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On φ - Classes of Modules

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To

My parents

My brothers and My sisters

and to all knowledge seekers

Abstract

Let R be a commutative ring with identity and let M be a unitary R-module. Let S(M) be the set of all submodules of M and ϕ :S(M)$\rightarrow$ S(M) $\bigcup$ $\{\emptyset\}$ be a function. Let N be a proper submodule of M. We say that N is ϕ-prime submodule of M if am $\in$ N - ϕ(N) where a $\in$ R, m $\in$ M implies that m $\in$ N or a $\in$ (N : M). This concept was introduced by N. Zamani (see [39]). In this thesis we generalize the concept of ϕ-prime submodule to ϕ-primary submodule and also, we introduce the concept of ϕ-primal submodule, and we introduce the concept of ϕ-compactly packed module.

The concept of n-primly ideals in a commutative ring was introduced by A.E.Ashour [8]. In this thesis we generalize it to the concept of n-primly submodule.

Let N be a proper submodule of M, we say that N is 2-absorbing submodule if abm $\in$ N where a,b $\in$ R, m $\in$ M implies that ab $\in$ (N : M) or am $\in$ N or bm $\in$ N. The concept of 2-absorbing submodule was introduced by A. Darani and F. Soheilnia (see[19]).

In this thesis, we generalize the concept of 2-absorbing submodule to ϕ-2-absorbing submodule and we introduce the concept of 2-absorbing compactly packed module.

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Contents

Abbreviation

ACC	**A**scending **C**hain **C**ondition
DCC	**D**escending **C**hain **C**ondition
CP	**C**ompactly **P**acked
FCP	**F**initely **C**ompactly **P**acked
ϕ-**CP**	ϕ-**C**ompactly **P**acked
ϕ-**FCP**	ϕ-**F**initely **C**ompactly **P**acked
2-abs.CP	**2-abs**orbing **C**ompactly **P**acked
2-abs.FCP	**2-abs**orbing **F**initely **C**ompactly **P**acked

Introduction

Let R be a commutative ring with identity and let M be a unitary R - module. A proper submodule N of M is compactly packed if for each family $\{P_\alpha\}_{\alpha\in\triangle}$ of prime submodules of M with N $\subseteq \bigcup_{\alpha\in\triangle} P_\alpha$, $N \subseteq P_\beta$ for some $\beta \in \triangle$. A module M is called compactly packed if every proper submodule of M is compactly packed. This concept was introduced by Al-Ani (see [1]), and generalized to primary compactly packed module by El-Atrash and Ashour (see [27]). Also it is generalized to primal compactly packed module by Al-Ashker, Ashour and Abu Mallouh (see [2]). Let S(M) be the set of all submodules of M and ϕ :S(M)$\rightarrow$ S(M)$\bigcup$ $\{\emptyset\}$ be a function. A proper submodule N of M is said to be a ϕ-prime (resp. a ϕ-primary) submodule if am $\in$ N - ϕ(N) for a $\in$ R, m $\in$ M implies that either m $\in$ N or a $\in$ (N : M) (resp. a $\in$ $\sqrt{(N:M)}$). In our work, we generalize the concept of ϕ-prime submodule to the concept of ϕ-primary submodule. Also we introduce the concept of ϕ-compactly packed module.

A proper submodule N of M is said to be 2-absorbing (resp. ϕ-2-absorbing) submodule if abm $\in$ N (resp. abm $\in$ N - ϕ(N)) with a,b $\in$ R, m $\in$ M implies that ab $\in$ (N : M) or am $\in$ N or bm $\in$ N. In our work, we generalize the concept of 2-absorbing submodule to ϕ-2-absorbing submodule. Also, we introduce the concept of 2-absorbing compactly packed module.

Let ϕ :J(R)$\longrightarrow$J(R)$\bigcup$ $\{\emptyset\}$ be a function with J(R) the set of all ideals of R. Let I be an ideal of R, an element a $\in$ R is called ϕ-prime to I if ra $\in$ I -ϕ(I) (with r $\in$ R) implies that r $\in$ I. We denote by S_ϕ(I) the set of all elements of R that are not ϕ-prime to I. I is called a ϕ-primal ideal of R if the set P = S_ϕ(I) $\bigcup$ ϕ(I) forms an ideal of R. The concept of ϕ-primal ideal over commutative ring was introduced by Darani (see[18]). In our work, we generalize the concept of ϕ-primal ideal to ϕ-primal submodule.

For a positive integer n, we say that an element s of a ring R is n-primary to I if the set s^{-n}I = {a $\in$ R : $s^n a$ $\in$ I} $\subseteq$ $\sqrt{I}$. Let n-adj(I) be the set of all elements in R that are not n-primary to I. I is called n-primly ideal if the set n-adj(I) forms an ideal of R. The concept of n-primly ideal over commutative ring was introduced and studied by Ashour (see[8]). In our work we generalize the concept of n-primly ideal to the concept of n-primly submodule.

The thesis consists of four Chapters. In Chapter one, we recall some basic concepts and results on rings and modules.

In the second Chapter, we study the classes of prime submodules, primary submodules, primal submodule and we introduce the concept of n-primly submodules as a generalization of the concept of n-primly ideals, also, we study the concept of 2-absorbing submodules

In Chapter three, we recall the concept of ϕ-prime submodules and we generalize it to ϕ-primary submodule and ϕ-2-absorbing submodules, also we introduce the concept of ϕ-primal submodules.

Finally, in Chapter four, we introduce the concept of ϕ-compactly packed modules, also we recall the concept of 2-absorbing submodules and we introduce the concept of 2-absorbing compactly packed modules.

We assume throughout this thesis that all rings are commutative rings with identity and all modules will be unitary.

Chapter 1

Basic Concepts

In this chapter we recall some basic definitions and results that we need throughout our thesis. These definitions and results are known and were introduced in [2, 17, 29, 37].

1.1 Basic Concept on Rings and Ideals

Definition 1.1.1. A *ring* R is a nonempty set together with two binary operations, addition denoted by a+b and multiplication denoted by a.b such that :

1. (R , +) is an abelian group.

2. (a.b)c = a(b.c) for all $a, b, c \in R$ (associative multiplication).

3. a.(b + c) = a.b + a.c and (b + c).a = b.a + c.a for all $a, b, c \in R$ (left and the right distributive laws).

If a.b = b.a for all $a, b \in R$, then R is said to be a commutative ring and if there is a necessarily unique element $1 \in R$ such that a.1 = 1.a = a for all $a \in R$, then R is a ring with unity (identity).(For a.b, we simply write ab).

Definition 1.1.2. A non-empty subset S of a ring R is a *subring* of R if S itself is a ring under the same operations of addition and multiplication defined on R.

Definition 1.1.3. A subring A of a ring R is called a (two-sided) *ideal* of R if for every $r \in R$ and every $a \in A$ both ra and ar are in A.

Theorem 1.1.4. *A nonempty subset A of a ring R is an ideal of R if*

1) $a + b \in A$ whenever $a, b \in A$.

*2) **ra** and **ar** are in A whenever $a \in A$ and $r \in R$.*

Definition 1.1.5. Let X be a subset of a ring R. If X $= \{a_1, a_2, ..., a_n\}$ then the ideal $< X > = \{r_1 a_1 + r_2 a_2 + ... + r_n a_n \mid r_i \in R\}$ is called *the ideal generated by X*. If X consists of has a single element, say a, then $< X > = < a >$ is called a *principal ideal.*

Definition 1.1.6. A *principal ideal ring* is a ring in which every ideal is principal.

Remark 1.1.7. If A and B are ideals of a ring R, the product of A and B is the ideal defined as: $AB = \{a_1b_1 + a_2b_2 + ... + a_nb_n : a_i \in A,\ b_i \in B,$ n a positive integer$\}$

Proposition 1.1.8. [35] *Let A, B, and C be ideals of a ring R such that $A = B \bigcup C$, then $A = B$ or $A = C$.*

Proof. Suppose that A $\neq$ B. Since A = B $\bigcup$ C, then $\exists$ a $\in$ A - B such that a $\in$ C. Not that A = (A - B) $\bigcup$ (A $\bigcap$ B). Now, let x be any arbitrary element in A. If x $\in$ A - B, then as before x $\in$ C. Let x $\in$ A $\bigcap$ B. then x - a $\in$ A - B. Since A and B are ideals. Thus x - a $\in$ C. Hence (x - a) - a = x $\in$ C, since C is an ideal. Thus A = C. □

Definition 1.1.9. A proper ideal P of a ring R is a *prime ideal* if for any a, b $\in R$, ab $\in$ P implies that either a $\in$ P or b $\in$ P.

Proposition 1.1.10. *An ideal P in a ring R is a prime ideal if and only if it satisfies the following property: If A and B are ideals in R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.*

Definition 1.1.11. Let I be an ideal of a ring R, the *radical of I*, denoted by $\sqrt{I}$, is the ideal $\sqrt{I} = \bigcap$ P, where the intersection runs over all prime ideals of R containing I. Equivalently, $\sqrt{I} = \{r \in R : r^n \in$ I, for some integer n $> 0\}$. An ideal I is said to be a *radical ideal* if $\sqrt{I} =$ I.

Definition 1.1.12. A proper ideal P of a ring R is a *primary ideal* if for any a,b $\in R$ such that ab $\in$ P, either a $\in$ P or $b^n \in$ P for some positive integer n.

Example 1.1.13. *Let $R = \mathbb{Z}$, p$\mathbb{Z}$ is primary ideal where p is prime number, but 6$\mathbb{Z}$, 10$\mathbb{Z}$, 14$\mathbb{Z}$, 15$\mathbb{Z}$ is not primary ideals*

Proposition 1.1.14. [32]*If Q is a primary ideal in a ring R. Then $\sqrt{Q}$ is a prime ideal.*

Proof. Since Q is a proper ideal in R, then $1 \notin Q$ and hence $1 \notin \sqrt{Q}$, so $\sqrt{Q}$ is proper ideal in R. Let ab $\in \sqrt{Q}$ and a $\notin \sqrt{Q}$, then $(ab)^n \in Q$ for some positive integer n, and hence $a^n b^n \in Q$. Since a $\notin \sqrt{Q}$, $a^n \notin$ Q. Since Q is primary, there is a positive integer k such that $(b^n)^k \in Q$, whence b $\in \sqrt{Q}$. Therefore $\sqrt{Q}$ is a prime ideal. □

Definition 1.1.15. [6] A proper ideal I of a ring R is called *weakly prime ideal* if for any a,b $\in R$ such that $0 \neq$ ab $\in$ I then a $\in$ I or b $\in$ I.

Definition 1.1.16. [16] A proper ideal I of a ring R is called *almost prime ideal* if for any a,b $\in R$ such that ab $\in$ I - I^2 then a $\in$ I or b $\in$ I.

Definition 1.1.17. [9] A proper ideal I of a ring R is called *weakly primary ideal* if for any a,b $\in R$ such that $0 \neq$ ab $\in$ I then a $\in$ I or b $\in \sqrt{I}$.

Definition 1.1.18. [15] A proper ideal I of a ring R is called *almost primary ideal* if for any a,b $\in R$ such that ab $\in$ I - I^2 then a $\in$ I or b $\in \sqrt{I}$.

The following diagram shows the relation between the pervious ideals.

Summary of Relation between ideals

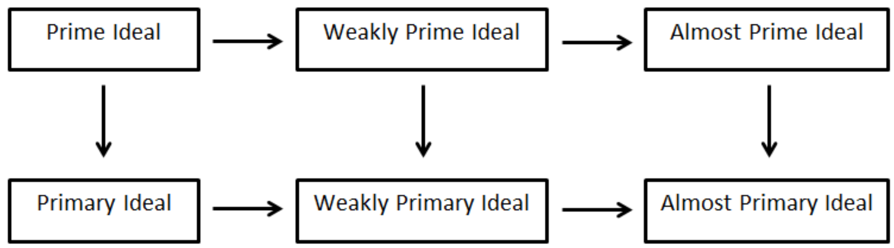


Figure 1 : Relation between ideals

Remark 1.1.19. From the above diagram, the converse relations are not necessarily true.

Example 1.1.20. *(1) let $R = \mathbb{Z}$, the ideals $4\mathbb{Z}$, $8\mathbb{Z}$ are primary, but not prime.*

(2) let $R = \mathbb{Z}_6$. Then 0 is a weakly prime ideal but 0 is not necessary prime in general because $0 = 2.3 \in I = \{0\}$, but $2, 3 \notin I$. Similarly, 0 is a weakly primary ideal that is not necessary primary ideal in general.

(3) Let $R = \mathbb{Z} \times \mathbb{Z}$. The ideal $P = 4\mathbb{Z} \times 0$ is a weakly primary ideal of R, which is not a weakly prime ideal of R because $(0, 0) \neq (2, 0)(2, 0) \in P$, but $(2, 0) \notin P$.

(4) let $R = \mathbb{Z}$. Let $I = 4\mathbb{Z}$ be an ideal of R, then $I^2 = 16\mathbb{Z}$. So I is almost primary ideal, which is not almost prime ideal of R because $2.6 \in I - I^2$, but $2 \notin I$ and $6 \notin I$.

(5) Let $R = \mathbb{Z}_{24}$. Let $I = <8>$ in $\mathbb{Z}_{24} = \{0, 8, 16\}$, then $I^2 = \{0, 8, 16\}$. So, I is almost prime ideal of R, but I is not weakly prime because $0 \neq 4.4 = 16 \in I$, but $4 \notin I$.

Definition 1.1.21. [5] Let R be ring. Let $\phi : \mathrm{I(R)} \rightarrow \mathrm{I(R)} \cup \{\emptyset\}$ be a function where I(R) is the set of all ideals of R. A proper ideal I of R is a *ϕ-prime ideal* if a, b $\in$ R with ab $\in$ I - ϕ(I) implies a $\in$ I or b $\in$ I.

Remarks 1.1.22. Let $\phi : I(R) \rightarrow I(R) \cup \{\emptyset\}$ be a function, where $I(R)$ be the set of all ideals in R.

(1) If $\phi(I) = 0$, $\forall\ I \in I(R)$, then we have I is weakly prime ideal.

(2) If $\phi(I) = I^2$, $\forall\ I \in I(R)$, then we have I is almost prime ideal.

Definition 1.1.23. [14] Let R be a commutative ring. A proper ideal I of R is called a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and abc $\in$ I, then ab $\in$ I or ac $\in$ I or bc $\in$ I.

Definition 1.1.24. A nonempty subset S of a ring R is said to be *multiplicatively closed* if ab $\in$ S whenever a, b $\in$ S.

Definition 1.1.25. [8] Let n be a positive integer. Let I be an ideal over a ring R. Let s be an element of R. Define s^{-n}I = {a $\in$ R : s^na $\in$ I }.

Remark 1.1.26. [8] Let I be an ideal over a ring R. Let s be an element of R. Then I $\subseteq$ s^{-1}I $\subseteq$ s^{-2}I $\subseteq$ s^{-3}I $\subseteq$ ...

Definition 1.1.27. [8] Let n be a positive integer. Let I be an ideal over a ring R. Let s $\in$ R. If s^{-n}I $\subseteq \sqrt{I}$ then s is said to be *n-primary to I.*

Definition 1.1.28. [8] Let n be a positive integer. Let I be an ideal over a ring R. The set of all elements that are not n-primary to I is called the *$n-adjoint$ set* for I and is denoted by $n - adj(I)$. That is, $n - adj(I)$ = { a $\in$ R : a^nb $\in$ I for some b $\in$ R -$\sqrt{I}$ }

Example 1.1.29. *Let $R = \mathbb{Z}$, then*

(1) $n - adj(2\mathbb{Z}) = 2\mathbb{Z}$.

(2) $1 - adj(p^2\mathbb{Z}) = p^2\mathbb{Z}$, *where p is a prime number.*

(3) $n - adj(p^2\mathbb{Z}) = p\mathbb{Z}$, $\forall\ n \geq 2$, *where p is a prime number.*

(4) $n - adj(6\mathbb{Z}) = 2\mathbb{Z} \cup 3\mathbb{Z}$, $\forall\ n \in \mathbb{N}$.

(5) $1 - adj(12\mathbb{Z}) = 4\mathbb{Z} \cup 3\mathbb{Z}$.

(6) $n - adj(12\mathbb{Z}) = 2\mathbb{Z} \cup 3\mathbb{Z}$ *for any positive integer* $n \geq 2$.

Definition 1.1.30. [8] Let n be a positive integer. An ideal I over a ring R is called *n-primly* if $n - adj(I)$ is an ideal of R.

Example 1.1.31. [8] *Let* $R = \mathbb{Z}$. *Then* $2\mathbb{Z}$, $4\mathbb{Z}$, $9\mathbb{Z}$ *and* $16\mathbb{Z}$ *are n-primly ideals of* $\mathbb{Z}$, *while* $6\mathbb{Z}$ *and* $12\mathbb{Z}$ *are not n-primly ideals of* $\mathbb{Z}$, *for every positive integer n.*

1.2 Basic Concept Of Modules and Submodules

Definition 1.2.1. Let R be a ring with unity. A *(left) R-module* is an additive abelian group M together with a function R × M → M (the image of (r,m) being denoted by rm) such that for all r, s ∈ R and m, m_1, m_2 ∈ M :

(i) $r(m_1 + m_2) = rm_1 + rm_2$.

(ii) (r + s)m = rm + sm.

(iii) r(sm) = (rs)m.

If in addition 1m = m for all m ∈ M (1 is the identity element of R), then M is said to be a *unitary R-module*.

A *right R-module* is defined similarly via a function M × R → M denoted (m,r) → mr where m ∈ M, r ∈ R, and satisfying the obvious analogues of (i) - (iii). Since we only deal with commutative rings in this thesis, then every left R-module M can be given the structure of a right R-module by defining mr = rm for r ∈ R, m ∈ M (commutativity is needed for (iii)). From now on, every module M is assumed to be both a left and a right $R-$module with mr = rm for r ∈ R, m ∈ M.

Definition 1.2.2. Let R be a ring, M an R-module and N a nonempty subset of M. N is called a *submodule* of M if N is an additive subgroup of M and rm ∈ N for all r ∈ R and m ∈ N.

Example 1.2.3. *(a) For any ring R, we can consider R as an R-module, and the ideals in the ring R as submodules of the module R.*

(b) If I is an ideal of R and N a submodule of M, then $IN = \{r_1m_1 + r_2m_2 + ... + r_nm_n : r_i \in I, m_i \in N, n \text{ a positive integer}\}$ *is a submodule of M.*

Definition 1.2.4. Let R be a ring, M an R-module and m ∈ M, the *cyclic submodule generated by m* is a submodule of M has the form Rm = $\{rm : r \in R\}$.

Definition 1.2.5. An R-module M is said to be *finitely generated* if there is a finite subset $\{x_1, ..., x_n\}$ of M such that M = $Rx_1 + ... + Rx_n$. In this case M is called generated by $x_1, ..., x_n$.

Definition 1.2.6. A mapping f from R-module M to R-module N is a *homomorphism* if $f(x+y) = f(x)+f(y)$ and $f(ax) = af(x)$, for all $x, y \in M$ and $a \in R$. A homomorphism $f : M \to N$ is *epimorphism* if it maps M onto N.

Definition 1.2.7. [37] Let M be an R-module and let $f : M \to N$ be a homomorphism, then $Ker(f) = \{x : x \in M \text{ and } f(x) = 0\}$ is a submodule of M called the *kernel of f*

Definition 1.2.8. [17] If N is a submodule of an R-module M, then M/N together with the operations: $(x + N) + (y + N) = (x + y) + N$ and $a(x + N) = ax + N$ for $x, y \in M$ and $a \in R$ is called the *factor module.*

Definition 1.2.9. [28] An R-module M is called a *multiplication module* if every submodule N of M is of the form IM, for some ideal I of R

Definition 1.2.10. [32] Let N be a submodule of an R-module M. *The residual of N by M*, denoted (N : M), is the ideal (N : M) =$\{r \in R : rM \subseteq N\}$.

Definition 1.2.11. [32] Let M be an R-module, A be an ideal of R and N be a submodule of M. the residual of N by A, denoted by (N : A), is the submodule (N : A) $= \{x : x \in M$ and $Ax \subseteq N\}$

Proposition 1.2.12. [32] *Let K, L an N be submodules of R-module M and let A and B be ideals of R. Then.*

(1) (L $\cap$ N : M) = (L : M) $\cap$ (N : M).

(2) A $\subseteq$ B implies (N : A) $\supseteq$ (N : B).

(3) ((N : A) : B) = (N : AB).

(4) (L $\cap$ N : A) = (L : A) $\cap$ (N : A).

(5) L $\subseteq$ N implies that (L :A) $\subseteq$ (N : A) and (L : K) $\subseteq$ (N : K).

Definition 1.2.13. [29] M is called a *faithful R-module* if (0 : M) = 0.

Lemma 1.2.14. [32] *Let R be a ring and M be an R-module. Let S be a multiplicatively closed set in R. Let T be the set of all pairs (x, s), where $x \in M$ and $s \in S$. Define a relation on T by $(x, s) \sim (x', s')$, if and only if there exists $t \in S$ such that $t(sx' - s'x) =$ 0. Then $\sim$ is an equivalence relation on T.*

Definition 1.2.15. [32] For $(x, s) \in T$ which defined in lemma 1.2.14, denote the equivalence classes of $\sim$ which contains (x, s) by $\frac{x}{s}$. Let $S^{-1}M$ denote the set of all equivalence classes of T with respect to this relation. We can make $S^{-1}M$ into an R-module by setting $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$ and $a(\frac{x}{s}) = \frac{ax}{s}$, where $x, y \in M$ and $t, s, a \in S$. The module $S^{-1}M$ is called *the module of fractions of M with respect to S* (or *quotient module of M*).

