إقـــرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

ON STABILITY AND SUPERSTABILITY OF SOME GENERALIZED CAUCHY FUNCTIONAL EQUATIONS

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DECLARATION

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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ON STABILITY AND SUPERSTABILITY OF SOME GENERALIZED CAUCHY FUNCTIONAL EQUATIONS

Master Thesis

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بناءً على موافقة شئون البحث العلمي والدراسات العليا بالجامعة الإسلامية بغزة على تشكيل لجنة الحكم على أطروحة الباحثة/ **ريهام جمال توفيق أبوغلوه** لنيل درجة الماجستير في **كلية *العلوم*** قسم **الرياضيات** وموضوعها:

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On stability and Superstability of some Generalized Cauchy Functional Equation

وبعد المناقشة العلنية التي تمت اليوم الأربعاء 21 ربيع الأول 1435هـ، الموافق 2014/01/22م الساعة العاشرة والنصف صباحاً بمبنى اللحيدان، اجتمعت لجنة الحكم على الأطروحة والمكونة من:

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وبعد المداولة أوصت اللجنة بمنح الباحثة درجة الماجستير في **كلية *العلوم*/** قسم **الرياضيات**.

واللجنة إذ تمنحها هذه الدرجة فإنها توصيها بتقوى الله ولزوم طاعته وأن تسخر علمها في خدمة دينها ووطنها.

والله ولي التوفيق ،،،

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Index of Symbols

$\mathbb{N}$	The set of natural numbers
$\mathbb{Q}^+$	The set of positive rational numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}^+$	The set of positive real numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\|a\|$	The absolute value of a
$\\|x\\|$	The norm of x
$d(x, y)$	The metric of x and y
A	Additive equation
E	Exponential equation
L	Logarithmic equation
M	Multiplicative equation

Abstract

In this thesis, we study some importants facts on Cauchy functional equations and we study the stability and superstability of some generalized Cauchy functional equations. We study the Hyers-Ulam, Th. M. Rassias and Gavruta Stability of a generalized Cauchy linear functional equation such that the Cauchy difference of a generalized Cauchy linear functional equation is bounded or unbounded.

After this, we study the Hyers-Ulam Stability of some Cauchy functional equations such that the Cauchy difference of a Cauchy functional equations is bounded. Also we study the superstability of some generalized Cauchy functional equations such that the Cauchy difference of a generalized Cauchy functional equations is bounded.

Contents

Introduction

Functional equations form a modern branch of mathematics. Functional equations are equations in which the unknowns are functions .There are three fundamental subjects in the study of functional equations [1]:

(1) Finding regular (particular) solutions.

(2) Finding general solutions.

(3) Stability problems.

We say that a function or a set of functions is **a particular solution** [2] of a functional equation or system if, and only if, it satisfies the functional equation or system in its domain of definition. Given a class of functions f, the **general solution** [2] of a functional equation or system is the totality of particular solutions in that class.
The field of functional equations includes differntial equations, difference equations and integral equations. Functional equation appeared in the literature around the same time as the modern theory of function. In (1747) and (1750), d'Alembert puplished three papers that were the first on functional equations. Functional equations were studied by d'Alembert (1747), Euler (1768), Poiasson(1804), Cauchy (1821), Abel (1823), Darboux (1875) and many others [1]. In (1902) [1] Hilbert suggested in connection with his 5th problem, that, while the theory of differential equations provides elegant and powerful techniques for solving functional equations. The differentiability assumptions are not inherently required. Motivated by Hilberrt's suggestions many researchers in

functional equations have treated various functional equations without any (or with mild) regularity assumotions. This effort has given rise to the modern theory of functional equations. The theory of functional equations forms a modern mathematical discipline, which has developed very ripidly in the last six decades. **To solve** a functional equation means to find all functions that satisfy the functional equation. In orded to obtain a solution, the functions must often be resticted to a specific nature. Many functional equations originated from applications. At present, problems in science and engineering are generally modeled by Cauchy functional equations. **The following functional equations, are referred to as Cauchy functional equations:** [3]

$$f(x+y) = f(x) + f(y), \qquad (Additive). \tag{1}$$

$$f(x+y) = f(x)f(y), \qquad (Exponential). \tag{2}$$

$$f(xy) = f(x) + f(y), \qquad (Logarithmic). \tag{3}$$

$$f(xy) = f(x)f(y), \qquad (Multiplicative). \tag{4}$$

The **stability problems** [3] is very important issue for various functional equations.We summarize the concept of stability problems as follows : when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of this functional equation ? If the problem accepts a solution, we say that the functional equation is stable. In this thesis we discuss in details some **stability** problems of some generalized Cauchy functional equations. Also we discuss in details some **superstability** problems of some generalized Cauchy functional equations. The **superstability** concept summed up in that definition [4]: Let f be a function from a Banach space to a Banach space and let $E_1(f)$ be the left side of the given functional equation and $E_2(f)$ the right side. We define the superstability of the given functional equation in the case that

$$\|E_1(f) - E_2(f)\| \leq \varepsilon$$

for some $\varepsilon > 0$ implies that either f is bounded or $E_1(f) = E_2(f)$.
This thesis is organized as follows.
Chapter one consists of two sections. The first one is called Preliminaries

where we will introduce some concepts that are nessessary for understanding this thesis and we will study some important facts on Cauchy functional equations, this includes the general solution of a Cauchy functional equations and some related definitions. The second section is called Independence Among Various Versions of The Cauchy Functional Equation where we will study a special case of a common solution between two of Cauchy functional equations.
Chapter two consists of three sections. In the first one we will study the Hyers-Ulam stability of the generalized Cauchy linear functional equation

$$f(x+y+z+a) = f(x) + f(y) + f(z) \tag{5}$$

from an abelian group X to a Banach space Y. In the second section we will study Th M-Rassias stability of (5) from a normed space X to a Banach space Y. In the third section we will study Gavruta stability of (5) from an abelian group X to a Banach space Y.
Chapter three consists of three sections. In the first one we will study the Hyers-Ulam Stability of $f(x+y) = f(x) + f(y)$ from a normed space E_1 to a Banach space E_2 and we will study Hyers-Ulam Stability of $f(xy) = f(x)+f(y)$ from a semigroup S to a Banach space Y. In the second section we will present the concept of the superstability and we will study the superstability of the generalized Cauchy functional equation $f(x+y) = f(x)g(y)+f(y)$ such that $g(y) \neq 1$ from a vector space V to a metric space W. In the third section we will study the superstability of the generalized Cauchy functional equation $f(xy) = f(x)g(y) + f(y)$ such that $g(y) \neq 1$ from a monoid S to a metric space W.

Chapter 1

On Cauchy functional equations

This chapter consists of two sections: In the first one we will introduce some concepts that are nessessary for understanding this thesis and we will study some important facts on Cauchy functional equations. In the second section we will study a special case of a common solution between two of Cauchy functional equations.

1.1 Preliminaries

In this section we give some facts about metric space, normed space, groups, semigroups, rings, homomorphism mappings and some facts on Cauchy functional equation.

Definition 1.1. [5] Let X be any nonempty set. A metric on X is a mapping $d : X \times X \longrightarrow \mathbb{R}$ which satisfies the following axioms: for all $x, y, z \in X$,
(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$. (Symmetry)

(iii) $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)
The pair (X, d) is called a metric space.

Notes. [5] For any $n \in \mathbb{N}$ and any $x_1, x_2, . \quad . \quad . x_n \in X$, we have

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + . \quad . \quad . + d(x_{n-1}, x_n).$$

Definition 1.2. [6] Let X be a vector space over the complex field $\mathbb{C}$ or(the real filed $\mathbb{R}$). If for any $x \in X$ there exists a real number $\|x\|$ satisfying the following four conditions, then $\|x\|$ is said to be the norm of x and X is called a normed space

(1) $\|x\| \geq 0$ for all $x \in X$.

(2) $\|x\| = 0$ if and only if $x = 0$.

(3) $\|x + y\| \leq \|x\| + \|y\|$ for all x and y in X. (Triangle inequality)

(4) $\|\lambda x\| = |\lambda| \, \|x\|$ for all x in X and all λ in the scalar field.

Notes. [6]

(i) A norm on X defines a metric d on X which is given by $d(x, y) = \|x - y\| \quad \forall x, y \in X$ and is called the metric induced by the norm.

(ii) For all $x, y \in X$, $|\|x\| - \|y\|| \leq \|x - y\|$.

Definition 1.3. [6] A sequence $\{x_n\}$ in a normed space X is said to be a Cauchy sequence if

$$\|x_n - x_m\| \longrightarrow 0 \quad \text{as } m \longrightarrow \infty \text{ and } n \longrightarrow \infty.$$

Definition 1.4. [6] A normed space X is said to be complete if every Cauchy sequence has a limit in X and a complete normed space is said to be Banach space.

Definition 1.5. [6] A function f defined on some set X with real or complex values is called bounded, if the set of its values is bounded. In other words, there exists a real number M such that $|f(x)| \leq M$ for all x in X.

Definition 1.6. [7] A semigroup is a set with a binary operation $(S, \cdot)$ such that

(i) $a \cdot b \in S$ for all $a, b \in S$.

(ii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all a, b and $c \in S$.

Notes. [7] If there is an element e such that $e \cdot a = a \cdot e = a$ for all a in S, then $(S, \cdot)$ is said to be a multiplication semigroup with an identity (e is said to be the identity element in S.)

A monoid is a semigroup with an identity element.

Definition 1.7. [7] A group is a monoid in which every element has an inverse element.

Definition 1.8. [8] A ring is a set R equipped with two binary operations $+$ and $\cdot$ called addition and multiplication, that map every pair of elements of R to a unique element of R. These operations must satisfy the following properties called ring axioms,which must be true for all a, b, c in R:

1. $(a + b) + c = a + (b + c)$.
2. There is an element 0 in R such that $0 + a = a$.
3. $a + b = b + a$.
4. For each a in R there exists $-a$ in R such that

$$a + (-a) = (-a) + a = 0.$$

5. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
6. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.
7. $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

Notes. [8]

A commutative ring is a ring in which the multiplication operation is commutative.

An entire ring is a commutative ring without zero divisors.

Definition 1.9. [8] A homomorphism ϕ from a group G to a group H is a mapping from G into H that preserves the group operation, that is,

$$\phi(ab) = \phi(a)\phi(b)$$

for all a,b in G.

The stability problem [3]

Very often instead of a functional equation, we consider a functional inequality and one can ask the following question: when can one assert that the solutions of the inequality lie near to the solutions of the equation? The stability of functional equations originated from the following fundamental question:when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of this functional equation ?If the problem accepts a solution, we say that the functional equation is **stable** and we called the the difference between the right side and the left side of equation which is approximately satisfies a functional equation is **Cauchy difference**. For example one of the stability problem had been formulated by S. M. Ulam, in(1940). Let G_1 be a group and let G_2 be a metric group with the metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if

$$f : G_1 \longrightarrow G_2$$

satisfies the inequality

$$d[f(xy), f(x)f(y)] < \delta$$

for all $x, y \in G_1$, then there exists a homomorphism

$$H : G_1 \longrightarrow G_2$$

with

$$d[f(x), H(x)] < \varepsilon$$

for all $x \in G_1$? If the answer is affirmative, we say that the functional equation for homomorphisms is stable.

These kinds of questions form the material of the stability theory. For Banach spaces, the above problem was solved by D.H. Hyers (1941) with $\delta = \varepsilon$ and

$$H(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

The stability problem for the functional equations was first raised by S. M. Ulam [9] in 1940. He discussed the number of unsolved problems related to the stability of functional equations. In the next year D. H. Hyer [10] gave the first affirmative answer of the Ulam problem for additive mapping $f(x + y) = f(x) + f(y)$ on a Banach space. A generalized version of the theorem of Hyer was given by Th. M. Rassias [11] in 1978 which allows Cauchy difference to be unbounded. The generalization given by Th. M. Rassias is called the Hyers- Ulam-Rassias stability.

Rassias result generated a lot of activities in the stability theory.

In 1979, J. Baker, J. Lawrence and F. Zorzitto [12] introduced that if f satisfies the inequality $\|E_1(f) - E_2(f)\| \leq \varepsilon$ then either f is bounded or $E_1(f) = E_2(f)$, where $E_1(f)$ is the left said of the given functional equation and $E_2(f)$ is the

right said of them, this concept is called the **superstability**.
In 1994, P. Gavruta [13] provided a further generalization of Th. M. Rassias theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by general function $\phi(x, y)$ for the existence of unique linear mapping.

Definition 1.10. [1] An additive function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called a linear (not dense) function if and only if it is of the form $f(x) = cx \quad (\forall\ x \in \mathbb{R}\ )$ where c is arbitrary constant.

Definition 1.11. [1] The graph of f, defined by $\{(x, f(x)), x \in \mathbb{R}\} \subset \mathbb{R}^2$, is dense in $\mathbb{R}^2$, that is, any point in $\mathbb{R}^2$ is the limit of a sequence of points in the graph of f.

Definition 1.12. [1] A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be rationally homogeneous if and only if $f(rx) = rf(x)$ for all $x \in \mathbb{R}$ and all rational numbers r.

Theorem 1.13. [1] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a solution of the additive Cauchy equation. Then f is rationally homogeneous. Moreover, f is linear on the set of rational numbers .

Theorem 1.14. [1] The graph of every non linear solution $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the additive Cauchy equation is everywhere dense in the palane $\mathbb{R}^2$.

Theorem 1.15. [1] If a real additive function f is either bounded from one side or monotonic, then it linear.

Theorem 1.16. [1] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function satisfying the additive Cauchy functional equation (1). Then f is linear, that is $f(x) = cx$ where c is an arbitrary constant.

Theorem 1.17. [1] The general solution of the exponential Cauchy equation (2):

$$f(x + y) = f(x)f(y)$$

is given by

$$f(x) = e^{A(x)} \quad \text{and } f(x) = 0 \quad \forall x \in \mathbb{R},$$

where $A : \mathbb{R} \longrightarrow \mathbb{R}$ is an additive function and e is Napierian base of logarithm.

Definition 1.18. [1] The general solution of the exponential Cauchy functional equation (2) is given by $f(z)$=$e^{A_1(z)+A_2(\overline{z})}$ and $f(z) = 0 \quad \forall z \in \mathbb{C}$, where A_1 and A_2 are additive functions, $\overline{z}$ denotes the complex conjugate of z.

Theorem 1.19. [1] The general solution of the multiplicative Cauchy equation (4):

$$f(xy) = f(x)f(y)$$

holds for all $x, y \in \mathbb{R}$, is given by

$$f(x) = 0,$$
$$f(x) = 1,$$
$$f(x) = e^{A(ln|x|)} \left| sgn(x) \right|,$$
$$f(x) = e^{A(ln|x|)} sgn(x).$$

Theorem 1.20. [1] Let $f : E_1 \longrightarrow E_2$ be a solution of the additive Cauchy equation such that E_1 and E_2 are two Banach spaces. Then f is rationally homogeneous.

Definition 1.21. [10] Let f be a function from an abelian group X to a Banach space Y. If f satisfies the functional equation

$$f(x + y + z + a) = f(x) + f(y) + f(z) \tag{1.1}$$

for all $x, y, z \in X$and a is an arbitrary number in X, then (1.1) is called a generalized Cauchy linear functional equation.

A particular case of the linear functional equation is

$$f(x + y + z) = f(x) + f(y) + f(z) \text{ at } \quad a = 0.$$

Theorem 1.22. [1] A function f from an abelian group X to a Banach space Y satisfies the functional equation (1.1) if, and only if, $f(x) = A(2x + a)$ for all $x \in X$, where A is an additive function.

Theorem 1.23. [1] If the functional equation (3):

$$f(xy) = f(x) + f(y)$$

holds for all $x, y \in \mathbb{R} - \{0\}$, then the general solution is given by $f(x) = A(ln\,|x|) \quad \forall x \in \mathbb{R} - \{0\}$.

Remark 1.24. If L is a logarithmic function from semi group S into a Banach space B then $L(x^n) = nL(x)$ for all $n \in \mathbb{N}$ and all $x \in S$.

