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On Scaling Reductions and Painlevé Hierarchies

حول الاختزال المتدرج وهرميات بانليفة

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Abstract

In this thesis we study scaling reduction of hierarchies of partial differential equations. We use scaling reduction of the Korteweg de Vries (KdV) to derive a thirty-fourth Painlevé hierarchy and we derive a fourth Painlevé hierarchy by scaling reduction of dispersive water wave (DWW). We also study generalized scaling reductions of Burgers hierarchy. Moreover we consider the mapping of nonisospectral KdV and DWW hierarchies onto isospectral KdV and DWW hierarchies.

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Contents

Introduction

The Painlevé equations were first discovered at the end of the 19th century by Painlevé and his colleagues when they classified second order ordinary differential equations (ODEs) according to their singularities. Painlevé and his school found that forty-four of fifty equations are reducible in the sense that they can be solved in terms of previously known functions, and the remaining six equations are requiring the introduction of new special functions to solve them. These six non-linear differential equations are called Painlevé equations and their solutions are called Painlevé transcendents (Clarkson, 2003; Ince, 1956).

The applications of Painlevé equations are very important in various areas of mathematics and physics. For example, in hydrodynamics and plasma physics, nonlinear waves, quantum gravity, quantum field theory, general relativity, nonlinear optics and fibre optics (Clarkson, 2003).

A Painlevé hierarchy is an infinite sequence of non-linear ordinary differential equations whose first member is a Painlevé equation. Kudryashov was the first to derive a first Painlevé hierarchy and Airault was the first to derive a second Painlevé hierarchy (Sakka et al., 2009; Airault, 1979; Kudryashov, 1997).

There are several ways to derive Painlevé hierarchies. Airault used higher order integrable partial differential equations (PDEs) to derive ODEs with Painlevé property to found a first member of the second Painlevé hierarchy, by similarity reduction of the Korteweg de Vries hierarchies (Gordoa and Pickering, 2011). (Gordoa et al., 2006) non-isospectral scattering problems is used to derive new second Painlevé

hierarchy and new fourth Painlevé hierarchy. After that a third Painlevé hierarchy was derived in (Sakka, 2008) by expansion of linear problems of the Painlevé equations in powers of the spectral variables.

In 1977 (Ablowitz and Segur, 1977) noted that there exists a connection between completely integrable partial differential equation and Painlevé equation. They noted that the similarity reduction of completely integrable partial differential equations gave rises to ODEs with the Painlevé property (Gordoa et al., 2014). For example, the second Painlevé equation (Clarkson, 2003)

$$\frac{d^2w}{dz^2} = 2w^3 + zw + \alpha \tag{0.1}$$

can be obtained from the modified Korteweg-de Vries (mKdV) equation

$$u_t - 6u^2u_x + u_{xxx} = 0 \tag{0.2}$$

through the scaling reduction

$$u(x,t) = \frac{w(z)}{(3t)^{\frac{1}{3}}}\ , \quad z = \frac{x}{(3t)^{\frac{1}{3}}}. \tag{0.3}$$

The goal of this thesis is to study scaling reduction of hierarchies of partial deferential equations and to derive thirty-fourth and fourth Painlevé hierarchies using scaling reduction.

This thesis is organized as follows:

In chapter one, we will give the basic definitions and theorems related to our study.

In chapter two, we will derive a thirty-fourth Painlevé hierarchy from the KdV hierarchy using scaling reductions, and we will derive a fourth Painlevé hierarchy from scaling reductions of dispersive water wave (DWW). Then we will study scaling reductions of Burgers hierarchy.

In chapter three, we show that the nonisospectral KdV hierarchy can be mapped onto isospectral KdV hierarchy, and its scattering problem is transformed into the nonisospectral scattering problem. Similarly, the nonisospectral DWW hierarchy can be mapped onto isospectral DWW hierarchy, and its scattering problem is

transformed into the nonisospectral scattering problem. Finally, we explain why the use of nonisospectral scattering problem and similarity reduction in KdV and DWW yields the same Painlevé hierarchy.

Chapter 1

Scaling Reduction

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Scaling Reduction

There exist many techniques to find the solution to a linear partial differential equation (PDE) like separation of variables, integral transform and change of variables. But nonlinear partial differential equation can not be solved by these methods. Therefore we need other methods to solve these equations such as similarity method.

A scaling reduction of a PDE is one that remains unchanged under scaling symmetries. The scaling reduction for a PDE in two variables can be found by solving an ordinary differential equation (ODE) and therefore make the solution much more simple. The main idea of a scaling reduction is that two or more independent variables can be linked by a transformation such that the PDE becomes ODE or a PDE with a reduced number of independent variables (Olver, 2014).

1.1 Solving PDE by scaling method

Consider a PDE in two independent variable t and x:

$$E(t, u, u_t, u_x \ldots,) = 0. \tag{1.1}$$

A scaling reduction of equation (1.1) can be found by solving an ODE.

To explain the similarity method in more details we consider the similarity variables

$$\tau = \beta^a t, \qquad z = \beta^b x, \qquad U(\tau, z) = \beta^c u(x,t), \tag{1.2}$$

where a, b, c are constants with a, b not both zero. Now, using the chain rule we find

$$U_\tau = \beta^{c-a} u_t, \tag{1.3}$$
$$U_z = \beta^{c-b} u_x, \tag{1.4}$$
$$U_{\tau\tau} = \beta^{c-2a} u_{tt}, \tag{1.5}$$
$$U_{\tau z} = \beta^{2c-a-b} u_{tx}, \tag{1.6}$$

and so on. Next we substitute these expressions in the given equation and we look for values of a, b, c, such that the equation for U is the same as that for u (Olver, 2014). Let us illustrate this by examples.

Example 1.1. *Consider the linear heat equation*

$$u_t = u_{xx}. \tag{1.7}$$

Assume

$$\tau = \beta^a t, \qquad z = \beta^b x, \qquad U(\tau, z) = \beta^c u(x,t). \tag{1.8}$$

By the chain rule we have

$$U_\tau = \beta^{c-a} u_t, \qquad U_z = \beta^{c-b} u_x, \qquad U_{zz} = \beta^{c-2b} u_{xx}. \tag{1.9}$$

Equation (1.9) gives

$$u_t = \beta^{a-c} U_\tau, \qquad u_{xx} = \beta^{2b-c} U_{zz}. \tag{1.10}$$

Substituting u_t and u_{xx} from (1.10) into (1.7) we obtain

$$\beta^{a-c}U_\tau - \beta^{2b-c}U_{zz} = 0,$$

that is

$$U_\tau - \beta^{2b-a}U_{zz} = 0.$$

Thus the equation for U is the same as that for u if $\beta^{2b-a} = 1$; that is $a = 2b$. As a result, if $u(t, x)$ is any solution to (1.7), so is the function

$$U(\tau, z) = \beta^c u(\beta^{-a}\tau, \beta^{-b}z), \qquad \text{where} \ \ a = 2b. \tag{1.11}$$

The relation (1.11) can be written as

$$u(t, x) = \beta^{-c}U(\beta^a t, \beta^b x).$$

Let $\beta = t^{-1/a}$. Then

$$u(t, x) = t^{\frac{c}{a}}U(1, t^{\frac{-1}{2}}x).$$

So, the scaling reduction has the form

$$u(t, x) = t^{\frac{c}{a}}v(\xi), \qquad \text{where} \qquad v(\xi) = U(1, t^{\frac{-1}{2}}x) \qquad \text{and} \qquad \xi = t^{-\frac{1}{2}}x,$$

where c is an arbitrary. By differentiating u with respect to t and x we obtain

$$\begin{aligned} u_t &= t^{\frac{c}{a}-1}\left[\frac{-1}{2}\xi v'(\xi) + \frac{c}{a}v(\xi)\right], \\ u_{xx} &= t^{\frac{c}{a}-1}v''(\xi). \end{aligned}$$

Substituting u_t and u_{xx} into (1.7) yields

$$t^{\frac{c}{a}-1}\left[\frac{-1}{2}\xi v'(\xi) + \frac{c}{a}v(\xi)\right] = t^{\frac{c}{a}-1}v''(\xi),$$

or

$$v''(\xi) + \frac{1}{2}\xi v'(\xi) - \frac{c}{a}v(\xi) = 0.$$

If $c = 0$, then the last equation becomes

$$v''(\xi) + \frac{1}{2}\xi v'(\xi) = 0. \tag{1.12}$$

Equation (1.12) *is a linear second-order ordinary differential equation for* $v(\xi)$ *and its solution is given by*

$$v(\xi) = c_1 erf(\frac{1}{2}\xi) + c_2,$$

where c_1, c_2 *are arbitrary constants and* $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$ *is the error function. It follows that a scaling reduction of the heat equation* (1.7) *is given by* $u(x,t) = c_2 + c_1 erf\left(\frac{x}{2\sqrt{t}}\right)$.

Example 1.2. *Consider the modified Korteweg-de Vries equation (mKdV)*

$$u_t - 6u^2 u_x + u_{xxx} = 0. \tag{1.13}$$

Assume

$$\tau = \beta^a t, \qquad z = \beta^b x, \qquad U = \beta^c u(t,x), \qquad \text{that is } \; U(\tau, z) = \beta^c u(\beta^{-a}\tau, \beta^{-b} z).$$

By differentiation we have

$$U_\tau = \beta^{c-a} u_t, \qquad U_z = \beta^{c-b} u_x, \qquad U_{zzz} = \beta^{c-3b} u_{xxx}. \tag{1.14}$$

From (1.14), *we get*

$$u_t = \beta^{a-c} U_\tau, \qquad u_x = \beta^{b-c} U_z, \qquad u_{xxx} = \beta^{3b-c} U_{zzz}. \tag{1.15}$$

Substituting u, u_t, u_x, u_{xxx} *from* (1.15) *into equation* (1.13) *yields*

$$\beta^{a-c} U_\tau - 6\beta^{b-3c} U^2 U_z + \beta^{3b-c} U_{zzz} = 0.$$

Thus the conditions for invariance are $a - c = b - 3c = 3b - c$; *that is,* $a = 3b$ *and* $c = -b$. *As a result we get*

$$u(x,t) = \beta^b U(\beta^a t, \beta^b x).$$

Let $\beta = (3t)^{-1/a}$. *Then we have*

$$u(x,t) = (3t)^{\frac{-1}{3}} U(\frac{1}{3}, (3t)^{\frac{-1}{3}} x).$$

Thus the scaling reduction has the form

$$u(x,t) = (3t)^{\frac{-1}{3}} v(\xi), \qquad v(\xi) = U(3^{-1}, (3t)^{\frac{-1}{3}} x), \qquad \xi = (3t)^{\frac{-1}{3}} x.$$

By differentiation u with respect to t and x, we have

$$u_t = (3t)^{-\frac{4}{3}} \left[-v(\xi) - \xi v'(\xi)\right], \quad u_x = (3t)^{-\frac{2}{3}} v'(\xi), \quad u_{xxx} = (3t)^{-\frac{4}{3}} v'''(\xi). \quad (1.16)$$

Substituting u and its derivatives from equation (1.16) into equation (1.13) yields

$$(3t)^{-\frac{4}{3}} \left[-v(\xi) - \xi v'(\xi)\right] - 6(3t)^{-\frac{4}{3}} v^2(\xi) v'(\xi) + (3t)^{-\frac{4}{3}} v'''(\xi) = 0.$$

Multiplying by $(3t)^{\frac{4}{3}}$ gives

$$v'''(\xi) - 6v^2(\xi)v'(\xi) - \xi v'(\xi) - v(\xi) = 0.$$

By integration we obtain

$$v''(\xi) = 2v^3(\xi) + \xi v(\xi) + \alpha, \qquad (1.17)$$

where α is a constant. Equation (1.17) is the second Painlevé equation (Clarkson, 2003).

Chapter 2

Scaling reductions and Painlevé hierarchies

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Scaling reductions and Painlevé hierarchies

In this chapter we will study scaling reductions of the Korteweg-de Vries, dispersive water wave and Burgers hierarchies.

2.1 The Korteweg-de Vries hierarchy

The Korteweg-de Vries (KdV) hierarchy is given by (Gordoa et al., 2013)

$$U_{t_{2n+1}} = \mathcal{R}^n[U]U_x + \sum_{i=1}^{n-1} \beta_i(t_{2n+1}) R^i[U]U_x,$$

where

$$\mathcal{R}[U] = \partial_x^2 + 4U + 2U_x\partial_x^{-1}. \tag{2.1}$$

The recursion operator $\mathcal{R}[U]$ of the KdV hierarchy is the quotient $\mathcal{R}[U] = \mathcal{B}_1[U]\mathcal{B}_0^{-1}[U]$ of the two Hamiltonian operators

$$\begin{aligned} \mathcal{B}_1[U] &= \partial_x^3 + 4U\partial_x + 2U_x \\ \mathcal{B}_0 &= \partial_x. \end{aligned}$$

The KdV hierarchy can be written as

$$U_{t_{2n+1}} = \mathcal{R}^n[U]U_x = \mathcal{B}_0[U]M_{n+1} = \mathcal{B}_1[U]M_n[U], \tag{2.2}$$

where $M_n[U]$ are defined recursively by

$$\begin{aligned} M_0 &= \frac{1}{2}, \\ M_{n+1}[U] &= \mathcal{B}_0^{-1}[U]\mathcal{B}_1[U]M_n[U]. \end{aligned} \tag{2.3}$$

In this section, we will derive the thirty-fourth Painlevé hierarchy from a scaling reduction of the KdV hierarchy (2.2) (Gordoa et al., 2013).