Definition 1.2.16. [32] Since R may be considered as an R-module we can form the quotient ring $S^{-1}R$. An element of $S^{-1}R$ has the form $\frac{a}{s}$, where $a \in R$ and $s \in S$. We can make $S^{-1}R$ into a ring by setting $(\frac{a}{s})(\frac{b}{t}) = \frac{ab}{st}$, where $a, b \in R$ and $s, t \in S$. the ring $S^{-1}R$ is called *the ring of fractions of R with respect to S* (*a quotient ring of R*).

Definition 1.2.17. [32] An R-module M is said to satisfy the *ascending chain condition* (ACC) on submodules (or to be *Noetherian*) if for every chain $N_1 \subseteq N_2 \subseteq N_3 \subseteq ...$ of submodules of M, there is an integer n such that $N_i = N_n$ for all i $\geq$ n.

Definition 1.2.18. [32] An R-module M is said to satisfy the *descending chain condition* (DCC) on submodules (or to be *Artinian*) if for every chain $N_1 \supseteq N_2 \supseteq N_3 \supseteq ...$ of submodules of M, there is an integer n such that $N_i = N_n$ for all i $\geq$ n.

Definition 1.2.19. An R-module M is said to satisfy the *maximal condition on submodules* if every nonempty collection of submodules of M contains a maximal element (with respect to set theoretic inclusion).

Definition 1.2.20. An R-module M is said to satisfy the *minimal condition on submodules* if every nonempty collection of submodules of M contains a minimal element (with respect to set theoretic inclusion).

Theorem 1.2.21. [37] *Let M be an R-module, then the following statements are equivalent:*

(i) *M satisfies the (ACC) on submodules.*

(ii) *M satisfies the maximal condition on submodules.*

(iii) *Every submodule of M is finitely generated.*

Theorem 1.2.22. [37] *Let M be an R-module, then the following statements are equivalent:*

(i) *M satisfies the (DCC) on submodules.*

(ii) *M satisfies the minimal condition on submodules.*

Chapter 2

Classes of Submodules

In this Chapter we recall the concepts of classes of prime, primary, and primal submodules and 2-absorbing submodule, also we introduce the concept of n-primly submodule as a generalization of n-primly ideals (see[8]).

2.1 Classes of Prime Submodules

Definition 2.1.1. [34] Let M be an R-module. A proper submodule N of M is said to be a *prime submodule* if whenever rm $\in$ N for r $\in$ R and m $\in$ M we get either m $\in$ N or rM $\subseteq$ N (equivalent r $\in$ (N : M)).

Proposition 2.1.2. [34] *If N is a prime submodule of an R-module M, then (N : M) is prime ideal in R.*

Proof. (N : M) is a proper ideal, since 1 $\notin$ (N : M). Let ab $\in$ (N : M) and b $\notin$ (N : M). Then bM $\nsubseteq$ N, that is there exists m $\in$ M with bm $\notin$ N. But a(bm) = (ab) m $\in$ N and N is prime, therefore aM $\subseteq$ N. Thus a $\in$ (N : M). □

Definition 2.1.3. [10] Let M be an R-module and N be a proper submodule of M. N is called a *weakly prime submodule* of M if, whenever r $\in$ R and m $\in$ M such that 0 $\neq$ rm $\in$ N, then either m $\in$ N or r $\in$ (N : M).

Remarks 2.1.4. [10] (1) Clearly, every prime submodule of a module is a weakly prime submodule. However, since 0 is always weakly prime (by definition), a weakly prime submodule need not be prime (see Example 1.1.20).
(2) If N is a weakly prime submodule of an R-module M, then (N : M) is not weakly prime ideal of R in general. For example, let M denote the cyclic $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z}$. Take N = $\{0\}$. Certainly N is a weakly prime submodule of M, but (N : M) = $8\mathbb{Z}$ is not a weakly prime ideal of R

Now, we find the condition that makes the residual of a weakly prime submodule is a weakly prime ideal, as in [10].

Proposition 2.1.5. [10] *Let R be a commutative ring with unity, M a faithful cyclic R-module with unity, and N a weakly prime submodule of M. Then (N : M) is a weakly prime ideal of R.*

Proof. Assume that M = Rx, where x $\in$ M, and let 0 $\neq$ ab $\in$ (N : M) with a $\notin$ (N : M). Then there exists r $\in$ R such that rax = a(rx) $\notin$ N, so ax $\notin$ N. As 0 $\neq$ abM $\subseteq$ N, it follows that 0 $\neq$ abx $\in$ N (for if abx = 0, then ab $\in$ (0 : x) $\subseteq$ (0 : M) = 0, a contradiction), so 0 $\neq$ abx = bax $\in$ N implies b $\in$ (N : M) since N is a weakly prime submodule of M. Thus (N : M) is weakly prime ideal. □

Theorem 2.1.6. [10] *Let R be a commutative ring with unity, M an R-module, and N a proper submodule of M. Then the following statements are equivalent.*
(i)N is a weakly prime submodule of M.
(ii) For m $\in$ M - N, (N : Rm) = (N : M) $\bigcup$ (0 : Rm).
(iii) For m $\in$ M - N, (N : Rm) = (N : M) or (0 : Rm) = (N : Rm).

Proof. (i $\Longrightarrow$ ii) Clearly, if m $\in$ M - N, then H = (N : M) $\cup$ (0 : Rm) $\subseteq$ (N : Rm). Let a $\in$ (N : Rm) where m $\in$ M - N. Then am $\in$ N. If am $\neq$ 0, then a $\in$ (N : M) since N is weakly prime, so a $\in$ H. If am = 0, then a $\in$ (0 : Rm), so a $\in$ H, and hence we have equality.

(ii $\Longrightarrow$ iii) It is obvious.

(iii $\Longrightarrow$ i) Suppose that 0 $\neq$ rm $\in$ N with r $\in$ R and m $\in$ M - N. Then r $\in$ (N : Rm) and r $\notin$ (0 : Rm). It follows from (iii) that r $\in$ (N : Rm) = (N : M), as required. □

Definition 2.1.7. [31] Let M be an R-module. A proper submodule N of M is called an *almost prime submodule* of M if, whenever r $\in$ R and m $\in$ M such that rm $\in$ N - (N : M)N, then either m $\in$ N or r $\in$ (N : M).

Remark 2.1.8. [15] Clearly, any weakly prime submodule is almost prime. However, the converse need not necessarily be true. For example, we consider the $\mathbb{Z}$-module M = $\mathbb{Z}_{24}$ and the proper submodule N of M generated by $\overline{8}$ = { $\overline{8}$, $\overline{16}$, $\overline{0}$ }. Then clearly (N : M)N = N. Indeed $\overline{0} = 0.\overline{8}$ and 0 $\in$ (N : M), $\overline{8} = 16.\overline{8}$ and 16 $\in$ (N : M), $\overline{16} = 16.\overline{16}$ and 16 $\in$ (N : M) and so N is almost prime. On the other hand, 0 $\neq 4.\overline{4}$ $\in$ N with $\overline{4} \notin$ N and 4 $\notin$ (N : M) and so N is not weakly prime.

Lemma 2.1.9. [15] *Let M be an R-module and let N be a proper submodule of M. Then* $(N/((N : M)N) : M/((N : M)N)) = (N : M)$.

Proof. First we proof $(N/((N : M)N) : M/((N : M)N)) \subseteq (N : M)$. Let $x \in (N/((N : M)N) : M/((N : M)N))$, then $x(M/((N : M)N)) \subseteq N/((N : M)N)$. Suppose $x \notin (N : M)$, so $xM \nsubseteq N$ which implies that $xM + ((N : M)N) \nsubseteq N + ((N : M)N)$, and hence $x(M + ((N : M)N)) \nsubseteq N + ((N : M))$, which is a contradiction. Now we proof the other inclusion, Let $x \in (N : M)$, then $xM \subseteq N$ and so $x(M/((N : M)N) \subseteq N/((N : M)N)$, hence, $x \in N/((N : M)N) : M/((N : M)N))$. □

Theorem 2.1.10. [31] *Let M be an R-module and N be a proper submodule of M. Then, N is almost prime in M if and only if $N/(N:M)N$ is weakly prime in $M/(N:M)N$.*

Proof. Suppose that N is almost prime in M. Let $r \in R$ and $m \in M$, such that $\overline{0} \neq r(m + (N : M)N) \in N/(N : M)N$ in $M/(N : M)N$. Then, $rm \in N - (N : M)N$ and so either $m \in N$ or $r \in (N : M)$. Hence, either $m + (N : M)N \in N/(N : M)N$ or $r \in (N : M) = (N/(N : M)N : M/(N : M)N)$ and so $N/(N : M)N$ is weakly prime in $M/(N : M)N$. Conversely, assume that $N/(N : M)N$ is weakly prime in $M/(N : M)N$ and let $r \in R$ and $m \in M$ such that $rm \in N - (N : M)N$. Then, $\overline{0} \neq r(m + (N : M)N) \in N/(N : M)N$ and hence either $m + (N : M)N \in N/(N : M)N$ and so $m \in N$ or $r \in (N/(N : M)N : M/(N : M)N) = (N : M)$. □

Theorem 2.1.11. [31] *Let N be an almost prime submodule of an R-module M. If K is a submodule of M with $K \subseteq N$, then N/K is an almost prime submodule of M/K.*

Proof. Let $r \in R$ and $m+K \in M/K$ such that $r(m+K) \in (N/K)-(N/K : M/K)N/K$. Then, $rm+K \in N/K-(N : M)N/K$ and so $rm \in N-(N : M)N$. As N is almost prime in M, either $m \in N$ or $r \in (N : M)$. Therefore, $m + K \in N/K$ or $r \in (N/K : M/K)$, and N/K is almost prime in M/K. □

Remark 2.1.12. The converse of above theorem is not be true in general. For example, for any non almost prime submodule N of an R-module M, we have $0 = N/N$ is a weakly prime (and so almost prime) submodule of M/N. For another non-trivial example, we consider the ring $R = K[x, y]$, where K is a field and ideals $P = (x, y^2)$, $I = (x, y)^2$. Then, P/I is weakly prime submodule of an R-module R/I, thus P/I is an almost prime submodule of an R-module R/I, while P is not so in R (see[4]).

Theorem 2.1.13. [31] *Let M be an R-module and N be a proper submodule of M. The following are equivalent:*

(1) N is an almost prime submodule.

(2) For $r \in R - (N : M)$, $(N : (r)) = N \bigcup ((N : M)N : (r))$.

(3) For $r \in R - (N : M)$, $(N : (r)) = N$ or $(N : (r)) = ((N : M)N : (r))$.

Proof. (1) $\Longrightarrow$ (2) Suppose that N is almost prime such that $r \notin (N : M)$. Let $m \in (N : (r))$ so that $rm \in N$. If $rm \notin (N : M)N$, then, N almost prime implies that $m \in N$. Suppose that $rm \in (N : M)N$. Then, $m \in ((N : M)N : (r))$ and so, $(N : (r)) \subseteq N \bigcup ((N : M)N : (r))$. The other containment holds for any submodule N.
(2) $\Longrightarrow$ (3) It is well known that if an ideal is the union of two ideals, then it is equal to one of them.
(3) $\Longrightarrow$ (1) Let $r \in R$ and $m \in M$ such that $rm \in N - (N : M)N$ and suppose $r \notin (N : M)$. By assumption, either $(N : (r)) = N$ or $(N : (r)) = ((N : M)N : (r))$. As $rm \notin (N : M)N$, then $m \notin ((N : M)N : (r))$ and so $m \in N$ as required. □

2.2 Classes of Primary Submodules

Definition 2.2.1. [37] Let M be an R-module. A proper submodule N of M is said to be a *primary submodule* if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r^n M \subseteq N$ for some positive integer n.

Directly from the definition every prime submodule is primary. The converse need not be true, (see Example 1.1.20).

Proposition 2.2.2. [32] *If N is a primary submodule of an R-module M, then $(N : M)$ is primary ideal in R, and hence $\sqrt{(N : M)}$ is prime ideal in R*

Proof. The proof of (N : M) is primary similar to that of Proposition 2.1.2, and $\sqrt{(N : M)}$ is prime follows from Proposition 1.1.14. □

Definition 2.2.3. [9] A proper submodule N of a module M over a commutative ring R is said to be a *weakly primary submodule* if whenever $0 \neq rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r^n M \subseteq N$ for some n $\in \mathbb{N}$.

Remark 2.2.4. Clearly, every primary submodule of a module is a weakly primary submodule. However, since 0 is always weakly primary (by definition), a weakly primary submodule need not be primary.

Theorem 2.2.5. [9] *Let R be a commutative ring, M an R-module, and N a proper submodule of M. Then the following statements are equivalent:*

(i) N is a weakly primary submodule of M.

(ii) For m $\in$ *M - N,* $\sqrt{(N : Rm)} = \sqrt{(N : M)} \bigcup$ *(0 : Rm).*

(iii) For m $\in$ *M - N,* $\sqrt{(N : Rm)} = \sqrt{(N : M)}$ *or (0 : Rm)* $= \sqrt{(N : Rm)}$*.*

Proof. (i) $\Longrightarrow$ (ii) Let a $\in \sqrt{(N : Rm)}$ where m $\in$ M - N. Then a^km $\in$ N for some k. If a^km $\neq 0$, then $a^k \in$ (N : M) since N is weakly primary, hence a $\in \sqrt{(N : M)}$. If a^km = 0, then assume that s is the smallest integer with a^sm = 0. If s = 1, then a $\in$ (0 : Rm). Otherwise, a $\in \sqrt{(N : M)}$, so $\sqrt{(N : Rm)} \subseteq \sqrt{(N : M)} \bigcup$ (0 : Rm) = H. For the other inclusion assume that b $\in$ H. Clearly, if b $\in$ (0 : Rm), then b $\in \sqrt{(N : Rm)}$. If b $\in \sqrt{(N : M)}$, then $b^t \in$ (N : M) $\subseteq$ (N : Rm) for some t, so b $\in \sqrt{(N : Rm)}$.

(ii) $\Longrightarrow$ (iii) It is obvious.

(iii) $\Longrightarrow$ (i) Suppose that 0 $\neq$ rm $\in$ N with r $\in$ R and m $\in$ M - N. Then r $\in$ (N : Rm) $\subseteq \sqrt{(N : Rm)}$ and r $\notin$ (0 : Rm). It follows from (iii) that r $\in \sqrt{(N : Rm)} = \sqrt{(N : M)}$, as required. □

Remark 2.2.6. [7] If N is a weakly primary submodule, the ideal (N :M) is not in general a weakly primary ideal of R, and also $\sqrt{(N : M)}$ is not in general a weakly prime ideal of R. For example, let M denotes the cyclic $\mathbb{Z}$module $\mathbb{Z}/6\mathbb{Z}$. Take N ={ 0 }. Certainly

N is a weakly primary submodule of M, but neither (N : M) $= 6\mathbb{Z}$ is a weakly primary ideal of R nor $\sqrt{(N : M)} = 6\mathbb{Z}$ is a weakly prime ideal of R.

However the following results hold.

Proposition 2.2.7. [7] *Let R be a commutative ring with unity, M a faithful cyclic R-module and N is a weakly primary submodule of M. Then* $(N : M)$ *is a weakly primary ideal of R.*

Proof. Assume $M = Rx$ and let $0 \neq ab \in (N : M)$ with $a \notin (N : M)$, then there exists $r \in R$ such that $a(rx) \notin N$, so $ax \notin N$. As $0 \neq abM \subseteq N$, it follows that $0 \neq abx \in N$ (for if $abx = 0$, then $ab \in (0 : x) = (0 : M) = 0$, a contradiction). Since N is a weakly primary submodule of M, $b^n \in (N : M)$ for some positive integer n. Thus $(N : M)$ is a weakly primary ideal of R. □

Proposition 2.2.8. [7] *Let R be a commutative ring, M a faithful cyclic R-module and N is a weakly primary submodule of M. Then* $\sqrt{(N : M)}$ *is a weakly prime ideal of R.*

Proof. Assume $M = Rx$ and let $0 \neq$ ab $\in \sqrt{(N : M)}$ with a $\notin \sqrt{(N : M)}$, then a $\notin$ (N : M). As in the proof of the previous proposition, $b^n \in$ (N : M) for some positive integer n. Thus b $\in \sqrt{(N : M)}$ and $\sqrt{(N : M)}$ is a weakly prime ideal of R. □

Definition 2.2.9. [33] Let M be an R-module and N a proper submodule of M, N is called an *almost primary submodule* of M if, whenever r $\in$ R, m $\in$ M such that rm $\in$ $N - (N : M)N$, then either m $\in$ N or r$\in \sqrt{(N : M)}$.

Remarks 2.2.10. [33] (1) In the above definition, when we set M as R and N as an ideal I of R, then the almost primary submodule is an almost primary ideal.

(2) every weakly primary submodule is almost primary.

(3) every almost prime submodule is almost primary. However, the converse is not necessarily true. For example, we consider M to be the $\mathbb{Z}$-module $\mathbb{Z}$ and let N = $< 4 >$. Here (N : M)N = ($< 4 > : \mathbb{Z}$)$< 4 >$ = $< 4 >^2$ = $< 16 >$. Furthermore if ab $\in N - (N : M)N$ for a $\in$ R and b $\in$ M, then ab $\in < 4 >$ - $< 16 >$. Hence, b $\in < 4 >$ or $a^n \in < 4 > =$ ($< 4 > : \mathbb{Z}$) for some natural n $\geq$ 2 and the result holds.

Proposition 2.2.11. [33] *Let N be a submodule of M with $(N : M) = \sqrt{(N : M)}$, then N is almost primary if and only if N is almost prime*

Proof. $\Longrightarrow$) Suppose N is almost primary. Let r $\in$ R, m $\in$ M with rm $\in N - (N : M)N$. If m $\notin$ M, then we get $r^n \in (N : M)$ for some positive integer n. Hence, r $\in \sqrt{(N : M)}$ = $(N : M)$, which means N is almost prime.

$\Longleftarrow$) It is trivial, because any almost prime submodule is always almost primary. □

Proposition 2.2.12. [33] *Let N be an almost primary submodule of an R-module M. If K is a submodule of M with $K \subseteq N$, then N/K is an almost primary submodule of M/K.*

Proof. Let r $\in$ R, m + K $\in$ M/K such that r(m + K) $\in$ N/K - (N/K : M/K)N/K. Since (N/K : M/K) = (N : M), then rm + K $\in$ N/K - (N : M)N/K. Hence rm $\in$ N - (N : M)N. Since N is almost primary in M, then either m $\in$ N or r $\in \sqrt{(N : M)}$. Therefore, m + K $\in$ N/K or r $\in \sqrt{(N : M)} = \sqrt{(N/K : M/K)}$, so N/K is almost primary in M/K. □

Theorem 2.2.13. [15] *Let R be a commutative ring, M be an R-module, and N be a proper submodule of M, then the following are equivalent:*

(i) N is almost primary submodule of M.

(ii) For r $\in$ R - $\sqrt{(N : M)}$, (N : (r)) = N $\bigcup$ ((N : M)N : (r)).

(iii) For r $\in$ R - $\sqrt{(N : M)}$, (N : (r)) = N or (N : (r)) = ((N : M)N : (r)).

Proof. (i) $\Longrightarrow$ (ii) Suppose N is almost primary such that r $\notin \sqrt{(N : M)}$. Let m $\in$ (N : (r)). So rm $\in$ N. If rm $\notin$ (N : M)N, then N is almost primary implies m $\in$ N, and if rm

$\in$ (N : M)N, then m $\in$ ((N : M)N : (r)). Hence (N : (r)) $\subseteq$ N $\bigcup$ ((N : M)N : (r)). The other inclusion holds trivially.

(ii) $\Longrightarrow$ (iii) It follows because (N : (r)) is the union of two ideals.

(iii) $\Longrightarrow$ (i) Let r $\in$ R - $\sqrt{(N : M)}$, m $\in$ M such that rm $\in$ N - (N : M)N. By assumption, either (N : (r)) = N or (N : (r)) = ((N : M)N : (r)). As rm $\notin$ (N : M)N, then m $\notin$ ((N : M)N : (r)) and as rm $\in$ N, then m $\in$ (N : (r)). Hence (N : (r)) = N, and so m $\in$ N as required. □

2.3 Classes of Primal Submodules

Definition 2.3.1. [24] Let M be an R-module and N a submodule of M. The element a $\in$ R is *(left) prime to N* if am $\in$ N (m $\in$ M) implies m $\in$ N. The subset A of R is *uniformly not prime to N*, if there exists an element u $\in M - N$ with Au $\subseteq$ N.

Example 2.3.2. [2] *Consider the submodule 6$\mathbb{Z}$ in the $\mathbb{Z}$-module $\mathbb{Z}$. 5 is prime to 6$\mathbb{Z}$ while 2 is not prime to 6$\mathbb{Z}$, and the set* $\{2, 4, 6, ...\}$ *is uniformly not prime to 6$\mathbb{Z}$.*

Definition 2.3.3. [24] Let M be an R-module and N a submodule of M. The *adjoint of N* is the set of all elements of R that are not prime to N and denoted by $adj(N)$. On other words, $adj(N)$ = {r $\in$ R : rm $\in$ N for some m $\in M - N$}.

Example 2.3.4. *In $\mathbb{Z}$-module $\mathbb{Z}$, adj(6$\mathbb{Z}$) is the set 2$\mathbb{Z}$ $\bigcup$ 3$\mathbb{Z}$ and adj(4$\mathbb{Z}$) is the ideal 2$\mathbb{Z}$.*

Definition 2.3.5. [24] Let M be an R-module. A proper submodule N of M is said to be *primal* if $adj(N)$ forms an ideal of R. In this case the adjoint of N will also be called the *adjoint ideal of N*.