Remark 1.25. For all $q \in \mathbb{Q}^+$, the equation (E) yields $f(q) = f(1)^q$.

Definition 1.26. [14] Let f and g be two functions from a vector space V to a metric space W . If f and g satisfy the functional equation

$$f(x+y) = f(x)g(y) + f(y), \quad g(y) \neq 1, \tag{1.2}$$

then this equation is called a generalized Cauchy functional equation.

Theorem 1.27. [14] The general solution of a generalized Cauchy functional equation (1.2) are*:*

$$\begin{cases} f = 0 \\ g : arbitrary; \end{cases} \begin{cases} f : constant \\ g = 0; \end{cases} \begin{cases} f(x) = \boldsymbol{A}(x) \\ g = 1; \end{cases} \begin{cases} f(x) = a(\boldsymbol{E}(x) - 1) \\ g(x) = \boldsymbol{E}(x); \end{cases}.$$

Definition 1.28. [14] Let f and g be two functions from a monoid S to a metric space W. If f and g satisfy the functional equation

$$f(xy) = f(x)g(y) + f(y), \quad g(y) \neq 1, \tag{1.3}$$

then this equation is called a generalized Cauchy functional equation.

Theorem 1.29. [14] The general solution of a generalized Cauchy functional equation (1.3) are*:*

$$\begin{cases} f = 0 \\ g : arbitrary; \end{cases} \begin{cases} f : constant \\ g = 0; \end{cases} \begin{cases} f(x) = \boldsymbol{L}(x) \\ g = 1; \end{cases} \begin{cases} f(x) = b(\boldsymbol{M}(x) - 1) \\ g(x) = \boldsymbol{M}(x); \end{cases}.$$

1.2 Independence Among Various Versions of The Cauchy Functional Equations

In this section we will study a special case of a common solution between two of Cauchy functional equations.

Lemma 1.30. [*15*] *Let X be a set, Y an entire ring and $f : X \longrightarrow Y$. Then $f = 0$ and $f = 2$ are the only solutions of the functional equation*

$$f(x) + f(y) = f(x)f(y). \tag{1.4}$$

Proof. Putting $y = x$ in (1.4), we obtain $2f(x) = f(x)^2$. Thus, for each $x \in X$, either $f(x) = 0$ or $f(x) = 2$. We proceed to show that either $f = 2$ or $f = 0$. Suppose that there exists x_0 such that $f(x_0) = 2$. Then in (1.4), $f(x) + 2 = 2f(x) \quad (x \in X)$, implying that $f = 2$. Now suppose that there exists y_0 such that $f(y_0) = 0$. Then in (1.4), $f(x) + 0 = f(x)(0) \quad (x \in X)$, implying that $f = 0$.

Lemma 1.31. [*15*] *Let Y be a set and $f : \mathbb{R}^+ \longrightarrow Y$. If f satisfies*

$$f(xy) = f(x + y), \tag{1.5}$$

then f is a constant function.

Proof. Putting $y = 1$ in (1.5), we get $f(x) = f(x + 1)$. Substituting $y + 1$ for y in (1.5), we obtain $f(xy + x) = f(x(y + 1)) = f(x + (y + 1)) = f(x + y + 1) = f(x + y) = f(xy)$.

Let z , w be distinct elements with $z > w$ and let $x = (z - w), \quad y = (z - w)^{-1}w$. Then, by the above equation,

$$\begin{aligned} f(z) = f(w + (z - w)) &= f((z - w)(z - w)^{-1}w + (z - w)) \\ &= f((z - w)(z - w)^{-1}w) = f(w). \end{aligned}$$

Therefore, f is constant.

Definition 1.32. [15] Let (α), (β) be two distinct equations taken from of (A), (M), (E) and (L). The pair (α, β) is ZI-independent over (X, Y) if the only common solution functions $f : X \longrightarrow Y$ to (α) and (β) are either the zero function or the identity function. In the case $X = Y$, we simply say $(\alpha, \beta)$is ZI-independent over X and ZI is abbreviation of Zero function and Identity function.

Theorem 1.33. [*15*] *The pairs* (A, E), (M, L) and (A, L) *are ZI-independent over* ($\mathbb{R}^+$,$\mathbb{C}$) *while* (M, E) *is not.*

Proof. In the pair (A, E), since by $(A), (E)$ and by equating the two equations, we have

$$f(x) + f(y) = f(x)f(y).$$

It thus follows from Lemma (1.30), that $f = 0$ or $f = 2$. By direct checking, $f = 2$ is not solution of (A),(E) because if $f = 2$, we have in (A) $2 = 2 + 2$

and this is a contradiction and if $f = 2$, we have in (E) $2 = (2)(2)$and this is a contradiction. Hence by Definition (1.32) the pair (A, E) are ZI-independent over $(\mathbb{R}^+, \mathbb{C})$. Similarly for the pair (M, L), since by $(M), (L)$ and by equating the two equations, we have

$$f(x) + f(y) = f(x)f(y).$$

It thus follows from Lemma (1.30), that $f = 0$ or $f = 2$. By direct checking, $f = 2$ is not solution of (L) because if $f = 2$, we have in (L) $2 = 2 + 2$ and this is a contradiction. Hence by Definition (1.32), the pair (M, L) are ZI-independent over $(\mathbb{R}^+$,$\mathbb{C})$. In the pair (A, L), since by $(A), (L)$ and by equating the two equations, we have

$$f(xy) = f(x + y).$$

It thus follows from Lemma (1.31), their solutions must be constant functions. So, we have $f(x) = c$, where c is constant $\forall x \in \mathbb{R}^+$. By $(A)$and (L), we have $c = c + c$ so, c must be only zero. Thus by Definition (1.32), the pair (A, L) is ZI-independent over$(\mathbb{R}^+$,$\mathbb{C})$. In the pair (M, E), since by (M),(E) and by equating the two equations, we have $f(xy) = f(x + y)$. It thus follows from Lemma (1.31), their solutions must be constant functions. So, we have $f(x) = c$, where c is constant $\forall x \in \mathbb{R}^+$. By (M) and (E) we have $c = c^2$ so, c must be only zero or one. Hence by Definition (1.32), the pair (M, E) are not ZI-independent over $(\mathbb{R}^+$,$\mathbb{C})$ because $f(x) = 1$ is a common solution function.

Theorem 1.34. [*15*] *The pair (E, L) is ZI-independent over $(\mathbb{R}^+$,$\mathbb{C})$.*

Proof. By Remark (1.25) the equation (E) yields $f(q) = f(1)^q$ for all $q \in \mathbb{Q}^+$. Now replacing x and y by 1 in (L), we obtain

$$f(1) = f(1) + f(1),$$

which implies $f(1) = 0$ and so, $f(q) = 0$ for all $q \in \mathbb{Q}^+$. Let φ be a positive irrational number. If $\varphi > 1$, then, by (E),

$$f(\varphi) = f(\varphi - 1 + 1) = f(\varphi - 1)f(1) = 0. \qquad (*)$$

If $\varphi < 1$, then, by (L),

$$0 = f(1) = f(\varphi \frac{1}{\varphi}) = f(\varphi) + f(\frac{1}{\varphi}).$$

So, we have

$$f(\varphi) = -f(\frac{1}{\varphi}).$$

Now since $\varphi < 1$ so $\frac{1}{\varphi} > 1$ and so, we have $f(\frac{1}{\varphi}) = 0$ by $(*)$. We deduce

$$f(\varphi) = -f(\frac{1}{\varphi}) = 0.$$

Therefore, f is the zero function which implies the ZI-independence of the pair (E, L).

Lemma 1.35. *[15] Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a solution of (A) and assume the image of f is not dense in $\mathbb{R}$.*

1. *If $f(1) = 1$, then f is the identity function.*
2. *If $f(1) = 0$, then f is the zero function.*

Proof. By Theorem (1.14), we deduce

$$f(x) = cx \quad (**)$$

for some constant c. If $f(1) = 1$, put $x = 1$ in $(**)$, we have, $1 = f(1) = c$, we obtain, $f(x) = x \ \forall x \in \mathbb{R}$, which is the identity function.
If $f(1) = 0$, put $x = 1$ in $(**)$, we have, $0 = f(1) = c$, we obtain, $f(x) = 0$ $\forall x \in \mathbb{R}$, which is the zero function.

Theorem 1.36. *[15] The pair of functional equations (A, M) is ZI-independent over ($\mathbb{R}^+$,K) where $\mathbb{C} \supset K = K_x + iK_y$ and either K_x or K_y is a non-dense subset of $\mathbb{R}$.*

Proof. Let $f : \mathbb{R}^+ \longrightarrow K$ be a function satisfying both (A) and (M). Putting $x = y = 1$ in (M), we have, $f(1) = f^2(1)$ and then $f(1) = 0$ or $f(1) = 1$. If $f(1) = 0$, put y=1 in (M), we have,

$$f(x) = f(x)f(1) = f(x)(0) = 0,$$

so by Definition (1.32), (A, M) is ZI-independent over ($\mathbb{R}^+$,K). If $f(1) = 1$, in this case, we express

$$f(x) = u(x) + iv(x),$$

where u and v are real valued functions on $\mathbb{R}^+$ and by Theorem (1.36) the image of u or the image of v can not be dense in $\mathbb{R}$. Now $f(1) = 1$ so,

$$1 = f(1) = u(1) + iv(1)$$

so, we have,

$$u(1) = 1, v(1) = 0. \tag{1.6}$$

To show that $u(x)$ and $v(x)$ are additive on $\mathbb{R}$, since

$u(x) = Re[f(x)] \;\; \forall \; x \in \; \mathbb{R}^+,$

$$\begin{aligned}
u(x+y) &= Re[f(x+y)] \;\; \forall \; x, y \in \; \mathbb{R}^+, \\
&= Re[f(x) + f(y)] \quad \text{because } f \text{ satisfying } (A) \\
&= Re[f(x)] + Re[f(x)] \\
&= u(x) + u(y).
\end{aligned}$$

We obtain that $u(x)$ is additive. By the same in the above we obtain $v(x)$ is additive. If the image of u is not dense, since u is additive and $u(1) = 1$ so, by Lemma (1.35), $u(x) = x$ for all x in $\mathbb{R}^+$ and since $f(x) = u(x) + iv(x)$ so, we obtain $f(x) = x + iv(x)$. Since f also satisfying (M), we have

$$\begin{aligned}
xy + iv(xy) &= (x + iv(x))(y + iv(y)) \\
&= xy - v(x)v(y) + i[xv(y) + yv(x)]
\end{aligned}$$

so, we have $v(x)v(y) = 0$ and so, $v(x) = 0$ for all $x \in \mathbb{R}^+$. Hence, $f(x) = x$, which implies that f is the identity function so, by Definition (1.32) the pair (A, M) is ZI-independent over $(\mathbb{R}^+, K)$. If the image of v is not dense in $\mathbb{R}$, since v is additive and $v(1) = 0$ so, by Lemma (1.35), we have v is zero function. So, we have $f(x) = u(x)$ for all $x \in \mathbb{R}^+$ so, f is a real valued function. By (M), for each $x \in \mathbb{R}^+$

$$f(x) = f(\sqrt{x}.\sqrt{x}) = f(\sqrt{x}).f(\sqrt{x}) = f(\sqrt{x})^2 \geq 0$$

and so, f is bounded from below and by Theorem (1.15), the image of f is not dense in $\mathbb{R}$. Now since f is not dense in $\mathbb{R}$ and additive and $f(1) = 1$ so, by Lemma (1.35) f is identity function. Therefore (A, M) is ZI-independent over $(\mathbb{R}^+, K)$ by Definition (1.32).

Chapter 2

The Stability of Generalized Cauchy Linear Functional Equation

This chapter consists of three sections:
In the first section we will prove the Hyers-Ulam stability of the generalized Cauchy linear functional equation (1.1) from an abelian group to a Banach space.
In the second section we will prove Th M-Rassias stability of the generalized Cauchy linear functional equation (1.1) from a normed space to a Banach space also we present corollary related to these results.
In the third section we will prove Gavruta stability of the generalized Cauchy linear functional equation (1.1) from an abelian group to a Banach space.

2.1 Hyers-Ulam Stability of Generalized Cauchy Linear Functional Equation

In this section we will prove the Hyers-Ulam stability of the generalized Cauchy linear functional equation $f(x+y+z+a) = f(x)+f(y)+f(z)$ from an abelian group to a Banach space such that the Cauchy difference of this function is bounded.

Definition 2.1. [10] Let f be a function from an abelian group to a Banach space and suppose that f satisfies the generalized Cauchy linear functional equation $f(x+y+z+a) = f(x)+f(y)+f(z)$ for all $x, y, z \in X$. We define the Hyers-Ulam Stability of this generalized Cauchy liner functional equation in the case that

$$\|f(x+y+z+a) - f(x) - f(y) - f(z)\| \leq \delta$$

for some $\delta > 0$, that is f which is the solution of the above inequality lies near to the solution of equation which satisfies the generalized Cauchy linear functional equation.

Theorem 2.2. *[10] Let $f : X \longrightarrow Y$ be a function from an abelian group X into a Banach space Y. If f satisfies the functional inequality*

$$\|f(x+y+z+a) - f(x) - f(y) - f(z)\| \leq \delta \tag{2.1}$$

for all $x, y, z \in X$ and for some $\delta > 0$, then the limit

$$C(x) = \lim_{n\longrightarrow\infty} \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right) \tag{2.2}$$

exists for all $x \in X$ and there exists a unique function $C{:}X \longrightarrow Y$ such that for all $x, y, z \in X$,

$$C(x+y+z+a) = C(x) + C(y) + C(z) \tag{2.3}$$

and

$$\|f(x) - C(x)\| \leq \frac{\delta}{2} \qquad \textit{for all } x \in X. \tag{2.4}$$

Proof. To establish this theorem we have to show that:
(i) $\{C_n(x)\} = \left\{\frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right)\right\}$ is a Cauchy sequence in Y.
(ii) $C(x) = \lim_{n\longrightarrow\infty} \frac{1}{3^n} f(3^n x + \frac{(3^n-1)a}{2})$ satisfies the generalized Cauchy linear functional equation:
$C(x+y+z+a) = C(x)+C(y)+C(z)$ for all $x, y, z \in X$.
(iii)Further $C(x)$ satisfies $\|f(x) - C(x)\| \leq \frac{\delta}{2}$ for all $x \in X$.
(iv) $C(x)$ is unique.
Now, letting $x = y = z$ in (2.1), we have

$$\|f(3x+a) - 3f(x)\| \leq \delta. \tag{2.5}$$

Replacing x by $3x+a$ in (2.5), we obtain

$$\|f(9x+4a)-3f(3x+a)\| \leq \delta. \tag{2.6}$$

Again in (2.6) replacing x with $3x+a$, we obtain

$$\|f(27x+13a)-3f(9x+4a)\| \leq \delta. \tag{2.7}$$

By (2.5)
$\delta \geq \|f(3x+a)-3f(x)\| = \left\|f(3x+\frac{3-1}{2}a)-3f(3^0x+\frac{(3^0-1)a}{2})\right\|$.
By (2.6)
$\delta \geq \|f(9x+4a)-3f(3x+a)\| = \left\|f(3^2x+\frac{3^2-1}{2}a)-3f(3x+\frac{(3-1)a}{2})\right\|$.
By (2.7)
$\delta \geq \|f(27x+13a)-3f(9x+4a)\| = \left\|f(3^3x+\frac{3^3-1}{2}a)-3f(3^2x+\frac{(3^2-1)a}{2})\right\|$.
By repeating the same previous steps k of times, we have

$$\left\|f\left(3^{k+1}x+\frac{(3^{k+1}-1)a}{2}\right)-3f\left(3^kx+\frac{(3^k-1)a}{2}\right)\right\| \leq \delta. \tag{2.8}$$

Now, by induction we will show that for every positive integer n,

$$\left\|f(x)-\frac{1}{3^n}f\left(3^nx+\frac{(3^n-1)a}{2}\right)\right\| \leq \frac{\delta}{2}\left(1-\frac{1}{3^n}\right) \tag{2.9}$$

For $n=1$, by (2.5)

$$\begin{aligned} \delta &\geq \|f(3x+a)-3f(x)\| \\ &= \|3f(x)-f(3x+a)\| \\ &= 3\left\|f(x)-\tfrac{1}{3}f(3x+\tfrac{(3-1)a}{2})\right\|. \end{aligned}$$

Then $\left\|f(x)-\frac{1}{3}f(3x+\frac{(3-1)a}{2})\right\| \leq \frac{\delta}{3} = \frac{\delta}{2}(1-\frac{1}{3})$.
Hence (2.9) is true for $n=1$.
Now suppose that (2.9) is true for $n=k$, then

$$\left\|f(x)-\frac{1}{3^k}f(3^kx+\frac{(3^k-1)a}{2})\right\| \leq \frac{\delta}{2}(1-\frac{1}{3^k}). \tag{2.10}$$

We will show that (2.9) is true for $n=k+1$, we have
$\left\|f(x)-\frac{1}{3^{k+1}}f\left(3^{k+1}x+\frac{(3^{k+1}-1)a}{2}\right)\right\|$

$= \left\| f(x) - \frac{1}{3^{k+1}} f\left(3^{k+1}x + \frac{(3^{k+1}-1)a}{2}\right) + \frac{1}{3^k} f\left(3^k x + \frac{(3^k-1)a}{2}\right) - \frac{1}{3^k} f\left(3^k x + \frac{(3^k-1)a}{2}\right) \right\|$

$\leq \frac{1}{3^{k+1}} \left\| f\left(3^{k+1}x + \frac{(3^{k+1}-1)a}{2}\right) - 3f\left(3^k x + \frac{(3^k-1)a}{2}\right) \right\| + \left\| \frac{1}{3^k} f\left(3^k x + \frac{(3^k-1)a}{2}\right) - f(x) \right\|$

$\leq \frac{1}{3^{k+1}}(\delta) + \frac{\delta}{2}\left(1 - \frac{1}{3^k}\right)$ by (2.8) and (2.10)

$= \frac{\delta}{2}\left(1 - \frac{1}{3^{k+1}}\right).$

Therefore, (2.9) is true for $n = k+1$ and hence, it is true for every $n \in \mathbb{N}$.