Lemma 2.1. *(Gordoa and Pickering, 2011) The change of variables* $\tilde{U} = U + C$, *where* C *is an arbitrary constant, in* $R^n[\tilde{U}]\tilde{U}_x$, *yields*

$$R^n[\tilde{U}]\tilde{U}_x = \sum_{j=0}^{n} \alpha_{n,j} C^{n-j} R^j[U]U_x, \tag{2.4}$$

where $R^0[U]$ *is the identity operator and the coefficients* $\alpha_{n,j}$ *are determined recursively by*

$$\alpha_{n,n} = 1, \tag{2.5}$$

$$\alpha_{n,j} = 4\alpha_{n-1,j} + \alpha_{n-1,j-1}, \tag{2.6}$$

$$\alpha_{n,0} = \frac{4n+2}{n}\alpha_{n-1,0}, \tag{2.7}$$

$\alpha_{0,0} = 1$, *and* $R^0[U] = 1$.

Proof. We must deal carefully with the operator ∂_x^{-1} which making the coefficient not binomial so that we added $\left(\frac{\alpha_{n,0}}{n+1}C^{n+1}\right)_x$ as a constant of integration. For $n = 1$, we have

$$\begin{aligned} R[\tilde{U}]\tilde{U}_x &= \left[\partial_x^2 + 4\tilde{U} + 2\tilde{U}_x\partial_x^{-1}\right]\tilde{U}_x \\ &= \tilde{U}_{xxx} + 6\tilde{U}\tilde{U}_x. \end{aligned}$$

Using $\tilde{U} = U + C$ we obtain

$$\begin{aligned} R[\tilde{U}]\tilde{U}_x &= U_{xxx} + 6(U+C)U_x \\ &= U_{xxx} + 6UU_x + 6CU_x \\ &= R[U]U_x + 6CR^0[U]U_x. \end{aligned}$$

Therefore the relation (2.4) is true for $n = 1$. Suppose (2.4)-(2.5) are true for n. Now for $n+1$ we have

$$R^{n+1}[\tilde{U}]\tilde{U}_x = R[\tilde{U}]R^n[\tilde{U}]\tilde{U}_x \tag{2.8}$$

Substituting $\tilde{U} = U + C$ in (2.8) and using the induction hypothesis we obtain

$$R^{n+1}[\tilde{U}]\tilde{U}_x = \left(\partial_x^2 + 4U + 2U_x\partial_x^{-1} + 4C\right)\left[R^n[U]U_x + \sum_{j=0}^{n-1}\alpha_{n,j}C^{n-j}R^j[U]U_x + \left(\frac{\alpha_{n,0}}{n+1}C^{n+1}\right)_x\right].$$

Hence

$$\begin{aligned} R^{n+1}[\tilde{U}]\tilde{U}_x &= \left(R[U]+4C\right)R^n[U]U_x + \left(R[U]+4C\right)\left[\sum_{j=0}^{n-1}\alpha_{n,j}C^{n-j}R^j[U]U_x\right] \\ &\quad + \left(\partial_x^2 + 4U + 2U_x\partial_x^{-1} + 4C\right)\left[\left(\frac{\alpha_{n,0}}{n+1}C^{n+1}\right)_x\right]. \end{aligned}$$

Simplifying we get

$$\begin{aligned} R^{n+1}[\tilde{U}]\tilde{U}_x &= R^{n+1}[U]U_x + 4CR^n[U]U_x + \sum_{j=0}^{n-1}\alpha_{n,j}C^{n-j}R^{j+1}[U]U_x + \sum_{j=0}^{n-1}4\alpha_{n,j}C^{n-j+1}R^j[U]U_x \\ &\quad + \frac{2\alpha_{n,0}}{n+1}C^{n+1}U_x, \\ &= R^{n+1}[U]U_x + \sum_{j=1}^{n}\alpha_{n,j-1}C^{n-j+1}R^j[U]U_x + \sum_{j=1}^{n}4\alpha_{n,j}C^{n-j+1}R^j[U]U_x \\ &\quad + 4\alpha_{n,0}C^{n+1}U_x + \frac{2\alpha_{n,0}}{n+1}C^{n+1}U_x. \end{aligned}$$

It follows that

$$\begin{aligned} R^{n+1}[\tilde{U}]\tilde{U}_x &= R^{n+1}[U]U_x + \sum_{j=1}^{n}\left(4\alpha_{n,j} + \alpha_{n,j-1}\right)C^{n+1-j}R^j[U]U_x + \left(4\alpha_{n,0} + \frac{2\alpha_{n,0}}{n+1}\right)C^{n+1}U_x \\ &= \sum_{j=0}^{n+1}\alpha_{n+1,j}C^{n+1-j}R^j[U]U_x, \end{aligned}$$

where

$$\begin{aligned}\alpha_{n+1,n+1} &= 1,\\ \alpha_{n+1,j} &= 4\alpha_{n,j}+\alpha_{n,j-1},\\ \alpha_{n+1,0} &= \frac{4n+6}{n+1}\alpha_{n,0}.\end{aligned}$$

Thus the relation is true for $n+1$ and the proof is finished. □

Proposition 2.1. *(Gordoa et al., 2013) There exists a choice of coefficient functions $\beta_i(t_{2n+1})$ and of the function $c(t_{2n+1})$ such that the substitution*

$$U=\frac{f(z)}{[2(2n+1)g_{n-1}t_{2n+1}]^{2/2n+1}}+h, \qquad z=\frac{x}{[2(2n+1)g_{n-1}t_{2n+1}]^{1/2n+1}}+c(t_{2n+1}), \tag{2.9}$$

where $g_{n-1}\neq 0$ and h are arbitrary constants, into the KdV hierarchy

$$U_{t_{2n+1}}=R^n[U]U_x+\sum_{i=1}^{n-1}\beta_i(t_{2n+1})R^i[U]U_x, \tag{2.10}$$

yields the generalized thirty-fourth Painlevé hierarchy

$$K[f](K[f])_{zz}-\frac{1}{2}(K[f]_z)^2+2f(K[f])^2+\frac{1}{2}(g_{n-1}+\alpha_n)^2=0, \tag{2.11}$$

where α_n is an arbitrary constant and

$$K[f]=M_n[f]+\sum_{i=0}^{n-1}B_iM_i[f]+g_{n-1}z. \tag{2.12}$$

In (2.12) the coefficients B_i are arbitrary constants and $M_j[f]$ is as given in (2.3) with dependent variable f and independent variable z.

Proof. Let

$$U=\frac{f(z)}{T^2}+h, \qquad z=\frac{x}{T}+c(t_{2n+1}), \text{ where } T=[2(2n+1)g_{n-1}t_{2n+1}]^{1/2n+1}. \tag{2.13}$$

Then we have

$$U=T^{-2}f(z)+h, \qquad x=T(z-c),$$

and

$$\frac{dz}{dt} = -2xg_{n-1}T^{-2n-2} + c'.$$

By the chain rule, we have

$$U_t = T^{-2}f'c' - 2(z-c)g_{n-1}T^{-(2n+3)}f' - 4g_{n-1}fT^{-(2n+3)}.$$

and

$$U_x = T^{-3}f_z.$$

Since $z = \frac{x}{T} + c(t_{2n+1})$, we have

$$\partial_x = T^{-1}\partial_z, \qquad \partial_x^{-1} = T\partial_z^{-1}.$$

Let

$$w(x,t) = T^{-2}f(z).$$

Then

$$w_x = T^{-3}f_z$$

and

$$\begin{aligned} R[w]w_x &= (\partial_x^2 + 4w + 2w_x\partial_x^{-1})T^{-3}f_z \\ &= (T^{-2}\partial_z^2 + 4T^{-2}f + 2T^{-2}f_z\partial_z^{-1})T^{-3}f_z. \end{aligned}$$

So that $R[w]w_x = T^{-5}R[f]f_z$. Therefore $R[w] = T^{-2}R[f]$. By (2.9) we have $U = w + h$. Now using Lemma 2.1, we obtain

$$R^n[U]U_x = \sum_{j=0}^{n} \alpha_{n,j}h^{n-j}R^j[w]w_x.$$

Using $R[w] = T^{-2}R[f]$, we have

$$\begin{aligned} R^n[U]U_x &= \sum_{j=0}^{n} \alpha_{n,j}h^{n-j}R^j[w]w_x \\ &= \sum_{j=0}^{n} \alpha_{n,j}h^{n-j}(T^{-2j}R^j[f])T^{-3}f_z. \end{aligned}$$

Thus (2.10) becomes

$$U_t = \sum_{j=0}^{n} \alpha_{n,j} h^{n-j} T^{-2j-3} R^j[f] f_z + \sum_{i=1}^{n-1} \beta_i(t) \sum_{j=0}^{i} \alpha_{i,j} h^{i-j} T^{-2j-3} R^j[f] f_z,$$

Substituting equation (2.9) in equation (2.10) gives

$$T^{-2} f' c' - 2(z-c) g_{n-1} T^{-2(n+3)} f' - 4 g_{n-1} f T^{-(2n+3)} = \sum_{j=0}^{n} \frac{\alpha_{n,j} h^{n-j}}{T^{2j+3}} \mathcal{R}^j[f] f_z + \sum_{i=1}^{n-1} \left(\beta_i \sum_{j=0}^{i} \frac{\alpha_{i,j}}{T^{2j+3}} \mathcal{R}^j[f] f_z \right)$$

Or

$$\sum_{j=0}^{n} \frac{\alpha_{n,j} h^{n-j}}{T^{2j+3}} \mathcal{R}^j[f] f_z + \sum_{i=1}^{n-1} \left(\beta_i \sum_{j=0}^{i} \frac{\alpha_{i,j}}{T^{2j+3}} \mathcal{R}^j[f] f_z \right) - \frac{1}{T^2} f_z c_(t_{2n+1})) - 2 \frac{g_{n-1}}{T^{2n+3}} f_z c + \frac{g_{n-1}}{T^{2n+3}} (4f + 2z f_z) = 0.$$

Then

$$\begin{aligned} \sum_{k=0}^{n} \gamma_k \mathcal{R}^k[f] f_z + \frac{g_{n-1}}{T^{2n+3}} (4f + 2z f_z) \quad = \quad & \sum_{j=0}^{n} \frac{\alpha_{n,j} h^{n-j}}{T^{2j+3}} \mathcal{R}^j[f] f_z + \sum_{i=1}^{n-1} \left(\beta_i \sum_{j=0}^{i} \frac{\alpha_{i,j} h^{i-j}}{T^{2j+3}} \mathcal{R}^j[f] f_z \right) \\ & - \frac{1}{T^2} f_z c_(t_{2n+1})) - 2 \frac{g_{n-1}}{T^{2n+3}} f_z c + \frac{g_{n-1}}{T^{2n+3}} (4f + 2z f_z) = 0. \end{aligned}$$

We solve the equations

$$\gamma_k = B_k / T^{2n+3}, \qquad k = n-1, \dots, 1$$

recursively for the coefficients β_k and the equation

$$\gamma_0 = B_0 / T^{2n+3},$$

for c, where all B_k are constants. The resulting equation can be written as

$$\mathcal{B}_1[f] K[f] = 0,$$

where $\mathcal{B}_1[f] = \partial_z^3 + 4f\partial_z + 2f_z$ and $K[f]$ is as given in (2.12) . This last equation admits (2.11) as a first integral, where α_n is an arbitrary constant of integration. □

Example 2.1. *Consider the fifth order KdV equation*

$$U_{t_5} = R^2[U] U_x + \beta_1(t_5) R[U] U_x.$$

Let $t_5 = t$. *Then using* $R[U] = \partial_x^2 + 4U + 2U_x\partial_x^{-1}$ *we obtain*

$$\begin{aligned} U_t &= (\partial_x^2 + 4U + 2U_x\partial_x^{-1})(\partial_x^2 + 4U + 2U_x\partial_x^{-1})U_x + \beta_1(t)(\partial_x^2 + 4U + 2U_x\partial_x^{-1})U_x \\ &= U_{xxxxx} + 10UU_{xxx} + 20U_xU_{xx} + 30U^2U_x + \beta_1(t)[U_{xxx} + 6UU_x]. \end{aligned} \tag{2.14}$$