Example 2.3.6. *In the ring $\mathbb{Z}$, the ideal 4$\mathbb{Z}$ is primal, and the ideal 6$\mathbb{Z}$ is not primal.*

Proposition 2.3.7. [12] *Let M be an R-module. If N is a primal submodule of M, then adj(N) is a prime ideal of R.*

Proof. Since $1 \notin adj(N)$, $adj(N)$ is a proper ideal. Let ab $\in adj(N)$ with a $\notin adj(N)$, there exists m $\in$ M - N with abm $\in$ N, since a $\notin adj(N)$, so we have bm $\in$ N which implies b $\in adj(N)$. □

Remark 2.3.8. [24] Let N be a proper submodule of an R-module M. If r $\in$ R and a $\in adj(N)$, then there exists m $\in$ M - N with am $\in$ N, and hence ram $\in$ N while m $\notin$ N and, as a consequence, ra $\in adj(N)$. Thus to prove $adj(N)$ is an ideal we only prove adj(N) is closed under the addition.

Proposition 2.3.9. [2] *Let N be a proper submodule of an R-module M. Then N is a primary submodule of M if and only if* $adj(N) = \sqrt{(N:M)}$.

Proof. Suppose N is a primary submodule of M. Let r $\in \sqrt{(N:M)}$, then r^nM $\subseteq$ N for some positive integer n. Pick m $\in$ M - N, then r^nm $\in$ N. If we choose n_0 be the smallest positive integer with r^{n_0}m $\in$ N, we have r(r^{n_0-1}m) $\in$ N while r^{n_0-1}m $\notin$ N, so r $\in adj(N)$. Thus $\sqrt{(N:M)} \subseteq adj(N)$. On the other hand, let r $\in adj(N)$, then there exists m $\in$ M - N with rm $\in$ N, since N is primary, then r^nM $\subseteq$ N for some positive integer n, and hence r $\in \sqrt{(N:M)}$. so we have $adj(N) \subseteq \sqrt{(N:M)}$. Thus $adj(N) = \sqrt{(N:M)}$. Conversely, assume $adj(N) = \sqrt{(N:M)}$, and let rm $\in$ N where r $\in$ R and m $\in$ M - N. Then r is not prime to N, therefore r $\in adj(N) = \sqrt{(N:M)}$. Thus r^kM $\subseteq$ N for some positive integer k, and N is primary. □

Corollary 2.3.10. [21] *(Primary $\Longrightarrow$ Primal)*

Let M be an R-module. If N is a primary submodule of M, then N is primal.

Proof. If N is a primary submodule of M, then by the previous proposition, $adj(N) = \sqrt{(N:M)}$ is an ideal in R, hence N is primal. □

Definition 2.3.11. [11] Let N be a submodule of an R-module M. An element r $\in$ R is called *weakly prime (simply wp) to N* if $0 \neq$ rm $\in$ N (m $\in$ M) implies that m $\in$ N. Otherwise r is not weakly prime (simply nwp) to N. Denote by W(N) the set of elements of R that are nwp to N.

Remarks 2.3.12. [11] Let N be a submodule of an R-module M. Then:

(1) 0 is always weakly prime to N.

(2) If r $\in$ R is prime to N, then r is wp to N, but the converse is not necessarily true. For example consider the $\mathbb{Z}$-module M = $\mathbb{Z}/24\mathbb{Z}$ and its submodule N = $8\mathbb{Z}/24\mathbb{Z}$. Denote each coset a + $24\mathbb{Z}$ in M by $\overline{a}$. Then, as $6.\overline{12} = \overline{0} \in$ N and $\overline{12} \in$ M - N, so 6 is not prime to N. But if $6.\overline{a} \in$ N for some $\overline{a} \in$ M, then 4 divides a. Hence $6.\overline{a} = \overline{0}$. This implies that 6 is wp to N.

Definition 2.3.13. [11] Let R be a commutative ring and let N be a proper submodule of an R-module M. N is called *weakly primal* if the set P = W(N) $\bigcup$ {0} forms an ideal of R. P is called the *(weakly) adjoint ideal of N* and we also say that N is a *P-weakly primal submodule of M.*

The following theorem provides a characterization of weakly primal submodules.

Theorem 2.3.14. [11] *Let P be an ideal of a commutative ring R, M an R-module and N a submodule of M. The following statements are equivalent:*

(a) N is P-weakly primal submodule.

(b) For every $r \notin P$ - $\{0\}$, $(N : r) = N \bigcup (0 : r)$*; and for every* $0 \neq r \in P$,

$N \bigcup (0 : r) \subsetneqq (N : r)$.

Proof. (a)$\Longrightarrow$ (b) Suppose that N is a P-weakly primal submodule of M. Then W(N) = P$\setminus$ {0}. Let r $\notin$ P$\setminus$ {0}, and choose an element m $\in$ (N : r). If rm = 0, then m $\in$ (0 :

r). If rm $\neq$ 0, since r weakly prime to N we get m $\in$ N. Hence m $\in$ N $\bigcup$ (0 : r), that is (N : r) $\subseteq$ N $\bigcup$ (0 : r). Therefore (N : r) = N $\bigcup$ (0 : r). Now assume that r $\in$ P$\setminus$ {0} = W(N). Then r is not weakly prime to N. So there exists m $\in$ M $\setminus$ N such that 0 $\neq$ rm $\in$ N. Hence m $\in$ (N : r) $\setminus$ (N $\bigcup$ (0 : r)).

(b)$\Longrightarrow$ (a) It follows from (b) that W(N) = P $\setminus$ {0}. Hence N is P-weakly primal. □

Proposition 2.3.15. [11] *Let R be a commutative ring, and let M be an R-module. If N is a P-weakly primal submodule of M then P is a weakly prime ideal of R.*

Proof. Suppose that r, s $\in$ R $\setminus$ P are such that rs $\neq$ 0. If there is m $\in$ M with 0 $\neq$ (rs)m $\in$ N, then 0 $\neq$ rm $\in$ (N : s) = N $\bigcup$ (0 : s) by Theorem 2.3.14. Hence 0 $\neq$ rm $\in$ N. As r $\notin$ P, r is weakly prime to N. Hence m $\in$ N, that is rs is weakly prime to N. So rs $\notin$ P. Consequently P is a weakly prime ideal of R. □

The concept of almost primal ideals in a commutative ring was introduced by A.Y. Darani in [23]. Let R be a ring and let I be a proper ideal of R. An element a $\in$ R is called almost prime to I if ra $\in$ I-I^2 (with r $\in$ R) implies that r $\in$ I. We denote by A(I) the set of all elements of R that are not almost prime to I. A proper ideal I is called almost primal if the set P = A(I) $\bigcup$ I^2 forms an ideal of R. This ideal P is an almost prime ideal of R, called the almost prime adjoint ideal of I. In this case we also say that I is a P-almost primal ideal.

Now we give some definitions and result in almost primal submodules.

Definition 2.3.16. Let M be an R-module and N a submodule of M. The element a $\in$ R is *(left) almost prime to N* if am $\in$ N -(N : M)N (m $\in$ M) implies m $\in$ N. Denote by A(N) the set of elements of R that are not almost prime to N.

Definition 2.3.17. Let R be a commutative ring and let N be a proper submodule of an R-module M. N is called *almost primal* if the set P = A(N) $\bigcup$ $(N : M)N$ forms an ideal of R. P is called the *(almost) adjoint ideal of N* and we also say that N is a *P-almost primal submodule of M.*

Theorem 2.3.18. *Let P be an ideal of a commutative ring R. Let N be a proper submodule of R-module M. The following are equivalent:*

(1) N is P-almost primal.

(2) For every $x \notin P$ *- (N : M)N, (N : x) = N* $\bigcup$ *((N : M)N : x)*

and for $x \in P$ *- (N : M)N, (N : x)* $\supsetneqq$ *N* $\bigcup$*((N : M)N : x).*

Proof. (1) $\Longrightarrow$ (2) Assume that N is P-almost primal then P-(N : M)N = A(N). Let x $\notin$ P - (N : M)N then x is almost prime to N. Clearly N $\cup$ ((N : M)N : x) $\subseteq$ (N : x). For every m $\in$ (N : x), if mx $\in$ (N : M)N then m $\in$ ((N : M)N : x) and if mx $\notin$ (N : M)N then x is almost prime to N, gives m $\in$ N. Hence m $\in$ N $\bigcup$ ((N : M)N : x), that is (N : x)$\subseteq$ N $\cup$ ((N : M)N : x). Therefor (N : x) = N $\cup$ ((N : M)N : x). Now assume that x $\in$ $P-(N:M)N$ then x is not almost prime to N so $\exists$ m $\in$ M-N such that xm $\in$ N - (N : M)N. So m $\in$ (N : x), but m $\notin$ ((N : M)N : x) nor m $\in$ N. Hence, (N : x) $\neq$ N $\bigcup$ ((N : M)N : x). However, it is clear that N $\bigcup$ ((N : M)N : x) $\subsetneqq$ (N : x).

(2) $\Longrightarrow$ (1) We want to prove that P-(N : M)N consists exactly of all elements of R that are not almost prime to N. Hence N is P-almost primal.

Let $x \notin P-(N:M)N$, then (N : x) = N $\cup$ ((N : M)N : x). We want to prove that x $\notin$ A(N). Let xm $\in$ N - (N : M)N with m $\in$ M. So, m $\in$ (N : x). By assumption, either (N : x) = N or (N : x) = ((N : M)N :x). As xm $\in$ N - (N : M)N, so m $\notin$ ((N : M)N : x). Thus, m $\in$ N and hence, x $\notin$ A(N). Conversely, let $x \in P-(N:M)N$, then (N :x) $\supsetneqq$ N $\bigcup$ ((N : M)N : x), so, $\exists$ m $\in$ (N : x) such that m $\notin$ (N $\bigcup$ ((N : M)N :x)). Therefore, m $\notin$ N and m $\notin$ ((N : M)N : x). Thus xm $\in$ $N-(N:M)N$ with m $\notin$ N, so x is not almost prime to N and hence x $\in$ A(N). $\square$

Proposition 2.3.19. *Let N be asubmodule of R-module M. If N is almost primal submodule, then $P = A(N) \bigcup (N : M)N$ is almost prime ideal of R.*

Proof. Suppose that r,s $\notin$ P, we show that either rs $\in P^2$ or rs $\notin$ P. Assume that rs $\notin P^2$. Let rsm $\in$ N-(N : M)N for some m $\in$ M. Then, by Theorem 2.3.18 gives that rm $\in$ (N : s)= N $\bigcup$((N : M)N : s) where rm $\notin$ ((N : M)N : s); hence rm $\in$ N which implies that rm $\in$ N -(N : M)N. Thus m $\in$ (N : r) = N $\bigcup$ ((N : M)N : r), and so m $\in$ N. Therefore, rs is almost prime to N and rs $\notin$ P as required. □

2.4 n-Primly Submodules

The concept of n - primly ideals in a commutative ring was introduced by A.E. Ashour in [8]. Let R be a ring and let I be an ideal of R. An element s $\in$ R is called n-primary to I if s^{-n}I = { a $\in$ R : s^na $\in$ I } $\subseteq \sqrt{I}$. A proper ideal I of R is said to be n-primly if the set of all elements of R that are not n-primary to I forms an ideal of R, this set is called n-adjoint set of an ideal I. In this section, we generalize this concept to the concept of n-primly submodule. Now we give new definitions and some results in n-primly submodule.

Definition 2.4.1. Let n be a positive integer. Let N be a submodule of an R-module M. Let s be an element of a ring R. Define the set $s^{-n}N = \{a \in M : s^n a \in N\}$.

Remarks 2.4.2. (1) Let N be asubmodule of an R-module M. Let s be an element of a ring **R** then $N \subseteq s^{-1}N \subseteq s^{-2}N \subseteq s^{-3}N \subseteq \cdots$, and the equality does not hold in general. If we take N = 8ℤ a submodule of ℤ-module ℤ, then $2^{-1}N$ = 4ℤ, $2^{-2}N$ = 2ℤ and $2^{-3}N$ = ℤ.

(2) Let N be a submodule of R-module M. Let s be an element of a ring R. Then $s^{-n}N$

does not mean that s^n has an inverse in R, however if s^n has an inverse in R, then $(s^{-n})N = \{s^{-n}b : b \in N\}$, and the set $s^{-n}N$ are the same.[To see this, let m $\in (s^{-n})N$, then m $= s^{-n}b$, $b \in N$. Now, $s^n m = s^n(s^{-n}b) =$ b $\in$ N. Thus m $\in s^{-n}N$. On the other hand, let m $\in s^{-n}N$ then $s^n m \in$ N. Hence m $= s^{-n}(s^n m) \in (s^{-n})N$].

Definition 2.4.3. [32] Let N be a submodule of an R-module M. The *M-radical of N*, denoted by *rad(N)*, is the intersection of all prime submodules of M containing N. We say that the submodule N is a radical submodule if $rad(N) =$ N.

Definition 2.4.4. Let n be a positive integer. Let N be asubmodule of an R-module M. Let s be an element of a ring R. If $s^{-n}N \subseteq rad(N)$ then s is said to be *n-primary to N*.

Definition 2.4.5. Let n be a positive integer. We say $s \in R$ is *not n-primary to N* if $\exists$ b $\in M - rad(N)$ with $s^n b \in N$.

Example 2.4.6. *In $\mathbb{Z}$-module $\mathbb{Z}$, Let N= $4\mathbb{Z}$ be a submodule. Then $2^{-1}N = 2\mathbb{Z}$, $2^{-2}N = \mathbb{Z}$ and $\sqrt{4\mathbb{Z}} = 2\mathbb{Z}$. Thus 2 is 1-primary to N, but it is not 2-primary to N.*

Definition 2.4.7. Let n be a positive integer. Let N be a submodule of an R-module M. A subset *A of R is not n-primary to N* if for every element a in the set A, a is not n-primary to N.

Definition 2.4.8. Let n be a positive integer. Let N be a submodule of R-module M. A subset A of R is *uniformly not n-primary to N* if $\exists$ b $\in M - rad(N)$ with $A^n b = \{a^n b : a \in A\} \subseteq N$.

Remark 2.4.9. Let n be a positive integer. Then every proper submodule is uniformly not n-primary to itself. Take b = 1, the identity in R then $N^n = \{ a^n \mid$ a $\in$ N $\} \subseteq$ N.

Proposition 2.4.10. *Let n be a positive integer. Let N be a submodule of R-module M. If A is uniformly not n-primary to N, then A is not n-primary to N.*

Proof. Let A be uniformly not n-primary to a submodule N. Then $\exists$ u $\in M - rad(N)$ with $A^n u \subseteq N$. Let $a \in A$ and $b = u$, then $a^n b \in N$ and $b \in M - rad(N)$. Thus a is not n-primary to N. Since a is an arbitrary element in A, then A is not n-primary to N. □

Remark 2.4.11. The converse of Proposition 2.4.10 is not true in general. For example let M = R = $\mathbb{Z}$ and let N = 6$\mathbb{Z}$. Then A = {2, 3} is not 1-primary to N However A is not uniformly not 1-primary to N because if Ab = $\{2,3\}$b $\subseteq 6\mathbb{Z}$ then b $\in$ {...,-18,-12,-6,0,6,12,18,...} $= 6\mathbb{Z} = \sqrt{6\mathbb{Z}}$.

Proposition 2.4.12. *Let n be a positive integer Let N be a prime submodule of R-module M and A a finite subset of R. For a positive integer n, A is uniformly not n-primary to N iff A is not n-primary to N.*

Proof. ($\Longrightarrow$) From Proposition 2.4.10.

($\Longleftarrow$) If $A = \{a_1, a_2, ..., a_m\}$ is not n-primary to N, then $\exists$ $b_1, b_2, ..., b_m \in$ M - rad(N) such that $a_i^n b_i \in$ N $\forall i \in \{1, 2, ..., m\}$. Since $b_1, b_2, ..., b_m \in M - rad(N), b_1, b_2, ..., b_m \in M - N$. But N is prime submodule then $a_i^n M \subseteq$ N $\forall$ $i \in \{1, 2, ..., m\}$. Let b $\in$ M - rad(N) then $a_i^n b \in N$, $\forall$ $i \in \{1, 2, ..., m\}$. Thus $A^n b \subseteq N$ for some b $\in$ M - rad(N). Hence A is uniformly not n-primary to N. □

Proposition 2.4.13. [8] *Let n be a positive integer. Let I be a proper ideal over a ring R. Then* $\sqrt[n]{I} = \{r \in R : r^n \in I\}$ *is uniformly not n-primary to I.*

Proof. Let $A = \sqrt[n]{I}$ then $\forall a \in A$, $a^n \in I$. Let b = 1, since $1 \in R - \sqrt{I}$, then $A^n b \subseteq I$. Thus A is uniformly not n-primary to I. □

Corollary 2.4.14. *Let n be a positive integer. Let N be a proper submodule of R-module M. Then* $\sqrt[n]{(N : M)} = \{r \in R : r^n M \subseteq N\}$ *is uniformly not n-primary to (N : M).*

Proposition 2.4.15. *Let N be a proper submodule of an R-module M then the ideal $(N : M)$ is uniformly not n-primary to N.*

Proof. By the definition of $(N : M)$, we have $(N : M)^n M \subseteq (N : M)M \subseteq$ N, where n is a positive integer, $(N : M)^n = \{a^n : a \in (N : M)\}$ and hence (N : M)u ⊆ N for any u ∈ M - rad(N). □

Definition 2.4.16. Let n be a positive integer. Let N be a submodule of R-module M. The set of all elements that are not n-primary to N is called the *n-adjoint set for N* and is denoted by n-adj(N). That is, n- adj(N) = $\{a \in R : a^n b \in N$ for some $b \in M - rad(N)\}$.

Proposition 2.4.17. *Let N be a submodule of an R-module M then $n - adj(N) \neq R$, for every positive integer n.*

Proof. If $n - adj(N) = R$ then $1 \in n - adj(N)$. Thus $b = 1b \in N$ for some element $b \in M - rad(N)$ which is a contradiction. □

Example 2.4.18. *Let $M = R = \mathbb{Z}$, then*

(a) 1-adj(4$\mathbb{Z}$) = 4$\mathbb{Z}$.

(b) n-adj(4$\mathbb{Z}$) = 2$\mathbb{Z}$, for every positive integer $n \geq 2$.

(c) n-adj(6$\mathbb{Z}$) = 2$\mathbb{Z} \bigcup$ 3$\mathbb{Z}$ for every positive integer n.

(d) 1-adj(8$\mathbb{Z}$) = 8$\mathbb{Z}$.

(e) 2-adj(8$\mathbb{Z}$) = 4$\mathbb{Z}$.

(f) n- adj(8$\mathbb{Z}$) = 2$\mathbb{Z}$, for every positive integer $n \geq 3$.

(g) 1-adj(9$\mathbb{Z}$) = 9$\mathbb{Z}$.

(h)n-adj(9$\mathbb{Z}$) = 3$\mathbb{Z}$, for every positive integer $n \geq 2$.

(k) 1-adj(12$\mathbb{Z}$) = 4$\mathbb{Z} \bigcup$ 3$\mathbb{Z}$.

(l) n-adj(12$\mathbb{Z}$) = 2$\mathbb{Z} \bigcup$ 3$\mathbb{Z}$, for every positive integer $n \geq 2$.

Remarks 2.4.19. Let n be a positive integer. Let N be a submodule of R-module M.

(i) From the above example, we can see that n-adj(N) is not necessarily an ideal of R. For example, n - adj(6Z) = 2Z $\bigcup$ 3Z, which is not an ideal of Z.

(ii) One can show, directly from the definition that for a proper submodule N,

N $\subseteq$ 1 - adj(N) $\subseteq$ 2 - adj(N) $\subseteq$ 3 - adj(N) $\subseteq$ ....

(iii) The equality in (ii) does not hold in general, since as in the previous example 1-adj(8Z) = 8Z, 2-adj(8Z) = 4Z and 3-adj(8Z) = 2Z.

Proposition 2.4.20. *Let R be a Noetherian ring. Let N be a submodule of an R-module M and suppose that $n - adj(N)$ are ideals of R for every positive integer n, then there exists a positive integer m such that $\bigcup_{n=1}^{\infty} n - adj(N) = m - adj(N)$.*

Proof. Since $1 - adj(N) \subseteq 2 - adj(N) \subseteq \dots$ and R is Noetherian ring, so $\exists m \in \mathbb{N}$ such that $n - adj(N) = m - adj(N)$ $\forall$ n $\geq$ m. Thus $\bigcup_{n=1}^{\infty} n - adj(N) = m - adj(n)$. □

Example 2.4.21. [8] *Since $\mathbb{Z}$ is Noetherian ring, then*

(a) $\bigcup_{n=1}^{\infty} n - adj(4\mathbb{Z}) = 2 - adj(4\mathbb{Z})$.

(b) $\bigcup_{n=1}^{\infty} n - adj(8\mathbb{Z}) = 3 - adj(8\mathbb{Z})$.

(c) $\bigcup_{n=1}^{\infty} n - adj(9\mathbb{Z}) = 2 - adj(9\mathbb{Z})$.

Proposition 2.4.22. *For any submodule N of R-module M, $1 - adj(N) \subseteq adj(N)$.*

Proof. Let a $\in 1 - adj(N)$ then $\exists$ b $\in M - rad(N)$ such that ab $\in$ N. Hence b $\in$ M - N. Therefore $a \in adj(N)$ □

Remark 2.4.23. Since in $\mathbb{Z}$-module $\mathbb{Z}$, $1 - adj(4\mathbb{Z}) = 4\mathbb{Z}$ and $adj(4\mathbb{Z}) = 2\mathbb{Z}$ then the equality in Proposition 2.4.22 does not hold in general.