Now, we will prove (i).

For a positive integer n,

$$\begin{aligned} \|C_{n+1}(x) - C_n(x)\| &= \left\| \frac{1}{3^{n+1}} f\left(3^{n+1}x + \frac{(3^{n+1}-1)a}{2}\right) - \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right) \right\| \\ &= \frac{1}{3^{n+1}} \left\| 3f\left(3^n x + \frac{(3^n-1)a}{2}\right) - f\left(3^{n+1}x + \frac{(3^{n+1}-1)a}{2}\right) \right\| \\ &\leq \frac{\delta}{3^{n+1}} \quad \text{by (2.8).} \end{aligned}$$

Now for $n, m \in \mathbb{N}$ and $n > m$, we have

$$\begin{aligned} \|C_n(x) - C_m(x)\| &= ||C_n(x) - C_{n-1}(x) + C_{n-1}(x) - C_{n-2}(x) \\ &+ C_{n-2}(x) + . \quad . \quad . - C_{m+1}(x) + C_{m+1}(x) - C_m(x)|| \\ &\leq \|C_n(x) - C_{n-1}(x)\| + \|C_{n-1}(x) - C_{n-2}(x)\| \\ &+ . \quad . \quad . + \|C_{m+1}(x) - C_m(x)\| \\ &\leq \frac{\delta}{3^n} + \frac{\delta}{3^{n-1}} + \frac{\delta}{3^{n-2}} + . \quad . \quad . + \frac{\delta}{3^{m+2}} + \frac{\delta}{3^{m+1}} \\ &= \delta\left[\frac{1}{3^n} + \frac{1}{3^{n-1}} + \frac{1}{3^{n-2}} + . \quad . \quad . + \frac{1}{3^{m+2}} + \frac{1}{3^{m+1}}\right] \\ &= \delta \sum_{k=m}^{n-1} \frac{1}{3^{k+1}}. \end{aligned}$$

$(As\ m \longrightarrow \infty)$, then $\delta \sum_{k=m}^{n-1} \frac{1}{3^{k+1}} \longrightarrow 0$. Therefore, by Definition (1.3), $C_n(x)$ is a Cauchy sequence .

Since, Y is a Banach space, the limit of this sequence exists and it is in Y. Define a mapping $C : X \longrightarrow Y$ by

$$C(x) = \lim_{n \longrightarrow \infty} \frac{1}{3^n} f\left(3^n x + \frac{(3^n - 1)a}{2}\right). \tag{2.11}$$

Now we will prove (ii).

$$\begin{aligned}
\text{From (i), } C_n(x+y+z+a) &= \frac{1}{3^n} f\left(3^n x + 3^n y + 3^n z + 3^n a + \frac{(3^n - 1)a}{2}\right) \\
&= \frac{1}{3^n} f\left(3^n x + 3^n y + 3^n z + \frac{(3^{n+1} - 1)a}{2}\right) \\
&= \frac{1}{3^n} f\left(3^n x + 3^n y + 3^n z + a + \frac{(3^n - 1)a}{2} + \frac{(3^n - 1)a}{2} + \frac{(3^n - 1)a}{2}\right).
\end{aligned}$$

So, we have

$$\|C_n(x+y+z+a) - C_n(x) - C_n(y) - C_n(z)\|$$

$$= \left\| \frac{1}{3^n} f\left(3^n x + \frac{(3^n - 1)a}{2} + 3^n y + \frac{(3^n - 1)a}{2} + 3^n z + \frac{(3^n - 1)a}{2} + a\right) \right.$$
$$\left. - \frac{1}{3^n} f\left(3^n x + \frac{(3^n - 1)a}{2}\right) - \frac{1}{3^n} f\left(3^n y + \frac{(3^n - 1)a}{2}\right) - \frac{1}{3^n} f\left(3^n z + \frac{(3^n - 1)a}{2}\right) \right\|$$

$$\leq \frac{\delta}{3^n} \quad \text{by (2.1).}$$

By taking the limit (as $n \longrightarrow \infty$), to get
$C(x+y+z+a) = C(x) + C(y) + C(z)$ for all $x, y, z \in X$.
We will prove (iii).
By taking the limit (as $n \longrightarrow \infty$) of both sides of (2.9) and using (2.11), to get

$$\|f(x) - C(x)\| \leq \frac{\delta}{2}.$$

Now to prove (iv) we will prove the uniqueness of the mapping $C : X \longrightarrow Y$. Suppose that there exists another mapping $H : X \longrightarrow Y$, which is the solution of (2.3) and satisfies the inequality (2.4) . Then, we get

$$\|C(x) - H(x)\| = \left\| \tfrac{1}{3^n} C\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} H\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\|$$

$$= \left\| \tfrac{1}{3^n} C\left(3^n x + \tfrac{(3^n-1)a}{2}\right) + \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} H\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\|$$

$$\leq \left\| \tfrac{1}{3^n} C\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\| + \left\| \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} H\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\|$$

$$\leq \frac{\delta}{2.3^n} + \frac{\delta}{2.3^n} = \frac{\delta}{3^n} \quad \text{by (2.4).}$$

By taking the limit $(as\ n \longrightarrow \infty)$ of both sides, we get $\|C(x) - H(x)\| = 0$ for all $x \in X$, which implies that $C(x) = H(x)$ for all $x \in X$.
This completes the proof of the theorem.

2.2 Th.M.Rassias Stability of Generalized Cauchy Linear Functional Equation

In this section we will prove Th M-Rassias stability of the generalized Cauchy linear functional equation $f(x+y+z+a) = f(x)+f(y)+f(z)$ from a normed space to a Banach space such that the Cauchy difference of this function is unbounded.

Definition 2.3. [11] Let f be a function from a normed space X to a Banach space Y and suppose that f satisfies the generalized Cauchy linear functional equation $f(x + y + z + a) = f(x) + f(y) + f(z)$ for all $x, y, z \in X$. We define the Th.M.Rassias Stability of this generalized Cauchy Linear functional equation in the case that

$$\|f(x+y+z+a) - f(x) - f(y) - f(z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for some $\theta > 0$, $0 \leq p < 1$ and for all $x, y, z \in X$, that is f which is the solution of the above inequality lies near to the solution of equation which satisfies the generalized Cauchy linear functional equation.

Theorem 2.4. [*11*] *Let f be a function from a normed space X into a Banach space Y. If f satisfies the functional inequality*

$$\|f(x+y+z+a)-f(x)-f(y)-f(z)\| \leq \theta(\|x\|^p+\|y\|^p+\|z\|^p) \quad (2.12)$$

for some $\theta > 0$, $0 \leq p < 1$ and for all $x, y, z \in X$, then the limit

$$C(x) = \lim_{n \longrightarrow \infty} \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right) \quad (2.13)$$

exists for all $x \in X$ and there exists a unique function $C : X \longrightarrow Y$ such that

$$C(x+y+z+a) = C(x)+C(y)+C(z) \quad (2.14)$$

and

$$\|f(x)-C(x)\| \leq \theta \sum_{i=0}^{+\infty} 3^{i(p-1)} \left\| x + \frac{1}{2}(1-\frac{1}{3^i}) \right\|^p \quad (2.15)$$

for all $x, y, z \in X$.

Proof. To establish this theorem we have to show that:

(i) $\{C_n(x)\} = \left\{\frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right)\right\}$ is a Cauchy sequence in Y.
(ii) $C(x) = \lim_{n \longrightarrow \infty} \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right)$ satisfies the generalized Cauchy linear functional equation :

$$C(x+y+z+a) = C(x)+C(y)+C(z) \quad \text{for all } x, y, z \in X.$$

(iii) Further $C(x)$ satisfies $\|f(x)-C(x)\| \leq \theta \sum_{i=0}^{+\infty} 3^{i(p-1)} \left\| x + \frac{1}{2}(1-\frac{1}{3^i}) \right\|^p$ for all $x \in X$.
(iv) $C(x)$ is unique.

Let $x = y = z$ in (2.12), to get

$$\|f(3x+a)-3f(x)\| \leq 3\theta\|x\|^p. \quad (2.16)$$

Replacing x with $3x+a$ in (2.16), we get

$$\|f(9x+4a)-3f(3x+a)\| \leq 3\theta\|3x+a\|^p. \quad (2.17)$$

Again substituting x with $3x+a$ in (2.17), to get

$$\|f(27x+13a)-3f(9x+4a)\| \leq 3\theta\|9x+4a\|^p. \tag{2.18}$$

By (2.16)

$$\begin{aligned}3\theta\left\|3^0x+\tfrac{(3^0-1)a}{2}\right\|^p &= 3\theta\|x\|^p\\ &\geq \|f(3x+a)-3f(x)\|\\ &= \left\|f(3x+\tfrac{3-1}{2}a)-3f(3^0x+\tfrac{(3^0-1)a}{2})\right\|.\end{aligned}$$

By (2.17)

$$\begin{aligned}3\theta\left\|3^1x+\tfrac{(3-1)a}{2}\right\|^p &= 3\theta\|3x+a\|^p\\ &\geq \|f(9x+4a)-3f(3x+a)\|\\ &= \left\|f(3^2x+\tfrac{3^2-1}{2}a)-3f(3x+\tfrac{(3-1)a}{2})\right\|.\end{aligned}$$

By (2.18)

$$\begin{aligned}3\theta\left\|3^2x+\tfrac{(3^2-1)a}{2}\right\|^p &= 3\theta\|9x+4a\|^p\\ &\geq \|f(27x+13a)-3f(9x+4a)\|\\ &= \left\|f(3^3x+\tfrac{3^3-1}{2}a)-3f(3^2x+\tfrac{(3^2-1)a}{2})\right\|.\end{aligned}$$

By repeating the same previous steps k of times, we have

$$\left\|f\left(3^{k+1}x+\frac{(3^{k+1}-1)a}{2}\right)-3f\left(3^kx+\frac{(3^k-1)a}{2}\right)\right\| \leq 3\theta\left\|3^kx+\frac{(3^k-1)a}{2}\right\|^p. \tag{2.19}$$

Now, by induction we prove that for any positive integer n,

$$\left\|f(x)-\frac{1}{3^n}f\left(3^nx+\frac{(3^n-1)a}{2}\right)\right\| \leq \theta\sum_{i=0}^{n-1}\frac{1}{3^i}\left\|3^ix+\left(\frac{(3^i-1)a}{2}\right)\right\|^p. \tag{2.20}$$

From (2.16), we have

$$\begin{aligned}3\theta\|x\|^p &\geq \|3f(x)-f(3x+a)\|\\ &= 3\left\|f(x)-\tfrac{1}{3}f(3x+\tfrac{(3-1)a}{2})\right\|.\end{aligned}$$

So that (2.20) is true for $n=1$. Now suppose that (2.20) is true for $n=k$, then

$$\left\|f(x)-\frac{1}{3^k}f(3^kx+\frac{(3^k-1)a}{2})\right\| \leq \theta\sum_{i=0}^{k-1}\frac{1}{3^i}\left\|3^ix+\left(\frac{(3^i-1)a}{2}\right)\right\|^p. \tag{2.21}$$

We will show that (2.20) is true for $n = k + 1$.

Then, we have

$\left\| f(x) - \frac{1}{3^{k+1}} f\left(3^{k+1}x + \frac{(3^{k+1}-1)a}{2}\right) \right\|$

$= \left\| f(x) - \frac{1}{3^{k+1}} f\left(3^{k+1}x + \frac{(3^{k+1}-1)a}{2}\right) + \frac{1}{3^k} f\left(3^k x + \frac{(3^k-1)a}{2}\right) - \frac{1}{3^k} f\left(3^k x + \frac{(3^k-1)a}{2}\right) \right\|$

$\leq \frac{1}{3^{k+1}} \left\| f\left(3^{k+1}x + \frac{(3^{k+1}-1)a}{2}\right) - 3f\left(3^k x + \frac{(3^k-1)a}{2}\right) \right\| + \left\| \frac{1}{3^k} f\left(3^k x + \frac{(3^k-1)a}{2}\right) - f(x) \right\|$

$\leq \frac{\theta}{3^k} \left\| 3^k x + \frac{(3^k-1)a}{2} \right\|^p + \theta \sum_{i=0}^{k-1} \frac{1}{3^i} \left\| 3^i x + \frac{(3^i-1)a}{2} \right\|^p$ by (2.19) , (2.21)

$= \theta \sum_{i=0}^{k} \frac{1}{3^i} \left\| 3^i x + \frac{(3^i-1)a}{2} \right\|^p.$

This shows that the inequality (2.20) is true for $n = k + 1$.

Thus, it is true for any positive integer n.