Equation (2.14) *can be written as*

$$U_t = \left(U_{xxxx} + 10UU_{xx} + 5U_x^2 + 10U^3\right)_x + \beta_1(t)\left(U_{xx} + 3U^2\right)_x. \tag{2.15}$$

By Proposition 2.1, Equation (2.15) *admits the scaling reduction*

$$U = \frac{f(z)}{T^2} + h, \qquad z = \frac{x}{T} + c, \qquad T = [10g_1t]^{\frac{1}{5}}. \tag{2.16}$$

From (2.16) *we have*

$$U = T^{-2}(f + hT^2), \qquad x = T(z - c),$$

and

$$\frac{dT}{dt} = 2g_1T^{-4}, \qquad \frac{dz}{dx} = T^{-1}, \qquad \frac{\partial z}{\partial t} = c' - 2g_1xT^{-6}.$$

By differentiating U *with respect to* t *and* x *we obtain*

$$U_t = -T^{-7}[4g_1f + 2g_1(z - c)f' - T^5c'f']$$

and

$$U_x = T^{-3}f', \qquad U_{xx} = T^{-4}f'', \qquad U_{xxx} = T^{-5}f''', \qquad U_{xxxx} = T^{-6}f^{(4)}, \qquad U_{xxxxx} = T^{-7}f^{(5)}.$$

Substituting into equation (2.15), *we get*

$$\begin{aligned} &-T^{-7}\left[4g_1f + 2g_1(z-c)f' - T^5c'f'\right] = \beta_1(t)T^{-1}\frac{d}{dz}\left[T^{-4}f'' + 3f^{-4}\left(f + hT^2\right)^2\right] \\ &+T^{-1}\frac{d}{dz}\left[T^{-6}f^{(4)} + 10T^{-6}f''\left(f + hT^2\right) + 5T^{-6}\left(f'\right)^2 + 10T^{-6}\left(f + hT^2\right)^3\right]. \end{aligned}$$

Then

$$\begin{aligned} &\frac{d}{dz}\left[f_{4z} + 10ff_{zz} + 5f_z^2 + 10hT^2f_{zz} + 10\left(f^3 + 3hT^2f^2 + 3h^2T^4f + h^3T^6\right)\right] \\ &+T^2\beta_1(t)\frac{d}{dz}\left[f'' + 3\left(f^2 + 2hT^2f + h^2T^4\right)\right] + 2g_1(z-c)f' + 4g_1f - T^3c'f' = 0. \end{aligned}$$

Therefore

$$\frac{d}{dz}\left[f_{4z}+10ff_{zz}+5f_z^2+10f^3+10hT^2(f_{zz}+3f^2)+30h^2T^4f+10h^3T^6\right]+2g_1(zf'+2f)$$
$$-f'(T^3c'+2g_1c)+T^2\beta_1(t)\frac{d}{dz}\left[f_{zz}+3f^26hT^2f+3h^2T^4\right]=0.$$

Thus

$$\frac{d}{dz}\left[f_{zzzz}+10ff_{zz}+5f_z^2+10f^3\right]+2g_1(zf_z+2f)+10hT^2\left[\frac{d}{dz}(f_{zz}+3f^2+3hT^2f_z)\right]$$
$$+T^2\beta_1(t)\left[\frac{d}{dz}\left(f_{zz}+3f^2\right)+6hT^2f_z\right]-f_z\left(T^5c'+2g_1c\right)=0.$$

That is

$$\frac{d}{dz}\left[f_{4z}+10ff_{zz}+5f_z^2+10f^3\right]+2g_1(zf_z+2f)+T^2\left[\beta_1(t)+10h\right]\frac{d}{dz}\left(f_{zz}+3f^2\right)$$
$$+\left[30h^2T^4+6hT^4\beta_1(t)-T^5c'-2g_1c\right]f_z=0. \tag{2.17}$$

We set

$$T^2\left[\beta_1(t)+10h\right]=B_1=\textit{constant}, \tag{2.18}$$

and

$$6hT^4(5h+\beta_1(t))-T^5c'-2g_1c=B_0=\textit{constant}. \tag{2.19}$$

Then, from equation (2.18) *we have*

$$\beta_1(t)=\frac{B_1}{T^2}-10h.$$

Substituting $\beta_1(t)$ *into equation* (2.19) *we have*

$$\begin{aligned}T^5c'+2g_1c &= -B_0+6hT^4\left(5h+\frac{B_1}{T^2}-10h\right)\\ &= -B_0+6hT^4\left(\frac{B_1}{T^2}-5h\right).\end{aligned}$$

Since $T^5=10g_1t$, *we get*

$$10g_1tc'+2g_1c=-B_0+6hB_1T^2-30h^2T^4. \tag{2.20}$$

By solving the differential equation (2.20) we have

$$c = \frac{-B_0}{2g_1} + \frac{hB_1}{g_1}T^2 - \frac{3h^2}{g_1}T^4 + \frac{\tilde{c}_0}{T},$$

where $\tilde{c}_0 = c_0(10g_1)^{\frac{-4}{5}}$, *and* c_0 *is a constant of integration.*

Equation (2.17) becomes

$$\frac{d}{dz}\left[f_{4z} + 10ff_{zz} + 5f_z^2 + 10f^3\right] + 2g_1(zf_z + 2f) + B_1\frac{d}{dz}\left(f_{zz} + 3f^2\right) + B_0f_z = 0, \tag{2.21}$$

or

$$\frac{d}{dz}\left[f_{4z} + 10ff_{zz} + 5f_z^2 + 10f^3 + B_1f_{zz} + 3B_1f^2 + B_0f\right) + 2g_1(zf_z + 2f) = 0. \tag{2.22}$$

From equation (2.12), using $n = 2$ *we find*

$$K[f] = f_{zz} + 3f^2 + B_1f + \frac{1}{2}B_0 + g_1z.$$

It follows that

$$\frac{d}{dz}K[f] = f_{zzz} + 6ff_z + B_1f_z + g_1, \tag{2.23}$$

$$\frac{d^2}{dz^2}K[f] = f_{zzzz} + 6f_z^2 + 6ff_{zz} + B_1f_{zz}. \tag{2.24}$$

Now will write equation (2.22) in terms of $K[f]$*. Substituting* $f_{zzzz} = \frac{d^2}{dz^2}K[f] - 6f_z^2 - 6ff_{zz} - B_1f_{zz}$ *in equation (2.22) yields*

$$\frac{d}{dz}\left[\frac{d^2}{dz^2}K[f] - f_z^2 + 4ff_{zz} + 10f^3 + 3B_1f^2 + B_0f\right] + 2g_1(zf_z + 2f) = 0$$

or

$$\frac{d^3}{dz^3}\left(K[f]\right) + 2f_zf_{zz} + 4ff_{zzz} + 30f^2f_z + 6B_1ff_z + B_0f_z + 2g_1zf_z + 4g_1f = 0. \tag{2.25}$$

Substituting $f_{zzz} = \frac{d}{dz}K[f] - 6ff_z - B_1f_z - g_1$ *in equation (2.25) gives*

$$\frac{d^3}{dz^3}\left(K[f]\right) + 4f\frac{d}{dz}K[f] + 6f^2f_z + 2B_1ff_z + 2f_zf_{zz} + B_0f_z + 2g_1zf_z = 0$$

or

$$\frac{d^3}{dz^3}\left(K[f]\right) + 4f\frac{d}{dz}K[f]_z + 2f_zK[f] = 0. \tag{2.26}$$

Multiplying both sides by $K[f]$ and integrating with respect to z, we get

$$K[f]K[f]_{zz} - \frac{1}{2}\left(K[f]_z\right)^2 + 2f\left(K[f]\right)^2 + \frac{1}{2}(g_1+\alpha_2)^2 = 0, \tag{2.27}$$

where α_2 is an arbitrary constant. The equation (2.27) is called the thirty-fourth Painlevé equation (Ince, 1956).

2.2 The dispersive water wave hierarchy

The dispersive water wave (DWW) hierarchy is a two-component hierarchy in $\mathbf{u} = (u,v)^T$ given by (Gordoa et al., 2013)

$$\mathbf{u}_{t_n} = \mathcal{R}^n[\mathbf{u}]\mathbf{u}_x + \sum_{i=1}^{n-1}\gamma_i R^i[u]u_x,$$

where

$$\mathcal{R}[\mathbf{u}] = \frac{1}{2}\begin{pmatrix} \partial_x(u\partial_x^{-1}) - \partial_x & 2 \\ 2v + v_x\partial_x^{-1} & u + \partial_x \end{pmatrix}. \tag{2.28}$$

The recursion operator $\mathcal{R}$ can be written as the quotient of two Hamiltonian operators $\mathcal{R}[\mathbf{u}] = \mathcal{B}_2[\mathbf{u}]\mathcal{B}_1^{-1}[\mathbf{u}]$, where

$$\mathcal{B}_2[\mathbf{u}] = \frac{1}{2}\begin{pmatrix} 2\partial_x & \partial_x u - \partial_x^2 \\ u\partial_x + \partial_x^2 & v\partial_x + \partial_x v \end{pmatrix},$$

and

$$\mathcal{B}_1[\mathbf{u}] = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}.$$

The DWW hierarchy can be written as

$$\mathbf{u}_{t_n} = \mathcal{B}_1 L_{n+1}[\mathbf{u}] = \mathcal{B}_2 L_n[\mathbf{u}], \tag{2.29}$$

where the quantities $L_n[\mathbf{u}]$ in (2.29) is defined by

$$\begin{aligned} L_0 &= (0,2)^T, \\ L_n[\mathbf{u}] &= \mathcal{B}_1^{-1}[\mathbf{u}]\mathcal{B}_2[\mathbf{u}]L_{n-1}[\mathbf{u}]. \end{aligned}$$

$$(2.30)$$

In this section we will derive a fourth Painlevé hierarchy from a scaling reduction of DWW hierarchy (2.29).

Lemma 2.2. *(Gordoa and Pickering, 2011) The change of variables $\tilde{\boldsymbol{u}} = (u+C, v)^T$, where C is an arbitrary constant, in $R^n[\tilde{\boldsymbol{u}}]\tilde{\boldsymbol{u}}_x$, yields*

$$R^n[\tilde{\boldsymbol{u}}]\tilde{\boldsymbol{u}}_x = \sum_{j=0}^{n} \alpha_{n,j} C^{n-j} R^j[\boldsymbol{u}]\boldsymbol{u}_x, \tag{2.31}$$

where where $R^0[u]$ is the identity operator and the coefficients $\alpha_{n,j}$ are determined recursively by

$$\begin{aligned} \alpha_{n,n} &= 1, \quad \alpha_{0,0} = 1 \\ \alpha_{n,j} &= \frac{1}{2}\alpha_{n-1,j} + \alpha_{n-1,j-1}, \\ \alpha_{n,0} &= \frac{1}{2}\frac{n+1}{n}\alpha_{n-1,0}, \end{aligned} \tag{2.32}$$

Proof. For $n = 1$, we have

$$\begin{aligned} R[\tilde{\mathbf{u}}]\tilde{\mathbf{u}}_x &= \frac{1}{2}\begin{pmatrix} \partial_x \tilde{u}\partial_x^{-1} - \partial_x & 2 \\ 2v + v_x\partial_x^{-1} & \tilde{u} + \partial_x \end{pmatrix}\begin{pmatrix} \tilde{u}_x \\ v_x \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (\partial_x \tilde{u}\partial_x^{-1} - \partial_x)\tilde{u}_x + 2v_x \\ (2v + v_x\partial_x^{-1})\tilde{u}_x + (\tilde{u} + \partial_x)v_x \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2\tilde{u}\tilde{u}_x - \tilde{u}_{xx} + 2v_x \\ 2v\tilde{u}_x + v_x\tilde{u} + v_{xx} + v_x\tilde{u} \end{pmatrix}. \end{aligned}$$

Making the change of variables $\tilde{u} = u + C$ we obtain

$$\begin{aligned} R[\tilde{\mathbf{u}}]\tilde{\mathbf{u}}_x &= \frac{1}{2}\begin{pmatrix} 2(u+C)u_x + u_{xx} + 2v_x \\ 2vu_x + (u+C)v_x + v_{xx} + (u+C)v_x \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2uu_x - u_{xx} + 2v_x + 2Cu_x \\ 2vu_x + v_x u + v_{xx} + uv_x + 2Cv_x \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2uu_x - u_{xx} + 2v_x \\ 2vu_x + v_x u + v_{xx} + uv_x \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2Cu_x \\ 2Cv_x \end{pmatrix} \\ &= R[\mathbf{u}]\mathbf{u}_x + CR^0[\mathbf{u}]\mathbf{u}_x \end{aligned}$$

Therefore, the relation (2.31) is true for $n = 1$. Suppose (2.31) is true for n. Before proving that (2.31) is true for $n+1$ we will show that $R[\tilde{\mathbf{u}}] = R[\mathbf{u}] + \frac{1}{2}CI$, where I is the identity matrix operator.