Proposition 2.4.24. *For a submodule N of an R-module M, we have $(N : M) \subseteq 1 - adj(N)$.*

Proof. Let r $\in$ (N : M) then rM $\subseteq$ N. Let b $\in$ M - $rad(N)$ then rb $\in$ N and hence r $\in$ $1 - adj(N)$. □

Proposition 2.4.25. [13] *Let N be a proper submodule of an R-module M, then* $(N : M) \subseteq \sqrt{(N : M)} \subseteq adj(N)$.

Proof. From definition of radical of ideal we have (N : M) $\subseteq \sqrt{(N : M)}$, so the proof reduces to proving that $\sqrt{(N : M)} \subseteq adj(N)$. Let $r \in \sqrt{(N : M)}$, then r^nM $\subseteq$ N for some positive integer n. Pick m $\in$ M - N, then r^nm $\in$ N. If we choose n_0 be the smallest positive integer with r^{n_0}m $\in$ N, we have r(r^{n_0-1}m) $\in$ N while r^{n_0-1}m $\notin$ N, Thus r $\in$ $adj(N)$. □

Proposition 2.4.26. [24] *Let N be a proper submodule of an R-module M. Then N is a prime submodule of M if and only if* $adj(N)$ = (N : M).

Proof. Suppose N is a prime submodule of M. By Proposition 2.4.25, it suffices to show that $adj(N) \subseteq$ (N : M). Let r $\in adj(N)$ then there exists m $\in$ M - N with rm $\in$ N, since N is prime, then rM $\subseteq$ N, and hence r $\in$ (N : M). Conversely, assume that $adj(N)$ = (N : M) and let rm $\in$ N where r $\in$ R and m $\in$ M - N. Then r is not prime to N, therefore r $\in adj(N)$ = (N : M). Thus rM $\subseteq$ N and N is prime. □

Theorem 2.4.27. *Let N be a proper submodule of R-module M. If N is a prime submodule then* $1 - adj(N) = adj(N)$.

Proof. Since (N : M) $\subseteq 1 - adj(N) \subseteq adj(N)$, by Propositions 2.4.24 and 2.4.22, and N is prime submodule then, by Proposition 2.4.26, (N : M) = $adj(N)$. Thus $1 - adj(N) = adj(N)$. □

Corollary 2.4.28. *If N is a prime submodule of R-module M, then* $adj(N) = 1 - adj(N) \subseteq 2 - adj(N) \subseteq 3 - adj(N) \subseteq ....$ *that is* $adj(N) \subseteq n - adj(N)$, *for every positive integer n.*

Proof. It is follows immediately from the Theorem 2.4.27 and Remarks 2.4.19. □

Definition 2.4.29. Let n be a positive integer. A submodule N of R-module M is called *n-primly* if $n - adj(N)$ is an ideal of R.

Example 2.4.30. *In $\mathbb{Z}$-module $\mathbb{Z}$, $4\mathbb{Z}$, $8\mathbb{Z}$ and $9\mathbb{Z}$ are n-primly submodule of $\mathbb{Z}$, while $6\mathbb{Z}$ and $12\mathbb{Z}$ are not n-primly submodules of $\mathbb{Z}$, for every positive integer n.*

Proposition 2.4.31. *Let n be a positive integer. Let N be a submodule of an R-module M. If $n - adj(N)$ is closed under addition then N is an n-primly submodule.*

Proof. We have to show that $n - adj(N)$ is an ideal of R. Since $n - adj(N)$ is closed under addition, it is enough to show that for every r $\in$ R and every a $\in n - adj(N)$, ra $\in n - adj(N)$. Let r $\in$ R and a $\in n - adj(N)$. Then $\exists$ b $\in$ M - rad(N) such that a^n b $\in$ N. Thus $r^n a^n b = (ra)^n b \in$ N. Therefore, ra $\in n - adj(N)$. □

Proposition 2.4.32. *Let n be a positive integer and Let N be a submodule of an R-module M then $\sqrt[n]{(N:M)} \subseteq n - adj(N)$.*

Proof. Let a $\in \sqrt[n]{(N:M)}$ then $a^n M \subseteq N$. therefore $\forall$ b $\in$ M - rad(N), we have $a^n b \in$ N. Thus a $\in n - adj(N)$. □

Proposition 2.4.33. *Let N be a submodule of an R-module M. If N is prime submodule then $\sqrt[n]{(N:M)} = n - adj(N)$.*

Proof. By Proposition 2.4.32 it is enough to show that $n - adj(N) \subseteq \sqrt[n]{(N:M)}$. Let a $\in n - adj(N)$ then a^n b $\in$ N, for some b $\in$ M - $rad(N)$, which implies b $\in$ M - N. But N is prime, so we have a^n M $\subseteq$ N. Thus a $\in \sqrt[n]{(N:M)}$ □

Proposition 2.4.34. *Let N be a proper submodule of R-module M. If N is prime submodule of M then N is 1-primly submodule of M.*

Proof. Since N is prime submodule of M, then by Proposition 2.4.26 and Theorem 2.4.27 we have $1 - adj(N) = adj(N)$ = (N : M), which is an ideal in R. Thus N is 1-primly submodule of M. □

Proposition 2.4.35. *Let n be a positive integer. Let N be a submodule of R-module M. If $n - adj(N)$ is uniformly not 1-primary to N then N is an n-primly submodule of M.*

Proof. According to Proposition 2.4.31, it is enough to show that $n - adj(N)$ is closed under addition. Let A = $n - adj(N)$. Let a, b ∈ A. Since A is uniformly not 1-primary to N, then ∃ u ∈ M - $rad(N)$ such that Au $\subseteq N$, so au ∈ N and bu ∈ N. Hence $a^m u \in$ N and $b^m u \in$ N for every positive integer m. Thus $(a+b)^n u = \sum_{k=0}^{n} a^{n-k} b^k u \in N$ with u ∈ M - $rad(N)$ This implies that a + b ∈ $n - adj(N)$. □

Definition 2.4.36. Let n be a positive integer. Let N be a submodule of R-module M. If $n - adj(N)$ is uniformly not n-primary to N then N is said to be *uniformly n-primly submodule of M.*

Theorem 2.4.37. *Let n be a positive integer. Let N be a submodule of an R-module M. If $n - adj(N)$ is a principal ideal of R, then N is a uniformly n-primly submodule of M.*

Proof. Let A = $n - adj(N)$ = Ra, for some a ∈ A. Then $a^n u \in$ N for some element u ∈ M - $rad(N)$. Let x ∈ A then x = ra for some r ∈ R. Since $a^n u \in$ N, then $x^n u = r^n a^n u$ ∈ N. Since x is an arbitrary element in A, then $A^n u \subseteq$ N. Hence A is uniformly not n-primary to N. Therefore, N is a uniformly n-primly submodule of M. □

Corollary 2.4.38. *Let n be a positive integer. Let R be a principal ideal ring then every n-primly submodule is uniformly n-primly.*

Example 2.4.39. *In $\mathbb{Z}$-module $\mathbb{Z}$, $2\mathbb{Z}$, $4\mathbb{Z}$, $8\mathbb{Z}$ and $9\mathbb{Z}$ are uniformly n-primly submodules of $\mathbb{Z}$.*

2.5 2-Absorbing Submodules

Definition 2.5.1. [19] Let R be a commutative ring with unity and M an R-module. A proper submodule N of M is said to be a *2-absorbing submodule* if whenever a, b $\in$ R and m $\in$ M with abm $\in$ N then ab $\in$ (N : M) or am $\in$ N or bm $\in$ N.

Definition 2.5.2. [19] Let R be a commutative ring and M an R-module. A proper submodule N of M is said to be *weakly 2-absorbing submodule* if whenever a, b $\in$ R, m $\in$ M with 0 $\neq$ abm $\in$ N then ab $\in$ (N : M) or am $\in$ N or bm $\in$ N.

Proposition 2.5.3. [19] *Let M be a module over a commutative ring R and N a submodule of M.*

(1) N is prime submodule $\Longrightarrow$ N is 2-absorbing submodule $\Longrightarrow$ N is weakly 2-absorbing submodule.

(2) N is weakly prime submodule $\Longrightarrow$ N is weakly 2-absorbing submodule.

Proof. Direct from the definitions. □

Example 2.5.4. *(1) For $\mathbb{Z}$-module $\mathbb{Z}$, the submodules $2\mathbb{Z}$, $4\mathbb{Z}$, $6\mathbb{Z}$, $9\mathbb{Z}$ are 2-absorbing submodule.*

(2) For $\mathbb{Z}$-module $\mathbb{Z}$, the submodules $4\mathbb{Z}$, $9\mathbb{Z}$ are 2-absorbing but not prime.

(3) For $\mathbb{Z}$-module $\mathbb{Z}$, the submodule $6\mathbb{Z}$ is 2-absorbing but not primary.

Remark 2.5.5. [20] Let R be a commutative ring and M an R-module. By above proposition, every 2-absorbing submodule is weakly 2-absorbing, but the converse does not necessarily hold. For example, consider the case where R = $\mathbb{Z}$, M = $\mathbb{Z}/30\mathbb{Z}$ and let N = $\{0\}$. Then 2.3.(5+30$\mathbb{Z}$) = 0 $\in$ N while 2.3 $\notin$ (N : M), 2(5+30$\mathbb{Z}$)$\notin$ N and 3.(5+30$\mathbb{Z}$) $\notin$ N. Therefore N is not 2-absorbing while it is weakly 2-absorbing.

Proposition 2.5.6. [20] *Let R be a commutative ring and let M be an R-module. Assume that N is a 2-absorbing submodule. Then*

(1) for every elements $a, b \in R$ and any submodule K of M. If $abK \subseteq N$ then $ab \in (N : M)$ or $aK \subseteq N$ or $bK \subseteq N$.

(2) $(N : M)$ is a 2-absorbing ideal of R.

Proof. (1) Assume that $ab \notin (N : M)$, $aK \nsubseteq N$ and $bK \nsubseteq N$. Then $ax \notin N$ and $by \notin N$ for some $x, y \in K$. As $abx \in N$ and $aby \in N$, we have $bx \in N$ and $ay \in N$. Now it follows that $ab(x + y) \in N$ then either $a(x + y) \in N$ or $b(x + y) \in N$. Consequently, either $ax \in N$ or $by \in N$ which are a contradiction.

(2) Suppose that $abc \in (N : M)$ then setting $K = cM$ so we have $abK \subseteq N$. As N is a 2-absorbing, it follows from (1) that $ab \in (N : M)$ or $aK \subseteq N$ or $bK \subseteq N$. Hence $ab \in (N : M)$ or $ac \in (N : M)$ or $bc \in (N : M)$. □

Theorem 2.5.7. [25] *Let $f : M \longrightarrow M'$ be an epimorphism of R-module.*

(1) If N' is 2-absorbing submodule of M' then $f^{-1}(N')$ is 2-absorbing submodule of M.

(2) If N is 2-absorbing submodule of M containing $ker(f)$ then $f(N)$ is 2-absorbing submodule of M'.

Proof. (1)Let $a, b \in R$ and $m' \in M'$ such that $abm \in f^{-1}(N')$ then $abf(m) \in N'$, but N' is 2-absorbing submodule of M', so $ab \in (N' : M')$ or $af(m) \in N'$ or $bf(m) \in N'$. If $ab \in (N' : M')$ then $abM' \subseteq N' \Longrightarrow abM = ab\ f^{-1}(M') = f^{-1}(abM') = \subseteq f^{-1}(N')$, so $ab \in (f^{-1}(N') : M)$. If $af(m) \in N'$ then $f(am) \in N' \Longrightarrow am \in f^{-1}(N')$. Similarly $bm \in f^{-1}(N)$. Thus $ab \in (f^{-1}(N') : M)$ or $am \in f^{-1}(N')$ or $bm \in f^{-1}(N')$ and hence $f^{-1}(N')$ is 2-absorbing submodule of M.

(2) Let $a, b \in R$, $m \in M$ and $abm' \in f(N)$, since f is epimorphism, there exists $m \in$ M such that $m' =$ f(m) and so f(abm) $\in$ f(N). Since $ker(f) \subseteq N$,we have $abm \in N$, since

N is 2-absorbing submodule then $ab \in (N : M)$ or $am \in N$ or $bm \in N$. If $abM \subseteq N$ then $f(abM) = abf(M) = abM' \subseteq f(N)$. Thus $ab \in (f(N) : M')$. If $am \in N$ then $f(am) = af(m) \in f(N)$, since $m' = f(m)$, then $am' \in f(N)$. If $bm \in N$ then as before $bm' \in f(N)$. Thus $ab \in (f(N) : M')$ or $am' \in f(N)$ or $bm' \in f(N)$. Hence $f(N)$ is a 2-absorbing submodule of M'. □

Proposition 2.5.8. [19] *Let R be a commutative ring, M an R-module. Let N, L be submodules of M with L ⊆ N. Then N is a 2-absorbing submodule of M if and only if N/L is a 2-absorbing submodule of M/L.*

Proof. Suppose first that N is a 2-absorbing submodule of M and let a, b ∈ R and m ∈ M be such that ab(m + L) ∈ N/L. Then abm ∈ N and as N is a 2-absorbing gives ab ∈ (N : M) or am ∈ N or bm ∈ N. Therefore ab ∈ (N/L : M/L) or a(m + L) ∈ N/L or b(m + L) ∈ N/L, that is N/L is a 2-absorbing submodule of M/L. Conversely, assume that N/L is a 2-absorbing submodule of M/L. Suppose that a, b ∈ R and m ∈ M such that abm ∈ N. Then we have ab(m + L) ∈ N/L. Therefore ab ∈ (N/L : M/L) or a(m + L) ∈ N/L or b(m + L) ∈ N/L, since N/L is 2-absorbing in M/L. Therefore ab ∈ (N : M) or am ∈ N or bm ∈ N. Thus N is a 2-absorbing submodule of M. □

Theorem 2.5.9. [25] *Let N be a submodule of an R-module M and let L be any submodule of M contained in N. If L and N/L are weakly 2-absorbing submodule of M and M/L respectively, then N is a weakly 2- absorbing submodule of M.*

Proof. Let a, b ∈ R, m ∈ M and 0 ≠ abm ∈ N. If 0 ≠ abm ∈ L then am ∈ L ⊂ N or bm ∈ L ⊂ N or ab ∈ (L : M) ⊆ (N : M). Thus N is weakly 2-absorbing submodule of M. Now assume that 0 ≠ abm ∉ L then 0 ≠ ab(m + L) ∈ N/L, but N/L is a weakly 2-absorbing submodule. Thus a(m + L) ∈ N/L or b(m + L) ∈ N/L or ab ∈ (N/L : M/L), so we have am ∈ N or bm ∈ N or ab ∈ (N : M). Hence N is weakly 2-absorbing submodule of M. □

Remarks 2.5.10. [39] (1) Let S be a multiplicatively close subset of R. Then by each submodule of S^{-1}M is of the form S^{-1}N for some submodule N of M (see [38]).

(2) It is well known that there is a one to one correspondence between the set of all prime submodules N of M with (N : M) $\bigcap$ S = $\emptyset$ and the set of all prime submodules of S^{-1}M, given by N $\rightarrow$ S^{-1}N (see [36]).

Definition 2.5.11. [39] Let N be a submodule of R-module M, we define N(S) = $\{x \in M : \exists s \in S, sx \in S\}$. Then N(S) is a submodule of M containing N and S^{-1}(N(S)) = S^{-1}N.

Definition 2.5.12. [39] Let $\phi :$ S(M) $\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function. We define $(S^{-1}\phi)$: $S(S^{-1}M) \rightarrow S(S^{-1}M) \bigcup\{\emptyset\}$ by $(S^{-1}\phi)(S^{-1}N) = S^{-1}(\phi(N(S)))$ if $\phi(N(S)) \neq \emptyset$ and $(S^{-1}\phi)(S^{-1}N) = \emptyset$ if $\phi(N(S)) = \emptyset$.

Remark 2.5.13. [39] Since we assume that $\phi(N) \subseteq$ N so we assume that $(S^{-1}\phi)(S^{-1}(N))$ $\subseteq S^{-1}N$. Also we note that $S^{-1}\phi_\emptyset = \phi_\emptyset$, $S^{-1}\phi_0 = \phi_0$, and whenever M is finitely generated $(S^{-1}\phi_i) = \phi_i$ for i = 1,2.

Definition 2.5.14. [39] For a submodule L of M, we define $\phi_L :$ S(M/L) $\rightarrow$ $S(M/L)$ $\cup\{\emptyset\}$ by $\phi_L(N/L) = (\phi(N) + L)/L$ for N $\supseteq$ L and $\phi_L(N/L) = \emptyset$ for $\phi(N) = \emptyset$.

N. Zamani in [39] gives relations between ϕ-prime submodules of R-module M and $(S^{-1}\phi)$-prime submodules of $S^{-1}M$. This leads us to give relations between 2-absorbing (weakly 2-absorbing) submodules of M and 2-absorbing (weakly 2-absorbing) submodules of S^{-1}M.

Theorem 2.5.15. *Let S be a multiplicatively closed subset of R and M be an R-module. Let $S^{-1}N$ be a submodule of $S^{-1}R$-module $S^{-1}M$. If N is a 2-absorbing submodule of M with $S^{-1}N \neq S^{-1}M$ then $S^{-1}N$ is a 2-absorbing submodule of $S^{-1}M$.*

Proof. Let r_1, r_2, r ∈ R, m ∈ M and s_1, s_2, s ∈ S such that $\frac{r_1r_2m}{s_1s_2s} \in S^{-1}$N then there exists u ∈ S such that ur_1r_2m ∈ N but N is 2-absorbing submodule, so we have ur_1m ∈ N or ur_2m ∈ N or r_1r_2 ∈ (N : M). Thus we have $\frac{r_1}{s_1}.\frac{m}{s} = \frac{ur_1m}{us_1s} \in S^{-1}$N or $\frac{r_2}{s_2}.\frac{m}{s} = \frac{ur_2m}{us_2s} \in S^{-1}$N or $\frac{r_1}{s_1}.\frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2} \in S^{-1}$(N $:_R$ M) ⊆ ($S^{-1}N :_{S^{-1}R} S^{-1}$M). □

Theorem 2.5.16. *Let S be a multiplicatively closed subset of R and M be an R-module. Let $S^{-1}N$ be a submodule of $S^{-1}R$-module $S^{-1}M$. If N is a weakly 2-absorbing submodule of M with $S^{-1}N \neq S^{-1}M$ then $S^{-1}N$ is a weakly 2-absorbing submodule of $S^{-1}M$.*

Proof. Assume that a, b ∈ R and s, t, l ∈ S and m ∈ M and $\frac{0}{1} \neq \frac{a}{s}\ \frac{b}{t}\ \frac{m}{l} \in S^{-1}$N which implies uabm ≠ 0 for all u ∈ S. Then there exists s_1 ∈ S such that 0 ≠ $s_1$abm ∈ N. Since N is weakly 2- absorbing, we have $s_1$am ∈ N or $s_1$bm ∈ N or ab ∈ (N : M). Hence $\frac{a}{s}\ \frac{m}{l} \in S^{-1}$N or $\frac{b}{t}\ \frac{m}{l} \in S^{-1}$N or $\frac{a}{s}\ \frac{b}{t} \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$. Thus S^{-1}N is a weakly 2-absorbing submodule of S^{-1}M. □

Chapter 3

ϕ - Classes of Submodules

In this Chapter, we recall the concept of ϕ-prime submodules and we generalize it to the concept of ϕ-primary and ϕ-2-absorbing submodules, also we introduce the concept of ϕ-primal submodules.

3.1 ϕ - Prime Submodules

Let R be a commutative ring with identity and M be a unitary R-module. Let S(M) be the set of all submodules of M, and let ϕ : S(M) $\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function.

Definition 3.1.1. [38] A proper submodule N of M is called ϕ - *prime submodule* if a $\in$ R, x $\in$ M with ax $\in$ N - $\phi(N)$ implies that a $\in$ (N : M) or x $\in$ N.

Definition 3.1.2. [30] Let R be a commutative ring. Let M be an R-module. Let S(M) be the set of all submodules of M. Define the following type of the functions ϕ_α : S(M) $\longrightarrow$ S(M) $\cup$ $\{\emptyset\}$ and the corresponding ϕ_α - prime submodules as follows :

If $\phi_\emptyset(N) = \emptyset$, $\forall$ N $\in$ S(M), then N is $\phi_\emptyset$-*prime submodule* iff N is prime submodule.

If $\phi_0(N) = \{\ 0\ \}$, $\forall$ N $\in$ S(M), then N is ϕ_0-*prime submodule* iff N is weakly prime submodule.

If $\phi_1(N)$ = N, ∀ N ∈ $S(M)$, then for any submodule N, N is ϕ_1-*prime submodule.*

If $\phi_2(N) = (N : M)N$, ∀ N ∈ S(M), then N is ϕ_2-*prime submodule* iff N is almost prime submodule.

If $\phi_3(N) = (N : M)^2N$, ∀ N ∈ S(M), then we say that N is ϕ_3-*prime submodule.*

Now, if $\phi_w(N) = \bigcap_{i=1}^{\infty}(N : M)^iN$, ∀ N ∈ S(M), then we say that N is ϕ_w-*prime submodule.*

and if $\phi_n(N) = (N : M)^{n-1}N$, ∀ N ∈ S(M) , then we say that N is ϕ_n-*prime submodule.*

Remarks 3.1.3. (1)[30] Since $N - \phi(N) = N - (N \cap \phi(N))$, so without loss of generality, throughout this thesis we will consider $\phi(N) \subseteq N$ for any N ∈ S(M).

(2) For functions ϕ, ψ :S(M)⟶ S(M)∪ {∅}, we write $\phi \le \psi$ if $\phi(N) \subseteq \psi(N)$ ∀ N ∈ S(M).

(3)Observe that $\phi_\emptyset \le \phi_0 \le \phi_w \le ... \le \phi_{n+1} \le \phi_n \le ... \le \phi_2 \le \phi_1$.