Now, we will prove (i).

$$\begin{aligned} \|C_{n+1}(x) - C_n(x)\| &= \left\| \frac{1}{3^{n+1}} f\left(3^{n+1}x + \frac{(3^{n+1}-1)a}{2}\right) - \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right) \right\| \\ &= \frac{1}{3^{n+1}} \left\| 3f\left(3^n x + \frac{(3^n-1)a}{2}\right) - f\left(3^{n+1}x + \frac{(3^{n+1}-1)a}{2}\right) \right\| \\ &\leq \frac{\theta}{3^n} \left\| 3^n x + \frac{(3^n-1)a}{2} \right\|^p \quad \text{by (2.19)} \\ &= \theta 3^{n(p-1)} \left\| x + \tfrac{1}{2}\left(1 - \tfrac{1}{3^n}\right) a \right\|^p. \end{aligned}$$

For n , m $\in \mathbb{N}$ and $n > m$, we have

$$\begin{aligned} \|C_n(x) - C_m(x)\| &= ||C_n(x) - C_{n-1}(x) + C_{n-1}(x) - C_{n-2}(x) \\ &+ C_{n-2}(x) + . \quad . \quad . - C_{m+1}(x) + C_{m+1}(x) - C_m(x)|| \\ &\leq \|C_n(x) - C_{n-1}(x)\| + \|C_{n-1}(x) - C_{n-2}(x)\| \\ &+ . \quad . \quad . + \|C_{m+1}(x) - C_m(x)\| \\ &\leq \theta 3^{(n-1)(p-1)} \left\| x + \tfrac{1}{2}\left(1 - \tfrac{1}{3^{n-1}}\right) a \right\|^p \end{aligned}$$

$$
\begin{aligned}
&+ \quad \theta 3^{(n-2)(p-1)} \left\|x + \tfrac{1}{2}\left(1 - \tfrac{1}{3^{n-2}}\right)a\right\|^p \\
&+ \quad . \quad . \quad . + \theta 3^{m(p-1)} \left\|x + \tfrac{1}{2}\left(1 - \tfrac{1}{3^m}\right)a\right\|^p \\
&= \quad \theta \sum_{k=m}^{n-1} \left\|x + \frac{1}{2}\left(1 - \frac{1}{3^k}\right)a\right\|^p .
\end{aligned}
$$

(As $m \longrightarrow \infty$), since $p < 1$, $\frac{1}{3} < 1$, we have, $\theta \sum_{k=m}^{n-1} \left\|x + \frac{1}{2}\left(1 - \frac{1}{3^k}\right)a\right\|^p \longrightarrow 0$. Therefore, by Definition (1.3), $C_n(x)$ is a Cauchy sequence .
Since Y is a complete normed space the limit of sequence $C_n(x)$ exists and is in Y. So we can define a mapping $C : X \longrightarrow Y$ by

$$
C(x) = \lim_{n \longrightarrow \infty} \frac{1}{3^n} f\left(3^n x + \frac{(3^n - 1)a}{2}\right). \tag{2.22}
$$

We will prove (ii)

$$
\begin{aligned}
\text{From (i), } C_n(x+y+z+a) &= \frac{1}{3^n} f\left(3^n x + 3^n y + 3^n z + 3^n a + \frac{(3^n - 1)a}{2}\right) \\
&= \frac{1}{3^n} f\left(3^n x + 3^n y + 3^n z + \frac{(3^{n+1} - 1)a}{2}\right) \\
&= \frac{1}{3^n} f\left(3^n x + 3^n y + 3^n z + a + \frac{(3^n - 1)a}{2} + \frac{(3^n - 1)a}{2} + \frac{(3^n - 1)a}{2}\right).
\end{aligned}
$$

So, we have

$$
\begin{aligned}
&\|C_n(x+y+z+a) - C_n(x) - C_n(y) - C_n(z)\| \\
&= \left\| \frac{1}{3^n} f\left(3^n x + \frac{(3^n - 1)a}{2} + 3^n y + \frac{(3^n - 1)a}{2} + 3^n z + \frac{(3^n - 1)a}{2} + a\right) \right. \\
&\left. - \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} f\left(3^n y + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} f\left(3^n z + \tfrac{(3^n-1)a}{2}\right) \right\| \\
&\le \frac{\theta}{3^n}\left(\left\|3^n x + \frac{(3^n - 1)a}{2}\right\|^p + \left\|3^n y + \frac{(3^n - 1)a}{2}\right\|^p + \left\|3^n z + \frac{(3^n - 1)a}{2}\right\|^p \right) \quad \text{by (2.12)} \\
&= 3^{n(p-1)}\theta \left(\left\|x + \tfrac{1}{2}\left(1 - \tfrac{1}{3^n}\right)a\right\|^p + \left\|y + \tfrac{1}{2}\left(1 - \tfrac{1}{3^n}\right)a\right\|^p + \left\|z + \tfrac{1}{2}\left(1 - \tfrac{1}{3^n}\right)a\right\|^p \right).
\end{aligned}
$$

Since $p < 1$, then (as $n \longrightarrow \infty$), $3^{n(p-1)} \longrightarrow 0$.
Therefore, $\lim_{n\longrightarrow\infty} \|C_n(x+y+z+a) - C_n(x) - C_n(y) - C_n(z)\| = 0$, that means $C(x+y+z+a) = C(x) + C(y) + C(z)$ for all $x, y, z \in X$.
We will prove (iii)
By taking the limit (as $n \longrightarrow \infty$) of both sides of (2.20) and using (2.22), to get

$$\|f(x) - C(x)\| \leq \theta \sum_{i=0}^{+\infty} 3^{i(p-1)} \left\| x + \frac{1}{2}\left(1 - \frac{1}{3^i}\right) a \right\|^p.$$

Finally, we prove the uniqueness of the mapping $C : X \longrightarrow Y$.
Suppose that there exists another mapping $H : X \longrightarrow Y$,which is the solution of (2.14) and satisfies the inequality(2.15), we get

$$\|C(x) - H(x)\| = \left\| \tfrac{1}{3^n} C\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} H\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\|$$

$$= \left\| \tfrac{1}{3^n} C\left(3^n x + \tfrac{(3^n-1)a}{2}\right) + \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} H\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\|$$

$$\leq \left\| \tfrac{1}{3^n} C\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\| + \left\| \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} H\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\|$$

$$\leq \frac{\theta \sum_{i=0}^{+\infty} 3^{i(p-1)} \left\| 3^n x + \frac{(3^n-1)a}{2} + \frac{1}{2}(1 - \frac{1}{3^i}) \right\|^p}{3^n} + \frac{\theta \sum_{i=0}^{+\infty} 3^{i(p-1)} \left\| 3^n x + \frac{(3^n-1)a}{2} + \frac{1}{2}(1 - \frac{1}{3^i}) \right\|^p}{3^n} \text{ by (2.15)}$$

$$= 2\theta 3^{n(p-1)} \sum_{i=0}^{+\infty} 3^{i(p-1)} \left\| x + \frac{a}{2} - \frac{a}{23^n} + \frac{1}{23^n}\left(1 - \frac{1}{3^i}\right) \right\|^p.$$

Since $\frac{1}{3} < 1$, $p < 1$, by taking the limit (as $n \longrightarrow \infty$) of both sides, we get $\|C(x) - H(x)\| = 0$ for all $x \in X$, which implies that $C(x) = H(x)$ for all $x \in X$. This completes the proof of the theorem.

Theorem 2.5. [11] *Let f be a function from a normed space X into a Banach space Y.If f satisfies the functional inequality*

$$\|f(x+y+z+a) - f(x) - f(y) - f(z)\| \leq \theta(\|x\|^p \|y\|^q \|z\|^r) \qquad (2.23)$$

for some $\theta > 0$ and $p + q + r \in [0, 1)$ for all $x, y, z \in X$, then there exists a *unique mapping* $C : X \longrightarrow Y$ *such that*

$$C(x+y+z+a) = C(x) + C(y) + C(z) \tag{2.24}$$

and

$$\|f(x) - C(x)\| \leq \frac{\theta}{3}\sum_{i=0}^{+\infty} 3^{i(p+q+r-1)} \left\| x + \frac{1}{2}\left(1 - \frac{1}{3^i}\right)\right\|^{p+q+r} \quad for\ all\ x \in X. \tag{2.25}$$

Proof. The proof is similar to the proof of Theorem (2.4).
To establish this theorem we have to show that:

(i) $\{C_n(x)\} = \left\{\frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right)\right\}$ is a Cauchy sequence in Y.
(ii) $C(x) = \lim_{n\longrightarrow\infty} \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right)$ satisfies the generalized Cauchy linear functional equation :

$$C(x+y+z+a) = C(x) + C(y) + C(z) \quad \text{for all } x, y, z \in X.$$

(iii) Further $C(x)$ satisfies $\|f(x) - C(x)\| \leq \frac{\theta}{3}\sum_{i=0}^{+\infty} 3^{i(p+q+r-1)} \left\| x + \frac{1}{2}\left(1 - \frac{1}{3^i}\right)\right\|^{p+q+r}$ for all $x \in X$.
(iv) $C(x)$ is unique.

Let $x = y = z$ in (2.23) , to get

$$\|f(3x+a) - 3f(x)\| \leq \theta\|x\|^{p+q+r}. \tag{2.26}$$

Replacing x with $3x + a$ in (2.26), we get

$$\|f(9x+4a) - 3f(3x+a)\| \leq \theta\|3x+a\|^{p+q+r}. \tag{2.27}$$

Again substituting x with $3x + a$ in (2.27), to get

$$\|f(27x+13a) - 3f(9x+4a)\| \leq \theta\|9x+4a\|^{p+q+r}. \tag{2.28}$$

By (2.26)
$\theta\left\|3^0 x + \frac{(3^0-1)a}{2}\right\|^{p+q+r} = \theta\|x\|^{p+q+r}$

$$\geq \|f(3x+a)-3f(x)\|$$
$$= \left\| f(3x+\tfrac{3-1}{2}a)-3f(3^0x+\tfrac{(3^0-1)a}{2})\right\|.$$

By (2.27)

$$\theta\left\|3^1x+\tfrac{(3-1)a}{2}\right\|^{p+q+r} = \theta\|3x+a\|^{p+q+r}$$
$$\geq \|f(9x+4a)-3f(3x+a)\|$$
$$= \left\|f(3^2x+\tfrac{3^2-1}{2}a)-3f(3x+\tfrac{(3-1)a}{2})\right\|.$$

By (2.28)

$$\theta\left\|3^2x+\tfrac{(3^2-1)a}{2}\right\|^{p+q+r} = \theta\|9x+4a\|^{p+q+r}$$
$$\geq \|f(27x+13a)-3f(9x+4a)\|$$
$$= \left\|f(3^3x+\tfrac{3^3-1}{2}a)-3f(3^2x+\tfrac{(3^2-1)a}{2})\right\|.$$

By repeating the same previous steps k of times, we have

$$\left\|f\left(3^{k+1}x+\tfrac{(3^{k+1}-1)a}{2}\right)-3f\left(3^kx+\tfrac{(3^k-1)a}{2}\right)\right\| \leq \theta\left\|3^kx+\tfrac{(3^k-1)a}{2}\right\|^{p+q+r}. \quad (2.29)$$

Now, by induction we prove that for any positive integer n,

$$\left\|f(x)-\frac{1}{3^n}f\left(3^nx+\frac{(3^n-1)a}{2}\right)\right\| \leq \frac{\theta}{3}\sum_{i=0}^{n-1}\frac{1}{3^i}\left\|3^ix+\left(\frac{(3^i-1)a}{2}\right)\right\|^{p+q+r}. \quad (2.30)$$

From (2.26), we have

$$\theta\|x\|^{p+q+r} \geq \|3f(x)-f(3x+a)\|$$
$$= 3\left\|f(x)-\tfrac{1}{3}f(3x+\tfrac{(3-1)a}{2})\right\|.$$

So that (2.30) is true for $n=1$.

Now assume that (2.30) is true for $n=k$, then

$$\left\|f(x)-\frac{1}{3^k}f\left(3^kx+\frac{(3^k-1)a}{2}\right)\right\| \leq \frac{\theta}{3}\sum_{i=0}^{k-1}\frac{1}{3^i}\left\|3^ix+\left(\frac{(3^i-1)a}{2}\right)\right\|^{p+q+r}. \quad (2.31)$$

We will show that (2.30) is true for $n=k+1$, we have

$$\left\|f(x)-\tfrac{1}{3^{k+1}}f\left(3^{k+1}x+\tfrac{(3^{k+1}-1)a}{2}\right)\right\|$$

$$= \left\|f(x)-\tfrac{1}{3^{k+1}}f\left(3^{k+1}x+\tfrac{(3^{k+1}-1)a}{2}\right)+\tfrac{1}{3^k}f\left(3^kx+\tfrac{(3^k-1)a}{2}\right)-\tfrac{1}{3^k}f\left(3^kx+\tfrac{(3^k-1)a}{2}\right)\right\|$$

$$\leq \tfrac{1}{3^{k+1}}\left\|f\left(3^{k+1}x+\tfrac{(3^{k+1}-1)a}{2}\right)-3f\left(3^kx+\tfrac{(3^k-1)a}{2}\right)\right\|+\left\|\tfrac{1}{3^k}f\left(3^kx+\tfrac{(3^k-1)a}{2}\right)-f(x)\right\|$$

$\leq \frac{\theta}{3^{k+1}}\left\|3^k x+\frac{(3^k-1)a}{2}\right\|^{p+q+r}+\frac{\theta}{3}\sum_{i=0}^{k-1}\frac{1}{3^i}\left\|3^i x+\frac{(3^i-1)a}{2}\right\|^{p+q+r}$ by (2.29) and (2.31)

$= \frac{\theta}{3}\sum_{i=0}^{k}\frac{1}{3^i}\left\|3^i x+\frac{(3^i-1)a}{2}\right\|^{p+q+r}.$

This shows that the inequality (2.30) is true for $n = k+1$.

Thus, it is true for any positive integer n.

Now, we will prove (i).

$$\begin{aligned}
\|C_{n+1}(x)-C_n(x)\| &= \left\|\frac{1}{3^{n+1}}f\left(3^{n+1}x+\frac{(3^{n+1}-1)a}{2}\right)-\frac{1}{3^n}f\left(3^n x+\frac{(3^n-1)a}{2}\right)\right\| \\
&= \frac{1}{3^{n+1}}\left\|3f\left(3^n x+\frac{(3^n-1)a}{2}\right)-f\left(3^{n+1}x+\frac{(3^{n+1}-1)a}{2}\right)\right\| \\
&\leq \frac{\theta}{3^{n+1}}\left\|3^n x+\frac{(3^n-1)a}{2}\right\|^{p+q+r} \text{ by (2.29)} \\
&= \theta 3^{n(p+q+r-1)-1}\left\|x+\frac{1}{2}\left(1-\frac{1}{3^n}\right)a\right\|^{p+q+r}.
\end{aligned}$$

For $n, m \in \mathbb{N}$ and $n > m$, we have

$$\begin{aligned}
\|C_n(x)-C_m(x)\| &= ||C_n(x)-C_{n-1}(x)+C_{n-1}(x)-C_{n-2}(x) \\
&+ C_{n-2}(x)+. \quad . \quad .-C_{m+1}(x)+C_{m+1}(x)-C_m(x)|| \\
&\leq \|C_n(x)-C_{n-1}(x)\|+\|C_{n-1}(x)-C_{n-2}(x)\| \\
&+ . \quad . \quad .+\|C_{m+1}(x)-C_m(x)\| \\
&\leq \theta 3^{(n-1)(p+q+r-1)-1}\left\|x+\tfrac{1}{2}\left(1-\tfrac{1}{3^{n-1}}\right)a\right\|^{p+q+r} \\
&+ \theta 3^{(n-2)(p+q+r-1)-1}\left\|x+\tfrac{1}{2}\left(1-\tfrac{1}{3^{n-2}}\right)a\right\|^{p+q+r} \\
&+ . \quad . \quad .+\theta 3^{m(p+q+r-1)-1}\left\|x+\tfrac{1}{2}\left(1-\tfrac{1}{3^m}\right)a\right\|^{p+q+r} \\
&= \theta\sum_{k=m}^{n-1}3^{k(p+q+r-1)-1}\left\|x+\frac{1}{2}\left(1-\frac{1}{3^k}\right)a\right\|^{p+q+r}.
\end{aligned}$$

(As $m \longrightarrow \infty$), since $p+q+r<1$, $\frac{1}{3}<1$, then,

$$\theta\sum_{k=m}^{n-1}3^{k(p+q+r-1)-1}\left\|x+\frac{1}{2}\left(1-\frac{1}{3^k}\right)a\right\|^{p+q+r} \longrightarrow 0.$$

Therefore, by Definition (1.3), $C_n(x)$ is a Cauchy sequence.