$$R[\tilde{\mathbf{u}}] = \frac{1}{2}\begin{pmatrix} \partial_x(\tilde{u}\partial_x^{-1}) - \partial_x & 2 \\ 2v + v_x\partial_x^{-1} & \tilde{u} + \partial_x \end{pmatrix}.$$

Making the change of variables $\tilde{\mathbf{u}} = (u + C, v)^T$, we get

$$R[\tilde{\mathbf{u}}] = \frac{1}{2}\begin{pmatrix} \partial_x u\partial_x^{-1} - \partial_x + \partial_x C\partial_x^{-1} & 2 \\ 2v + v_x\partial_x^{-1} & u + \partial_x + C \end{pmatrix}$$

or

$$R[\tilde{\mathbf{u}}] = \frac{1}{2}\begin{pmatrix} \partial_x u\partial_x^{-1} - \partial_x & 2 \\ 2v + v_x\partial_x^{-1} & u + \partial_x \end{pmatrix} + \frac{1}{2}\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}.$$

Hence

$$R[\tilde{\mathbf{u}}] = R[\mathbf{u}] + \frac{1}{2}CI. \tag{2.33}$$

Now we will proof that (2.31) is true for $n+1$. We must deal carefully with the operator ∂_x^{-1} which makes the coefficient not binomial so that we added $\left(\frac{\alpha_{n,0}}{n+1}C^{n+1}\right)_x$ as a constant of integration

$$R^{n+1}[\tilde{\mathbf{u}}]\tilde{\mathbf{u}}_x = R[\tilde{\mathbf{u}}]R^n[\tilde{\mathbf{u}}]\tilde{\mathbf{u}}_x.$$

Using $R^{n+1}[\tilde{\mathbf{u}}] = RR^n[\tilde{\mathbf{u}}]$, equations (2.31) and (2.33), we obtain

$$R^{n+1}[\tilde{\mathbf{u}}]\tilde{\mathbf{u}}_x = \left[R[\mathbf{u}] + \frac{1}{2}CI\right]\left(R^n[\mathbf{u}]\mathbf{u}_x + \sum_{j=0}^{n-1}\alpha_{n,j}C^{n-j}R^j[\mathbf{u}]\mathbf{u}_x + \left(\frac{\alpha_{n,0}}{n+1}C^{n+1}\right)_x\right)$$

Thus

$$\begin{aligned} R^{n+1}[\tilde{\mathbf{u}}]\tilde{\mathbf{u}}_x &= R^{n+1}[\mathbf{u}]\mathbf{u}_x + \frac{1}{2}CR^n[\mathbf{u}]\mathbf{u}_x + \sum_{j=0}^{n-1}\alpha_{n,j}C^{n-j}R^{j+1}[\mathbf{u}]\mathbf{u}_x + \sum_{j=0}^{n-1}\frac{1}{2}\alpha_{n,j}C^{n+1-j}R^j[\mathbf{u}]\mathbf{u}_x \\ &\quad + \frac{1}{2}\left(\frac{\alpha_{n,0}}{n+1}\right)C^{n+1}\mathbf{u}_x. \end{aligned}$$

Collecting similar terms, we obtain

$$\begin{aligned} R^{n+1}[\tilde{\mathbf{u}}]\tilde{\mathbf{u}}_x &= R^{n+1}[\mathbf{u}]\mathbf{u}_x + \frac{1}{2}CR^n[\mathbf{u}]\mathbf{u}_x + \sum_{j=1}^{n}\alpha_{n,j-1}C^{n-j+1}R^j[\mathbf{u}]\mathbf{u}_x + \sum_{j=1}^{n-1}\frac{1}{2}\alpha_{n,j}C^{n+1-j}R^j[\mathbf{u}]\mathbf{u}_x \\ &\quad + \frac{1}{2}\left(\alpha_{n,0} + \frac{\alpha_{n,0}}{n+1}\right)C^{n+1}\mathbf{u}_x. \end{aligned}$$

As a result, we get

$$\begin{aligned} R^{n+1}[\tilde{\mathbf{u}}]\tilde{\mathbf{u}}_x &= R^{n+1}[\mathbf{u}]\mathbf{u}_x + \sum_{j=1}^{n}(\frac{1}{2}\alpha_{n,j} + \alpha_{n,j-1})C^{n-j}R^j[\mathbf{u}]\mathbf{u}_x + \frac{1}{2}\left(\frac{n+2}{n+1}\right)\alpha_{n,0}C^{n+1}\mathbf{u}_x \\ &= \sum_{j=0}^{n+1}\alpha_{n+1,j}C^{n+1-j}R^j[u]u_x, \end{aligned}$$

where

$$\begin{aligned} \alpha_{n+1,n+1} &= 1, \\ \alpha_{n+1,j} &= \frac{1}{2}\alpha_{n,j} + \alpha_{n,j-1}, \\ \alpha_{n+1,0} &= \frac{1}{2}\left(\frac{n+2}{n+1}\right)\alpha_{n,0}. \end{aligned}$$

Therefore the relation is true for $n+1$ and the proof is completed.

□

Proposition 2.2. *(Gordoa et al., 2013) There exists a choice of coefficient functions $\gamma_j(t_n)$ and of the function $c(t_n)$ such that the substitution*

$$u = \frac{f(z)}{[\frac{1}{2}(n+1)g_n t_n]^{1/(n+1)}} + h, \quad v = \frac{g(z)}{[\frac{1}{2}(n+1)g_n t_n]^{2/(n+1)}}, \quad z = \frac{x}{[\frac{1}{2}(n+1)g_n t_n]^{1/(n+1)}} + c(t_n), \tag{2.34}$$

where $g_n \neq 0$ and h are arbitrary constants, into the DWW hierarchy

$$\boldsymbol{u}_{t_n} = R^n[u]u_x + \sum_{i=1}^{n-1} \gamma_i R^i[u]u_x \tag{2.35}$$

yields the fourth Painlevé hierarchy in $\mathbf{f} = (f, g)^T$

$$0 = 2K + fL + g_n - 2\alpha_n - L_z, \tag{2.36}$$

$$0 = \left(K + \frac{1}{2}g_n - \alpha_n\right)^2 - \frac{1}{4}\beta_n^2 - gL^2 - K_zL. \tag{2.37}$$

Here α_n and β_n are arbitrary constants and K and L are the components of $\mathbf{K}[\mathbf{f}]$, $\mathbf{K} = (K, L)^T$, *and*

$$\mathbf{K}[\mathbf{f}] = L_n + \sum_{i=0}^{n-1} B_i L_i[f] + g_n \begin{pmatrix} 0 \\ z \end{pmatrix}. \tag{2.38}$$

In (2.38), coefficients B_i are arbitrary constants and $L_j[f]$ is as given in (2.30) with dependent variables f and g and independent variable z.

Proof. Let

$$u = \frac{f(z)}{T} + h, \qquad v = \frac{g(z)}{T^2} \qquad z = \frac{x}{T} + c(t_n), \quad \text{where } T = \left[\frac{1}{2}(n+1)g_n t_n\right]^{1/(n+1)}. \tag{2.39}$$

Then

$$u = T^{-1}f(z) + h, \qquad v = T^{-2}g, \qquad x = T(z - c),$$

and

$$\frac{dz}{dt} = -\frac{1}{2}xg_nT^{-(n+2)} + c'.$$

Differentiating $\mathbf{u}$ with respect to t yields

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}g_2T^{-(n+2)} + T^{-1}f'c' - \frac{1}{2}T^{-1}g_nT^{-(n+2)} + \frac{1}{2}g_nT^{-(n+2)}cf' \\ T^{-2}g'c' - \frac{1}{2}g_ng'T^{-(2n+1)}z + \frac{1}{2}g_ng'T^{-(2n+1)}c \end{pmatrix}. \tag{2.40}$$

Differentiating $\mathbf{u}$ with respect to x we get

$$\mathbf{u}_x = \begin{pmatrix} T^{-2}(t)f_z \\ T^{-3}(t)g_z \end{pmatrix}.$$

Thus $\mathbf{u}_x$ can be written as

$$\mathbf{u}_x = \begin{pmatrix} T^{-2} & 0 \\ 0 & T^{-3} \end{pmatrix} \mathbf{f}_z.$$

Let

$$\mathbf{w}(x,t) = (w_1, w_2)^T,$$

where

$$w_1 = T^{-1} f(z), \quad w_2 = T^{-2}(t) g(z), \quad z = T^{-1}x + c(t).$$

Hence

$$\partial_x = T^{-1}\partial_z, \quad \partial_x^{-1} = T\partial_z^{-1}, \quad w_x = u_x$$

and

$$R[\mathbf{w}]\mathbf{w}_x = \frac{1}{2} \begin{pmatrix} \partial_x w_1 \partial_x^{-1} - \partial_x & 2 \\ 2w_2 + (w_2)_x \partial_x^{-1} & w_1 + \partial_x \end{pmatrix} \begin{pmatrix} T^{-2} & 0 \\ 0 & T^{-3} \end{pmatrix} \mathbf{f}_z.$$

Substituting $w_1 = T^{-1} f(z)$ and $w_2 = T^{-2} g(z)$ and using $\partial_x = T^{-1}\partial_z$, $\partial_x^{-1} = T$ we have

$$\begin{aligned} R[\mathbf{w}]\mathbf{w}_x &= \frac{1}{2} \begin{pmatrix} T^{-1}\partial_z T^{-1} f T \partial_z^{-1} - T^{-1}\partial_z & 2 \\ 2T^{-2} g + T^{-3} g_z T \partial_z^{-1} & T^{-1} f + T^{-1}\partial_z \end{pmatrix} \begin{pmatrix} T^{-2} & 0 \\ 0 & T^{-3} \end{pmatrix} \mathbf{f}_z. \\ &= \frac{1}{2} \begin{pmatrix} T^{-1}[\partial_z f \partial_z^{-1} - \partial_z] & 2 \\ T^{-2}[2g + g_z \partial_z^{-1}] & T^{-1}[f + \partial_z] \end{pmatrix} \begin{pmatrix} T^{-2} & 0 \\ 0 & T^{-3} \end{pmatrix} \mathbf{f}_z. \end{aligned}$$

It follows that

$$R[\mathbf{w}]\mathbf{w}_x = \begin{pmatrix} T^{-3} & 0 \\ 0 & T^{-4} \end{pmatrix} R[\mathbf{f}]\mathbf{f}_z,$$

and hence

$$R[\mathbf{w}] = \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} R[\mathbf{f}],$$

$$\mathbf{u} = (w_1 + h, w_2)^T$$

Now using Lemma 2.2, we obtain

$$\begin{aligned} R^n[\mathbf{u}]\mathbf{u}_x &= \sum_{j=0}^{n} \alpha_{n,j} h^{n-j} R^j[\mathbf{w}]\mathbf{w}_x \\ &= \sum_{j=0}^{n} \alpha_{n,j} h^{n-j} \left[\begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix}^j R^j[\mathbf{f}] \right] \begin{pmatrix} T^{-2} & 0 \\ 0 & T^{-3} \end{pmatrix} \mathbf{f}_z. \end{aligned}$$

Then

$$R^n[\mathbf{u}]\mathbf{u}_x = \sum_{j=0}^{n} \alpha_{n,j} h^{n-j} \mathbf{T}_{j+2} R^j[\mathbf{f}]\mathbf{f}_z,$$

where

$$\mathbf{T}_j = \begin{pmatrix} T^{-j}(t) & 0 \\ 0 & T^{-(j+1)} \end{pmatrix}.$$

Thus

$$\mathbf{u}_t = \sum_{j=0}^{n} \alpha_{n,j} h^{n-j} \mathbf{T}_{j+2} R^j[\mathbf{f}]\mathbf{f}_z + \sum_{i=1}^{n-1} \gamma_i \sum_{j=0}^{i} \alpha_{i,j} h^{i-j} \mathbf{T}_{j+2} R^j[\mathbf{f}]\mathbf{f}_z.$$