Proposition 3.1.4. [30] *Let R be a commutative ring with unity and N be a submodule of an R-module M.*

(1) Let ψ_1, ψ_2 : S(M)⟶ S(M) ∪{∅} be functions with $\psi_1 \le \psi_2$. Then N is ψ_1-prime implies N is ψ-prime.

(2) N is prime ⟹ N is weakly prime ⟹ N is ϕ_w-prime ⟹ N is ϕ_{n+1}-prime ⟹ ϕ_n-prime(n≥2)⟹ N is almost prime.

Proof. (1) Assume that N is ϕ_1 - prime. Let rm ∈ N - ϕ_2(N) for r ∈ R, m ∈ M then rm ∈ N - ϕ_1(N). Since N is ϕ_1 - prime, r ∈ (N : M) or m ∈ N. Hence N is ϕ_2 - prime.

(2) This follows from (1) and the ordering of the $\phi_{\alpha'}$ s given in Definition 3.1.2, and Remarks 3.1.3. □

Theorem 3.1.5. [39] *Let ϕ : S(M) → S(M) ∪ {∅} be a function. Let N be a ϕ-prime submodule of M. If (N : M)N $\nsubseteq \phi(N)$, then N is a prime submodule of M.*

Proof. Let a $\in$ R and x $\in$ M be such that ax $\in$ N. If ax $\notin \phi$(N), then since N is ϕ - prime, we have a $\in$ (N : M) or x $\in$ N. So let ax $\in \phi(N)$. In this case, we may assume that aN $\subseteq \phi(N)$, because if aN $\nsubseteq \phi(N)$ then there exists p $\in$ N such that ap $\notin \phi(N)$, so that a(x+p) $\in$ N - $\phi(N)$. Therefore a $\in$ (N : M) or x + p $\in$ N and hence a $\in$ (N : M) or x $\in$ N. Second we may assume that (N : M)x $\in \phi(N)$. If this is not the case, there exists u $\in$ (N : M) such that ux $\notin \phi(N)$ and so (a+ u)x $\in$ N - $\phi(N)$. Since N is a ϕ -prime submodule, we have a + u $\in$ (N : M) or x $\in$ N. Thus a $\in$ (N : M) or x $\in$ N. Now since (N : M)N $\nsubseteq \phi(N)$, there exists r $\in$ (N : M) and p $\in$ N such that rp $\notin \phi(N)$. So (a + r)(x + p) $\in$ N - $\phi(N)$, and hence a + r $\in$ (N : M) or x + p $\in$ N. Therefore a $\in$ (N : M) or x $\in$ N. Thus N is prime submodule. □

Since every prime submodule is primary, then the following result holds.

Corollary 3.1.6. [39] *Let ϕ : S(M) $\to$ S(M) $\cup$ $\{\emptyset\}$ be a function. Let N be a ϕ-prime submodule of M. If (N : M)N $\nsubseteq \phi(N)$, then N is a primary submodule of M.*

Corollary 3.1.7. [39] *Let N be a weakly prime submodule of M such that $(N :_R M)N \neq$ 0. Then N is a prime submodule of M.*

Proof. In the above theorem set $\phi = \phi_0$. □

Theorem 3.1.8. [39] *Let $R = R_1 \times R_2$ such that each R_i (i = 1, 2) is a commutative ring with identity. Let M_i be R_i-module (i = 1, 2), and $M = M_1 \times M_2$ be an R-module with $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$, where $r_i \in R_i$, $m_i \in M_i$ (i = 1, 2). Then we have:*
(1) If N_1 is a prime submodule of M_1, then $N_1 \times M_2$ is a prime submodule of M.
(2) If N_2 is a prime submodule of M_2, then $M_1 \times N_2$ is a prime submodule of M.

Proof. Because the prove of (1) and (2) are similar, so we will prove part (1). Hence suppose that N_1 is a prime submodule of M_1 and let $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$

$\in N_1 \times M_2$. then $r_1 m_1 \in N_1$. Since N_1 is prime, so $r_1 \in (N_1 : M_1)$ or $m_1 \in N_1$. Thus $(m_1, m_2) \in N_1 \times M_2$ or $(r_1, r_2) \in (N_1 : M_1) \times R_2$. Hence $(m_1, m_2) \in N_1 \times M_2$ or (r_1, r_2) $\in (N_1 \times M_2 : M_1 \times M_2)$. Hence, $N_1 \times M_2$ is prime. □

Example 3.1.9. [39] *The above theorem is not true for correspondence ϕ-prime submodules in general, for example if N_1 is a ϕ_0-prime submodule of M_1 then $N_1 \times M_2$ is not necessarily a ϕ_0-prime submodule of $M_1 \times M_2$. Let $R_1 = R_2 = M_1 = M_2 = Z_6$ and suppose that $N_1 = \{0\}$ then evidently N_1 is a ϕ_0-prime submodule of M_1. However, $0 \neq$ $(2,1)(3,1) \in N_1 \times M_2$ and $(3, 1) \notin N_1 \times M_2$. Also as $(2, 1)(2, 1) \notin N_1 \times M_2$, $(2, 1)M$ $\nsubseteq N_1 \times M_2$. Thus $N_1 \times N_2$ is not ϕ_0-prime submodule of M.*

Theorem 3.1.10. [30] *Let $R = R_1 \times R_2$ such that each R_i (i = 1, 2) is a commutative rings with identity. Let M_i be R_i-module ($\forall$ i = 1, 2) and $M = M_1 \times M_2$ be an R - module with $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$, where $r_i \in R_i$, $m_i \in M_i$ ($\forall$ i = 1, 2), and let ψ_i : S(M)$\rightarrow$ S(M) $\cup \{\emptyset\}$ be a functions, (i = 1, 2), and let $\phi = \psi_1 \times \psi_2$. Then each of the following types are ϕ-prime submodules of $M_1 \times M_2$,*

(i) $N_1 \times N_2$ where N_i is a proper submodule of M_i, with $\psi_i(N_i) = N_i$ (i = 1, 2).

(ii) $P_1 \times M_2$ where P_1 is a prime submodule of M_1.

(iii) $P_1 \times M_2$ where P_1 is a ψ_1-prime submodule of M_1 and $\psi_2(M_2) = M_2$.

(iv) $M_1 \times P_2$ where P_2 is a prime submodule of M_2.

(v) $M_1 \times P_2$ where P_2 is a ψ_2-prime submodule of M_2 and $\psi_1(M_1) = M_1$.

Proof. (i) It is clear, since $N_1 \times N_2$ - $\phi(N_1 \times N_2) = \emptyset$.

(ii) If P_1 is a prime submodule of M_1, then by Theorem 3.1.8, $P_1 \times M_2$ as a prime submodule of $M_1 \times M_2$. Thus $P_1 \times M_2$ is ϕ-prime submodule of $M_1 \times M_2$.

(iii) Let P_1 be a ψ_1-prime submodule of M_1 and $\psi_2(M_2) = M_2$. Let $(r_1, r_2) \in R$ and $(x_1, x_2) \in M$ such that $(r_1, r_2)(x_1, x_2) = (r_1 x_1, r_2 x_2) \in P_1 \times M_2$ - $\phi(P_1 \times M_2) = P_1 \times M_2$ -

$\psi_1(P_1) \times \psi_2(M_2) = P_1 \times M_2$ - $\psi_1(P_1) \times M_2 = (P_1 - \psi_1(P_1)) \times M_2$. Since P_1 is ψ_1 - prime submodule then $r_1 \in (P_1 :_{R_1} M_1)$ or $x_1 \in P_1$. Therefore $(r_1, r_2) \in (P_1 :_{R_1} M_1) \times R_2 = (P_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)$ or $(x_1, x_2) \in P_1 \times M_2$. So $P_1 \times M_2$ is a ϕ-prime submodule of $M_1 \times M_2$. Parts (iv), (v) are proved similar to (ii), (iii) respectively. □

Theorem 3.1.11. [39] *Let N be a proper submodule of M and let ϕ : S(M) $\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function. Then the following are equivalent:*

(i) N is a ϕ - prime submodule of M;

(ii) for x $\in$ M - N, (N : x) = (N : M) $\bigcup$ ($\phi(N)$: x);

(iii) for x $\in$ M - N, (N : x) = (N : M) or (N : x) = ($\phi(N)$: x);

(iv) for any ideal I of R and any submodule L of M, if IL $\subseteq$ N - $\phi(N)$,

then I $\subseteq$ (N : M) or L $\subseteq$ N.

Proof. (i)$\Longrightarrow$(ii) Let x $\in$ M - N and a $\in$ (N : x) - ($\phi(N)$: x). Then ax $\in N - \phi(N)$. Since N is a ϕ-prime submodule of M then a $\in$ (N : M). As we may assume that $\phi(N) \subseteq N$, the other inclusion always holds.

(ii)$\Longrightarrow$ (iii) If an ideal is the union of two ideals, it is equal to one of them.

(iii)$\Longrightarrow$(iv) Let I be an ideal of R and L be a submodule of M such that IL $\subseteq$ N. Suppose I $\nsubseteq$ (N : M) and L $\nsubseteq$ N. We show that IL $\subseteq \phi(N)$. Let a $\in$ I and x $\in$ L. We have two cases, first let a $\notin$ (N : M), since ax $\in$ N, we have (N : x) $\neq$ (N : M). Hence by our assumption (N : x) = ($\phi(N)$: x). So ax $\in \phi(N)$. Now assume that a $\in$ I $\cap$ (N : M). Let u $\in$ I - (N : M). Then a + u $\in$ I - (N : M). So by the first case, for each x $\in$ L we have ux $\in \phi(N)$ and (a + u)x $\in \phi(N)$. This gives that ax $\in \phi(N)$. Thus in any case ax $\in \phi(N)$, therefore IL $\subseteq \phi(N)$.

(iv)$\Longrightarrow$ (i) Let ax $\in$ N - $\phi(N)$. By considering the ideal $< a >$ and the submodule $< x >$, the result follows. □

Theorem 3.1.12. [39] *Let M be an R-module and let* $\phi : S(M) \to S(M) \cup \{\emptyset\}$*. Let N be a* ϕ*-prime submodule of M.*

(i) If $L \subseteq N$ *is a submodule of M, then N/L is a* ϕ_L*-prime submodule of M/L, where* ϕ_L *is defined as in Definition 2.5.14.*

(ii) Suppose that S is a multiplicatively closed subset of R such that $S^{-1}N \neq S^{-1}M$ *and* $S^{-1}(\phi(N)) \subseteq (S^{-1}\phi)(S^{-1}N)$*, and* $(N : M) \cap S = \emptyset$*. Then* $S^{-1}N$ *is an* $(S^{-1}\phi)$*-prime submodule of* $S^{-1}M$*.*

Proof. (i) Let a $\in$ R and $\bar{x} \in$ M/L with a$\bar{x} \in N/L - \phi_L(N/L)$, where $\bar{x}$ = x + L, for some x $\in$ M. By the definition of ϕ_L, this gives that ax $\in$ N - $(\phi(N) + L)$. So we have ax $\in$ N - $\phi(N)$, which gives that a $\in$ (N : M) or x $\in$ N. Therefore a $\in$ (N/L : M/L) or $\overline{x} \in$ N/L and so N/L is ϕ_L - prime submodule.

(ii) Let a/s $\in S^{-1}R$ and x/t $\in S^{-1}M$ with ax/st $\in S^{-1}N - (S^{-1}\phi)(S^{-1}N)$. Then by our assumption ax/st $\in S^{-1}N$-$S^{-1}(\phi(N))$. Therefore there exists u $\in$ S such that uax $\in$ N - $\phi(N)$, (note that for each v $\in$ S, vax $\notin \phi(N)$). Since N is ϕ-prime and (N : M) $\bigcap S = \emptyset$, we have ua $\in$ (N : M) or x $\in$ N. Therefore a/s $\in S^{-1}$(N : M) $\subseteq$ (S^{-1}N : S^{-1}M) or x/t $\in S^{-1}N$. Hence $S^{-1}N$ is an $(S^{-1}\phi)$-prime submodule of $S^{-1}M$. □

The concept of (n-1, n)-ϕ-prime submodule of an R-module M was introduced by M.Ebrahimpour and R.Nebooei (see [26]), which is the generalization of ϕ-prime submodule of M.

Now, we restrict some results to ϕ-prime submodule of M.

Proposition 3.1.13. *Let* $R = R_1 \times R_2 \times ... \times R_n$ *where* R_i *is a commutative ring with identity* $\forall$ $i \in \{ 1,2,...,n \}$ *and* $M = M_1\times M_2\times ... \times M_n$ *be an R - module, where* M_i *is an* R_i *- module, for* $i \in \{1, 2, ..., n\}$ *and* $(r_1,r_2,...,r_n)(m_1,m_2,...,m_n) = (r_1m_1,r_2m_2,...,r_nm_n)$ *where* $r_i \in R_i$ *and* $m_i \in M_i$*,* $\forall$ $i \in \{ 1,2,...,n\}$*. Let* $N = N_1\times N_2\times ...\times N_n$*, where* N_i

is a submodule of M_i $\forall$ $i \in \{$ *1,2,...,n* $\}$ *and let* ψ_i : $S(M_i) \longrightarrow S(M_i) \cup \{\emptyset\}$ *and* $\phi(N)$ $= \psi_1(N_1) \times \psi_2(N_2) \times \ldots \times \psi_n(N_n)$. *If N is* ϕ*-prime submodule of M, then* N_i *is a* ψ_i *- prime submodule of* M_i*, for each* $i \in \{$*1,2,...,n*$\}$ *with* $N_i \neq M_i$.

Proof. Let $N_i \neq M_i$, $x_i \in M_i$ and $a_i \in R_i$ such that $a_i x_i \in N_i - \psi_i(N_i)$. Thus (1,...,1,$a_i$,...,1). (0,...,0,$x_i$,...,0)= (0,...,0,$a_i x_i$,...,0) $\in$ N - $\phi(N)$, but N is ϕ-prime submodule. Therefore, (1,...,1,a_i ,...,1) $\in$ (N : M) or (0,...,0,x_j,0,...,0) $\in$ N. So we have $a_i \in (N_i : M_i)$ or $x_i \in N_i$. Thus N_i is ψ_i - prime submodule for each i. □

Corollary 3.1.14. *Let* $R = R_1 \times R_2 \times \ldots \times R_n$ *where* R_i *is a commutative ring with identity* $\forall$ $i \in \{$ *1,2,...,n* $\}$ *and* $M = M_1 \times M_2 \times \ldots \times M_n$ *be an R - module, where* M_i *is an* R_i *- module, for* $i \in \{1, 2, ..., n\}$ *and* $(r_1,r_2,\ldots,r_n)(m_1,m_2,\ldots,m_n) = (r_1m_1,r_2m_2,\ldots,r_nm_n)$ *where* $r_i \in R_i$ *and* $m_i \in M_i$, $\forall$ $i \in \{$ *1,2,...,n*$\}$. *Let* $N = N_1 \times N_2 \times \ldots \times N_n$*, where* N_i *is a submodule of* M_i $\forall$ $i \in \{$ *1,2,...,n* $\}$ *and let* ψ_i : $S(M_i) \longrightarrow S(M_i) \cup \{\emptyset\}$ *and* $\phi(N) = \psi_1(N_1) \times \psi_2(N_2) \times \ldots \times \psi_n(N_n)$. *Let N be a* ϕ_n*-prime submodule of M. Then* N_i *is a* ϕ_n*-prime submodule of* M_i*, for each i with* $N_i \neq M_i$ *and* $n \geq 2$.

Proof. We have $\phi_n(P)$ = (P:M)$^{n-1}$P = $(P_1$:M$)^{n-1}P_1 \times (P_2$:M$)^{n-1}P_2 \times \ldots \times (P_n$:M$)^{n-1}P_n$ $= \phi_n(P_1) \times \phi_n(P_2) \times \ldots \times \phi_n(P_n)$. So the result follows by Proposition 3.1.13. □

3.2 ϕ - Primary Submodules

The concept of ϕ - primary ideals in a commutative ring was introduced by A.Y. Darani in [22] and M. Batanineh and S. Kuhail in [15]. Let R be a commutative ring with identity. Let $\phi : \mathbb{J}(R) \to \mathbb{J}(R) \cup \{\emptyset\}$ be a function, where $\mathbb{J}(R)$ denotes the set of all ideals of R. A proper ideal I of R is called ϕ-primary ideal if for x, y $\in$ R, xy $\in$ I - ϕ(I) implies x $\in$ I or y $\in$ rad(I). Now we generalize some results of ϕ-prime submodule in [39] to ϕ-primary submodule and we also generalize the concept of ϕ-primary ideal in [22] to ϕ - primary submodule. Let R be a commutative ring with identity and M be a unitary R-module. Let S(M) be the set of all submodules of M, and ϕ : S(M) $\longrightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function. Then we have the following definition.

Definition 3.2.1. [15] A proper submodule N of M is called ϕ - *primary submodule* if a $\in$ R, x $\in$ M with ax $\in$ N - $\phi(N)$ implies that x $\in$ N or $a^k \in$ (N : M), for some positive integer k. In other word, x $\in$ N or a $\in$ $\sqrt{(N : M)}$.

Definition 3.2.2. Let R be a commutative ring. Let M be an R-module. Let S(M) be the set of all submodules of M. Define the following type of the functions ϕ_α : S(M) $\longrightarrow$ S(M) $\cup$ $\{\emptyset\}$ and the corresponding ϕ_α - primary submodules as follows :

If $\phi_\emptyset(N) = \emptyset$, $\forall$ N $\in$ S(M), then N is $\phi_\emptyset$-*primary* submodule iff N is primary submodule.

If $\phi_0(N) = \{\ 0\ \}$, $\forall$ N $\in$ S(M), then N is ϕ_0-*primary submodule* iff N is weakly primary submodule.

If $\phi_1(N) =$ N, $\forall$ N $\in S(M)$, then for any submodule N, N is ϕ_1-*primary submodule.*

If $\phi_2(N) = (N : M)N$, $\forall$ N $\in$ S(M), then N is ϕ_2-*primary submodule* iff N is almost primary submodule.

If $\phi_3(N) = (N : M)^2N$, $\forall$ N $\in$ S(M), then we say that N is ϕ_3-*primary submodule.*

Now, if $\phi_w(N) = \bigcap_{i=1}^{\infty}(N : M)^iN$, $\forall$ N $\in$ S(M), then we say that N is ϕ_w-*primary submodule.*

and if $\phi_n(N) = (N : M)^{n-1}N$, $\forall$ N $\in$ S(M), then we say that N is ϕ_n-*primary submodule.*

Proposition 3.2.3. *Let R be a commutative ring and N be a submodule of R-module M.*

(1)Let ψ_1, ψ_2 : S(M)$\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be functions with $\psi_1 \leq \psi_2$. Then N is ψ_1-primary implies N is ψ_2-primary.

(2)Let ϕ : S(M)$\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be functions. If N is ϕ-prime then N is ϕ-primary.

(3)N is primary $\Longrightarrow$ N is weakly primary $\Longrightarrow$ N is ϕ_w-primary $\Longrightarrow$ N is ϕ_{n+1}-primary $\Longrightarrow$ ϕ_n-primary $(n \geq 2)$ $\Longrightarrow$ N is almost primary.

Proof. (1)Assume that N is ϕ_1-primary. Let rm $\in N - \phi_2(N)$ for r $\in$ R, m $\in$ M then rm $\in N - \phi_1(N)$. Since N is ϕ_1-primary, $r^k \in$ (N : M) for some $k \in \mathbb{N}$ or m $\in$ N. Hence N is ϕ_2-primary.

(2)Assume that N is ϕ-prime. Let $rm \in N - \phi(N)$ for r $\in$ R, m $\in$ M. Then r $\in$ (N : M) or m $\in$ N, since (N : M) $\subseteq \sqrt{(N:M)}$ then r $\in \sqrt{(N:M)}$ or m $\in$ N. Thus N is ϕ-primary.

(3) This follows from (1) and the ordering of the ϕ_α' s given in Definition 3.2.2. □

Theorem 3.2.4. *Let R be a commutative ring and M be an R-module. Let ϕ: S(M)$\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function. Let N be a ϕ-primary submodule of M. If (N : M)N $\nsubseteq \phi(N)$ then N is a primary submodule of M.*

Proof. Let a $\in$ R and x $\in$ M be such that ax $\in$ N. If ax $\notin \phi$(N), then since N is ϕ-primary, we have a$^k \in (N:M)$ for some k $\in \mathbb{N}$ or x $\in$ N. So let ax $\in \phi(N)$. In this case we may assume that aN $\subseteq \phi(N)$, because if aN $\nsubseteq \phi(N)$ then there exists p $\in$ N such that ap $\notin \phi(N)$, so that a(x+p) $\in$ N - $\phi(N)$. Therefore a $\in \sqrt{(N:M)}$ or x + p $\in$ N and hence a $\in \sqrt{(N:M)}$ or x $\in$ N. Second we may assume that $(N:M)x \in \phi(N)$. If this is not the case, there exists u $\in (N:M)$ such that ux $\notin \phi(N)$ and so (a+ u)x $\in$ N - $\phi(N)$. Since N is a ϕ -primary submodule, we have a + u $\in \sqrt{(N:M)}$ or x $\in$ N. So a $\in \sqrt{(N:M)}$ or x $\in$ N. Now since $(N:M)$N $\nsubseteq \phi(N)$, there exist r $\in (N:M)$ and p $\in$ N such that rp $\notin \phi(N)$. So (a + r)(x + p) $\in$ N - $\phi(N)$, and hence a + r $\in \sqrt{(N:M)}$ or x + p $\in$ N. Therefore a $\in \sqrt{(N:M)}$ or x $\in$ N. Thus N is primary submodule. □

Corollary 3.2.5. *Let N be a weakly primary submodule of M such that $(N:M)N \neq 0$. Then N is a primary submodule of M.*

Proof. In the above theorem, set $\phi = \phi_0$. □

Remark 3.2.6. Suppose that N is a ϕ-primary submodule of M such that $\phi(N) \subseteq (N : M)N$ (resp. $\phi(N) \subseteq (N : M)^2N$) and that N is not a primary submodule. Then by Theorem 3.2.4, we have $\phi(N) = (N : M)N$ (resp. $\phi(N) = (N : M)^2N$). In particular if N is a weakly primary (resp. ϕ_3 - primary) submodule but not primary submodule then $(N : M)N = 0$ (resp. $(N : M)N = (N : M)^2N$).