Since Y is a complete normed space the limit of sequence $C_n(x)$ exists and is in Y. So we can define a mapping $C : X \longrightarrow Y$ by

$$C(x) = \lim_{n\longrightarrow\infty} \frac{1}{3^n} f\left(3^n x + \frac{(3^n - 1)a}{2}\right). \tag{2.32}$$

We will prove (ii).

$$\begin{aligned}
\text{From (i), } C_n(x+y+z+a) &= \frac{1}{3^n} f\left(3^n x + 3^n y + 3^n z + 3^n a + \frac{(3^n-1)a}{2}\right) \\
&= \frac{1}{3^n} f\left(3^n x + 3^n y + 3^n z + \frac{(3^{n+1}-1)a}{2}\right) \\
&= \frac{1}{3^n} f\left(3^n x + 3^n y + 3^n z + a + \frac{(3^n-1)a}{2} + \frac{(3^n-1)a}{2} + \frac{(3^n-1)a}{2}\right).
\end{aligned}$$

So, we have

$$\|C_n(x+y+z+a) - C_n(x) - C_n(y) - C_n(z)\|$$

$$= \Big\| \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2} + 3^n y + \tfrac{(3^n-1)a}{2} + 3^n z + \tfrac{(3^n-1)a}{2} + a\right)$$
$$- \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} f\left(3^n y + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} f\left(3^n z + \tfrac{(3^n-1)a}{2}\right)\Big\|$$

$$\leq \frac{\theta}{3^n}\left(\left\|3^n x + \frac{(3^n-1)a}{2}\right\|^p \left\|3^n y + \frac{(3^n-1)a}{2}\right\|^q \left\|3^n z + \frac{(3^n-1)a}{2}\right\|^r\right) \qquad \text{by (2.23)}$$

$$= 3^{n(p+q+r-1)}\theta\left(\left\|x + \tfrac{1}{2}\left(1 - \tfrac{1}{3^n}\right)a\right\|^p \left\|y + \tfrac{1}{2}\left(1 - \tfrac{1}{3^n}\right)a\right\|^q \left\|z + \tfrac{1}{2}\left(1 - \tfrac{1}{3^n}\right)a\right\|^r\right).$$

Since $p+q+r<1$, then (as $n \longrightarrow \infty$), $3^{n(p+q+r-1)} \longrightarrow 0$.
Therefore, $\lim_{n\longrightarrow\infty} \|C_n(x+y+z+a) - C_n(x) - C_n(y) - C_n(z)\| = 0$, that means $C(x+y+z+a) = C(x) + C(y) + C(z)$ for all $x, y, z \in X$.
Now we will prove (iii).
By taking the limit (as $n \longrightarrow \infty$) of both sides of (2.30) and using (2.32), to get

$$\|f(x) - C(x)\| \leq \frac{\theta}{3}\sum_{i=0}^{+\infty} 3^{i(p+q+r-1)} \left\|x + \frac{1}{2}\left(1 - \frac{1}{3^i}\right)\right\|^{p+q+r} \quad \text{for all } x \in X.$$

We will prove (iv) the uniqueness of the mapping $C : X \longrightarrow Y$.
Suppose that there exists another mapping $H : X \longrightarrow Y$, which is the solution of (2.24) and satisfies the inequality (2.25) .

$$\|C(x) - H(x)\| = \left\| \tfrac{1}{3^n} C\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} H\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\|$$

$$= \left\| \tfrac{1}{3^n} C\left(3^n x + \tfrac{(3^n-1)a}{2}\right) + \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} H\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\|$$

$$\leq \left\| \tfrac{1}{3^n} C\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\| + \left\| \tfrac{1}{3^n} f\left(3^n x + \tfrac{(3^n-1)a}{2}\right) - \tfrac{1}{3^n} H\left(3^n x + \tfrac{(3^n-1)a}{2}\right) \right\|$$

$$\leq \frac{\theta \sum_{i=0}^{+\infty} 3^{i(p+q+r-1)} \left\| 3^n x + \tfrac{(3^n-1)a}{2} + \tfrac{1}{2}(1 - \tfrac{1}{3^i}) \right\|^{p+q+r}}{3^{n+1}}$$

$$+ \frac{\theta \sum_{i=0}^{+\infty} 3^{i(p+q+r-1)} \left\| 3^n x + \tfrac{(3^n-1)a}{2} + \tfrac{1}{2}(1 - \tfrac{1}{3^i}) \right\|^{p+q+r}}{3^{n+1}} \quad \text{by } (2.25)$$

$$= 2\theta 3^{n(p+q+r-1)-1} \sum_{i=0}^{+\infty} 3^{i(p+q+r-1)} \left\| x + \frac{a}{2} - \frac{a}{23^n} + \frac{1}{23^n}\left(1 - \frac{1}{3^i}\right) \right\|^{p+q+r}.$$

Since $p + q + r < 1$, by taking the limit (as $n \longrightarrow \infty$) of both sides we get, $\|C(x) - H(x)\| = 0$ for all $x \in X$, which implies that $C(x) = H(x)$ for all $x \in X$. This completes the proof of the theorem.

Corollary 2.6. [*11*] *Let f be a mapping from a normed space X into a Banach space Y . If f satisfies the functional inequality*
$\|f(x+y+z+a) - f(x) - f(y) - f(z)\| \leq \theta(\|x\|^p \|y\|^p \|z\|^p)$ *for some* $\theta > 0$ *and* $3p \in [0, 1)$ *for all* $x, y, z \in X$, *then there exists a unique mapping*

$$C : X \longrightarrow Y$$

such that

$$C(x+y+z+a) = C(x) + C(y) + C(z)$$

and $\|f(x) - C(x)\| \leq \dfrac{\theta}{3} \sum_{i=0}^{+\infty} 3^{i(3p-1)} \left\| x + \dfrac{1}{2}\left(1 - \dfrac{1}{3^i}\right) \right\|^{3p}$ *for all* $x \in X$.

Proof. In Theorem (2.5), let $p = q = r$ we will get the result.

2.3 Gavruta Stability of Generalized Cauchy Linear Functional Equation

In this section we will prove Gavruta stability of the generalized Cauchy linear functional equation $f(x + y + z + a) = f(x) + f(y) + f(z)$ from an abelian group into a Banach space such that the Cauchy difference of this function is unbounded.

Definition 2.7. [13] Let f be a function from an abelian group X to a Banach space Y and suppose that f satisfies the generalized Cauchy linear functional equation $f(x+y+z+a) = f(x)+f(y)+f(z)$ for all $x, y, z \in X$. We define the Gavruta Stability of this generalized Cauchy linear functional equation in the case that

$$\|f(x+y+z+a) - f(x) - f(y) - f(z)\| \leq \Phi(x, y, z),$$

where Φ be a mapping from $X \times X \times X$ to $[0, +\infty)$ satisfying

$$\sum_{i=0}^{\infty} \frac{1}{3^{i+1}} \Phi\left(3^i x + \frac{(3^i - 1)a}{2}, 3^i y + \frac{(3^i - 1)a}{2}, 3^i z + \frac{(3^i - 1)a}{2}\right) < +\infty$$

for all $x, y, z \in X$, that is f is the solution of the above inequality lies near to the solution of equation which satisfies the generalized Cauchy linear functional equation.

Theorem 2.8. [*13*] *Let X be an abelian group and Y be a Banach space and let*

$\Phi : X \times X \times X \longrightarrow [0, +\infty)$ *be a mapping satisfying*

$$\sum_{i=0}^{\infty} \frac{1}{3^{i+1}} \Phi\left(3^i x + \frac{(3^i - 1)a}{2}, 3^i y + \frac{(3^i - 1)a}{2}, 3^i z + \frac{(3^i - 1)a}{2}\right) < +\infty$$

for all $x, y, z \in X$. If a function $f : X \longrightarrow Y$ is a solution of the functional inequality

$$\|f(x+y+z+a) - f(x) - f(y) - f(z)\| \leq \Phi(x, y, z) \tag{2.33}$$

for all $x, y, z \in X$, then there exists a unique mapping $C : X \longrightarrow Y$ such that

$$C(x+y+z+a) = C(x) + C(y) + C(z) \tag{2.34}$$

and

$$\|f(x) - C(x)\| \leq \sum_{i=0}^{+\infty} \frac{1}{3^{i+1}} \Phi\left(3^i x + \frac{(3^i-1)a}{2}, 3^i x + \frac{(3^i-1)a}{2}, 3^i x + \frac{(3^i-1)a}{2}\right) \tag{2.35}$$

for all $x \in X$.

Proof. To establish this theorem we have to show that:
(i) $\{C_n(x)\} = \left\{\frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right)\right\}$ is a Cauchy sequence in Y.
(ii) $C(x) = \lim\limits_{n \longrightarrow \infty} \dfrac{1}{3^n} f\left(3^n x + \dfrac{(3^n-1)a}{2}\right)$ satisfies the generalized Cauchy linear functional equation: $C(x+y+z+a) = C(x) + C(y) + C(z)$ for all $x, y, z \in X$.
(iii) $C(x)$ satisfies
$\|f(x) - C(x)\| \leq \sum\limits_{i=0}^{+\infty} \dfrac{1}{3^{i+1}} \Phi\left(3^i x + \dfrac{(3^i-1)a}{2}, 3^i x + \dfrac{(3^i-1)a}{2}, 3^i x + \dfrac{(3^i-1)a}{2}\right)$
for all $x \in X$.
(iv) $C(x)$ is unique.
Now, letting $x = y = z$ in (2.33), we have

$$\|f(3x+a) - 3f(x)\| \leq \Phi(x, x, x). \tag{2.36}$$

Replacing x by $3x + a$ in (2.36), we obtain

$$\|f(9x+4a) - 3f(3x+a)\| \leq \Phi(3x+a, 3x+a, 3x+a). \tag{2.37}$$

Again in (2.37) replacing x with $3x + a$, we obtain

$$\|f(27x+13a) - 3f(9x+4a)\| \leq \Phi(9x+4a, 9x+4a, 9x+4a). \tag{2.38}$$

By (2.36)

$$\begin{aligned} \Phi(3^0x + \tfrac{(3^0-1)a}{2}, 3^0x + \tfrac{(3^0-1)a}{2}, 3^0x + \tfrac{(3^0-1)a}{2}) &= \Phi(x, x, x) \\ &\geq \|f(3x+a) - 3f(x)\| \\ &= \left\|f(3x + \tfrac{3-1}{2}a) - 3f(3^0x + \tfrac{(3^0-1)a}{2})\right\|. \end{aligned}$$

By (2.37)

$$\begin{aligned}\Phi(3x+\tfrac{(3-1)a}{2},3x+\tfrac{(3-1)a}{2},3x+\tfrac{(3-1)a}{2}) &= \Phi(3x+a,3x+a,3x+a)\\ &\geq \|f(9x+4a)-3f(3x+a)\|\\ &= \left\|f(3^2x+\tfrac{3^2-1}{2}a)-3f(3x+\tfrac{(3-1)a}{2})\right\|.\end{aligned}$$

By (2.38)

$$\begin{aligned}\Phi(3^2x+\tfrac{(3^2-1)a}{2},3^2x+\tfrac{(3^2-1)a}{2},3^2x+\tfrac{(3^2-1)a}{2}) &= \Phi(9x+4a,9x+4a,9x+4a)\\ &\geq \|f(27x+13a)-3f(9x+4a)\|\\ &= \left\|f(3^3x+\tfrac{3^3-1}{2}a)-3f(3^2x+\tfrac{(3^2-1)a}{2})\right\|.\end{aligned}$$

By repeating the same previous steps k of times, we have

$$\begin{aligned}&\left\|f(3^{k+1}x+\frac{(3^{k+1}-1)a}{2})-3f(3^kx+\frac{(3^k-1)a}{2})\right\|\\ &\leq \Phi(3^kx+\frac{(3^k-1)a}{2},3^kx+\frac{(3^k-1)a}{2},3^kx+\frac{(3^k-1)a}{2}).\end{aligned} \tag{2.39}$$

Now, by induction we show that for any positive integer n,

$$\left\|f(x)-\frac{1}{3^n}f\left(3^nx+\frac{(3^n-1)a}{2}\right)\right\|$$

$$\leq \sum_{i=0}^{n-1}\frac{1}{3^{i+1}}\Phi\left(3^ix+\frac{(3^i-1)a}{2},3^ix+\frac{(3^i-1)a}{2},3^ix+\frac{(3^i-1)a}{2}\right). \tag{2.40}$$

From (2.36), we have

$$\begin{aligned}\Phi(x,x,x) &\geq \|3f(x)-f(3x+a)\|\\ &= 3\left\|f(x)-\tfrac{1}{3}f(3x+\tfrac{(3-1)a}{2})\right\|.\end{aligned}$$

So that (2.40) is true for $n=1$.
Now assume that (2.40) is true for $n=k$, we will prove that it is true for $n=k+1$.
Then, we have

$$\left\|f(x)-\tfrac{1}{3^{k+1}}f(3^{k+1}x+\tfrac{(3^{k+1}-1)a}{2})\right\|$$

$$=\left\|f(x)-\tfrac{1}{3^{k+1}}f(3^{k+1}x+\tfrac{(3^{k+1}-1)a}{2})+\tfrac{1}{3^k}f(3^kx+\tfrac{(3^k-1)a}{2})-\tfrac{1}{3^k}f(3^kx+\tfrac{(3^k-1)a}{2})\right\|$$

$$\leq \tfrac{1}{3^{k+1}}\left\|f(3^{k+1}x+\tfrac{(3^{k+1}-1)a}{2})-3f(3^kx+\tfrac{(3^k-1)a}{2})\right\|+\left\|\tfrac{1}{3^k}f(3^kx+\tfrac{(3^k-1)a}{2})-f(x)\right\|$$

$$\leq \frac{1}{3^{k+1}}\Phi(3^kx+\frac{(3^k-1)a}{2},3^kx+\frac{(3^k-1)a}{2},3^kx+\frac{(3^k-1)a}{2})$$
$$+\sum_{i=0}^{k-1}\frac{1}{3^{i+1}}\Phi(3^ix+\frac{(3^i-1)a}{2},3^ix+\frac{(3^i-1)a}{2},3^ix+\frac{(3^i-1)a}{2}) \quad \text{by } (2.39),\ (2.40)$$
$$=\sum_{i=0}^{k}\frac{1}{3^{i+1}}\Phi(3^ix+\frac{(3^i-1)a}{2},3^ix+\frac{(3^i-1)a}{2},3^ix+\frac{(3^i-1)a}{2}).$$

So, (2.40) is true for $n=k+1$ and the induction is true for every $n\in\mathbb{N}$.

We will prove (i).

$$\begin{aligned}\|C_{n+1}(x)-C_n(x)\| &= \left\|\tfrac{1}{3^{n+1}}f(3^{n+1}x+\tfrac{(3^{n+1}-1)a}{2})-\tfrac{1}{3^n}f(3^nx+\tfrac{(3^n-1)a}{2})\right\|\\ &= \tfrac{1}{3^{n+1}}\left\|3f(3^nx+\tfrac{(3^n-1)a}{2})-f(3^{n+1}x+\tfrac{(3^{n+1}-1)a}{2})\right\|\\ &\leq \tfrac{1}{3^{n+1}}\Phi(3^nx+\tfrac{(3^n-1)a}{2},3^nx+\tfrac{(3^n-1)a}{2},3^nx+\tfrac{(3^n-1)a}{2}) \quad \text{by } (2.39).\end{aligned}$$

Now for n , m $\in\mathbb{N}$ and $n>m$, we have

$$\begin{aligned}\|C_n(x)-C_m(x)\| &= ||C_n(x)-C_{n-1}(x)+C_{n-1}(x)-C_{n-2}(x)\\ &+ C_{n-2}(x)+\ldots-C_{m+1}(x)+C_{m+1}(x)-C_m(x)||\\ &\leq \|C_n(x)-C_{n-1}(x)\|+\|C_{n-1}(x)-C_{n-2}(x)\|\\ &+ \ldots+\|C_{m+1}(x)-C_m(x)\|\\ &\leq \tfrac{1}{3^n}\Phi(3^{n-1}x+\tfrac{(3^{n-1}-1)a}{2},3^{n-1}x+\tfrac{(3^{n-1}-1)a}{2},3^{n-1}x+\tfrac{(3^{n-1}-1)a}{2})\\ &+ \tfrac{1}{3^{n-1}}\Phi(3^{n-2}x+\tfrac{(3^{n-2}-1)a}{2},3^{n-2}x+\tfrac{(3^{n-2}-1)a}{2},3^{n-2}x+\tfrac{(3^{n-2}-1)a}{2})\\ &+ \ldots\\ &+ \tfrac{1}{3^{m+1}}\Phi(3^mx+\tfrac{(3^m-1)a}{2},3^mx+\tfrac{(3^m-1)a}{2},3^mx+\tfrac{(3^m-1)a}{2}).\end{aligned}$$

Hence, for n , m $\in\mathbb{N}$ and $n>m$, we get

$$\|C_n(x)-C_m(x)\|\leq\sum_{k=m}^{n-1}\frac{1}{3^{k+1}}\Phi\left(3^kx+\frac{(3^k-1)a}{2},3^kx+\frac{(3^k-1)a}{2},3^kx+\frac{(3^k-1)a}{2}\right).$$

Since $\frac{1}{3}<1$ and

$$\sum_{i=0}^{\infty}\frac{1}{3^{i+1}}\Phi\left(3^i x+\frac{(3^i-1)a}{2},3^i y+\frac{(3^i-1)a}{2},3^i z+\frac{(3^i-1)a}{2}\right)<+\infty$$

for all $x,y,z\in X$ and $n\in\mathbb{N}$, then (as $m\longrightarrow\infty$)

$$\sum_{k=m}^{n-1}\frac{1}{3^{k+1}}\Phi\left(3^k x+\frac{(3^k-1)a}{2},3^k x+\frac{(3^k-1)a}{2},3^k x+\frac{(3^k-1)a}{2}\right)\to 0.$$

Therefore, by Definition (1.3), $C_n(x)$ is a Cauchy sequence.

