will prove that if $k > 1$, then $B_{(\alpha_1,\alpha_2,...,\alpha_{k-1})} \subseteq \{x \in \mathbb{N} : -x + B_{(\alpha_1,\alpha_2,...,\alpha_k) \in r_{\alpha_k}}\}$. So, let $x \in B_{(\alpha_1,\alpha_2,...,\alpha_{k-1})}$. We claim that $C = \{y \in \mathbb{N} : supp(y) \subseteq 2\mathbb{N}\} \subseteq \{z \in \mathbb{N} : -z + (-x + B_{(\alpha_1,\alpha_2,...,\alpha_k)}) \in P_{\alpha_k} + q\}$ so, let $y \in C$. We claim that $\{2^s : s \in A_{\alpha_k} s > n_{k-1}\} \subseteq \{h \in \mathbb{N} : -h + (-y + (-x + B_{(\alpha_1,\alpha_2,...,\alpha_k)})) \in q\}$. Let $t = \max\{\max supp(y), \max supp(x)\}$, and $s \in A_{\alpha_k}$. We claim that $C \cap 2^{t+1}\mathbb{N} \subseteq -2^s + (-y + (-x + B_{(\alpha_1,\alpha_2,...,\alpha_k)}))$, let $w \in C \cap 2^{t+1}\mathbb{N}$. Since each of w and y in C, then $supp(w) \subseteq 2\mathbb{N}$ and $supp(y) \subseteq 2\mathbb{N}$. Since $t \geq \max supp(y)$, then $supp(w) \cap supp(y) = \phi$ and $supp(w) \cap supp(x) = \phi$. Moreover, since $supp(y) \subseteq 2\mathbb{N}$. $supp(x+y) \cap 2\mathbb{N}+1 = (supp(x) \cap 2\mathbb{N}+1) \cup (supp(y) \cap 2\mathbb{N}+1)$. In the same way $supp(x+w) \cap 2\mathbb{N}+1 = (supp(x) \cap 2\mathbb{N}+1) \cup (supp(w) \cap 2\mathbb{N}) + 1$.

Hence $supp(x+y+2^s+w) \cap 2\mathbb{N}+1 = (supp(x) \cap 2\mathbb{N}+1) \cup (supp(x) \cap 2\mathbb{N}+1) \cup (supp(2^s) \cap 2\mathbb{N}+1) \cup (supp(w) \cap 2\mathbb{N}+1) = \{n_1, n_2, ..., n_{k-1}\} \cup \phi \cup s \cup \phi = \{n_1, n_2, ..., n_k\}$, where $n_k = s$. Thus $x + y + 2^s + w \in B_{(\alpha_1,\alpha_2,...,\alpha_k)}$. Therefore, $C \cap 2^{t+1}\mathbb{N} \subseteq -2^s + (-y + (-x + B_{(\alpha_1,\alpha_2,...,\alpha_k)}))$ Since q is an idempotent, then $2^{t+1}\mathbb{N} \in q$. But $C \in q$, then $C \cap 2^{t+1}\mathbb{N} \in q$ and consequentely, $-2^s + (-y + (-x + B_{(\alpha_1,\alpha_2,...,\alpha_k)})) \in q$. Hence $\{2^s : s \in A_{\alpha_k} s > n_{k-1}\} \subseteq \{h \in \mathbb{N} : -h + (-y + (-x + B_{(\alpha_1,\alpha_2,...,\alpha_k)})) \in q\}$.

Since $p_{\alpha_k} \in c\ell\{2^n : n \in A_\alpha\}$. then $\{2^s : s \in A_{\alpha_k} s > n_{k-1}\} \in p_{\alpha_k}$. So, $\{-y + (-x + B_{(\alpha_1,\alpha_2,...,\alpha_k)}) \in p_{\alpha_k} + q\}$. Thus $C \subseteq \{z \in \mathbb{N} : -z + (-x + B_{(\alpha_1,\alpha_2,...,\alpha_k)}) \in P_{\alpha_k} + q\}$. Since $C \in q$ then $\{z \in \mathbb{N} : -z + (-x + B_{(\alpha_1,\alpha_2,...,\alpha_k)}) \in P_{\alpha_k} + q\} \in q$. Therefore $-x + B_{(\alpha_1,\alpha_2,...,\alpha_k)} \in q + P_{\alpha_k} + q = r_{\alpha_k}$.....$(*)$