Theorem 3.2.7. [9] *Let $R = R_1 \times R_2$ where each R_i is a commutative ring with identity. Let M_i be R_i-module $\forall\ i \in \{1,2\}$, and $M = M_1 \times M_2$ be an R-module with $(r_1, r_2)(m_1, m_2)$ $= (r_1m_1, r_2m_2)$, where $r_i \in R_i$, $m_i \in M_i$. Then,*

(1)If N_1 is a primary submodule of M_1, then $N_1 \times M_2$ is a primary submodule of M.

(2) If N_2 is a primary submodule of M_2, then $M_1 \times N_2$ is a primary submodule of M.

Proof. Because the proof of (1) and (2) are similar, so we will prove part(1). Hence, suppose that N_1 is a primary submodule of M_1 and let $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$ $\in N_1 \times M_2$, where $(r_1,r_2) \in$ R and $(m_1,m_2) \in$ M. Then $r_1m_1 \in N_1$. Since N_1 is primary, so $r_1^k \in (N_1 :_{R_1} M_1)$ for some k $\in \mathbb{N}$ or $m_1 \in N_1$. So $(m_1, m_2) \in N_1 \times M_2$ or $(r_1, r_2)^k$ $\in (N_1 :_{R_1} M_1) \times (M_2 :_{R_2} M_2) = (N_1 \times M_2 :_{R_1\times R_2} M_1 \times M_2)$ Hence $(m_1, m_2) \in N_1 \times M_2$ or $(r_1, r_2)^k \in (N_1 \times M_2 : M)$ for some k $\in \mathbb{N}$, so $N_1 \times M_2$ is primary submodule of $M_1 \times M_2$. □

Remark 3.2.8. The above theorem is not true for correspondence ϕ - primary submodules in general, for example if N_1 is a ϕ_0-primary submodule of M_1 then $N_1 \times M_2$ is not necessarily a ϕ_0-primary submodule of $M_1 \times M_2$. Let $R_1 = R_2 = M_1 = M_2 = Z_6$, and suppose $N_1 = 0$. Then evidently N_1 is a ϕ_0-primary submodule of M_1. However, (2, 1)(3, 1) $\in N_1 \times M_2$, and (3, 1) $\notin N_1 \times M_2$. Also $(2, 1)^k(2, 1) \notin N_1 \times M_2$ for any k $\in \mathbb{N}$, $(2, 1)^k$ M $\nsubseteq N_1 \times M_2$.

Proposition 3.2.9. *Let R_1 and R_2 be two commutative rings, M_1 and M_2 be R_1 and R_2*

- modules respectively. Let $M = M_1 \times M_2$ and ϕ : S(M) $\rightarrow$ S(M) $\cup \{\emptyset\}$ be a function. Suppose that N_1 is a weakly primary submodule of M_1 such that $\{0\} \times M_2 \subseteq \phi(N_1 \times M_2)$. Then $N_1 \times M_2$ is a ϕ-primary submodule of $M_1 \times M_2$.

Proof. Let $(r_1, r_2)(x_1, x_2) = (r_1x_1, r_2x_2) \in N_1 \times M_2$ - $\phi(N_1 \times M_2)$, but $N_1 \times M_2$ - $\phi(N_1 \times M_2)$ $\subseteq N_1 \times M_2$ - $\{0\} \times M_2 = (N_1 - \{0\}) \times M_2$. We have $r_1x_1 \in N_1 - \{0\}$ and by the assumption on N_1 we have $r_1^k \in (N_1 :_{R1} M_1)$ for some positive integer k or $x_1 \in N_1$. This gives that $(r_1, r_2)^k = (r_1^k, r_2^k) \in (N_1 :_{R_1} M_1) \times R_2 = (N_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)$ for some positive integer k or $(x_1, x_2) \in N_1 \times M_2$. Therefore $N_1 \times M_2$ is a ϕ-primary submodule of $M_1 \times M_2$. □

Proposition 3.2.10. *Let R_1 and R_2 be two commutative rings, M_1 and M_2 be R_1 and R_2 - modules respectively. Let $M = M_1 \times M_2$ and ϕ : S(M) $\rightarrow$ S(M) $\cup \{\emptyset\}$ be a function. such that $\phi_w \leq \phi$. Then for any weakly primary submodule N_1 of M_1, $N_1 \times M_2$ is a ϕ - primary submodule of $M_1 \times M_2$.*

Proof. If N_1 is a primary submodule of M_1, then $N_1 \times M_2$ is primary submodule of M, (see Theorem 3.2.7), and so a ϕ - primary submodule of $M_1 \times M_2$. Suppose that N_1 is not a primary submodule of M_1. Then by Remark 3.2.6, we have $(N_1 :_{R_1} M_1)N_1 = \{0\}$. This gives that

$(N_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)^i(N_1 \times M_2) = [(N_1 :_{R_1} M_1)^i N_1] \times M_2 = \{0\} \times M_2$, for all i ≥ 1 and hence we have $\{0\} \times M_2 = \cap_{i=1}^{\infty}(N_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)^i(N_1 \times M_2) = \phi_w(N_1 \times M_2) \subseteq \phi(N_1 \times M_2)$, and by Proposition 3.2.9, we have $N_1 \times M_2$ is a ϕ-primary submodule of $M_1 \times M_2$. □

Theorem 3.2.11. *Let $R = R_1 \times R_2$ such that each R_i is a commutative ring with identity. Let M_i be R_i-module $\forall$ i $\in$ {1,2}, and $M = M_1 \times M_2$ with $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$, be an R-module, where $r_i \in R_i$, $m_i \in M_i$ $\forall$ i $\in$ {1,2}, and let ψ_i : S(M)$\rightarrow$ S(M) $\cup \{\emptyset\}$ be a functions, $\phi = \psi_1 \times \psi_2$. Then each of the following types are ϕ-primary submodules of*

$M_1 \times M_2$,

(i) $N_1 \times N_2$ *where* N_i *is a proper submodule of* M_i, *with* $\psi_i(N_i) = N_i$.

(ii) $P_1 \times M_2$ *where* P_1 *is a primary submodule of* M_1.

(iii) $P_1 \times M_2$ *where* P_1 *is a* ψ_1*-primary submodule of* M_1 *and* $\psi_2(M_2) = M_2$.

(iv) $M_1 \times P_2$ *where* P_2 *is a primary submodule of* M_2.

(v) $M_1 \times P_2$ *where* P_2 *is a* ψ_2*-primary submodule of* M_2 *and* $\psi_1(M_1) = M_1$.

Proof. (i) is clear, since $N_1 \times N_2$ - $\phi(N_1 \times N_2) = \emptyset$

(ii) If P_1 is a primary submodule of M_1, then by Theorem 3.2.7, $P_1 \times M_2$ a primary submodule of $M_1 \times M_2$, and thus $P_1 \times M_2$ is ϕ-primary submodule of M.

(iii) Let P_1 be a ψ_1-primary submodule of M_1 and $\psi_2(M_2) = M_2$. Let $(r_1, r_2) \in R$ and $(x_1, x_2) \in M$ be such that $(r_1, r_2)(x_1, x_2) = (r_1x_1, r_2x_2) \in P_1 \times M_2$ - $\phi(P_1 \times M_2) = P_1 \times M_2$ - $\psi_1(P_1) \times \psi_2(M_2) = P_1 \times M_2$ - $\psi_1(P_1) \times M_2 = (P_1 - \psi_1(P_1)) \times M_2$. So $r_1x_1 \in P_1$ - $\psi_1(P_1)$ but P_1 is ψ_1 - primary submodule, so $r_1^k \in (P_1 :_{R_1} M_1)$ for some k $\in \mathbb{N}$ or $x_1 \in P_1$. Therefore $(r_1, r_2)^k \in (P_1 :_{R_1} M_1) \times R_2 = (P_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)$ or $(x_1, x_2) \in P_1 \times M_2$. So $P_1 \times M_2$ is a ϕ-primary submodule of $M_1 \times M_2$.

Parts (iv), (v) are proved similar to (ii), (iii) respectively. □

Theorem 3.2.12. *Let N be a proper submodule of M and let* ϕ *: S(M)* $\rightarrow$ *S(M)* $\cup \{\emptyset\}$ *be a function. Then the following are equivalent:*

(i) N is ϕ *- primary submodule of M.*

(ii) For $r \in R$ - $\sqrt{(N:M)}$, *(N : (r))* $=$ *N* $\cup$ *(*$\phi(N)$ *: (r)).*

(iii) For $r \in R$ - $\sqrt{(N:M)}$, *(N : (r))* $=$ *N or (N : (r))* $=$ *(*$\phi(N)$ *: (r)).*

Proof. (i)$\Longrightarrow$(ii) Suppose that N is ϕ - primary such that r $\notin \sqrt{(N:M)}$. Let m $\in$ (N : (r)). So rm $\in$ N. If rm $\notin \phi(N)$, then N is ϕ - primary implies m $\in$ N, and if rm $\in \phi(N)$, then m $\in$ ($\phi(N)$: (r)). Hence (N : (r)) $\subseteq$ N $\cup$ ($\phi(N)$: (r)). The other inclusion hold trivially, since $\phi(N) \subseteq$ N.

(ii) $\Longrightarrow$ (iii) It is clear because (N : (r)) is an ideal of R.

(iii) $\Longrightarrow$(i) Let r $\in$ R, m $\in$ M such that rm $\in$ N - $\phi(N)$. If r $\notin \sqrt{(N:M)}$, then by assumption, either (N : (r)) = N or (N : (r)) = ($\phi(N)$: (r)). As rm $\notin \phi(N)$, then m $\notin$ ($\phi(N)$: (r)) and as rm $\in$ N, then m $\in$ (N : (r)). Hence (N : (r)) = N, and so m $\in$ N as required. ☐

Theorem 3.2.13. *Let M be an R-module and let N be a proper submodule of M. If for any ideal I of R and submodule K of M with IK $\subseteq$ N and IK $\nsubseteq \phi(N)$, we have I $\subseteq \sqrt{(N:M)}$ or K $\subseteq$ N, then N is ϕ - primary submodule of M.*

Proof. Suppose that rm $\in$ N - $\phi(N)$ for r $\in$ R and m $\in$ N. Then $(r)(m) = (rm) \subseteq N - \phi(N)$. By the assumption, either $(m) \subseteq N$ or $(r) \subseteq \sqrt{(N:M)}$. Therefore, $m \in N$ or $r \in \sqrt{(N:M)}$ and N is ϕ-primary submodule of M. ☐

Proposition 3.2.14. *Let N be a submodule of M with (N : M) = $\sqrt{(N:M)}$, then N is ϕ-primary if and only if N is ϕ-prime.*

Proof. Trivial from the definitions of ϕ-prime and ϕ-primary submodules. ☐

Theorem 3.2.15. *Let M be an R-module and let ϕ : S(M) $\longrightarrow$ S(M) $\cup \{\emptyset\}$. Let P be a ϕ-primary submodule of M.*

(i) If L $\subseteq$ P is a submodule of M, then P/L is a ϕ_L-primary submodule of M/L, where ϕ_L is defined as in Definition 2.5.14.

(ii) Suppose that S is a multiplicatively closed subset of R such that $S^{-1}P \neq S^{-1}M$ and $S^{-1}(\phi(P)) \subseteq (S^{-1}\phi)(S^{-1}P)$ and (P : M) $\cap$ S = $\emptyset$. Then $S^{-1}P$ is an $(S^{-1}\phi)$-primary submodule of $S^{-1}M$.

Proof. (i) Let a $\in$ R and $\bar{x} \in$ M/L with a $\bar{x} \in$ P/L - $\phi_L(P/L)$, where $\bar{x}$ = x + L, for some x $\in$ M. By the definition of ϕ_L, this gives that ax $\in$ P - $\phi(P)$, which gives that a^k

$\in (P : M)$ for some k $\in \mathbb{N}$ or x $\in$ P. Therefor $a^k \in (P/L : M/L)$ for some k $\in \mathbb{N}$ or $\overline{x} \in$ P/L and so P/L is ϕ_L-primary submodule.

(ii) Let a/s $\in S^{-1}R$ and x/t $\in S^{-1}M$ with ax/st $\in S^{-1}P$ - $(S^{-1}\phi)(S^{-1}P)$. Then by our assumption ax/st $\in S^{-1}P$ - $S^{-1}(\phi(P))$. Therefore there exists u $\in$ S such that uax $\in$ P - $\phi(P)$, (note that for each v $\in$ S, vax $\notin \phi(P)$). Since P is ϕ - primary and $(P : M) \bigcap S = \emptyset$, we have $(ua)^k \in (P : M)$ for some k $\in \mathbb{N}$ or x $\in$ P. Therefore $(a/s)^k \in S^{-1}((P :_R M))$ $\subseteq (S^{-1}P :_{S^{-1}R} S^{-1}M)$ for some k $\in \mathbb{N}$ (because (P : M) $\subseteq (S^{-1}P : S^{-1}M)$) or x/t $\in S^{-1}P$. Hence $S^{-1}P$ is an $(S^{-1}\phi)$-primary submodule of $S^{-1}M$. □

3.3 ϕ-2-Absorbing Submodules

In this section, we introduce the concept of ϕ-2-absorbing submodules which is a generalization to concept of 2-absorbing submodules. Let R be a commutative ring with identity and M be a unitary R-module. Let S(M) be the set of all submodules of M, and ϕ : S(M) $\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function.

Definition 3.3.1. A proper submodule N of M is called *ϕ-2-absorbing submodule* if r,s $\in$ R, m $\in$ M with rsm $\in$ N - $\phi(N)$ implies that rs $\in$ (N : M) or rm $\in$ N or sm $\in$ N.

Definition 3.3.2. Let R be a commutative ring. Let M be an R-module. Let S(M) be the set of all submodules of M. Define the following type of the functions ϕ_α : S(M) $\longrightarrow$ S(M) $\cup$ $\{\emptyset\}$ and the corresponding ϕ_α-2-absorbing submodules as follows :

If $\phi_\emptyset(N) = \emptyset$, $\forall$ N $\in$ S(M), then N is *$\phi_\emptyset$-2-absorbing submodule* iff N is 2-absorbing submodule,

If $\phi_0(N) = \{ 0 \}$, $\forall$ N $\in$ S(M), then N is *ϕ_0-2-absorbing submodule* iff N is weakly 2-absorbing submodule,

If $\phi_1(N) =$ N, $\forall$ N $\in S(M)$, then for any submodule N, N is *ϕ_1-2-absorbing submodule*, If $\phi_2(N) = (N : M)N$, $\forall$ N $\in$ S(M), then we say that N is *ϕ_2-2-absorbing submodule*,

If $\phi_3(N) = (N : M)^2N$, $\forall$ N $\in$ S(M), then we say that N is *ϕ_3-2-absorbing submodule*,

Now, if $\phi_w(N) = \bigcap_{i=1}^{\infty}(N : M)^iN$, $\forall$ N $\in$ S(M), then we say that N is *ϕ_w-2-absorbing submodule*,

and if $\phi_n(N) = (N : M)^{n-1}N$, $\forall$ N $\in$ S(M) , then we say that N is *ϕ_n-2-absorbing submodule.*

Remarks 3.3.3. (1) every 2-absorbing submodule is ϕ-2-absorbing submodule but the converse need not be true in general. For example, let M = $\mathbb{Z}_8$ be a $\mathbb{Z}$ module and let N = $\{0\}$. N is ϕ_0(weakly)-2-absorbing submodule but not 2-absorbing submodule because 2.2.2 = 0 $\in$ N and 4 $\notin$ N and 4 $\notin$ (N : M) = $\{$8n : n $\in \mathbb{Z}\}$.

(2) Observe that $\phi_\emptyset \leq \phi_0 \leq \phi_w \leq ... \leq \phi_{n+1} \leq \phi_n \leq ... \leq \phi_2 \leq \phi_1$.

Proposition 3.3.4. *Let R be a commutative ring and N be a submodule of R - module M.*

(1) Let ψ_1, ψ_2 : S(M)$\longrightarrow$ S(M)$\cup\{\emptyset\}$ be functions with $\psi_1 \leq \psi_2$. Then N is ψ_1-2-absorbing submodule implies N is ψ_2-2-absorbing.

(2) N is 2-absorbing $\Longrightarrow$ N is weakly 2-absorbing $\Longrightarrow$ N is ϕ_w- 2-absorbing $\Longrightarrow$ N is ϕ_{n+1}-2-absorbing $\Longrightarrow$ ϕ_n-2-absorbing(n$\geq$2)$\Longrightarrow$ N is ϕ_2-2-absorbing.

Proof. (1) Assume that N is ϕ_1-2-absorbing submodule of M. Let rsm $\in N - \phi_2(N)$ for r,s $\in$ R, m $\in$ M then rsm $\in N - \phi_1(N)$. Since N is ϕ_1 -2-absorbing, rs $\in$ (N : M) or rm $\in$ N or sm $\in$ N. Hence N is ϕ_2-2-absorbing submodule of M.

(2) This follows from (1) and the ordering of the $\phi_{\alpha'}$ s given in Definition 3.3.2 and Remarks 3.3.3. □

Theorem 3.3.5. *Let ϕ : S(M) $\to$ S(M) $\cup$ $\{\emptyset\}$ be function. Let N be a ϕ-2-absorbing submodule of M. If (N : M)N $\nsubseteq$ $\phi(N)$, then N is a 2-absorbing submodule of M.*

Proof. Let r,s $\in$ R and m $\in$ M be such that rsm $\in$ N. If rsm $\notin \phi$(N) and since N is ϕ -2-absorbing then we have rs $\in (N :_R M)$ or rm $\in$ N or sm $\in$ N. So let rsm $\in \phi(N)$. In this case we may assume that rsN $\subseteq \phi(N)$. Because if rsN $\nsubseteq \phi(N)$, then there exists p $\in$ N such that rsp $\notin \phi(N)$, so that rs(m + p) $\in$ N - $\phi(N)$. Therefore rs $\in (N : M)$ or r(m + p) $\in$ N or s(m +p) $\in$ N and hence rs $\in (N : M)$ or rm $\in$ N or sm $\in$ N. Second we may assume that $(N : M)m \in \phi(N)$. If this is not the case, there exists u $\in (N : M)$ such that um $\notin \phi(N)$ and so (r + u)sm $\in N - \phi(N)$. Since N is a ϕ -2-absorbing submodule, we have (r + u)s $\in (N : M)$ or (r + u)m $\in$ N or sm $\in$ N. Thus rs $\in (N : M)$ or rm $\in$ N or sm $\in$ N. Now since $(N :_R M)$N $\nsubseteq \phi(N)$, there exist v $\in (N : M)$ and p $\in$ N such that vp $\notin \phi(N)$. So (r + v)s(m + p) $\in$ N - $\phi(N)$, and hence (r +v)s $\in (N : M)$ or (r + v)(m + p) $\in$ N or s(m + p) $\in$ N. Therefore rs $\in (N : M)$ or rm $\in$ N or sm $\in$ N. Thus N is 2-absorbing submodule. □

Corollary 3.3.6. *Let N be a weakly 2-absorbing submodule of M such that $(N :_R M)N$ $\neq$ 0. Then N is a 2-absorbing submodule of M.*

Proof. In the above Theorem set $\phi = \phi_0$. □

Remark 3.3.7. Suppose that N is a ϕ-2-absorbing submodule of M such that $\phi(N) \subseteq$ (N : M)N and N is not 2-absorbing submodule then by Theorem 4.3.5, we have $\phi(N)$ = (N : M)N. In particular if N is weakly 2-absorbing submodule but not 2-absorbing then (N : M)N = 0.

Theorem 3.3.8. [25] *Let $R = R_1 \times R_2$ such that each R_i is a commutative ring with identity. Let M_i be R_i-module $\forall$ i $\in$ {1,2} and $M = M_1 \times M_2$ be an R-module with $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$, where $r_i \in R_i$, $m_i \in M_i$ $\forall$ i $\in$ {1,2}. Then we have:*

(1) If N_1 is a 2-absorbing submodule of M_1, then $N_1 \times M_2$ is a 2-absorbing submodule of M.

(2) If N_2 is a 2-absorbing submodule of M_2, then $M_1 \times N_2$ is a 2-absorbing submodule of M.