Using $\mathbf{u}_t$ as in equation (2.40) we obtain

$$\begin{pmatrix} -\frac{1}{2} g_n T^{-(n+2)} + T^{-1} f' c' - \frac{1}{2} T^{-1} g_n T^{-(n+2)} + \frac{1}{2} g_n T^{-(n+2)} c f' \\ T^{-2} g' c' - \frac{1}{2} g_n g' T^{-(2n+1)} z + \frac{1}{2} g_n g' T^{-(2n+1)} c \end{pmatrix} = \sum_{j=0}^{n} \alpha_{n,j} h^{n-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z$$
$$+ \sum_{i=1}^{n-1} \gamma_i \left(\sum_{j=0}^{i} \alpha_{i,j} h^{i-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z \right),$$

or

$$\sum_{j=0}^{n} \alpha_{n,j} h^{n-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z + \sum_{i=1}^{n-1} \gamma_i \left(\sum_{j=0}^{i} \alpha_{i,j} h^{i-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z \right)$$
$$- \begin{pmatrix} -\frac{1}{2} g_n T^{-(n+2)} + T^{-1} f' c' - \frac{1}{2} T^{-1} g_n T^{-(n+2)} + \frac{1}{2} g_n T^{-(n+2)} c f' \\ T^{-2} g' c' - \frac{1}{2} g_n g' T^{-(2n+1)} z + \frac{1}{2} g_n g' T^{-(2n+1)} c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then

$$\sum_{j=0}^{n} \alpha_{n,j} h^{n-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z + \sum_{i=1}^{n-1} \gamma_i \left(\sum_{j=0}^{i} \alpha_{i,j} h^{i-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z \right) - \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{-2} \end{pmatrix} \begin{pmatrix} f' \\ g' \end{pmatrix} c'$$
$$+ \frac{1}{2} g_n \begin{pmatrix} T^{-(n+2)} & 0 \\ 0 & T^{-(2n+1)} \end{pmatrix} \begin{pmatrix} (zf)_z \\ 2g + z g_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore

$$\sum_{j=0}^{n} \alpha_{n,j} h^{n-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z + \sum_{i=1}^{n-1} \gamma_i \left(\sum_{j=0}^{i} \alpha_{i,j} h^{i-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z \right) - \mathbf{T}_1 \mathrm{f}_z c_{t_n} - \frac{1}{2} g_n \mathbf{T}_{j+2} \mathrm{f}_z c$$
$$+ \frac{1}{2} g_n \mathbf{T}_{j+2} \begin{pmatrix} (zf)_z \\ 2g + z g_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

That is

$$\sum_{k=0}^{n} \Gamma_k \mathcal{R}^k[\mathbf{f}]\mathbf{f}_z + \frac{1}{2} g_n \mathbf{T}_{j+2} \begin{pmatrix} (zf)_z \\ 2g + z g_z \end{pmatrix} = \sum_{j=0}^{n} \alpha_{n,j} h^{n-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z + \sum_{i=1}^{n-1} \gamma_i \left(\sum_{j=0}^{i} \alpha_{i,j} h^{i-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z \right)$$
$$- \mathbf{T}_1 \mathrm{f}_z c_{t_n} - \frac{1}{2} g_n \mathbf{T}_{j+2} \mathrm{f}_z c + \frac{1}{2} g_n \mathbf{T}_{n+2} \begin{pmatrix} (zf)_z \\ 2g + z g_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $\mathcal{R}[\mathbf{f}]$ is obtained from $\mathcal{R}[\mathbf{u}]$ by replacing u by f and ∂_x by ∂_z. We recall that each $\alpha_{i,i} = 1$, and so in particular $\Gamma_n = \mathrm{T}_{n+2}$. We solve the equations

$$\Gamma_k = B_k / \mathrm{T}_{n+2}, \qquad k = n-1, \ldots, 1$$

recursively for the coefficients γ_k and the equation

$$\Gamma_0 = B_0 / \mathrm{T}_{n+2}$$

for c, where all B_k are constants. The resulting ODE equation can be written

$$\mathcal{B}_2[f] K[f] = 0,$$

where $\mathcal{B}_2[f] = \partial_z^3 + 4f\partial_z + 2f_z$ and $K[f]$ is as given in (2.38). The last equation admits (2.54) as a first integral, where α_n and β_n are arbitrary constants of integration. □

Example 2.2. *The second nontrivial dispersive water wave.*

Let $t_2 = t$*. Then the hierarchy* (2.35) *for* $n = 2$ *gives*

$$\begin{aligned}\begin{pmatrix} u \\ v \end{pmatrix}_t &= \mathcal{R}^2[\boldsymbol{u}]\boldsymbol{u}_x + \gamma(t)\mathcal{R}[\boldsymbol{u}]\boldsymbol{u}_x \\ &= \frac{1}{4}\begin{pmatrix} u_{xxx} - 3uu_{xx} + 6uv_x + 6vu_x + 3u^2u_x + 3u_x^2 \\ v_{xxx} + 6vu_x + 3uv_{xx} + 3u_xv_x + 6uu_xv + 3u^2v_x \end{pmatrix} \\ &\quad + \frac{1}{2}\gamma(t)\begin{pmatrix} 2uu_x + 2v_x - u_{xx} \\ 2vu_x + 2v_x + v_{xx} \end{pmatrix}.\end{aligned} \tag{2.41}$$

Equation (2.41) *can be written as*

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \frac{1}{4}\begin{pmatrix} u_{xx} - 3uu_x + u^3 + 6uv \\ v_{xx} + 3v^2 + 3uv_x + 3u^2v \end{pmatrix}_x + \frac{1}{2}\gamma(t)\begin{pmatrix} 2v + u^2 - u_x \\ 2vu_x + v_x \end{pmatrix}_x. \tag{2.42}$$

By Proposition 2.2 *equation* (2.42) *admits the scaling reduction*

$$u = \frac{f(z)}{T} + h, \qquad v = \frac{g(z)}{T^2}, \qquad z = \frac{x}{T} + c, \qquad T = \left(\frac{3}{2}g_2t\right)^{1/3}. \tag{2.43}$$

From (2.43)*, we have*

$$\frac{dT}{dt} = \frac{1}{2}g_2T^{-2}, \qquad \frac{dz}{dx} = T^{-1}, \qquad \partial_x = T^{-1}\frac{d}{dz}, \qquad \frac{dz}{dt} = c' - \frac{1}{2}g_2xT^{-4}.$$

By differentiating u *with respect to* x *we obtain*

$$u_x = T^{-2}f', \qquad u_{xx} = T^{-3}f'', \qquad u_{xxx} = T^{-4}f'''.$$

By differentiating v *with respect to* x *we have*

$$v_x = T^{-3}g', \qquad v_{xx} = T^{-4}g'', \qquad v_{xxx} = T^{-5}g'''.$$

Differentiating u *and* v *with respect to* t *gives*

$$u_t = -T^{-4}\left[\frac{1}{2}g_2f + \frac{1}{2}g_2f' - T^3c'f'\right],$$

and

$$v_t = -T^{-5}\left[g_2g + \frac{1}{2}g_2(z-c)g' - T^3g'c'\right].$$

Substituting u_t, v_t into equation (2.42) we get,

$$-\begin{pmatrix} T^{-4}\left[\frac{1}{2}g_2 f + \frac{1}{2}g_2 f' - T^3 c' f'\right] \\ T^{-5}\left[g_2 g + \frac{1}{2}g_2(z-c)g' - T^3 g' c'\right] \end{pmatrix} = \frac{1}{4}T^{-1}\frac{d}{dz}\begin{pmatrix} T^{-3}\left[f_{zz} - 3f_z(f+hT) + (f+hT)^3 + 6(f+hT)g\right] \\ T^{-4}\left[g_{zz} + 3g^2 + 3g_z(f+hT) + 3g(f+hT)^2\right] \end{pmatrix}$$
$$+\frac{1}{2}\gamma_1(t)T^{-1}\frac{d}{dz}\begin{pmatrix} T^{-2}\left[2g + (f+hT)^2 - f_z\right] \\ T^{-3}\left[2g(f+hT)\right] \end{pmatrix}.$$

Multiplying both sides by $\begin{pmatrix} T^4 & 0 \\ 0 & T^5 \end{pmatrix}$, *gives*

$$-\begin{pmatrix} \frac{1}{2}g_2 f + \frac{1}{2}g_2 f' - T^3 c' f' \\ g_2 g + \frac{1}{2}g_2(z-c)g' - T^3 g' c' \end{pmatrix} = \frac{1}{4}\frac{d}{dz}\begin{pmatrix} \left[f_{zz} - 3f_z(f+hT) + (f+hT)^3 + 6(f+hT)g\right] \\ \left[g_{zz} + 3g^2 + 3g_z(f+hT) + 3g(f+hT)^2\right] \end{pmatrix}$$
$$+\frac{1}{2}\gamma_1(t)T^1\frac{d}{dz}\begin{pmatrix} T^{-2}\left[2g + (f+hT)^2 - f_z\right] \\ T^{-3}\left[2g(f+hT)\right] \end{pmatrix}.$$

Therefore

$$\frac{1}{4}\frac{d}{dz}\begin{pmatrix} f_{zz} - 3f_z f + f^3 + 6gf + hT\left[3f^2 - 3f_z + 6g + 3hTf\right] + h^3T^3 \\ g_{zz} + 3g^2 + 3g_z f + 3gf^2 + hT\left[3g_z + 6fg + 3hTg\right] \end{pmatrix} + \frac{1}{2}g_2\begin{pmatrix} zf_z + f \\ zg_z + 2g \end{pmatrix}$$
$$+\frac{1}{2}\gamma_1(t)T\frac{d}{dz}\begin{pmatrix} 2g + f^2 + 2fhT - f_z + h^2T^2 \\ 2gf + 2hTg + g_z \end{pmatrix} - \left[T^3c' + \frac{1}{2}g_2 c\right]\begin{pmatrix} f' \\ g' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus

$$\frac{1}{4}\frac{d}{dz}\begin{pmatrix} f_{zz} - 3f_z f + f^3 + 6gf \\ g_{zz} + 3g^2 + 3g_z f + 3gf^2 \end{pmatrix} + \frac{3}{4}hT\begin{pmatrix} \frac{d}{dz}\left[f^2 - f_z + 2g\right] + hTf_z \\ \frac{d}{dz}\left[g_z + 2fg\right] + hTg_z \end{pmatrix}$$
$$+\frac{1}{2}\gamma_1(t)T\begin{pmatrix} \frac{d}{dz}\left[f^2 - f_z + 2g\right] + 2hTf_z \\ \left[g_z + 2fg\right] + 2hTg_z \end{pmatrix} + \frac{1}{2}g_2\begin{pmatrix} zf_z + f \\ zg_z + 2g \end{pmatrix} - \left[T^3c' + \frac{1}{2}g_2 c\right]\begin{pmatrix} f' \\ g' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So

$$\frac{1}{4}\frac{d}{dz}\begin{pmatrix} f_{zz} - 3f_z f + 3f^3 + 6gf \\ g_{zz} + 3g^2 + 3g_z f + 3gf^2 \end{pmatrix} + \frac{1}{2}T\left[\gamma_1(t) + \frac{3}{2}h\right]\frac{d}{dz}\begin{pmatrix} f^2 - f_z + 2g \\ g_z + 2fg \end{pmatrix}$$
$$+\left[\frac{3}{4}h^2T^2 + hT^2\gamma_1(t) - T^3c' - \frac{1}{2}g_2 c\right]\begin{pmatrix} f_z \\ g_z \end{pmatrix} + \frac{1}{2}g_2\begin{pmatrix} zf_z + f \\ zg_z + 2g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$(2.44)$$

We set

$$T\left[\gamma_1(t)+\frac{3}{2}h\right]=B_1=constant. \tag{2.45}$$

$$hT^2\left(\frac{3}{4}h+\gamma_1(t)\right)-T^3c'-\frac{1}{2}g_2c=B_0=constant. \tag{2.46}$$

Then, from equation (2.45) we have

$$\gamma_1(t_2)=\frac{B_1}{T}-\frac{3}{2}h.$$

Substituting $\gamma_1(t)$ in equation (2.46) we have

$$\begin{aligned} T^3c'+\frac{1}{2}g_2c &= -B_0+hT^2\left(\frac{3}{4}h+\frac{B_1}{T}-\frac{3}{2}h\right) \\ &= -B_0+hT^2\left(\frac{B_1}{T}-\frac{3}{4}h\right). \end{aligned}$$

Since $T^3=\frac{3}{2}g_2t$, we get

$$\frac{3}{2}g_2tc'+\frac{1}{2}g_2c=-B_0+hB_1T-\frac{3}{4}T^2h^2. \tag{2.47}$$

By solving the differential equation (2.47) we have

$$c=\frac{-2B_0}{g_2}+\frac{hT}{g_2}B_1-\frac{h^2}{2g_2}T^2+\frac{\tilde{c_0}}{T},$$

where $\tilde{c}_0=c_0(\frac{3}{2}g_2)^{\frac{-1}{3}}$, and c_0 is the constant of integration.

Equation (2.44) becomes

$$\frac{1}{4}\frac{d}{dz}\begin{pmatrix} f_{zz}-3f_zf+3f^3+6gf \\ g_{zz}+3g^2+3g_zf+3gf^2 \end{pmatrix}+\frac{1}{2}B_1\frac{d}{dz}\begin{pmatrix} f^2-f_z+2g \\ g_z+2fg \end{pmatrix}+B_0\begin{pmatrix} f_z \\ g_z \end{pmatrix}$$

$$+\frac{1}{2}g_2\begin{pmatrix} zf_z+f \\ zg_z+2g \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$\begin{pmatrix} f_{zz}-3f_zf+3f^3+6gf+2B_1f^2-2B_1f_z+4B_1g+4B_0f \\ g_{zz}+3g^2+3g_zf+3gf^2+2B_1g_z+4B_1fg+4B_0g \end{pmatrix}_z+2g_2\begin{pmatrix} zf_z+f \\ zg_z+2g \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.48}$$

From equation (2.38), using $n = 2$ we find

$$\begin{pmatrix} K \\ L \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2fg + g_z \\ 2g + f^2 - f_z \end{pmatrix} + B_1 \begin{pmatrix} g \\ f \end{pmatrix} + B_0 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + g_2 \begin{pmatrix} 0 \\ z \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} K \\ L \end{pmatrix}_z = \begin{pmatrix} f_z g + g_z f + \frac{1}{2} g_{zz} + B_1 g_z \\ g_z + f f_z - \frac{1}{2} f_{zz} + B_1 f_z + g_2 \end{pmatrix}.$$

Now will write equation (2.48) in terms of $(K, L)^T$.