Since Y is a complete normed space, we can define a mapping $C:X\longrightarrow Y$ by

$$C(x)=\lim_{n\longrightarrow\infty}\frac{1}{3^n}f\left(3^n x+\frac{(3^n-1)a}{2}\right). \tag{2.41}$$

We will prove (ii).

For all $x,y,z\in X$ and for all positive integer n, we have

From (i) $$C_n(x+y+z+a)=\frac{1}{3^n}f\left(3^n x+3^n y+3^n z+3^n a+\frac{(3^n-1)a}{2}\right)$$

$$=\frac{1}{3^n}f\left(3^n x+3^n y+3^n z+\frac{(3^{n+1}-1)a}{2}\right)$$

$$=\frac{1}{3^n}f\left(3^n x+3^n y+3^n z+a+\frac{(3^n-1)a}{2}+\frac{(3^n-1)a}{2}+\frac{(3^n-1)a}{2}\right).$$

So, we have

$$\|C_n(x+y+z+a)-C_n(x)-C_n(y)-C_n(z)\|$$

$$=\left\|\tfrac{1}{3^n}f\left(3^n x+\tfrac{(3^n-1)a}{2}+3^n y+\tfrac{(3^n-1)a}{2}+3^n z+\tfrac{(3^n-1)a}{2}+a\right)-\tfrac{1}{3^n}f\left(3^n x+\tfrac{(3^n-1)a}{2}\right)\right.$$
$$\left.-\tfrac{1}{3^n}f\left(3^n y+\tfrac{(3^n-1)a}{2}\right)-\tfrac{1}{3^n}f\left(3^n z+\tfrac{(3^n-1)a}{2}\right)\right\|$$

$$\le\tfrac{1}{3^n}\Phi\left(3^n x+\tfrac{(3^n-1)a}{2},3^n y+\tfrac{(3^n-1)a}{2},3^n z+\tfrac{(3^n-1)a}{2}\right)\quad\text{by (2.33).}$$

Since

$$\sum_{i=0}^{\infty}\frac{1}{3^{i+1}}\Phi\left(3^i x+\frac{(3^i-1)a}{2},3^i y+\frac{(3^i-1)a}{2},3^i z+\frac{(3^i-1)a}{2}\right)<+\infty$$

for all $x,y,z\in X$ and $n\in\mathbb{N}$, then $\lim_{n\longrightarrow\infty}\|C_n(x+y+z+a)-C_n(x)-C_n(y)-C_n(z)\|=0$, that means $C(x+y+z+a)=C(x)+C(y)+C(z)$ for all $x,y,z\in X$.

Now we will prove (iii).

By taking the limit (as $n \longrightarrow \infty$) of both sides of (2.40) and using (2.41), we get $\|f(x)-C(x)\| \leq \sum_{i=0}^{+\infty} \frac{1}{3^{i+1}} \Phi\left(3^i x+\frac{(3^i-1)a}{2}, 3^i x+\frac{(3^i-1)a}{2}, 3^i x+\frac{(3^i-1)a}{2}\right)$.
Finally we will prove the uniqueness of mapping $C : X \longrightarrow Y$. Suppose that there exists another mapping $H : X \longrightarrow Y$, which is the solution of (2.34) and satisfies the inequality (2.35), we get

$$\|C(x)-H(x)\| = \left\|\tfrac{1}{3^n}C(3^n x+\tfrac{(3^n-1)a}{2}) - \tfrac{1}{3^n}H\left(3^n x+\tfrac{(3^n-1)a}{2}\right)\right\|$$

$$\leq \left\|\tfrac{1}{3^n}C(3^n x+\tfrac{(3^n-1)a}{2}) - \tfrac{1}{3^n}f(3^n x+\tfrac{(3^n-1)a}{2})\right\| + \left\|\tfrac{1}{3^n}f(3^n x+\tfrac{(3^n-1)a}{2}) - \tfrac{1}{3^n}H(3^n x+\tfrac{(3^n-1)a}{2})\right\|$$

$$\leq \tfrac{1}{3^n}\sum_{i=0}^{+\infty}\frac{1}{3^{i+1}}\Phi\left(3^i(3^n x+\frac{(3^n-1)a}{2})+\frac{(3^i-1)a}{2}\right.$$

$$\left., 3^i(3^n x+\tfrac{(3^n-1)a}{2})+\tfrac{(3^i-1)a}{2}, 3^i(3^n x+\tfrac{(3^n-1)a}{2})+\tfrac{(3^i-1)a}{2}\right)$$

$$+\tfrac{1}{3^n}\sum_{i=0}^{+\infty}\frac{1}{3^{i+1}}\Phi\left(3^i(3^n x+\frac{(3^n-1)a}{2})+\frac{(3^i-1)a}{2}\right.$$

$$\left., 3^i(3^n x+\tfrac{(3^n-1)a}{2})+\tfrac{(3^i-1)a}{2}, 3^i(3^n x+\tfrac{(3^n-1)a}{2})+\tfrac{(3^i-1)a}{2}\right) \text{ by (2.35)}$$

$$=\frac{2}{3^n}\sum_{i=0}^{+\infty}\frac{1}{3^{i+1}}\Phi\left(3^{n+i}x+\frac{(3^{n+i}-1)a}{2}, 3^{n+i}x+\frac{(3^{n+i}-1)a}{2}, 3^{n+i}x+\frac{(3^{n+i}-1)a}{2}\right)$$
$$=\frac{2}{3^{n+1}}\sum_{i=0}^{+\infty}\frac{1}{3^i}\Phi\left(3^{n+i}x+\frac{(3^{n+i}-1)a}{2}, 3^{n+i}x+\frac{(3^{n+i}-1)a}{2}, 3^{n+i}x+\frac{(3^{n+i}-1)a}{2}\right)$$
$$=2\sum_{i=n}^{+\infty}\frac{1}{3^{i+1}}\Phi\left(3^i x+\frac{(3^i-1)a}{2}, 3^i x+\frac{(3^i-1)a}{2}, 3^i x+\frac{(3^i-1)a}{2}\right).$$

Since $\frac{1}{3}<1$, $\sum_{i=0}^{\infty}\frac{1}{3^{i+1}}\Phi\left(3^i x+\frac{(3^i-1)a}{2}, 3^i y+\frac{(3^i-1)a}{2}, 3^i z+\frac{(3^i-1)a}{2}\right) < +\infty$ for all $x, y, z \in X$ and $n \in \mathbb{N}$, by taking the limit (as $n \longrightarrow \infty$) of both sides, we get $\|C(x)-H(x)\| = 0$ for all $x \in X$, which implies that $C(x) = H(x)$ for all $x \in X$. This completes the proof of the theorem.

Chapter 3

Hyers-Ulam Stability and Superstability of Generalized Cauchy Functional Equations

This chapter consists of three sections. In the first section we will prove the Hyers-Ulam Stability of the functional equation $f(x+y) = f(x) + f(y)$, where f is a function from a normed vector space to a Banach space. Moreover we will prove the Hyers-Ulam Stability of $f(xy) = f(x) + f(y)$, where f is a function from a semigroup to a Banach space.

In the second section we will present the concept of the superstability and we will prove the superstability of the generalized Cauchy functional equation $f(x+y) = f(x)g(y) + f(y)$, where f and g are functions from a vector space to a metric space and $g(y) \neq 1$.

In the third section we will prove the superstability of the generalized Cauchy functional equation $f(xy) = f(x)g(y) + f(y)$, where f and g are functions from a monoid to a metric space and $g(y) \neq 1$.

3.1 Hyers-Ulam Stability of Additive and Logarithmic Cauchy Functional Equations

In this section we will prove the Hyers-Ulam Stability of the functional equation $f(x+y) = f(x)+f(y)$, where f is a function from a normed space to a Banach space and we will prove the Hyers-Ulam Stability of $f(xy) = f(x)+f(y)$, where f is a function from a semigroup to a Banach space such that the Cauchy difference of this functions is bounded.

Theorem 3.1. [*10*] *Let E_1 be a normed vector space and E_2 be a Banach space. Suppose that $f : E_1 \longrightarrow E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \tag{3.1}$$

for all x,y in E_1, where $\varepsilon > 0$ is a constant. Then the limit $A(x) = \lim_{n\longrightarrow\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E_1$ and $A : E_1 \longrightarrow E_2$ is a unique additive mapping satisfying

$$\|f(x) - A(x)\| \leq \varepsilon \quad \textit{for all } x \in E_1. \tag{3.2}$$

Proof. To establish this theorem we have to show that:

(i) $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence for every fixed $x \in E_1$.

(ii) If $A(x) = \lim_{n\longrightarrow\infty} \left\{\frac{f(2^n x)}{2^n}\right\}$ then A is additive on E_1.

(iii) Further A satisfies $\|f(x) - A(x)\| \leq \varepsilon$, for all $x \in E_1$.

(iv) A is unique.

Now to show (i), letting $y = x$ in (3.1), we have

$$\|f(2x) - 2f(x)\| \leq \varepsilon \tag{3.3}$$

for any $x \in E_1$. Replacing x by $2^{k-1}x$ in (3.3), (where k is a positive integergreater than or equal to 1), we obtain $\left\|f(2^k x) - 2f(2^{k-1}x)\right\| \leq \varepsilon$ for all $x \in E_1$, and $k = 1, 2, . \;. \;., n$ where $n \in \mathbb{N}$.

Multiplying both sides of the above inequality by $\frac{1}{2^k}$, then taking the sum of both sides from $k = 1$ to $k = n$, to get

$$\sum_{k=1}^{n} \frac{1}{2^k} \left\| f(2^k x) - 2f(2^{k-1}x) \right\| \leq \sum_{k=1}^{n} \frac{1}{2^k} \varepsilon. \tag{3.4}$$

Since $\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$, which yields

$$\sum_{k=1}^{n} \frac{1}{2^k} \left\| f(2^k x) - 2f(2^{k-1}x) \right\| \leq (1 - \frac{1}{2^n})\varepsilon. \tag{3.5}$$

Now

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{2^k} \left[f(2^k x) - 2f(2^{k-1}x) \right] &= \frac{1}{2}[f(2x) - 2f(x)] + \frac{1}{2^2}[f(2^2 x) - 2f(2x)] \\ &+ \tfrac{1}{2^3}[f(2^3 x) - 2f(2^2 x)] \\ &+ . \qquad . \qquad . \\ &+ \tfrac{1}{2^n}[f(2^n x) - 2f(2^{n-1}x)]\end{aligned}$$

$$= \left[\tfrac{1}{2} f(2x) - f(x) + \tfrac{1}{2^2} f(2^2 x) - \tfrac{1}{2} f(2x) + \tfrac{1}{2^3} f(2^3 x) - \tfrac{1}{2^2} f(2^2 x) \right.$$
$$\left. +. \qquad . \qquad . + \tfrac{1}{2^{n-1}} f(2^{n-1}x) - \tfrac{1}{2^{n-2}} f(2^{n-2}x) + \tfrac{1}{2^n} f(2^n x) - \tfrac{1}{2^{n-1}} f(2^{n-1}x) \right]$$

$$= \left[\tfrac{1}{2^n} f(2^n x) - f(x) \right].$$

By using (3.5), we obtain

$$\begin{aligned}\left\| \frac{1}{2^n} f(2^n x) - f(x) \right\| &= \left\| \sum_{k=1}^{n} \frac{1}{2^k} [f(2^k x) - 2f(2^{k-1}x)] \right\| \\ &\leq \sum_{k=1}^{n} \frac{1}{2^k} \left\| [f(2^k x) - 2f(2^{k-1}x)] \right\| \\ &\leq \varepsilon(1 - \tfrac{1}{2^n}).\end{aligned}$$

Therefore,

$$\left\| \frac{1}{2^n} f(2^n x) - f(x) \right\| \leq \varepsilon(1 - \frac{1}{2^n}) \text{ for all } x \in E_1 \text{ and } n \in \mathbb{N}. \tag{3.6}$$

Now if $n > m > 0$, then $n - m$ is a natural number and n can be replaced by $n - m$ in (3.6), we obtain

$$\left\| \frac{f(2^{n-m}x)}{2^{n-m}} - f(x) \right\| \leq \varepsilon \left(1 - \frac{1}{2^{n-m}} \right).$$

Multiplying both sides by $\frac{1}{2^m}$ and simplifying, we get

$$\left\| \frac{f(2^{n-m}x)}{2^n} - \frac{f(x)}{2^m} \right\| \leq \varepsilon \left(\frac{1}{2^m} - \frac{1}{2^n} \right) \text{for all } x \in E_1.$$

Now we replace x by $2^m x$, to have

$$\left\| \frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m} \right\| \leq \varepsilon \left(\frac{1}{2^m} - \frac{1}{2^n} \right).$$

If $m \longrightarrow \infty$ then, $\left(\frac{1}{2^m} - \frac{1}{2^n}\right) \longrightarrow 0$ and therefore, $\lim\limits_{m \longrightarrow \infty} \left\| \frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m} \right\| = 0.$

Hence $\left\{ \frac{f(2^n x)}{2^n} \right\}$ is a Cauchy sequence in the Banach space E_2.

So the limit of this sequence exists.

Define $A : E_1 \longrightarrow E_2$ by

$$A(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{2^n} \tag{3.7}$$

(ii) Now we show that $A : E_1 \longrightarrow E_2$ defined by (3.7) is additive.

Consider

$$\begin{aligned} \|A(x+y) - A(x) - A(y)\| &= \left\| \lim_{n \longrightarrow \infty} \left\{ \frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} \right\} \right\| \\ &= \left\| \lim_{n \longrightarrow \infty} \frac{1}{2^n} \left\{ f(2^n(x+y)) - f(2^n x) - f(2^n y) \right\} \right\| \\ &= \lim_{n \longrightarrow \infty} \frac{1}{2^n} \| f(2^n x + 2^n y) - f(2^n x) - f(2^n y) \| \\ &\leq \lim_{n \longrightarrow \infty} \frac{\varepsilon}{2^n} \text{ by (3.1)} \\ &= 0. \end{aligned}$$

Therefore, $A(x+y) = A(x) + A(y)$ for all $x, y \in E_1$.

Now we will prove (iii).

Consider

$$\begin{aligned}
\|A(x) - f(x)\| &= \left\| \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{2^n} - f(x) \right\| \\
&= \lim_{n \longrightarrow \infty} \left\| \frac{f(2^n x)}{2^n} - f(x) \right\| \\
&\leq \lim_{n \longrightarrow \infty} \varepsilon \left(1 - \frac{1}{2^n}\right) \text{ by (3.6)} \\
&= \varepsilon.
\end{aligned}$$

Hence, we obtain $\|A(x) - f(x)\| \leq \varepsilon$ for all $x \in E_1$.