Proof. Because the proof of (1) and (2) are similar, So we only prove (1). Hence suppose that N_1 is a 2-absorbing submodule of M_1 and let r_1,$s_1 \in R_1$, r_2, $s_2 \in R_2$, $m_1 \in M_1$ and $m_2 \in M_2$ such that $(r_1, r_2)(s_1, s_2)(m_1, m_2) = (r_1s_1m_1, r_2s_2m_2) \in N_1 \times M_2$. then $r_1s_1m_1 \in N_1$. Since N_1 is 2-absorbing submodule of M_1, So $r_1s_1 \in (N_1 : M_1)$ or $r_1m_1 \in N_1$ or $s_1m_1 \in N_1$. So $(r_1, r_2)(s_1, s_2) = (r_1s_1,r_2s_2) \in (N_1 : M_1) \times (M_2 : M_2) = (N_1 \times M_2 : M_1 \times M_2)$ or $(r_1, r_{2)}(m_1, m_2) \in N_1 \times M_2$ or $(s_1, s_{2)}(m_1, m_2) \in N_1 \times M_2$. Hence $N_1 \times M_2$ is 2-absorbing submodule of M. □

Example 3.3.9. *The above theorem is not true for correspondence ϕ - 2-absorbing submodules in general, for example if N_1 is a ϕ_0-2-absorbing submodule of M_1 then $N_1 \times M_2$ is not necessarily a ϕ_0 - 2-absorbing submodule of $M_1 \times M_2$. Let $R_1 = R_2 = M_1 = M_2 = \mathbb{Z}_8$ and suppose that $N_1 = \{0\}$ then evidently N_1 is a ϕ_0-2-absorbing submodule of M_1. However, $0 \neq (2,1)(2,1)(2,1) \in N_1 \times M_2$ and $(2, 1)(2,1) = (4,1) \notin (N_1 \times M_2 : M_1 \times M_2)$ and $(2,1)(2,1) \notin N_1 \times M_2$. Thus $N_1 \times M_2$ is not ϕ_0-absorbing submodule of M.*

Proposition 3.3.10. *Let R_1 and R_2 be two commutative rings, M_1 and M_2 be R_1 and R_2 - modules respectively. Let $M = M_1 \times M_2$ and define $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function. Suppose that N_1 is a weakly 2-absorbing submodule of M_1 such that $\{0\} \times M_2 \subseteq \phi(N_1 \times M_2)$. Then $N_1 \times M_2$ is a ϕ-2-absorbing submodule of $M_1 \times M_2$.*

Proof. Let $r_1, s_1 \in R_1$, $r_2, s_2 \in R_2$, $x_1 \in M_1$ and $x_2 \in M_2$. Let $(r_1, r_2)(s_1, s_2)(x_1, x_2) = (r_1s_1x_1, r_2s_2x_2) \in N_1 \times M_2$ - $\phi(N_1 \times M_2)$. Since $N_1 \times M_2$ - $\phi(N_1 \times M_2) \subseteq N_1 \times M_2$ - $\{0\} \times M_2 = (N_1 - \{0\}) \times M_2$, so we have $r_1s_1x_1 \in N_1 - \{0\}$ and by the assumption on N_1 we have $r_1s_1 \in (N_1 :_{R1} M_1)$ or $r_1x_1 \in N_1$ or $s_1x_1 \in N_1$. If $r_1s_1 \in (N_1 :_{R_1} M_1)$ then $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2) \in (N_1 :_{R_1} M_1) \times R_2 = (N_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)$. If r_1x_2

$\in N_1$ then $(r_1, r_2)(x_1, x_2) = (r_1x_1, r_2x_2) \in N_1 \times M_2$. If $s_1x_1 \in N_1$ then $(s_1, s_2)(x_{1,2}) = (s_1x_1, s_2x_2) \in N_1 \times M_2$. Therefore $N_1 \times M_2$ is ϕ-2-absorbing submodule of M. □

In the next theorem we give characterizations of ϕ -2-absorbing submodules.

Theorem 3.3.11. *Let N be a proper submodule of M and let ϕ : S(M) $\to$ S(M) $\cup$ $\{\emptyset\}$ be a function. Then the following are equivalent:*

(i) N is a ϕ-2-absorbing submodule of M;

(ii) for any r,s $\in$ R, with rs $\notin$ (N : M), we have $(N : rs) = (N : r) \bigcup (N : s) \bigcup (\phi(N) : rs)$;

(iii) for any r,s $\in$ R, with rs $\notin$ (N : M), we have, $(N : rs) = (N : r)$ or $(N : rs) = (\phi(N) : s)$ or (N : rs) = (ϕ(N) : rs).

Proof. (i)$\Longrightarrow$(ii). Let m $\in$ (N : rs) then rsm $\in$ N. If rsm $\notin$ ϕ(N) then N is a ϕ-2-absorbing submodule of M implies rm $\in$ N or sm $\in$ N, that is m $\in$ (N : r) or m $\in$ (N : s). If rsm $\in$ ϕ(N) then m $\in$ (ϕ(N) : rs). As we may assume that ϕ(N) $\subseteq$ N, the other inclusion always hold.

(ii) $\longrightarrow$ (iii) If an ideal is the union of two ideals, it is equal to one of them.

(iii)$\Longrightarrow$(i)Let rsm $\in$ N - ϕ(N) with rs $\notin$ (N : M) then m $\in$ (N : rs) and m $\notin$ (ϕ(N) :rs), so m $\in$ (N : r) or m $\in$ (N : s) that is, rm $\in$ N or sm $\in$ N. □

Theorem 3.3.12. *Let M be an R-module and let ϕ : S(M) $\to$ S(M) $\cup$ $\{\emptyset\}$ be a function. Let N be a ϕ-2-absorbing submodule of M.*

(i) If $L \subseteq N$ is a submodule of M, then N/L is a ϕ_L-2-absorbing submodule of M/L, where ϕ_L is defined as in Definition 2.5.14.

(ii) Suppose that S is a multiplicatively closed subset of R such that $S^{-1}N \neq S^{-1}M$ and $S^{-1}(\phi(N)) \subseteq (S^{-1}\phi)(S^{-1}N)$ with $(N :_R M) \cap S = \emptyset$. Let $S^{-1}\phi : S(S^{-1}M) \longrightarrow S(S^{-1}M) \bigcup \{\emptyset\}$. Then $S^{-1}N$ is an $(S^{-1}\phi)$-2-absorbing submodule of $S^{-1}M$.

Proof. (i) Let r,s $\in$ R and $\bar{x} \in$ M/L with rs$\bar{x} \in$ N/L - $\phi_L(N/L)$, where $\bar{x} =$ x + L, for some x $\in$ M. By the definition of ϕ_L, this gives that rsx $\in$ N - $(\phi(N) + L)$. So we have rsx $\in$ N - $\phi(N)$, which gives that rs $\in (N : M)$ or rx$\in$ N or sx $\in$ N. Therefore rs $\in (N/L : M/L)$ or $r\bar{x} \in$ N/L or $s\bar{x} \in$ N/L and so N/L is ϕ_L - 2-absorbing submodule.

(ii) Let a/s , b/w $\in S^{-1}R$ and x/t $\in S^{-1}M$ with abx/swt $\in S^{-1}N$ - $(S^{-1}\phi)(S^{-1}N)$. Then by our assumption abx/swt $\in S^{-1}N$ - $S^{-1}(\phi(N))$. Therefore there exists u $\in$ S such that uabx $\in$ N - $\phi(N)$, (note that for each v $\in$ S, vabx $\notin \phi(N)$). Since N is ϕ-2-absorbing submodule and $(N : M) \bigcap S = \emptyset$, we have uab $\in$ (N : M) or ax $\in$ N or bx $\in$ N. Therefore ab/sw $\in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$ or ax/st $\in S^{-1}N$ or bx/wt $\in S^{-1}N$. Hence $S^{-1}N$ is an $(S^{-1}\phi)$-2-absorbing submodule of $S^{-1}M$. □

Proposition 3.3.13. *Let $R = R_1 \times R_2 \times ... \times R_n$ and $M = M_1\times M_2\times ... \times M_n$ be an R - module, where R_i is a commutative ring and M_i is an R_i - module, for each $i \in \{1, 2, ..., n\}$. Let $N = N_1\times N_2\times ...\times N_n$ be a ϕ - 2-absorbing submodule of M, where N_i is a submodule of M_i and let $\psi_i : S(M_i) \longrightarrow S(M_i)\cup \{\emptyset\}$ $\forall$ $i \in \{1,2,...,n\}$ and $\phi(N) = \psi_1(N_1) \times \psi_2(N_2) \times ... \times\psi_n(N_n)$. Then N_j is a ψ_j -2-absorbing submodule of M_j, for each j with $N_j \neq M_j$.*

Proof. Let $x_j \in M_j$ and $a_j, b_j \in R_j$ such that $a_jb_j\, x_j \in N_j$ - $\psi_j(N_j)$. Thus (1,1,...,1,a_j,...,1). (1,1,...,1,b_j,...,1).(0,0,...,0,x_j,...,0)= (0,0,...,0,$a_jb_jx_j$,...,0)$\in N$ - $\phi(N)$, but N is ϕ-2-absorbing submodule. Therefore, (1,1,...,1,a_j ,...,1).(1,1,...,1,b_j,...,1) $\in$ (N : M) or (1,1,...,1,a_j,...,1). (0,0,...,0,x_j,0,...,0) $\in$ N or (1,1,...,1,b_j,...,1)(0,0,...,0,x_j,0,...,0) $\in$ N. So we have $a_jb_j \in (N_j : M_j)$ or $a_jx_j \in N_j$ or $b_jx_j \in N_j$. Thus N_j is ψ_j-2-absorbing submodule for each j. □

Corollary 3.3.14. *Let $R = R_1 \times R_2 \times ... \times R_n$ and $M = M_1 \times M_2 \times ... \times M_n$ an R-module and $N = N_1 \times N_2 \times ... \times N_n$, where R_i is a commutative ring and M_i is an R_i - module and N_i is a submodule of M_i, for $i \in \{1, 2, ..., n\}$. Let N be a ϕ_n -2-absorbing* *submodule of M. Then N_j is a ϕ_n -2-absorbing submodule of M_j, for each j with $N_j \neq M_j$ and $n \geq 2$.*

Proof. We have $\phi_n(N)$ = (N:M)$^{n-1}$N = $(N_1$:M$)^{n-1}N_1 \times (N_2$:M$)^{n-1}N_2 \times$ ... $\times(N_n$:M$)^{n-1}N_n$ = $\phi_n(N_1) \times \phi_n(N_2) \times ... \times \phi_n(N_n)$. So the result follows by Proposition 3.3.13. □

3.4 ϕ - Primal Submodules

The concept of ϕ - primal ideals in a commutative ring was introduced by A.Y. Darani in [18]. Let R be a commutative ring with identity. Let $\phi : \mathbb{J}(R) \rightarrow \mathbb{J}(R) \cup \{\emptyset\}$ be a function where $\mathbb{J}$(R) denotes the set of all ideals of R. Let I be an ideal of R. An element a ∈ R is called ϕ - prime to I if ra ∈ I - $\phi(I)$ (with r ∈ R) implies that r ∈ I. We denote by S_ϕ(I) the set of all elements of R that are not ϕ - prime to I. I is called a ϕ - primal ideal of R if the set P = S_ϕ(I) $\bigcup$ ϕ(I) forms an ideal of R. In this case P is called the ϕ - prime adjoint ideal (simply adjoint ideal) of I, and I is called a P-ϕ-primal ideal of R. Now we generalize the concept of ϕ -primal ideals to ϕ - primal submodules. Let M be R-module, let S(M) be the set of all submodule of M and ϕ : S(M) → S(M) ∪ {∅} be a function.

Definition 3.4.1. Let N be a submodule of R-module M and ϕ : S(M) → S(M) ∪ {∅} be a function. An element r ∈ R is called *ϕ-prime to N* if rm ∈ N - $\phi(N)$ (with m ∈ M) implies that m ∈ N. Otherwise r is not ϕ-prime to N.

Remarks 3.4.2. Let N be a submodule of R-module M. Denote by $S_\emptyset$(N) the set of all elements of R that are not ϕ- prime to N, then

(1) If an element of R is prime to N then it is ϕ - prime to N, so $S_\emptyset$(N) ⊆ $adj(N)$ = S(N).

(2) The converse of (1) is not necessarily true in general. For example assume that ϕ =

ϕ_0 where ϕ_0(N) = 0 for every submodule N then by Remark 2.3.12 we have $adj(N) \nsubseteq S_{\emptyset}$(N).

Definition 3.4.3. Let R be a commutative ring and let ϕ : S(M) $\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function. A proper submodule N of M is said to be a *ϕ-primal* if the set P = S_ϕ(N) $\bigcup$ ϕ(N) forms an ideal of R.

Definition 3.4.4. Let R be a commutative ring. Let M be an R-module. Let S(M) be the set of all submodules of M. Define the following type of the functions ϕ_α : S(M) $\longrightarrow$ S(M) $\cup$ $\{\emptyset\}$ and the corresponding ϕ_α - primal submodules as follows :

If $\phi_\emptyset(N) = \emptyset$, $\forall$ N $\in$ S(M), then N is *$\phi_\emptyset$-primal submodule* iff N is primal submodule.

If $\phi_0(N) = \{\ 0\ \}$, $\forall$ N $\in$ S(M), then N is *ϕ_0-primal submodule* iff N is weakly primal submodule.

If $\phi_1(N)$ = N, $\forall$ N $\in$ $S(M)$, then for any submodule N, N is *ϕ_1-primal submodule.*

If $\phi_2(N) = (N : M)N$, $\forall$ N $\in$ S(M), then N is *ϕ_2-primal submodule* iff N is almost primal submodule.

If $\phi_3(N) = (N : M)^2N$, $\forall$ N $\in$ S(M), then we say that N is *ϕ_3-primal submodule.*

Now, if $\phi_w(N) = \bigcap_{i=1}^{\infty}(N : M)^iN$, $\forall$ N $\in$ S(M), then we say that N is *ϕ_w-primal submodule.*

and if $\phi_n(N) = (N : M)^{n-1}N$, $\forall$ N $\in$ S(M), then we say that N is *ϕ_n-primal submodule.*

Theorem 3.4.5. *Let P be an ideal of a commutative ring R. Let N be a proper submodule of R-module M. Let ϕ : S(M) $\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function. Then the following are equivalent:*

(1) N is P-ϕ-primal submodule.

(2) For every $x \notin$ P - ϕ(N), $(N : x) = N \bigcup (\phi(N) : x)$ and for $x \in$ P - ϕ(N),

$(N : x) \supsetneqq N \bigcup (\phi(N) : x)$.

(3) For every $x \notin$ P - ϕ(N), $(N : x) = N$ or $(N : x) = (\phi(N) : x)$ and for

$x \in P$ - $\phi(N)$, $(N : x) \supsetneqq N$ and $(N : x) \supsetneqq (\phi(N) : x)$.

Proof. (1 → 2) Assume that N is P-ϕ-primal submodule then P - ϕ(N) consists entirely of elements of R that are not ϕ - prime to N. Let x $\notin$ P - ϕ(N) then x is ϕ-prime to N. Clearly N $\cup$ $(\phi(N) : x)$ $\subseteq$ (N : x). On the other hand, for every m $\in$ (N : x), if mx $\in$ $\phi(N)$ then m $\in$ $(\phi(N) : x)$ and if mx $\notin$ $\phi(N)$ then x is ϕ - prime to N gives m $\in$ N. Hence m $\in$ N $\bigcup$ $(\phi(N) : x)$, that is (N : x) $\subseteq$ N $\bigcup$ $(\phi(N) : x)$. Therefor (N : x) = N $\bigcup$ $(\phi(N) : x)$. Now assume that x $\in$ P - $\phi(N)$ then x is not ϕ - prime to N, so $\exists$ m $\in$ M - N such that mx $\in$ N-ϕ(N). Hence m $\in$ (N : x) - (N $\bigcup$ ($\phi(N)$: x)). Thus (N : x) $\supsetneqq$ N $\bigcup$ $(\phi(N) : x)$
(2 → 3) It is clear because (N : x) is an ideal in R.
(3 → 1) We want to prove that P-$\phi(N)$ consists exactly of all elements of R that are not ϕ-prime to N. Hence N is P-ϕ-primal.
Let x $\notin$ P-$\phi(N)$, then (N : x) = N $\cup$ ($\phi(N)$: x). We want to prove that x $\notin$ S_ϕ(N). Let xm $\in$ N - $\phi(N)$ with m $\in$ M. So, m $\in$ (N : x). By assumption, either (N : x) = N or (N : x) = ($\phi(N)$:x). As xm $\in$ N-$\phi(N)$, so m $\notin$ ($\phi(N)$: x). Thus, m $\in$ N and hence, x $\notin$ S_ϕ(N). Conversely, let x $\in$ P - $\phi(N)$, then (N :x) $\supsetneqq$ N $\bigcup$ ($\phi(N)$: x), so, $\exists$ m $\in$ (N : x) such that m $\notin$ (N $\bigcup$ ($\phi(N)$: x)). Therefore, m $\notin$ N and m $\notin$ ($\phi(N)$: x). Thus xm $\in$ N-$\phi(N)$ with m $\notin$ N, so x is not ϕ-prime to N and hence x $\in$ S_ϕ(N). Hence N is P-ϕ-primal submodule. □

Proposition 3.4.6. *Let R be a commutative ring and M be R-module. Let ϕ : S(M) → S(M) $\cup$ $\{\emptyset\}$ be a function. If N is ϕ-primal submodule of M then $P = S_\emptyset(N) \bigcup \phi(N)$ is ϕ-prime ideal of R.*

Proof. Suppose that r,s $\notin$ P, we show that either rs $\in$ $\phi(P)$ or rs $\notin$ P. Assume that rs $\notin$ $\phi(P)$. Let rsm $\in$ N - $\phi(N)$ for some m $\in$ M. Then, by Theorem 3.4.5 gives that rm $\in$ (N : s)= N $\bigcup$ ($\phi(N)$: s) where rm $\notin$ ($\phi(N)$: s); hence rm $\in$ N which implies that rm $\in$ N

-$\phi(N)$. Thus m $\in$ (N : r) = N $\bigcup$ ($\phi(N)$: r), and so m $\in$ N. Therefore, rs is ϕ-prime to N and rs $\notin$ P as required. $\square$

Notation 3.4.7. Let N be a ϕ-primal submodule of R-module M. By Proposition 3.4.6, P = $S_{\emptyset}(N) \bigcup \phi(N)$ is ϕ-prime ideal of R. In this case P is called the ϕ-prime adjoint ideal and N is called a *P-ϕ-primal submodule* of M.

The concepts " primal submodule" and " ϕ-primal submodule " are different. In fact, neither implies the other. We will show this by the following examples, in Example 3.4.8 below we give a primal submodule that is not ϕ - primal. An example of ϕ- primal submodule which is not primal is given in Example 3.4.9.

Example 3.4.8. [18],[11] *Consider the submodule $N = 8\mathbb{Z}/24\mathbb{Z}$ of $\mathbb{Z}$-module $M = \mathbb{Z}/24\mathbb{Z}$. Denote each coset $a + 24\mathbb{Z}$ in M by $\overline{a}$. Let $\phi = \phi_0$ (weakly primal).*

(1) since $0 \neq 2.\overline{4} \in N$ and $0 \neq 4.\overline{2} \in N$ with $\overline{2}$, $\overline{4} \in$ M-N we have 2,4 $\in S_{\phi_0}(N)$. If $6.\overline{a} \in N$ for some $\overline{a} \in M$ then 4 divides a and hence $6.\overline{a} = 0$. This shows that 2+4 = 6 is ϕ_0-prime to N so $6 \notin S_{\phi_0}(N)$. Therefor $S_{\phi}(N) \bigcup \phi(N)$ is not an ideal of $\mathbb{Z}$, that is N is not ϕ-primal submodule of M.

(2) Now set $P = 2\mathbb{Z}/24\mathbb{Z}$ then every element of P is not prime to N. Assume that $\overline{a} \notin P$, if $\overline{a}.\overline{n} \in N$ for some $\overline{n} \in M$ then 8 divides n, that is $\overline{n} \in N$. Hence $\overline{a}$ is prime to N so $\overline{a} \notin S(N) = adj(N)$. So we have S(N) = P, that is N is P-primal submodule. This example show that a primal submodule need not necessarily be ϕ-primal.

Example 3.4.9. [11] *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_6$ and denote every integer a modulo 6 by $\overline{a}$. Consider the submodule $N = \{0\}$ of M and let $\phi = \phi_0$ then:*

(1) 0 is weakly prime to N so $S_{\phi_0}(N) = \emptyset$. Thus N is weakly primal submodule of M.

(2) Since $2.\overline{3} = \overline{0} \in N$ and $3.\overline{2} = \overline{0} \in N$, so 2,3 $\in S_{\emptyset}(N)$ while 3-2 = 1 is prime to N, so

we have 1 $\notin$ S(N). Therefore N is not a primal submodule of M.

This example shows that ϕ -primal submodule need not necessarily be primal.

Theorem 3.4.10. *Let M be R-module and ϕ : S(M) $\rightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function. Let N and L be submodules of M with $L \subseteq \phi(N)$ then N is a ϕ-primal submodule of M iff N/L is a ϕ_L-primal submodule of M/L, where ϕ_L defined as in Definition 2.5.14.*

Proof. Assume N is P-ϕ-primal submodule. Suppose that a + L is an element of M/L that is not ϕ_L - prime to N/L, so there exists b $\in$ M - N with (a + L)(b + L) $\in$ N/L - ϕ_L(N/L). In this case ab $\in$ N - ϕ(N) with b $\in$ M - N implies that a is not ϕ - prime to N. Hence a $\in$ S_ϕ(N) $\subseteq$ P and so a + L $\in$ P/L. Now assume that c + L $\in$ P/L then c $\in$ P = S_ϕ(N) $\cup$ ϕ(N). If c $\in$ ϕ(N) then c + L $\in$ ϕ_L(N/L), so assume that c $\in$ S_ϕ(N), that is c is not ϕ - prime to N then cd $\in$ N - ϕ(N) for some d $\in$ M - N. Consequently, (c + L)(d + L) $\in$ N/L -(ϕ(N)/L) = N/L - ϕ_L(N/L) with d + L $\in$ M/L - N/L. This implies that c + L is not ϕ_L - prime to N/L, so c + L $\in$ S_{ϕ_L}(N/L). We have already shown that P/L = S_{ϕ_L}(N/L) $\bigcup$ ϕ_L(N/L). Therefore N/L is ϕ_L - primal.