Substituting $\begin{pmatrix} g_{zz} \\ f_{zz} \end{pmatrix} = \begin{pmatrix} 2K_z - 2f_{zg} - 2g_z f - 2B_1 g_z \\ -2L_z + 2g_z + 2ff_z + 2B_1 f_z + 2g_2 \end{pmatrix}$ *in equation (2.48) yields*

$$2\begin{pmatrix} -L_z \\ K_z \end{pmatrix}_z + \begin{pmatrix} 2g_{zz} + f_z\left[-f_z + 3f^2 + 6g + 4B_1 f + 4B_0 + 2g_2 z\right] - ff_{zz} + 6g_z f + 4B_1 g_z + 2g_2 f \\ g_{zz} f + g_z\left[-f_z + 3f^2 + 6g + 4B_1 f + 4B_0 + 2g_2 z\right] - 2g f_{zz} + 6gff_z f + 4B_1 g f_z + f g_2 5 \end{pmatrix}$$
$$+ \begin{pmatrix} 2g_2 \\ 0 \end{pmatrix}_z = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$2\begin{pmatrix} -L_z \\ K_z \end{pmatrix}_z + \begin{pmatrix} 2f_z L + f\left[-f_{zz} + 2g_z + 2ff_z + 2B_1 f_z + 2g_2\right] + \left[2g_{zz} + 4B_1 g_z + 4g_z f + 4g f_z\right] \\ 2g_z L + g\left[-2f_{zz} + 4g_z + 4ff_z + 4B_1 f_z + 4g_2\right] + f\left[g_{zz} + 2g_z f + 2B_1 g_z + 2g f_z\right] \end{pmatrix}$$
$$+ \begin{pmatrix} 2g_2 \\ 0 \end{pmatrix}_z = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows that

$$2\begin{pmatrix} -L_z \\ K_z \end{pmatrix}_z + \begin{pmatrix} 2f_z L + fL_z + 4K_z \\ 2g_z L + 4gL_z + 2Kf_z \end{pmatrix} + \begin{pmatrix} 2g_2 \\ 0 \end{pmatrix}_z = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Multiplying both sides by $\dfrac{1}{2}$ and integrating with respect to z, we get

$$\begin{pmatrix} 2K + fL + g_2 - 2\alpha_2 - L_z \\ \left(K + \frac{1}{2}g_2 - \alpha_2\right)^2 - \frac{1}{4}\beta_2^2 - gL^2 - K_z L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.49}$$

where α_2 and β_2 are arbitrary constants. The equation (2.49) is called the fourth Painlevé equation.

2.3 The Burgers hierarchy

The Burgers hierarchy is given by (Gordoa et al., 2013)

$$U_{t_{n+1}} = \mathcal{R}^n[U]U_x, \qquad \mathcal{R}[U] = \partial_x(\partial_x + \frac{1}{2})\partial_x^{-1}, \tag{2.50}$$

or alternatively

$$U_{t_{n+1}} = \partial_x \mathcal{L}_n[U] = \partial_x \mathcal{T}^n[U]U, \qquad \mathcal{T}[U] = \partial_x + \frac{1}{2}U, \tag{2.51}$$

where $\mathcal{L}_n[U]$ are defined by

$$\begin{aligned}
\mathcal{L}_{-1}[U] &= 2, \\
\mathcal{L}_0[U] &= \mathcal{T}^0[U]U = U, \\
\mathcal{L}_1[U] &= \mathcal{T}[U]U = U_x + \frac{1}{2}U^2, \\
\mathcal{L}_n[U] &= \mathcal{T}^n[U]U.
\end{aligned}$$

In this section we will study a scaling reduction of Burgers hierarchy (2.51) and this reduction leads to a hierarchy of ODEs.

Lemma 2.3. *(Gordoa and Pickering, 2011) The change of variables $\tilde{U} = U + C$, where C is an arbitrary constant, in $\mathcal{L}_n[\tilde{U}]$, yields*

$$\mathcal{L}_n[\tilde{U}] = \sum_{j=-1}^{n} \binom{n+1}{j+1} \left(\frac{1}{2}C\right)^{n-j} \mathcal{L}_j[U], \tag{2.52}$$

where we define $\mathcal{L}_{-1}[U] = 2$.

Proof. We will prove this Lemma by induction. For $n = 1$, we have

$$\begin{aligned}
\mathcal{L}_1[\tilde{U}] &= \mathcal{T}[\tilde{U}]\tilde{U} \\
&= (\partial_x + \frac{1}{2}\tilde{U})\tilde{U} \\
&= \tilde{U}_x + \frac{1}{2}\tilde{U}^2.
\end{aligned}$$

Since $\tilde{U} = U + C$, we have

$$\begin{aligned}\mathcal{L}_1[\tilde{U}] &= (U+C)_x + \frac{1}{2}(U+C)^2 \\ &= U_x + \frac{1}{2}U^2 + UC + \frac{1}{2}C^2 \\ &= \mathcal{L}_1[U] + \mathcal{L}_0[U]C + \frac{1}{2}C^2.\end{aligned}$$

This show that (2.52) is true for $n = 1$. Suppose (2.52) is true for n, which can be written as

$$\mathcal{L}_n[\tilde{U}] = \sum_{j=-1}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j} \mathcal{T}^j[U]U,$$

and

$$\mathcal{T}[\tilde{U}] = \partial_x + \frac{1}{2}\tilde{U}$$

Making the change of variables $\tilde{U} = U + C$, we obtain

$$\begin{aligned}\mathcal{T}[\tilde{U}] &= \partial_x + \frac{1}{2}(U+C) \\ &= \mathcal{T}[U] + \frac{1}{2}C.\end{aligned}$$

Now for $n+1$, we have

$$\mathcal{L}_{n+1}[\tilde{U}] = \mathcal{T}[\tilde{U}]\mathcal{T}^n[\tilde{U}]\tilde{U}.$$

Making the change of variables $\tilde{U} = U + C$, we obtain

$$\begin{aligned}\mathcal{L}_{n+1}[\tilde{U}] &= \left[T[U] + \frac{1}{2}C\right]\left[\sum_{j=-1}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j} T^j[U]U\right] \\ &= T[U]\left[\sum_{j=-1}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j} T^j[U]U\right] + \frac{1}{2}C\left[\sum_{j=-1}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j} T^j[U]U\right] \\ &= \left[\sum_{j=-1}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j} T^{j+1}[U]U\right] + \left[\sum_{j=-1}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j+1} T^j[U]U\right].\end{aligned}$$

Hence, from (2.52) we obtain

$$\mathcal{L}_{n+1}[\tilde{U}] = \left[\sum_{j=0}^{n+1} \begin{pmatrix} n+1 \\ j \end{pmatrix} \left(\frac{1}{2}C\right)^{n-(j-1)} \mathcal{L}_j[U]\right] + \left[\sum_{j=-1}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j+1} \mathcal{L}_j[U]\right].$$

Collecting similar terms, we obtain

$$
\begin{aligned}
\mathcal{L}_{n+1}[\tilde{U}] &= \begin{pmatrix} n+1 \\ 0 \end{pmatrix} \left(\frac{1}{2}C\right)^{n+2} \mathcal{L}_{-1}[U] + \begin{pmatrix} n+1 \\ n+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-(n+1-1)} \mathcal{L}_{n+1}[U] \\
&\quad + \left[\sum_{j=0}^{n} \begin{pmatrix} n+1 \\ j \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j+1)} \mathcal{L}_j[U]\right] + \left[\sum_{j=0}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j+1} \mathcal{L}_j[U]\right]. \\
&= \left(\frac{1}{2}C\right)^{n+2} \mathcal{L}_{-1}[U] + \left(\frac{1}{2}C\right)^{0} \mathcal{L}_{n+1}[U] \\
&\quad + \sum_{j=0}^{n} \left[\begin{pmatrix} n+1 \\ j+1 \end{pmatrix} + \begin{pmatrix} n+1 \\ j \end{pmatrix}\right] \left(\frac{1}{2}C\right)^{n-j+1} \mathcal{L}_j[U].
\end{aligned}
$$

So

$$
\begin{aligned}
\mathcal{L}_{n+1}[\tilde{U}] &= \begin{pmatrix} n+2 \\ 0 \end{pmatrix} \left(\frac{1}{2}C\right)^{n+2} \mathcal{L}_{-1}[U] + \begin{pmatrix} n+2 \\ n+2 \end{pmatrix} \left(\frac{1}{2}C\right)^{0} \mathcal{L}_{n+1}[U] \\
&\quad + \sum_{j=0}^{n} \begin{pmatrix} n+2 \\ j+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j+1} \mathcal{L}_j[U].
\end{aligned}
$$

That is

$$
\mathcal{L}_{n+1}[\tilde{U}] = \sum_{j=-1}^{n+1} \begin{pmatrix} n+2 \\ j+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j+1} \mathcal{L}_j[U].
$$

Therefore the relation is true for $n+1$ and the proof is complete. □

Proposition 2.3. *(Gordoa et al., 2013) There exists a choice of coefficient functions $\beta_i(t_{n+1})$ and of the function $c(t_{n+1})$ such that the substitution*

$$
U = \frac{f(z)}{[(n+1)g_{n-1}t_{n+1}]^{1/(n+1)}} + h, \qquad z = \frac{x}{[(n+1)g_{n-1}t_{n+1}]^{1/(n+1)}} + c(t_{n+1}), \quad (2.53)
$$

where $g_{n-1} \neq 0$ and d are arbitrary constants, into the hierarchy

$$
U_{t_{n+1}} = R^n[U]U_x + \sum_{i=1}^{n-1} \beta_i(t_{n+1}) R^i[U]U_x, \qquad (2.54)
$$

yields the hierarchy of ODEs

$$
\mathcal{L}_n[f] + \sum_{i=-1}^{n-1} B_i \mathcal{L}_i[f] + g_{n-1} z f = 0, \qquad (2.55)
$$

where

$$\mathcal{L}_n[f] = \sum_{j=-1}^{n} \left(\begin{array}{c} n+1 \\ j+1 \end{array} \right) \left(\frac{1}{2}h\right)^{n-j} \mathcal{L}_j[U],$$

with dependent variable f and independent variable z, and where the coefficient B_i are arbitrary constant.

Proof. Let

$$U = \frac{f(z)}{T} + h, \qquad z = \frac{x}{T} + c(t_{n+1}), \qquad \text{where } T = [(n+1)g_{n-1}t_{n+1}]^{1/n+1}. \quad (2.56)$$

Then

$$U = T^{-1}f(z) + h, \qquad x = T(z-c), \qquad \frac{dz}{dx} = \frac{1}{T},$$

and

$$\frac{\partial z}{\partial t} = -xg_{n-1}T^{-(n+2)} + c'.$$

It follows that

$$U_x = T^{-1}f_z.$$

$$U_t = -g_{n-1}fT^{-(n+2)} - g_{n-1}f'zT^{-(n+2)} + g_{n-1}f'T^{-(n+2)} + T^{-1}f'c'_{t_{n+1}}. \qquad (2.57)$$

Since $z = \frac{x}{T} + c(t_{n+1})$, we have

$$\partial_x = T^{-1}\partial_z, \qquad \partial_x^{-1} = T\partial_z^{-1}.$$

Let

$$w(x,t) = T^{-1}f(z).$$

Then

$$w_x = T^{-2}f_z$$

and

$$\begin{aligned} R[w]w_x &= \partial_x\mathcal{L}_n[w]w_x \\ &= \partial_x[\partial_x + \frac{1}{2}w]\partial_x^{-1}T^{-2}f_z \end{aligned}$$

So that $R[w]w_x = T^{-3}R[f]f_z$. Therefore $R[w] = T^{-1}R[f]$. By (2.53) we have $U = w + h$. Now using Lemma 2.3, we obtain

$$\begin{aligned} R^n[U]U_x &= \partial_x \mathcal{L}_n[w]w_x, \\ &= \partial_x \sum_{j=0}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}h\right)^{n-j} \mathcal{L}_j[w]w_x, \\ &= \sum_{j=0}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}h\right)^{n-j} T^{-(j+2)}R^j[f]f_z \end{aligned}$$