Now we will see by induction on k that for each $\langle \alpha_1, \alpha_2, ..., \alpha_k \rangle$, $B_{(\alpha_1,\alpha_2,...,\alpha_k)} \in r_{\alpha_1} + r_{\alpha_2} + ... + r_{\alpha_k}$. At $k = 1$ we will show that $C \subseteq \{x \in \mathbb{N} : -x + B_{(\alpha_1)} \in P_{\alpha_1+q}\}$

so, let $y \in C$. We claim that $\{2^s : s \in A_{\alpha_1}\} \subseteq \{z \in \mathbb{N} : -z+(-y+B_{(\alpha_1)}) \in q\}$. Let $t = \max supp(y)$, let $s \in A_{\alpha_1}$ and let $w \in C \cap 2^{t+1}\mathbb{N}$. Since each of w and y in C then $supp(w) \subseteq 2\mathbb{N}$ and $supp(y) \subseteq 2\mathbb{N}$. Since $s \in A_{\alpha_1}$ then $supp(y+2^s+w) \cap 2\mathbb{N}+1 = \{s\}$ so, $y+2^s+w \in B_{\alpha_1}$. Thus $C \cap 2^{t+1}\mathbb{N} \subseteq -2^s+(-y+B_{\alpha_1})$ and Consequently $-2^s+(-y+B_{\alpha_1}) \in q$. Hence $\{2^s : s \in A_{\alpha_1}\} \subseteq \{z \in \mathbb{N} : -z+(-x+B_{(\alpha_1)}) \in q\}$. Since $\{2^s : s \in A_{\alpha_1}\} \in p_{\alpha_1}$ then $\{z \in \mathbb{N} : -z+(-y+B_{(\alpha_1)}) \in q\} \in p_{\alpha_1}$ which implise that $-y+B_{(\alpha_1)} \in p_{\alpha_1}+q$. So, $C \subseteq \{x \in \mathbb{N} : -x+B_{(\alpha_1)} \in P_{\alpha_1+q}\}$. Since $C \in q$ then $\{x \in \mathbb{N} : -x+B_{(\alpha_1)} \in P_{\alpha_1+q}\} \in q$. Therefore $B_{(\alpha_1)} \in r_{\alpha_1}$.

Suppose the statement is true for $n = k-1$; that is, for each $\langle \alpha_1, \alpha_2, ..., \alpha_{k-1} \rangle$, $B_{(\alpha_1,\alpha_2,...,\alpha_{k-1})} \in r_{\alpha_1}+r_{\alpha_2}+...+r_{\alpha_{k-1}}$. For $n = k$ let $\langle \alpha_1, \alpha_2, ..., \alpha_k \rangle$. From $*$, $B_{(\alpha_1,\alpha_2,...,\alpha_{k-1})} \subseteq \{x \in \mathbb{N} : -x+B_{(\alpha_1,\alpha_2,...,\alpha_k) \in r_{\alpha_k}}\}$. so, $B_{(\alpha_1,\alpha_2,...,\alpha_k)} \in r_{\alpha_1}+r_{\alpha_2}+ ... + r_{\alpha_k}$.

Let $C = \{x \in \mathbb{N} : supp(x) \subseteq 2\mathbb{N}\}$. Note that for each $\langle \alpha_1, \alpha_2, ..., \alpha_k \rangle$, $C \cap B_{(\alpha_1,\alpha_2,...,\alpha_k)} = \phi$. Otherwise if there is x in the intesection then, $supp(x) \cap 2\mathbb{N}+1 = \phi$ since $x \in C$ and $supp(x) \cap 2\mathbb{N}+1 = \{n_1, n_2, ..., n_{k-1}\}$. Since $w \in B_{(\alpha_1,\alpha_2,...,\alpha_k)}$ contradiction.

To complete the proof we show that if $\langle \alpha_1, \alpha_2, ..., \alpha_k \rangle \neq \langle \delta_1, \delta_2, ..., \delta_l \rangle$, then $\overline{B_{(\alpha_1,\alpha_2,...,\alpha_k)}} \cap \overline{B_{(\delta_1,\delta_2,...,\delta_l)}} \cap \mathbb{H} = \phi$. If $k \neq l$, then $B_{(\alpha_1,\alpha_2,...,\alpha_k)} \cap B_{(\delta_1,\delta_2,...,\delta_l)} = \phi$, otherwise if there is w in this intersection, then $|supp(w) \cap 2\mathbb{N}+1| = k$ and $|supp(w) \cap 2\mathbb{N}+1| = l$, contradiction. So assume that $k = l$ and pick $i \in \{1, 2, ..., k\}$, such that $\alpha_i \neq \delta_i$. Since $A_{\alpha_i} \cap A_{\delta_i}$ is finite, then pick $m \in \mathbb{N}$ such that $A_{\alpha_i} \cap A_{\delta_i} \subseteq \{1, 2, ..., m\}$. Then $B_{(\alpha_1,\alpha_2,...,\alpha_k)} \cap B_{(\delta_1,\delta_2,...,\delta_l)} \cap 2^{m+1}\mathbb{N} = \phi$. To show this, suppose by contrary there is y in the intersection, then $\min supp(y) = m+1$. But $y \in B_{(\alpha_1,\alpha_2,...,\alpha_k)} \cap B_{(\delta_1,\delta_2,...,\delta_l)}$ then there is $n_i \in supp(y)$ such that $n_i \in A_{\alpha_i} \cap A_{\delta_i}$ so, $n_i \leq m$. which is a contradiction.

$\square$

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