Conversely, suppose that N/L is ϕ_L - primal in M/L with the adjoint ideal P/L. For every a $\in$ P - ϕ(N) we have a + L $\in$ P/L - ϕ_L(N/L) = S_{ϕ_L}(N/L), so a + L is not ϕ_L-prime to N/L, thus (a + L)(b + L) $\in$ N/L - ϕ_L(N/L) for some b + L $\in$ M/L - N/L. In this case b $\in$ M - N and ab $\in$ N - ϕ(N) implies that a is not ϕ-prime to N. On the other hand, assume that c $\in$ R is not ϕ-prime to N then cd $\in$ N - ϕ(N) for some d $\in$ M - N so we have (c + L)(d + L) $\in$ N/L - ϕ_L(N/L) with d + L $\notin$ N/L, that is c + L is not ϕ_L-prime to N/L. Hence c + L $\in$ P/L - ϕ_L(N/L), so we have c $\in$ P - ϕ(N). It follows that P = S_ϕ(N) $\bigcup$ ϕ(N) which implies that N is P-ϕ-primal submodule of M. □

Remark 3.4.11. [11] Let R be a commutative ring, M an R-module and S a multiplicatively closed set in R. If K is a submodule of S^{-1}M, define K $\cap$ M = v^{-1}(K) = {m $\in$ M : m/1

$\in$ K}, where v : M $\to$ S^{-1}M is the natural mapping m $\mapsto$ m/1. Clearly, K $\cap$ M is a submodule of M.

Proposition 3.4.12. *Let R be a commutative ring and S a multiplicatively closed subset of R. Let ϕ : S(M) $\longrightarrow$ S(M) $\cup$ $\{\emptyset\}$ be a function. Let N be a P-ϕ-primal submodule of an R-module M with P $\bigcap$ S = $\emptyset$.*

(1) Let λ = a/s $\in$ $S^{-1}N$ - $S^{-1}(\phi(N))$ (with a $\in$ M, s $\in$ S), where $S^{-1}N$ and $S^{-1}(\phi(N))$ defined in Definition 2.5.12, then a $\in$ N.

(2) If $S^{-1}(\phi(N)) \neq S^{-1}N$ then N = $S^{-1}N \bigcap M$.

Proof. (1) Assume that λ = a/s $\in$ $S^{-1}N$ - $S^{-1}(\phi(N))$ then a/s = b/t for some b $\in$ N, t $\in$ S. In this case, since us $\in$ S and b $\in$ N, then uta = usb $\in$ N for some u $\in$ S. If uta $\in$ $\phi(N)$ then a/s = uta/uts $\in$ $S^{-1}(\phi(N))$ which is a contradiction, so we have uta $\in$ N - $\phi(N)$. If a $\notin$ N then ut is not ϕ-prime to N, so ut $\in$ P $\bigcap$ S which contradicts the hypothesis. Therefore a $\in$ N.

(2) Let m $\in$ S^{-1}N $\bigcap$ M then m/1 $\in$ $S^{-1}N$, so $\exists$ s $\in$ S such that sm $\in$ N. If sm $\notin$ ϕ(N) and m $\notin$ N then s is not ϕ-prime to N, so s $\in$ P $\bigcap$ S, which a contradiction. Thus m $\in$ N. If sm $\in$ ϕ(N) then m/1 = sm/s $\in$ $S^{-1}(\phi(\mathrm{N}))$ which implies that m $\in$ $S^{-1}(\phi(\mathrm{N})) \bigcap$ M. Therefore (S^{-1}N $\bigcap$ M) = N $\bigcup$ (S^{-1} (ϕ(N) $\bigcap$ M), so (S^{-1}N $\bigcap$ M) = N or (S^{-1} N $\bigcap$ M) = (($S^{-1}\phi$(N)) $\bigcap$ M). But S^{-1} N $\neq$ $S^{-1}(\phi(\mathrm{N}))$, so S^{-1} N $\bigcap$ M $\neq$ S^{-1} (ϕ(N)) $\bigcap$ M. Thus S^{-1}N $\bigcap$ M = N. $\square$

Chapter 4

Classes of compactly packed modules

Let R be a commutative ring with identity and let M be a unitary R - module. A proper submodule N of M is compactly packed if for each family $\{P_\alpha\}_{\alpha\in\triangle}$ of prime submodules of M with N $\subseteq \bigcup_{\alpha\in\triangle} P_\alpha$, $N \subseteq P_\beta$ for some $\beta \in \triangle$. A module M is called compactly packed (CP) if every proper submodule of M is compactly packed (CP). This concept was introduced by Al-Ani (see [1]), and generalized to primary compactly packed module by El-Atrash and Ashour (see [27]). Also it generalized to primal compactly packed module by Al-Ashker, Ashour and Abu Mallouh (see [2]). Let S(M) be the set of all submodules of M and ϕ :S(M)$\rightarrow$ S(M)$\bigcup$ $\{\emptyset\}$ be a function.

In this Chapter, we introduce the concept of ϕ-compactly packed modules and 2-absorbing compactly packed modules.

4.1 ϕ-Compactly Packed Modules

In this Section, we introduce the concept of ϕ-compactly packed modules.

Definition 4.1.1. Let M be an R-module. Let ϕ : S(M) $\longrightarrow$ S(M) $\bigcup$ $\{\emptyset\}$ be a function.

A proper submodule N of M is *ϕ-compactly packed* (ϕ-CP) if for every family $\{P_\alpha\}_{\alpha\in\triangle}$ of ϕ-prime submodule of M with N $\subseteq \bigcup_{\alpha\in\triangle} P_\alpha$, N $\subseteq P_\beta$ for some $\beta \in \triangle$. A module M is called ϕ-CP if every proper submodule of M is ϕ-CP.

Definition 4.1.2. Let M be an R-module. Let ϕ : S(M) $\longrightarrow$ S(M) $\bigcup$ $\{\emptyset\}$ be a function. A proper submodule N of M is *ϕ-finitely compactly packed* (ϕ-FCP) if for every family $\{P_\alpha\}_{\alpha\in\triangle}$ of ϕ-prime submodule of M with N $\subseteq \bigcup_{\alpha\in\triangle} P_\alpha$, there exists α_1, α_2,..., $\alpha_n \in \triangle$ such that N $\subseteq \bigcup_{i=1}^{n} P_{\alpha_i}$. A module M is called ϕ-FCP if every proper submodule of M is ϕ-FCP.

Remarks 4.1.3. (1) Let M be R-module and let ϕ : S(M) $\rightarrow$ S(M) $\cup$ $\{\emptyset\}$. Directly from the definitions, every ϕ-CP module is ϕ-FCP, however the converse is not true. For example, let V be a vector space with dimension greater than 2 over the field F = $\mathbb{Z}/2\mathbb{Z}$. Then every submodule of V is prime, so every submodule of V is ϕ-prime. Let e_1,e_2 be distinct vectors of basis for V. Let V_1 = Fe_1, V_2 = Fe_2, V_3 = F$(e_1 + e_2)$ and L = V_1 + V_2, so L = $\{0, e_1, e_2, e_1 + e_2\}$. Thus V_1, V_2, V_3 are ϕ-prime submodule of V with property that L $\subseteq \bigcup_{i=1}^{3} V_i$ but L $\nsubseteq V_i$, $\forall$ i $\in \{1, 2, 3\}$. Thus L is ϕ-FCP, however L is not ϕ-CP.
(2) If M is an R-module that contains only a finite number of ϕ-prime submodule then M is ϕ-FCP.

Proposition 4.1.4. *Let R be a commutative ring with unity and let M be an R-module with unity. If every proper submodule of M is cyclic then M is ϕ-CP.*

Proof. Let N be a proper submodule of M and $\{N_\alpha\}_{\alpha\in\lambda}$ be a family of ϕ-prime submodule of M such that N $\subseteq \bigcup_{\alpha\in\lambda} N_\alpha$. Since N is cyclic, then N = Ra for some a $\in$ N. However, a $\in$ N $\subseteq \bigcup_{\alpha\in\lambda} N_\alpha$, therefore a $\in N_\beta$ for some $\beta \in \lambda$. Thus N = Ra $\subseteq N_\beta$ and hence M is ϕ-CP. □

Theorem 4.1.5. *If M is ϕ-FCP module which has at least one maximal submodule, then M satisfies the ACC on ϕ-prime submodules.*

Proof. Let $N_1 \subseteq N_2 \subseteq ...$ be an ascending chain of ϕ-prime submodules of M and let L $= \bigcup_{i=1}^{\infty} N_i$. We claim that L is a proper submodule of M. Assume on contrary that L = M and let H be a maximal submodule of M, then H $\subseteq \bigcup_{i=1}^{\infty} N_i$. Since M is ϕ-FCP, there exist n_1, n_2, ..., n_k such that H $\subseteq \bigcup_{j=1}^{k} N_{n_j} = N_m$ where m = $max\{n_1, n_2, ..., n_k\}$. Therefore H = N_m, hence N_m is maximal and $N_i = N_m$ for all i $\geq$ m. Thus $N_m = \bigcup_{i=1}^{\infty} N_i$ = M which contradicts N_m is ϕ-prime. Thus L is a proper submodule of M. Since M is ϕ-FCP, there exists m_1, m_2, ..., m_t such that L $\subseteq \bigcup_{j=1}^{t} N_{m_j} = N_s$ where s = max$\{m_1, m_2, ..., m_t\}$. Hence $N_i \subseteq N_s$ for all i, so $N_i = N_s$ $\forall$ i $\geq$ s. therefore the (ACC) is satisfied for ϕ-prime submodule. □

Since every finitely generated module and every multiplication module have a proper maximal submodule (see[13]), then we have the following corollary.

Corollary 4.1.6. *Let M be a ϕ-FCP R-module. If M is a finitely generated or a multiplication module, then M satisfies the ACC on ϕ-prime submodule.*

4.2 2-Absorbing Compactly Packed Modules

In this Section, we introduce the concept of 2-absorbing compactly packed module.

Definition 4.2.1. Let M be an R-module. A proper submodule N of M is called *2-absorbing compactly packed* (2-abs.CP) if for each family $\{P_\alpha\}_{\alpha \in \triangle}$ of 2-absorbing submodules of M with N $\subseteq \bigcup_{\alpha \in \triangle} P_\alpha$, N $\subseteq P_\beta$ for some $\beta \in \triangle$. A module M is called 2-abs.CP if every proper submodule of M is 2-abs.CP.

Definition 4.2.2. Let M be an R-module. A proper submodule N of M is called *2-absorbing finitely compactly packed* (2-abs.FCP) if for each family $\{P_\alpha\}_{\alpha\in\triangle}$ of 2-absorbing submodules of M with N $\subseteq \bigcup_{\alpha\in\triangle} P_\alpha$, there exists α_1, α_2, ..., α_n in $\triangle$ such that N $\subseteq \bigcup_{i=1}^{n} P\alpha_i$ A module M is called 2-abs.FCP if every proper submodule of M is 2-abs.FCP.

Proposition 4.2.3. *Every 2-abs.CP (resp. 2-abs.FCP) module is CP (resp. FCP)*

Proof. Since by Proposition 2.5.3, every prime submodule is 2-absorbing submodule, then the proof done. □

Remark 4.2.4. It is clear from the definitions that every 2-abs.CP module is 2-abs.FCP module but the converse is not true in general see Remark 4.1.3.

Proposition 4.2.5. *Let M be an R-module. If every submodule of M is cyclic then M is 2-abs.CP.*

Proof. Let N be a proper submodule of M and $\{N_\alpha\}_{\alpha\in\lambda}$ be a family of 2-absorbing submodules of M such that N $\subseteq \bigcup_{\alpha\in\lambda} N_\alpha$. Since N is cyclic, then N = Ra for some a $\in$ N. However, a $\in$ N $\subseteq \bigcup_{\alpha\in\lambda} N_\alpha$, therefore a $\in N_\beta$ for some $\beta \in \lambda$. Thus N = Ra $\subseteq N_\beta$ and hence M is 2-abs-CP. □

Theorem 4.2.6. *If M is a 2-abs.CP module which has at least one maximal submodule then M satisfies the (ACC) on 2-absorbing submodule.*

Proof. Let $N_1 \subseteq N_2 \subseteq ...$ be an ascending chain of 2-absorbing submodules of M. If N_k = M for some k, then the result follows immediately, so assume that none of N_k 's is M, and let L = $\bigcup_{i=1}^{\infty} N_i$. We claim that L is a proper submodule of M. Assume on contrary that L = M and let H be a maximal submodule of M, then H $\subseteq \bigcup_{i=1}^{\infty} N_i$. Since M is 2-abs-CP, there exist a positive integer m such that H $\subseteq N_m$. Therefore H = N_m, hence

N_m is maximal, and $N_i = N_m$ for all i $\geq$ m. Therefore $N_m = \bigcup_{i=1}^{\infty} N_i =$ M which is impossible, thus L is a proper submodule of M. Since M is 2-abs-CP, L $\subseteq N_j$ for some j and hence $N_i \subseteq N_j$ for all i, thus $N_i = N_j$ for all i $\geq$ j. Therefore M satisfies the (ACC) on 2-absorbing submodule. □

Theorem 4.2.7. *If M is a 2-abs.FCP module which has at least one maximal submodule then M satisfies the (ACC) on 2-absorbing submodule.*

Proof. The proof is similarly as Proposition 4.1.6. □

Since every finitely generated or multiplication R-module has a proper maximal submodule (see [13]), the following corollary holds.

Corollary 4.2.8. *Let M is (2-abs.FCP) R-module. If M is finitely generated or a multiplication R-module then M satisfies the (ACC) on 2-absorbing submodule.*

Definition 4.2.9. [3] A module M is called a *Bezout module* if every finitely generated submodule of M is cyclic.

Theorem 4.2.10. *Let M be a Bezout R-module. If M satisfies the (ACC) on submodules then M is 2-abs.CP.*

Proof. Let N be a proper submodule of M. By Theorem 1.2.21, N is finitely generated submodule and hence it is cyclic,because M is Bezout module. Thus by Proposition 4.2.5, M is 2-abs.CP. □

Proposition 4.2.11. *Let $f : M \longrightarrow M'$ be an epimorphism R-module. If M is 2-abs.CP then so is M'. The converse is true if any 2-absorbing submodule of M containing $ker(f)$.*

Proof. ($\Longrightarrow$) Let M is 2-abs.CP module and N' be a proper submodule of M' suppose that $N' \subseteq \bigcup_{\alpha \in \triangle} P_\alpha$, where P_α is a 2-absorbing submodule of M' for each $\alpha \in \triangle$.

Since f is an epimorphism R-module, $\phi^{-1}(N') \subseteq \phi^{-1}(\bigcup_{\alpha\in\triangle} P_\alpha)$. Thus $\phi^{-1}(N') \subseteq \bigcup_{\alpha\in\triangle}$ $\phi^{-1}(P_\alpha)$. Since P_α is 2-absorbing submodule for each $\alpha \in \triangle$, by Theorem 2.5.7 $\phi^{-1}(P_\alpha)$ is 2-absorbing submodule of M for each $\alpha \in \triangle$. But M is 2-abs.CP, thus there exists β $\in \triangle$ such that $\phi^{-1}(N') \subseteq \phi^{-1}(P_\beta)$. Therefore $N' \subseteq P_\beta$ for some $\beta \in \triangle$ and hence N' is 2-abs.CP. Thus M' is 2-abs.CP.

($\Longleftarrow$) Suppose that M' is 2-abs.CP and $ker(f)$ contained in any 2-absorbing submodule. Let N $\subseteq \bigcup_{\alpha\in\triangle} P_\alpha$ where N is a submodule of M and P_α is 2-absorbing submodule of M for each $\alpha \in \triangle$ so $\phi(N) \subseteq \phi(\bigcup_{\alpha\in\triangle} P_\alpha)$, thus $\phi(N) \subseteq \bigcup_{\alpha\in\triangle} \phi(P_\alpha)$. But $ker(f) \subseteq P_\alpha$ for each $\alpha \in \triangle$. Therefore by Theorem 4.2.7 $\phi(P_\alpha)$ is 2-absorbing submodule of M' for each $\alpha \in \triangle$. Since M' is 2-abs.CP module, $\phi(N) \subseteq \phi(P_\beta)$ for some $\beta \in \triangle$. Thus for any x $\in$ N, $\phi(x) \in \phi(N) \subseteq \phi(P_\beta)$, so $\exists$ b $\in P_\beta$ such that $\phi(x) = \phi(b)$, so we have $\phi(x-b) = 0$, that is x-b $\in ker(f) \subseteq P_\beta$, so we have x-b $\in P_\beta$. Since b $\in P_\beta$ we have x $\in P_\beta$. Therefore N $\subseteq P_\beta$ and hence N is 2-abs.CP. Thus M is 2-abs.CP. □

Proposition 4.2.12. *Let $f : M \longrightarrow M'$ be an epimorphism. If M is 2-abs.FCP then so is M'.*

Proof. The proof is similar to the Proposition 4.2.11. □

Definition 4.2.13. [16] Let M be an R-module, and let S be a multiplicatively closed subset of R. An *S-component of M* is denoted by M_s and defined as M_s = {a : a $\in$ R and as $\in$ M for some s $\in$ S }

Proposition 4.2.14. *Let M be an R-module and S a multiplicatively closed subset of R. If M is 2-abs.CP (resp. 2-abs.FCP) then so is M_s.*

Proof. Suppose that N $\subseteq \bigcup_{\alpha\in\lambda} W_\alpha$ where N is a proper submodule of M_s and W_α is 2-absorbing submodule of M_s for each $\alpha \in \lambda$. Define ϕ : M $\longrightarrow M_s$ as follow $\phi(m) = \frac{m}{1}$

for any m $\in$ M. Thus ϕ is an epimorphism, so ϕ^{-1}(N) $\subseteq \bigcup_{\alpha\in\lambda} \phi^{-1}(W_\alpha)$ for each $\alpha \in \lambda$. Since W_α is 2-absorbing submodule of M_s and ϕ is an epimorphism, we have $\phi^{-1}(W_\alpha)$ is 2-absorbing submodule of M for each $\alpha \in \lambda$, but M is 2-abs.CP (see Theorem 2.5.7), then $\phi^{-1}(N) \subseteq \phi^{-1}(W_\beta)$ for some $\beta \in \lambda$. Therefore $(\phi^{-1}(N))_s \subseteq (\phi^{-1}(W_\beta))_s$. Now we need only to prove that $(\phi^{-1}(T))_s =$ T for any submodule T of M_s. Let $\frac{x}{s} \in (\phi^{-1}(T))_s$ where x $\in \phi^{-1}$(T) and s $\in$ S then $\phi(x) \in$ T. Therefore $\frac{x}{1} \in$ T, hence $\frac{x}{s} = \frac{1}{s}.\frac{x}{1} \in$ T. Thus $(\phi^{-1}(T))_s \subseteq$ T .... (1).

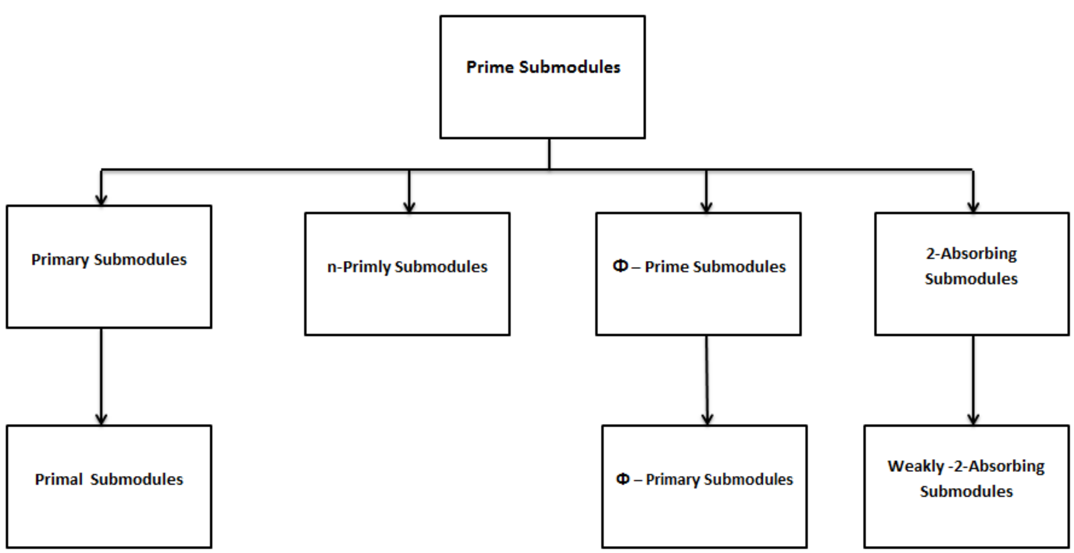
On the other hand, let $\frac{x}{s} \in$ T then $\frac{1}{s}.\frac{x}{1} \in$ T and hence $\frac{x}{1} \in$ T, that is $\phi(x) \in$ T so x $\in$ ϕ^{-1}(T) and $\frac{x}{s} \in (\phi^{-1}(T))_s$. Thus T $\subseteq (\phi^{-1}(T))_s$ ....(2).

thus from (1) and (2) we have $(\phi^{-1}(T))_s =$ T for any submodule T of M_s. Now since $(\phi^{-1}(N))_s \subseteq (\phi^{-1}(W_\beta))_s$ for some $\beta \in \lambda$, N $\subseteq W_\beta$ for some $\beta \in \lambda$. Thus N is 2-abs.CP and hence M_s is 2-abs.CP. □

Conclusion

In this thesis, we introduced the concept of n-primly submodules and we generalized the concept of ϕ-prime submodules to ϕ-primary submodules and ϕ-2-absorbing submodules. Also we introduced the concept of ϕ-primal submodules, ϕ-compactly packed modules and 2-absorbing compactly packed modules. For future study, we recommend to study the ϕ-classes of submodules in non commutative rings. At the end of thesis, we conclude the relations between submodules.

Summary of Relation between Submodules



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