(iv) Finally we prove that A is unique. Suppose A is not unique, then there exists another additive function $B : E_1 \longrightarrow E_2$ such that $\|B(x) - f(x)\| \leq \varepsilon$ for all $x \in E_1$.

$$\begin{aligned}
\text{Note that } \|B(x) - A(x)\| &= \|B(x) - f(x) + f(x) - A(x)\| \\
&\leq \|B(x) - f(x)\| + \|f(x) - A(x)\| \\
&\leq \varepsilon + \varepsilon. \\
&= 2\varepsilon.
\end{aligned}$$

Further, since A and B are additive, we have by Theorem (1.13) a solution of the additive Cauchy equation is rationally homogeneous, that is $A(nx) = nA(x)$ for all $x \in E_1$ and all rational numbers n. So, we have

$$\begin{aligned}
\|A(x) - B(x)\| &= \left\| \frac{nA(x)}{n} - \frac{nB(x)}{n} \right\| \\
&= \left\| \frac{A(nx)}{n} - \frac{B(nx)}{n} \right\| \\
&= \frac{1}{n} \|A(nx) - B(nx)\| \\
&\leq \frac{2\varepsilon}{n}.
\end{aligned}$$

Taking the limit of both sides, we get

$$\|A(x) - B(x)\| = \lim_{n \longrightarrow \infty} \|A(x) - B(x)\| \leq \lim_{n \longrightarrow \infty} \frac{2\varepsilon}{n} = 0.$$

Hence,

$$A(x) = B(x) \quad \text{for all } x \in E_1.$$

Therefore, the additive map A is unique and the proof of the theorem is now complete.

Theorem 3.2. [*16*] *Let S be a semigroup and Y a Banach space. Further, let $f : S \longrightarrow Y$ be a mapping satisfying*

$$\|f(xy) - f(x) - f(y)\| \leq \varepsilon \tag{3.8}$$

for all x, y in S, where $\varepsilon > 0$ is a constant. Then the limit $L(x) = \lim_{n \longrightarrow \infty} \dfrac{f(x^{2^n})}{2^n}$ exists for all x in S and $L : S \longrightarrow Y$ is a unique mapping satisfying

$$\|f(x) - L(x)\| \leq \varepsilon \quad \textit{for all } x \textit{ in } S.$$

Proof. To establish this theorem we have to show that:

(i) $\left\{\dfrac{f(x^{2^n})}{2^n}\right\}$ is a Cauchy sequence for every fixed $x \in S$.

(ii) If $L(x) = \lim_{n \longrightarrow \infty} \left\{\dfrac{f(x^{2^n})}{2^n}\right\}$ then L is Logarithmic on S.

(iii) Further L satisfies $\|f(x) - L(x)\| \leq \varepsilon$, for all $x \in S$.

(iv) L is unique.

We will show (i). Let $y = x$ in (3.8), we have

$$\left\|f(x^2) - 2f(x)\right\| \leq \varepsilon \quad \text{for all } x \in S. \tag{3.9}$$

Replacing x by $x^{2^{k-1}}$ in (3.9), (where k is a positive integer greater than or equal to 1), we obtain

$$\left\|f(x^{2^k}) - 2f(x^{2^{k-1}})\right\| \leq \varepsilon \text{ for all } x \in S, \text{ and } k \in \mathbb{N}.$$

Multiplying both sides of the above inequality by $\dfrac{1}{2^k}$, then taking the sum of both sides from $k = 1$ to $k = n$, to get,

$$\sum_{k=1}^{n} \frac{1}{2^k}\left\|f(x^{2^k}) - 2f(x^{2^{k-1}})\right\| \leq \sum_{k=1}^{n}\frac{1}{2^k}\varepsilon = (1-\frac{1}{2^n})\varepsilon. \qquad (3.10)$$

Now

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{2^k}\left[f(x^{2^k}) - 2f(x^{2^{k-1}})\right] &= \frac{1}{2}[f(x^2)-2f(x)] + \frac{1}{2^2}[f(x^{2^2})-2f(x^2)] \\ &+ \tfrac{1}{2^3}[f(x^{2^3})-2f(x^{2^2})] \\ &+ . \quad . \quad . \\ &+ \tfrac{1}{2^n}[f(x^{2^n})-2f(x^{2^{n-1}})]\end{aligned}$$

$= \Big[\tfrac{1}{2}f(x^2) - f(x) + \tfrac{1}{2^2}f(x^{2^2}) - \tfrac{1}{2}f(x^2) + \tfrac{1}{2^3}f(x^{2^3}) - \tfrac{1}{2^2}f(x^{2^2})$
$+. \quad . \quad . + \tfrac{1}{2^{n-1}}f(x^{2^{n-1}}) - \tfrac{1}{2^{n-2}}f(x^{2^{n-2}}) + \tfrac{1}{2^n}f(x^{2^n}) - \tfrac{1}{2^{n-1}}(x^{2^{n-1}})\Big]$
$= \left[\tfrac{1}{2^n}f(x^{2^n}) - f(x)\right].$

By using (3.10), we obtain

$$\begin{aligned}\left\|\tfrac{1}{2^n}f(x^{2^n}) - f(x)\right\| &= \left\|\sum_{k=1}^{n}\frac{1}{2^k}[f(x^{2^k}) - 2f(x^{2^{k-1}})]\right\| \\ &\leq \sum_{k=1}^{n}\frac{1}{2^k}\left\|[f(x^{2^k}) - 2f(x^{2^{k-1}})]\right\| \\ &\leq \varepsilon(1-\tfrac{1}{2^n}).\end{aligned}$$

Therefore,

$$\left\|\tfrac{1}{2^n}f(x^{2^n}) - f(x)\right\| \leq \varepsilon(1-\frac{1}{2^n}) \qquad (3.11)$$

for all $x \in S$ and $n \in \mathbb{N}$. Now if $n > m > 0$ then $n-m$ is a natural number and n can be replaced by $n-m$ in (3.11), to obtain

$$\left\|\frac{f(x^{2^{n-m}})}{2^{n-m}} - f(x)\right\| \leq \varepsilon\left(1-\frac{1}{2^{n-m}}\right).$$

Multiplying both sides by $\dfrac{1}{2^m}$ and simplifying, we get

$$\left\|\frac{f(x^{2^{n-m}})}{2^n} - \frac{f(x)}{2^m}\right\| \leq \varepsilon\left(\frac{1}{2^m}-\frac{1}{2^n}\right) \quad \text{for all } x \in S.$$

Now we replace x by x^{2^m}, to have

$$\left\|\frac{f(x^{2^m})^{2^{n-m}}}{2^n} - \frac{f(x^{2^m})}{2^m}\right\| = \left\|\frac{f(x^{2^n})}{2^n} - \frac{f(x^{2^m})}{2^m}\right\| \leq \varepsilon\left(\frac{1}{2^m}-\frac{1}{2^n}\right).$$

If $m \longrightarrow \infty$ then $\left(\frac{1}{2^m} - \frac{1}{2^n}\right) \longrightarrow 0$ therefore,

$$\lim_{m \longrightarrow \infty} \left\| \frac{f(x^{2^n})}{2^n} - \frac{f(x^{2^m})}{2^m} \right\| = 0.$$

Hence,

$$\left\{ \frac{f(x^{2^n})}{2^n} \right\}$$

is a Cauchy sequence in the Banach space Y. Hence the limit of this sequence exists. Define $L : S \longrightarrow Y$ by

$$L(x) = \lim_{n \longrightarrow \infty} \frac{f(x^{2^n})}{2^n} \tag{3.12}$$

(ii) Now we will show that $L : S \longrightarrow Y$ defined in (3.12) is Logarithmic. Consider

$$\begin{aligned}
\|L(xy) - L(x) - L(y)\| &= \left\| \lim_{n \longrightarrow \infty} \left\{ \frac{f((xy)^{2^n})}{2^n} - \frac{f(x^{2^n})}{2^n} - \frac{f(y^{2^n})}{2^n} \right\} \right\| \\
&= \left\| \lim_{n \longrightarrow \infty} \frac{1}{2^n} \left\{ f((xy)^{2^n}) - f(x^{2^n}) - f(y^{2^n}) \right\} \right\| \\
&= \lim_{n \longrightarrow \infty} \frac{1}{2^n} \| f(x^{2^n} y^{2^n}) - f(x^{2^n}) - f(y^{2^n}) \| \\
&\leq \lim_{n \longrightarrow \infty} \frac{\varepsilon}{2^n} \qquad \text{by } (3.8) \\
&= 0.
\end{aligned}$$

Therefore,
$L(xy) = L(x) + L(y)$ for all $x, y \in S$.

(iii) Our next goal is to show that

$$\|L(x) - f(x)\| \leq \varepsilon.$$

Thus consider,

$$\begin{aligned}\|L(x) - f(x)\| &= \left\| \lim_{n\longrightarrow\infty} \frac{f(x^{2^n})}{2^n} - f(x) \right\| \\ &= \lim_{n\longrightarrow\infty} \left\| \frac{f(x^{2^n})}{2^n} - f(x) \right\| \\ &\leq \lim_{n\longrightarrow\infty} \varepsilon \left(1 - \frac{1}{2^n}\right) \text{ by } (3.11) \\ &= \varepsilon.\end{aligned}$$

Hence, we obtain

$$\|L(x) - f(x)\| \leq \varepsilon \qquad \text{for all } x \in S.$$

(iv) Finally we will prove that L is unique. Suppose L is not unique, then there exists another Logarithmic function

$$B : S \longrightarrow Y$$

such that

$$\|B(x) - f(x)\| \leq \varepsilon \quad \text{for all } x \in S.$$

Note that

$$\begin{aligned}\|B(x) - L(x)\| &= \|B(x) - f(x) + f(x) - L(x)\| \\ &\leq \|B(x) - f(x)\| + \|f(x) - L(x)\| \\ &\leq \varepsilon + \varepsilon \\ &= 2\varepsilon.\end{aligned}$$

Further, since L and B are logarithmic, by Remark (1.24), we have $L(x^n) = nL(x)$.
So, we have

$$\begin{aligned}\|L(x) - B(x)\| &= \left\| \frac{nL(x)}{n} - \frac{nB(x)}{n} \right\| \\ &= \left\| \frac{L(x^n)}{n} - \frac{B(x^n)}{n} \right\|\end{aligned}$$

$$=\frac{1}{n}\left\|L(x^n)-B(x^n)\right\|$$

$$\leq \frac{2\varepsilon}{n}.$$

Taking the limit of both sides, we get

$$\lim_{n\longrightarrow\infty}\|L(x)-B(x)\| \leq \lim_{n\longrightarrow\infty}\frac{2\varepsilon}{n}=0.$$

Hence,

$$L(x)=B(x) \quad \text{for all } x\in S.$$

Therefore, the Logarithmic map L is unique. This completes the proof of the theorem.

3.2 Superstability of the Generalized Cauchy Functional Equation $f(x+y)=f(x)g(y)+f(y)$

In this section we will study the superstability of the generalized Cauchy functional equation $f(x+y)=f(x)g(y)+f(y), \quad g(y)\neq 1$.

Definition 3.3. [4] Let f and g be a complex valued functions from a vector space V to a metric space W, where W is a subset of complex number and let f,g satisfy the generalized Cauchy functional equation $f(x+y)=f(x)g(y)+f(y)$ for all $x,y\in V$. We define the superstability of this equation in the case that $|f(x+y)-f(x)g(y)-f(y)|\leq \varepsilon$ for some $\varepsilon>0$ and for all $x,y\in V$ implies that f and g are both bounded or $f(x+y)=f(x)g(y)+f(y), \quad g(y)\neq 1$.

Theorem 3.4. [17] *Let V be a vector space and let $f,g:V\longrightarrow W$ be a complex valued functions with $g\neq 1$ and W is a subset of complex number. Suppose that f and g satisfy the inequality*

$$|f(x+y)-f(x)g(y)-f(y)|\leq \varepsilon \; for \; some \; \varepsilon>0 \; and \; for \; all \; x,y\in V. \quad (3.13)$$

Then, one of the following conditions holds:

(i) If $f = 0$, then g is arbitrary.

(ii) If $f \neq 0$ is bounded or $f(0) \neq 0$, then g is also bounded.

(iii) If f is unbounded, then $f(0) = 0$, g is also unbounded and $f(x + y) = f(x)g(y) + f(y)$ for all $x, y \in V$.

Proof.

(i) Let $f = 0$ in (3.13), we have $|0 - 0g(y) - 0| \leq \varepsilon$ then, we see that g is arbitrary.

(ii) Suppose that f is bounded and $f \neq 0$. Then, there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in V$ Since $x,y \in V$, we have $x + y \in V$ because V is a vector space and so, we have $|f(x + y)| \leq M$ for all $x, y \in V$. From (3.13), it folows that
$|f(x)g(y)| - |f(x + y)| - |f(y)| \leq |f(x + y) - f(x)g(y) - f(y)| \leq \varepsilon$.

Then,

$$\begin{aligned} |f(x)g(y)| &\leq \varepsilon + |f(y)| + |f(x + y)| \\ &\leq \varepsilon + M + M \\ &= \varepsilon + 2M \quad \text{for all } x, y \in V. \end{aligned} \tag{3.14}$$

Since $f \neq 0$, then there exists a point x_0 such that $f(x_0) \neq 0$.

Putting $x = x_0$ in (3.14), we have

$$|f(x_0)g(y)| \leq \varepsilon + 2M. \tag{3.15}$$

Since $|f(x_0)g(y)| = |f(x_0)||g(y)|$ so, (3.15) becomes $|f(x_0)||g(y)| \leq \varepsilon + 2M$. And dividing both sides by$|f(x_0)|$, we have $|g(y)| \leq \dfrac{\varepsilon + 2M}{|f(x_0)|}$ for all $y \in V$.

This shows that g is bounded.

Now suppose that $f(0) \neq 0$ we will show that g is bounded.

Putting $x = 0$ in (3.13), yields

$|f(y) - f(0)g(y) - f(y)| \leq \varepsilon.$

So, $|f(0)|\,|g(y)| = |f(0)g(y)| \leq \varepsilon.$

Dividing both sides by $|f(0)|$, to get $|g(y)| \leq \dfrac{\varepsilon}{|f(0)|}$ for all $y \in V$.
This shows that g is bounded and this completes the proof of (ii).
(iii) Suppose that f is unbounded, then for all $n \in \mathbb{N}$ there exist $x_n \in V$ such that $|f(x_n)| > n$. Hence $|f(x_n)| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Putting $x = x_n$ in (3.13), we have

$$|f(x_n + y) - f(x_n)g(y) - f(y)| \leq \varepsilon. \tag{3.16}$$

And then dividing both sides by $|f(x_n)|$, to get

$$\frac{|f(x_n + y) - f(x_n)g(y) - f(y)|}{|f(x_n)|} \leq \frac{\varepsilon}{|f(x_n)|}, \text{ so that}$$

$$\left|\frac{f(x_n + y)}{f(x_n)} - g(y) - \frac{f(y)}{f(x_n)}\right| \leq \frac{\varepsilon}{|f(x_n)|}. \tag{3.17}$$

Taking the limit (as $n \longrightarrow \infty$) of both sides of (3.17), we get

$$\left|\lim_{n\longrightarrow\infty} \frac{f(x_n + y)}{f(x_n)} - \lim_{n\longrightarrow\infty} g(y) - \lim_{n\longrightarrow\infty} \frac{f(y)}{f(x_n)}\right| \leq \lim_{n\longrightarrow\infty} \frac{\varepsilon}{|f(x_n)|}.$$

$$\text{So, } \lim_{n\longrightarrow\infty} \frac{f(x_n + y)}{f(x_n)} - g(y) - 0 = 0, \quad \text{that is}$$

$$g(y) = \lim_{n\longrightarrow\infty} \frac{f(x_n + y)}{f(x_n)}. \tag{3.18}$$

Substituting x by $x + x_n$ in (3.13), to get

$$|f(x + x_n + y) - f(x + x_n)g(y) - f(y)| \leq \varepsilon.$$

And then dividing both sides by $|f(x_n)|$, we get

$$\left|\frac{f(x+x_n+y)}{f(x_n)} - \frac{f(x+x_n)g(y)}{f(x_n)} - \frac{f(y)}{f(x_n)}\right| \leq \frac{\varepsilon}{|f(x_n)|}. \tag{3.19}$$