Thus (2.54) becomes

$$U_t = \sum_{j=0}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}h\right)^{n-j} T^{-(j+2)}R^j[f]f_z + \sum_{i=1}^{n-1} \beta_i(t) \sum_{j=0}^{i} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}h\right)^{i-j} T^{-(j+2)}R^j[f]f_z.$$

Substituting equation (2.53) into equation (2.54) gives

$$\begin{aligned} \frac{1}{T}f_z c_{t_{n+1}} + \frac{g_{n-1}}{T^{n+2}}f_z c - \frac{g_{n-1}}{T^{n+2}}z f_z &= \sum_{j=0}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}h\right)^{n-j} \frac{1}{T^{j+2}}\mathcal{R}_j[f]f_z \\ &\quad + \sum_{i=0}^{n-1} \beta_i \left(\sum_{j=0}^{i} \begin{pmatrix} i+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}h\right)^{i-j} \frac{1}{T^{j+2}}\mathcal{R}_j[f]f_z \right). \end{aligned}$$

Then

$$\sum_{j=0}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}h\right)^{n-j} \frac{1}{T^{j+2}}\mathcal{R}_j[f]f_z + \sum_{i=0}^{n-1} \beta_i \left(\sum_{j=0}^{i} \begin{pmatrix} i+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}h\right)^{i-j} \frac{1}{T^{j+2}}\mathcal{R}_j[f]f_z \right)$$
$$-\frac{1}{T}f_z c'_{t_{n+1}} - \frac{g_{n-1}}{T^{n+2}}f_z c + \frac{g_{n-1}}{T^{n+2}}(zf)_z = 0.$$

That is

$$\begin{aligned} \sum_{k=0}^{n} \gamma_k \mathcal{R}_k[f]f_z + \frac{g_{n-1}}{T^{n+2}}f_z c + \frac{g_{n-1}}{T^{n+2}}(zf)_z &= \sum_{j=0}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}h\right)^{n-j} \frac{1}{T^{j+2}}\mathcal{R}_j[f]f_z \\ &\quad + \sum_{i=0}^{n-1} \beta_i \left(\sum_{j=0}^{i} \begin{pmatrix} i+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}h\right)^{i-j} \frac{1}{T^{j+2}}\mathcal{R}_j[f]f_z \right) \\ &\quad - \frac{1}{T}f_z c'_{t_{n+1}} - \frac{g_{n-1}}{T^{n+2}}f_z c + \frac{g_{n-1}}{T^{n+2}}(zf)_z = 0, \end{aligned}$$

where $\mathcal{L}_{-1}$ is constant and $\gamma_n = 1/T^{n+2}$. We solve the equations

$$\gamma_k = B_k/T^{n+2}, \qquad k = n-1, \dots, 1,$$

recursively for the coefficients γ_k and the equation

$$\gamma_0 = B_0/T^{n+2}$$

for c, where all B_k are constants. Integrating the resulting ODE then yields (2.55), where we include a constant of integration as the term $B_{-1}\mathcal{L}_{-1}[f] = 2B_{-1}$. □

Proposition 2.4. *(Gordoa et al., 2013) The hierarchy (2.55) is linearizable by the Cole-Hopf transformation $f = \dfrac{2\varphi z}{\varphi}$ to the hierarchy of ODEs*

$$\partial_z^{n+1}\varphi + \sum_{i=1}^{n-1} B_i \partial_z^{i+1}\varphi + g_{n-1} z\varphi = 0. \tag{2.58}$$

Proof. To prove equation (2.58) we want to verify that

$$\mathcal{L}_n[f] = \frac{2}{\varphi}\partial_z^{n+1}\varphi. \tag{2.59}$$

We will use induction to prove this. First of all we have

$$\mathcal{L}_n[f] = \mathrm{T}^n[f]f = (\partial_z + \frac{1}{2}f)^n f.$$

Let $f = 2\varphi_z/\varphi$. Now for $n = 1$ we obtain

$$\begin{aligned}
\mathcal{L}_1[f] &= \mathrm{T}[f]f \\
&= (\partial_z + \frac{1}{2}f)f \\
&= f_z + \frac{1}{2}f^2.
\end{aligned}$$

Using $f = 2\varphi_z/\varphi$ we have

$$\mathcal{L}_1[f] = 2\frac{\varphi_{zz}}{\varphi} - 2\left(\frac{\varphi_z}{\varphi}\right)^2 + \frac{1}{2}\left(2\frac{\varphi_z}{\varphi}\right)^2.$$

Thus

$$\mathcal{L}_1[f] = \frac{2}{\varphi}\partial_z^2\varphi, \tag{2.60}$$

and we have the relation (2.59) is true for $n = 1$. Suppose the relation (2.59) is true for n. For $n+1$ we have

$$\begin{aligned}
\mathcal{L}_{n+1}[f] &= \mathrm{T}[f]\mathrm{T}^n[f]f \\
&= (\partial_z + \frac{1}{2}f)(\frac{2}{\varphi}\partial_z^{n+1}\varphi) \\
&= \partial_z(\frac{2}{\varphi}\partial_z^{n+1}\varphi) + \frac{1}{\varphi}f\partial_z^{n+1}\varphi \\
&= \frac{2}{\varphi}\partial_z^{n+2}\varphi - 2\frac{\varphi_z}{\varphi^2}\partial_z^{n+1}\varphi + 2\frac{\varphi_z}{\varphi^2}\partial_z^{n+1}\varphi.
\end{aligned}$$

Hence we have

$$\mathcal{L}_{n+1}[f] = \frac{2}{\varphi}\partial_z^{n+2}\varphi. \tag{2.61}$$

That is the relation (2.59) is true for $n+1$. Substituting $\mathcal{L}_n$, $\mathcal{L}_i$ and f in hierarchy (2.55) gives

$$\frac{2}{\varphi}\partial_z^{n+1}\varphi + \sum_{i=1}^{n-1} B_i\left(\frac{2}{\varphi}\partial_z^{i+1}\varphi\right) + g_{n-1}z\left(\frac{2\varphi_z}{\varphi}\right) = 0.$$

Multiplying both side by $\dfrac{\varphi}{2}$ we obtain

$$\partial_z^{n+1}\varphi + \sum_{i=1}^{n-1} B_i\partial_z^{i+1}\varphi + g_{n-1}z\varphi_z = 0.$$

□

Example 2.3. *Consider the second nontrivial member of the Burgers hierarchy*

$$U_{t_3} = R^2[U]U_x + \beta_1 R[U]U_x. \tag{2.62}$$

We have

$$R^2[U]U_x = \partial_x\mathcal{L}_2[U] = \partial_x\left(U_{xx} + \frac{3}{2}UU_x + \frac{1}{4}U^3\right), \tag{2.63}$$

and

$$R[U]U_x = \partial_x\mathcal{L}_1[U] = \partial_x\left(U + \frac{1}{2}U^2\right). \tag{2.64}$$

Let $t_3 = t$. From equations (2.63)-(2.64) we have

$$U_t = \left(U_{xx} + \frac{3}{2}UU_x + \frac{1}{4}U^3\right)_x + \left(U + \frac{1}{2}U^2\right)_x. \tag{2.65}$$

By Proposition 2.3 equation (2.65) admits the scaling reduction

$$U = \frac{f(z)}{T} + h, \qquad z = \frac{x}{T} + c(t), \qquad T = [3g_1 t]^{\frac{1}{3}}. \tag{2.66}$$

It follows that

$$U = T^{-1}(f + hT), \quad x = T(z - c), \quad \frac{dz}{dx} = T^{-1},$$

and

$$\frac{dT}{dt} = g_1 T^{-2}, \quad \frac{\partial}{\partial_x} = T^{-1}\frac{d}{dz}, \quad \frac{\partial z}{\partial t} = -xg_1 T^{-4} + c'.$$

By differentiating U with respect to x and t we obtain

$$U_x = T^{-2} f', \quad U_{xx} = T^{-3} f'', \quad U_{xxx} = T^{-4} f''',$$

and

$$U_t = -T^{-4}\left[g_1(f + f'z) - f'(g_1 c + T^3 c')\right].$$

Substituting into equation (2.65) gives

$$-T^{-4}\left[g_1(f + f'z) - f'(g_1 c + T^3 c')\right] = T^{-1}\frac{d}{dz}\left[T^{-3}f'' + \frac{3}{2}T^{-3}f'(f + hT)\frac{1}{4}T^{-3}(f + hT)^3\right]$$
$$+\beta_1 T^{-1}\frac{d}{dz}\left[T^{-2}f' + \frac{1}{2}T^{-2}(f + hT)^2\right].$$

Then

$$\frac{d}{dz}\left[f_{zz} + \frac{3}{2}f_z f + \frac{3}{2}f_z hT + \frac{1}{4}f^3 + \frac{3}{4}f^2 hT + \frac{3}{4}fh^2T^2 + \frac{1}{4}h^3T^3\right]$$
$$+\beta_1(t)T\frac{d}{dz}\left[f_z + \frac{1}{2}f^2 + fhT + \frac{1}{2}h^2T^2\right] + g_1(f + zf_z) - f_z(g_1 c + T^3 c') = 0.$$

Therefore

$$\frac{d}{dz}\left[f_{zz} + \frac{3}{2}f_z f + \frac{1}{4}f^3\right] + g_1(f + zf_z) + \frac{3}{2}hT\frac{d}{dz}\left[f_z + \frac{1}{2}f^2\right] + \frac{3}{4}h^2T^2 f_z$$
$$+\beta_1(t)T\frac{d}{dz}\left[f_z + \frac{1}{2}f^2\right] + \beta_1(t)T^2 h f_z - f_z(g_1 c + T^3 c') = 0.$$

That is

$$\frac{d}{dz}\left[f_{zz} + \frac{3}{2}f_z f + \frac{1}{4}f^3\right] + g_1(f + zf_z) + \left[\frac{3}{2}hT + \beta_1(t)T\right]\frac{d}{dz}(f_z + \frac{1}{2}f^2)$$
$$+\left[\frac{3}{4}h^2T^2 + \beta_1(t)T^2 h - T^3 c' - g_1 c\right] f_z = 0.$$

(2.67)

We set

$$T\left[\beta_1(t)+\frac{3}{2}h\right]=B_1=constant, \tag{2.68}$$

and

$$hT^2(\frac{3}{4}h+\beta_1(t))-T^3c'-g_1c=B_0=constant. \tag{2.69}$$

Then, from equation (2.68), we get

$$\beta_1(t)=\frac{B_1}{T}-\frac{3}{2}h. \tag{2.70}$$

Substituting for $\beta_1(t)$ from (2.70) in (2.69), we have

$$T^3c'+g_1c=-B_0+hT^2(\frac{B_1}{T}-\frac{3}{4}h). \tag{2.71}$$

Since $T^3=3g_1t$, equation (2.71) becomes

$$c'+\frac{1}{3t}c=\frac{-1}{3g_1t}B_0+B_1t^{\frac{-2}{3}}(3g_1)^{\frac{-2}{3}}h-\frac{3}{4}h^2t^{\frac{-1}{3}}(3g_1)^{\frac{-1}{3}}. \tag{2.72}$$

By solving the ODE (2.72), we have

$$c=\frac{-B_0}{g_1}+\frac{B_1h}{2g_1}T-\frac{h^2}{4g_1}T^2+\frac{\tilde{c}}{T},$$

where $\tilde{c}=c_0(3g_1)^{\frac{-1}{3}}$, and c_0 is a constant of integration.

Equation (2.67) becomes

$$\frac{d}{dz}\left[f_{zz}+\frac{3}{2}f_zf+\frac{1}{4}f^3\right]+g_1(f+zf_z)+B_1\frac{d}{dz}(f_z+\frac{1}{2}f^2)+B_0f_z=0.$$

Integration with respect to z gives

$$f_{zz}+\frac{3}{2}f_zf+\frac{1}{4}f^3+g_1zf+B_1(f_z+\frac{1}{2}f^2)+B_0f+2B_{-1}=0, \tag{2.73}$$

where $2B_{-1}$ is a constant of integration. Equation (2.73) is linearizable by the Cole-Hopf transformation $f=\dfrac{2\varphi z}{\varphi}$ to

$$\varphi_{zzz}+B_1\varphi_{zz}+B_0\varphi_z+B_{-1}\varphi+g_1z\varphi_z=0.$$

Chapter 3

Nonisospectral scattering problems and scaling reductions

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Nonisospectral scattering problems and scaling reductions

Two operators are called isospectral if they have the same spectrum this means they have the same sets of eigenvalues.
In this chapter we will show that nonisospectral hierarchies and their scattering problems can be mapped onto standard PDE hierarchies and their isospectral scattering problems.