Taking the limit (as $n \longrightarrow \infty$) of both sides, we get

$$\left|\lim_{n\longrightarrow\infty} \frac{f(x+x_n+y)}{f(x_n)} - \lim_{n\longrightarrow\infty} \frac{f(x+x_n)g(y)}{f(x_n)} - \lim_{n\longrightarrow\infty} \frac{f(y)}{f(x_n)}\right| \leq \lim_{n\longrightarrow\infty} \frac{\varepsilon}{|f(x_n)|}.$$

Hence,

$$\lim_{n\longrightarrow\infty} \frac{f(x+x_n+y)}{f(x_n)} - \lim_{n\longrightarrow\infty} \frac{f(x+x_n)g(y)}{f(x_n)} = 0, \quad \text{that is}$$

$$\lim_{n\longrightarrow\infty} \frac{f(x+y+x_n)}{f(x_n)} = \lim_{n\longrightarrow\infty} \frac{f(x+x_n)g(y)}{f(x_n)}.$$

By (3.18) and the last equation, we have

$$g(x+y) = g(x)g(y) \qquad \text{for all } x, y \in V. \tag{3.20}$$

Now g is satisfying exponential Cauchy functional equation so, by Definition (1.18), we have $g(x)$ is exponential and $g(x) = 0$.
If $g = 0$, then from (3.13), we have
$|f(x+y) - f(y)| \leq \varepsilon$ for all $x, y \in V$.
So, we have $|f(x+y)| \leq \varepsilon + |f(y)|$ for all $x, y \in V$
and so, $|f(x)| = |f(x+0)| \leq \varepsilon + |f(0)|$ for all $x \in V$.
Since $\varepsilon + |f(0)| > 0$, this shows that f is bounded which is reduces a contradiction to the assumption of (iii).
Therefore, $g \neq 0$ and so, it must be exponential. Hence, g is unbounded.
Then we can choose a sequence $(y_n) \in V$ such that $|g(y_n)| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Putting $y = y_n$ in (3.13), to get

$$|f(x+y_n) - f(x)g(y_n) - f(y_n)| \leq \varepsilon.$$

Then dividing both sides by $|g(y_n)|$, we have

$$\left|\frac{f(x+y_n)}{g(y_n)} - f(x) - \frac{f(y_n)}{g(y_n)}\right| \leq \frac{\varepsilon}{|g(y_n)|}.$$

Taking the limit (as $n \longrightarrow \infty$) of both sides, we get

$$\left|\lim_{n\longrightarrow\infty} \frac{f(x+y_n)}{g(y_n)} - f(x) - \lim_{n\longrightarrow\infty} \frac{f(y_n)}{g(y_n)}\right| \leq \lim_{n\longrightarrow\infty} \frac{\varepsilon}{|g(y_n)|}.$$

Hence,

$$\lim_{n\longrightarrow\infty} \frac{f(x+y_n)}{g(y_n)} - f(x) - \lim_{n\longrightarrow\infty} \frac{f(y_n)}{g(y_n)} = 0.$$

We obtain $f(x) = \lim_{n\longrightarrow\infty} \frac{f(x+y_n)}{g(y_n)} - \lim_{n\longrightarrow\infty} \frac{f(y_n)}{g(y_n)}$.

Let $x = 0$, to get, $f(0) = \lim_{n\longrightarrow\infty} \frac{f(y_n)}{g(y_n)} - \lim_{n\longrightarrow\infty} \frac{f(y_n)}{g(y_n)} = 0$ and we get,

$$f(x) = \lim_{n\longrightarrow\infty} \frac{f(x+y_n) - f(y_n)}{g(y_n)} \qquad (3.21)$$

Substituting y by $y + y_n$ in (3.13) and using (3.20), we obtain

$$|f(x+y+y_n) - f(x)g(y+y_n) - f(y+y_n)| \leq \varepsilon$$

and

$$|f(x+y+y_n) - f(x)g(y)g(y_n) - f(y+y_n)| \leq \varepsilon.$$

Dividing both sides by $|g(y_n)|$, to get

$$\left|\frac{f(x+y+y_n)}{g(y_n)} - f(x)g(y) - \frac{f(y+y_n)}{g(y_n)}\right| \leq \frac{\varepsilon}{|g(y_n)|}.$$

Taking the limit (as $n \longrightarrow \infty$) of both sides, we get

$$\lim_{n\longrightarrow\infty} \frac{f(x+y+y_n)}{g(y_n)} - f(x)g(y) - \lim_{n\longrightarrow\infty} \frac{f(y+y_n)}{g(y_n)} = 0.$$

Hence,

$$f(x)g(y) = \lim_{n\longrightarrow\infty} \frac{f(x+y+y_n) - f(y+y_n)}{g(y_n)}$$

$$= \lim_{n\longrightarrow\infty} \frac{f(x+y+y_n) - f(y_n) + f(y_n) - f(y+y_n)}{g(y_n)}$$

$$= \lim_{n\longrightarrow\infty} \frac{f(x+y+y_n) - f(y_n)}{g(y_n)} - \lim_{n\longrightarrow\infty} \frac{f(y+y_n) - f(y_n)}{g(y_n)}$$

$$= f(x+y) - f(y) \quad \text{by (3.21)}.$$

This completes the proof of the theorem.

3.3 Superstability of the Generalized Cauchy Functional Equation $f(xy)$=$f(x)g(y)$+$f(\mathbf{y})$

In this section we investigate the superstability of the generalized cauchy functional equations $f(xy) = f(x)g(y) + f(y)$.

Definition 3.5. [4]
Let f and g be a complex valued functions from a monoid S to a metric space W, where W is a subset of complex number and let f and g satisfy the generalized Cauchy functional equation $f(xy) = f(x)g(y) + f(y)$ for all $x, y \in S$. We define the superstability of this equation in the case that

$$|f(xy) - f(x)g(y) - f(y)| \leq \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y \in S$ implies that either f and g are both bounded or $f(xy) = f(x)g(y) + f(y), \quad g(y) \neq 1$.

Theorem 3.6. [17] *Let S be a monoid and let $f, g : S \longrightarrow W$ be a complex valued functions with $g \neq 1$ and W is a subset of complex number. Suppose that f and g satisfy the inequality*

$$|f(xy) - f(x)g(y) - f(y)| \leq \varepsilon \; for \; some \; \varepsilon > 0. \tag{3.22}$$

Then, one of the following conditions holds:

(i) If $f = 0$, then g is arbitrary.

(ii) If $f \neq 0$ is bounded or $f(1) \neq 0$, then g is also bounded.

(iii) If f is unbounded, then $f(1) = 0$, g is also unbounded and $f(xy) = f(x)g(y) + f(y)$ for all $x, y \in S$.

Proof.

(i) Let $f = 0$ in (3.22), we have $|0 - 0g(y) - 0| \leq \varepsilon$ then, we see that g is arbitrary.

(ii)Suppose that f is bounded and $f \neq 0$. Then, there exists a constant $N > 0$ such that $|f(x)| \leq N$ for all $x \in S$. Since $x, y \in S$ we have $xy \in V$ because S is a monoid and so we have $|f(xy)| \leq N$.
From (3.22), it folows that
$|f(x)g(y)| - |f(xy)| - |f(y)| \leq |f(xy) - f(x)g(y) - f(y)| \leq \varepsilon$.
Then

$$\begin{aligned} |f(x)g(y)| &\leq \varepsilon + |f(y)| + |f(xy)| \\ &\leq \varepsilon + N + N \\ &= \varepsilon + 2N \quad \text{for all } x, y \in S. \end{aligned} \tag{3.23}$$

Since $f \neq 0$ then there exists a point x_0 such that $f(x_0) \neq 0$.
Putting $x = x_0$ in (3.23), we have

$$|f(x_0)||g(y)| = |f(x_0)g(y)| \leq \varepsilon + 2N. \tag{3.24}$$

Dividing both side by $|f(x_0)|$, we have $|g(y)| \leq \dfrac{\varepsilon + 2N}{|f(x_0)|}$. This shows that g is bounded for all $y \in S$.

Now suppose that $f(1) \neq 0$ we will show g is bounded.
Putting $x = 1$ in (3.22) , yields

$|f(y) - f(1)g(y) - f(y)| \leq \varepsilon.$

So, $|f(1)g(y)| \leq \varepsilon$. Dividing both sides by $|f(1)|$, we obtain

$$|g(y)| \leq \frac{\varepsilon}{|f(1)|} \qquad \text{for all } y \in S.$$

Hence, g is bounded when $f(1) \neq 0$ and this completes the proof of (ii).

(iii) Finally, we suppose that f is unbounded, then for all $n \in N$ there exist $x_n \in S$ such that $|f(x_n)| > n$. Hence $|f(x_n)| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Putting $x = x_n$ in (3.22), we have

$$|f(x_n y) - f(x_n)g(y) - f(y)| \leq \varepsilon. \tag{3.25}$$

And then dividing both sides by $|f(x_n)|$, we have

$$\frac{|f(x_n y) - f(x_n)g(y) - f(y)|}{|f(x_n)|} \leq \frac{\varepsilon}{|f(x_n)|}.$$

So, we have

$$\left| \frac{f(x_n y)}{f(x_n)} - g(y) - \frac{f(y)}{f(x_n)} \right| \leq \frac{\varepsilon}{|f(x_n)|.} \tag{3.26}$$

Taking the limit (as $n \longrightarrow \infty$) of both sides in (3.26), we get

$$\left| \lim_{n\longrightarrow\infty} \frac{f(x_n y)}{f(x_n)} - \lim_{n\longrightarrow\infty} g(y) - \lim_{n\longrightarrow\infty} \frac{f(y)}{f(x_n)} \right| \leq \lim_{n\longrightarrow\infty} \frac{\varepsilon}{|f(x_n)|}.$$

Therefore,

$$\lim_{n\longrightarrow\infty} \frac{f(x_n y)}{f(x_n)} - g(y) - 0 = 0, \quad \text{that is}$$

$$g(y) = \lim_{n\longrightarrow\infty} \frac{f(x_n y)}{f(x_n)}. \tag{3.27}$$

Substituting x by xx_n in (3.22), we get

$$|f(xx_n y) - f(xx_n)g(y) - f(y)| \leq \varepsilon.$$

And then dividing both sides by $|f(x_n)|$, to get

$$\left| \frac{f(xx_n y)}{f(x_n)} - \frac{f(xx_n)g(y)}{f(x_n)} - \frac{f(y)}{f(x_n)} \right| \leq \frac{\varepsilon}{|f(x_n)|}. \tag{3.28}$$

Taking the limit (as $n \longrightarrow \infty$) of both sides, we get

$$\left|\lim_{n\longrightarrow\infty}\frac{f(xx_ny)}{f(x_n)}-\lim_{n\longrightarrow\infty}\frac{f(xx_n)g(y)}{f(x_n)}-\lim_{n\longrightarrow\infty}\frac{f(y)}{f(x_n)}\right|\leq\lim_{n\longrightarrow\infty}\frac{\varepsilon}{|f(x_n)|}.$$

So,

$$\lim_{n\longrightarrow\infty}\frac{f(xx_ny)}{f(x_n)}-\lim_{n\longrightarrow\infty}\frac{f(xx_n)g(y)}{f(x_n)}-0=0.$$

Hence,

$$\lim_{n\longrightarrow\infty}\frac{f(xyx_n)}{f(x_n)}=\lim_{n\longrightarrow\infty}\frac{f(xx_n)g(y)}{f(x_n)}.$$

By (3.27) and the last equation, we have

$$g(xy)=g(x)g(y)\qquad\text{for all } x,y\ \in S. \tag{3.29}$$

Now g is satisfying multiplication Cauchy equation so by Theorem (1.19) we have $g(x)$ is exponential, $g(x)=0$ and $g(x)=1$.

If $g=1$, hence, this reduces a contradiction to the assumption $g\neq 1$ in theorem.

If $g=0$, then from (3.22), we have

$$|f(xy)-f(y)|\leq\varepsilon\qquad\text{for all } x,y\in S.$$

$$\textit{So,}\text{ we have } |f(xy)|\leq\varepsilon+|f(y)|\quad\text{for all } x,y\in S.$$

Let $y=1$, we have $|f(x)|\leq\varepsilon+|f(1)|$ for all $x\in S$.

This shows that f is bounded and hence, this reduces a contradiction to the assumption in (iii). Therefore, g is exponential and hence, it is unbounded.

Choose a sequence (y_n) such that $|g(y_n)|\longrightarrow\infty$ as $n\longrightarrow\infty$.

Putting $y=y_n$ in (3.22), to get

$$|f(xy_n)-f(x)g(y_n)-f(y_n)|\leq\varepsilon.$$

Then dividing both sides by $|g(y_n)|$, we have

$$\left|\frac{f(xy_n)}{g(y_n)} - f(x) - \frac{f(y_n)}{g(y_n)}\right| \leq \frac{\varepsilon}{|g(y_n)|}.$$

Taking the limit (as $n \longrightarrow \infty$) of both sides, we get

$$\left|\lim_{n\longrightarrow\infty} \frac{f(xy_n)}{g(y_n)} - f(x) - \lim_{n\longrightarrow\infty} \frac{f(y_n)}{g(y_n)}\right| \leq \lim_{n\longrightarrow\infty} \frac{\varepsilon}{|g(y_n)|}$$

Hence,

$$\lim_{n\longrightarrow\infty} \frac{f(xy_n)}{g(y_n)} - f(x) - \lim_{n\longrightarrow\infty} \frac{f(y_n)}{g(y_n)} = 0.$$

We obtain, $f(x)= \lim_{n\longrightarrow\infty} \frac{f(xy_n)}{g(y_n)} - \lim_{n\longrightarrow\infty} \frac{f(y_n)}{g(y_n)}$.

Let $x = 1$, to get, $f(1) = \lim_{n\longrightarrow\infty} \frac{f(y_n)}{g(y_n)} - \lim_{n\longrightarrow\infty} \frac{f(y_n)}{g(y_n)} = 0$ and we get,

$$f(x) = \lim_{n\longrightarrow\infty} \frac{f(xy_n) - f(y_n)}{g(y_n)}. \tag{3.30}$$

Substituting y by yy_n in (3.22) and using (3.29), we obtain

$$|f(xyy_n) - f(x)g(yy_n) - f(yy_n)| \leq \varepsilon$$
$$\text{and} \quad |f(xyy_n) - f(x)g(y)g(y_n) - f(yy_n)| \leq \varepsilon.$$

And then dividing both sides by $|g(y_n)|$, to get

$$\left|\frac{f(xyy_n)}{g(y_n)} - f(x)g(y) - \frac{f(yy_n)}{g(y_n)}\right| \leq \frac{\varepsilon}{|g(y_n)|}.$$

Taking the limit (as $n \longrightarrow \infty$) of both sides, we get

$$\lim_{n\longrightarrow\infty} \frac{f(xyy_n)}{g(y_n)} - f(x)g(y) - \lim_{n\longrightarrow\infty} \frac{f(yy_n)}{g(y_n)} = 0.$$

Hence,

$$f(x)g(y) = \lim_{n\longrightarrow\infty} \frac{f(xyy_n) - f(yy_n)}{g(y_n)}$$

$$= \lim_{n\longrightarrow\infty} \frac{f(xyy_n) - f(y_n) + f(y_n) - f(yy_n)}{g(y_n)}$$

$$= \lim_{n\longrightarrow\infty} \frac{f(xyy_n) - f(y_n)}{g(y_n)} - \lim_{n\longrightarrow\infty} \frac{f(yy_n) - f(y_n)}{g(y_n).}$$

$$= f(xy) - f(y) \text{ by } (3.30).$$

This completes the proof of the theorem.

Conclusion

In this thesis, we have studied the stability and superstability of some generalized Cauchy functional equations. We can be trust that, if a functional equation is stable and linear then the Cauchy difference of this function must be bounded or unbounded but if a functional equation is stable and nonlinear then the Cauchy difference of this function must be only bounded. We are also confident that the domain and range of a functional equation play a big role in the indication that a functional equation is stable or superstable or not. Stability and superstability in functional equation it would be interesting to investigate how to find solution of a functional inequality which approximately satisfies a functional equation also it allows to find another solutions of a functional equation.

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