3.1 A nonisospectral Korteweg-de Vries hierarchy

The KdV hierarchy can be solved by the Inverse Scattering Transform (IST), having the corresponding scattering problem

$$\psi_{xx} + (U - \lambda)\psi = 0, \tag{3.1}$$

where λ is the constant spectral parameter.
Consider the nonisospectral KdV hierarchy

$$u_\tau = \mathrm{R}^n[u]u_\xi + \sum_{j=0}^{n-1} B_j(\tau)\mathrm{R}^j[u]u_\xi + g_{n-1}(\tau)(4u + 2\xi u_\xi) + g_n(\tau), \tag{3.2}$$

where R$[u]$ is given in (2.1) with U replaced by u and ∂_x by ∂_ξ. Equation (3.2) is important since its stationary reduction when $g_{n-1}(\tau)$, $g_n(\tau)$ and all $B_j(\tau)$ are constants yields the equation

$$\mathrm{R}^n[u]u_\xi + \sum_{j=0}^{n-1} B_j \mathrm{R}^j[u]u_\xi + g_{n-1}(4u + 2\xi u_\xi) + g_n = 0, \tag{3.3}$$

which is a generalized thirty-fourth Painlevé hierarchy if $g_n = 0$. In this section we will show that nonisospectral KdV hierarchy can be mapped to an isospectral KdV hierarchy.

Proposition 3.1. *(Gordoa et al., 2014) The nonisospectral KdV hierarchy (3.2) is equivalent to the isospectral KdV hierarchy*

$$U_t = \mathrm{R}^n[U]U_x + \sum_{i=1}^{n-1} \beta_i(t)\mathrm{R}^i[U]U_x, \tag{3.4}$$

under a change of variables of the form

$$U = g(t)^2 u(\xi, \tau) + m(t), \qquad \xi = g(t)x + h(t), \qquad \tau = k(t). \tag{3.5}$$

Proof. By differentiating U with respect to t and x we get

$$\begin{aligned} U_t &= g^2 \frac{\partial u}{\partial t} + 2gg'u + m' && (3.6)\\ &= g^2 \left[u_\tau \frac{\partial \tau}{dt} + u_\xi \frac{\partial \xi}{dt} \right] + 2gg'u + m'. && (3.7) \end{aligned}$$

Hence

$$U_t = g^2[u_\tau k' + (g'x + h')u_\xi] + 2gg'u + m'.$$

In a similar way we can found

$$U_x = g^2[gu_\xi] = g^3 u_\xi, \qquad u_\xi = g^{-3}U_x.$$

Since $\xi = gx + h$, we have

$$\partial_x = g\partial_\xi, \qquad \partial_x^{-1} = g^{-1}\partial_\xi^{-1}.$$

Let $w(x,t) = g^2 u(\xi,\tau)$. Then

$$w_x = g^3 u_\xi,$$

and

$$\begin{aligned} R[w]w_x &= (\partial_x^2 + 4w + 2w_x\partial_x^{-1})g^3 u_\xi \\ &= (g^2\partial_\xi^2 + 4g^2 u + 2g^2 u_\xi \partial_\xi^{-1})g^3 u_\xi. \end{aligned}$$

So that $R[w]w_x = g^5 R[u]u_\xi$. Therefore $R[w] = g^2 R[u]$.

By (3.5) we have $U = w + m$. Now using Lemma 2.1, we obtain

$$R^n[U]U_x = \sum_{j=0}^{n} \alpha_{n,j} m^{n-j} R^j[w]w_x.$$

Using $R[w] = g^2 R[u]$, we have

$$\begin{aligned} R^n[U]U_x &= \sum_{j=0}^{n} \alpha_{n,j} m^{n-j} R^j[w]w_x \\ &= \sum_{j=0}^{n} \alpha_{n,j} m^{n-j} (g^{2j} R^j[u]) g^3 u_\xi. \end{aligned}$$

Thus (3.4) becomes

$$U_t = \sum_{j=0}^{n} \alpha_{n,j} m^{n-j} g^{2j+3} R^j[u]u_\xi + \sum_{i=1}^{n-1} \beta_i(t) \sum_{j=0}^{i} \alpha_{i,j} m^{i-j} g^{2j+3} R^j[u]u_\xi.$$

Using

$$U_t = g^2 k' u_\tau + g^2(g'x + h')u_\xi + 2gg'u + m',$$

we get

$$\begin{aligned} g^2 k' u_\tau + [g^2 h' + gg'(\xi - h)]u_\xi + 2gg'u + m' &= \sum_{j=0}^{n} \alpha_{n,j} m^{n-j} g^{2j+3} R^j[u]u_\xi \\ &\quad + \sum_{i=1}^{n-1} \beta_i(t) \sum_{j=0}^{i} \alpha_{i,j} m^{i-j} g^{2j+3} R^j[u]u_\xi. \end{aligned}$$

Which can be written as

$$g^2 k' u_\tau = g^{2n+3} \mathrm{R}^n[u]u_\xi + \sum_{i=0}^{n-1} B_i k(t) g^{2n+3} R^i[u]u_\xi - gg'u_\xi(2u + \xi u_\xi) - m'. \quad (3.8)$$

Let

$$k'(t) = g^{2n+1}, \quad m'(t) = -g_n(k(t))g^{2n+3}, \quad g'(t) = -2g_{n-1}(k(t))g^{2n+2}. \tag{3.9}$$

Then substituting k', m', and g' in equation (3.8), yields

$$g^{2n+3}u_\tau = g^{2n+3}\mathrm{R}^n[u]u_\xi + \sum_{i=0}^{n-1} B_i k(t) g^{2n+3} R^i[u]u_\xi + 2g_{n-1}(k(t))g^{2n+3}u_\xi(2u+\xi u_\xi) + g_n(k(t))g^{2n+3}.$$

Multiplying both sides by g^{-2n-3} gives

$$u_\tau = \mathrm{R}^n[u]u_\xi + \sum_{i=0}^{n-1} B_i k(t) R^i[u]u_\xi + 2g_{n-1}(k(t))u_\xi(2u + \xi u_\xi) + g_n(k(t)).$$

Witting the coefficient as a function of τ gives

$$u_\tau = \mathrm{R}^n[u]u_\xi + \sum_{i=0}^{n-1} B_i(\tau)\mathrm{R}^i[u]u_\xi + g_{n-1}(\tau)(4u + 2\xi u_\xi) + g_n(\tau).$$

□

Proposition 3.2. *(Gordoa et al., 2014) Extending the change of variables (3.5) with*

$$\psi(x,t) = \varphi(\xi,\tau), \qquad \mu(k(t)) = \frac{\lambda - m(t)}{g(t)^2}, \tag{3.10}$$

the scattering problem (3.1) is transformed into the nonisospectral scattering problem

$$\varphi_{\xi\xi} + [u - \mu(\tau)]\varphi = 0, \tag{3.11}$$

where $\mu(\tau)$ satisfies the nonisospectral condition

$$\frac{d\mu}{d\tau} = 4g_{n-1}(\tau)\mu(\tau) + g_n(\tau). \tag{3.12}$$

Proof. Since

$$\psi(x,t) = \varphi(\xi,\tau), \qquad \lambda = g^2(t)\mu(k(t)) + m(t),$$

we have

$$\psi_x = \varphi_\xi \frac{d\xi}{dx} = g(t)\varphi_\xi,$$

and

$$\psi_{xx} = g^2(t)\varphi_{\xi\xi}.$$

Substituting ψ_x and ψ_{xx} in equation (3.1) we have

$$g(t)^2\varphi_{\xi\xi} + \left[g(t)^2u(\xi,\tau) + m(t) - \mu(k(t))g(t)^2 - m(t)\right]\varphi = 0.$$

It follows that

$$g(t)^2\left[\varphi_{\xi\xi} + (u(\xi,\tau) - \mu(k(t)))\varphi\right] = 0,$$

or

$$\varphi_{\xi\xi} + \left[u - \mu(\tau)\right]\varphi = 0.$$

To proof that $\mu(\tau)$ satisfies the nonisospectral condition (3.12) we calculate

$$\begin{aligned}\frac{d}{dt}\mu &= \frac{\partial\mu}{\partial\tau}\frac{\partial\tau}{dt}\\ &= \frac{-m'(t)g(t)^2 - 2g(t)g'(t)(\lambda - m(t))}{g(t)^4}.\end{aligned}$$

Then

$$\frac{\partial\mu}{\partial\tau}k'(t) = \frac{-2g(t)g'(t)[\lambda - m(t)]}{g(t)^4} - \frac{m'(t)g(t)^2}{g(t)^4}.$$

It follows that

$$\frac{\partial\mu}{\partial\tau}k'(t) = -2\left(\frac{\lambda - m(t)}{g(t)^2}\right)\left(\frac{g'(t)}{g(t)}\right) - \frac{m'(t)}{g(t)^2}. \tag{3.13}$$

Substituting $g'(t)$, $m'(t)$ and $k'(t)$ from (3.9) into equation (3.13) we have

$$\frac{\partial\mu}{\partial\tau}g^{2n+1}(t) = \left(-2\frac{\lambda - m(t)}{g(t)^2}\right)\left(\frac{-2g_{n-1}(k(t))g(t)^{2n+2}}{g(t)}\right) + \frac{g_n(k(t))g(t)^{2n+3}}{g(t)^2}$$

$$\begin{aligned}\frac{\partial\mu}{\partial\tau}g^{2n+1}(t) &= \left(-2\frac{\lambda - m(t)}{g(t)^2}\right)\left(\frac{-2g_{n-1}(k(t))g(t)^{2n+2}}{g(t)}\right) + \frac{g_n(k(t))g(t)^{2n+3}}{g(t)^2}\\ &= 4\left[\lambda - m(t)\right]g_{n-1}(k(t))g^{2n-1}(t) + g_n(k(t))g^{2n+1}(t).\end{aligned}$$

Substituting $\lambda = \mu(k(t))g^2(t) + m(t)$ gives

$$\frac{\partial\mu}{\partial\tau}g^{2n+1}(t) = g(t)^{2n+1}\left[4\mu(k(t))g_{n-1}(k(t)) + g_n(k(t))\right].$$

Multiplying by $g^{-2n-1}(t)$ and writing the coefficients as functions of τ. we obtain

$$\frac{\partial\mu}{\partial\tau} = 4g_{n-1}(\tau)\mu(\tau) + g_n(\tau).$$

□

3.2 Scaling reductions of the Korteweg-de Vries hierarchy

We have studied in Chapter 2 the scaling reduction of KdV. In this section we will explain the relationship between the derivation of Painlevé hierarchies using nonisospectral scattering problems and scaling reductions and show that the same Painlevé hierarchies are found using the two method.

Corollary 3.1. *(Gordoa et al., 2014) For the special case of equation* (3.4) *and transformation* (3.5) *having* $g(t) = 1$, $m(t) = -g_n t$ *(*g_n *constant),* $k(t) = t$ *and* $h(t)$ *and all* $\beta_k(t)$ *such that all* $B_k(t)$, $k = 0, 1, \ldots n - 1$ *are constants, the corresponding equivalent nonisospectral equation is*

$$u_t = \mathrm{R}^n[u]u_\xi + \sum_{j=0}^{n-1} B_j(t)\mathrm{R}^j[u]u_\xi + g_n. \tag{3.14}$$

Proof. If $g(t) = 1$, $m(t) = -g_n t$ and $k(t) = t$, the transformation (3.5) becomes

$$U = u(\xi, \tau) - g_n t, \qquad \xi = x + h, \qquad \tau = t. \tag{3.15}$$

It follows that

$$U_t = u_\tau - g_n, \qquad U_x = u_\xi.$$

By using Proposition 3.1, substituting U_t and U_x in equation (3.4) we have

$$u_\tau - g_n = \mathrm{R}^n[U]u_\xi + \sum_{i=1}^{n-1} \beta_i(t)\mathrm{R}^i[U]u_\xi,$$

Since $\tau = k(t) = t$, we obtain

$$u_t = \mathrm{R}^n[U]u_\xi + \sum_{i=1}^{n-1} \beta_i(t)\mathrm{R}^i[U]u_\xi + g_n.$$

□

Corollary 3.2. *(Gordoa et al., 2014) For the special case of equation* (3.4) *and transformation* (3.5) *having*

($g(t) = 1/[2(2n+1)g_{n-1}t]^{1/(2n+1)}$) ($g_{n-1}$ constant), $m(t) = d$ (constant), $k(t) = \log(t)/[2(2n+1)g_{n-1}]$ and $h(t)$ and all $\beta_k(t)$ such that all $B_k(\upsilon)$, $k = 0, 1, \ldots n-1$ are constants, the corresponding equivalent nonisospectral equation is

$$u_\tau = \mathrm{R}^n[u]u_\xi + \sum_{j=0}^{n-1} B_j(t)\mathrm{R}^j[u]u_\xi + g_{n-1}(4u + 2\xi u_\xi). \tag{3.16}$$

Proof. Let

$$2(2n+1)g_{n-1}t = T$$

By differentiating $g(t), k(t), m(t)$ and $h(t)$ with respect to t we obtain

$$g'(t) = -2g_{n-1}[2(2n+1)g_{n-1}t]^{-2n-2/(2n+1)} = -2g_{n-1}T^{-2n-2/(2n+1)},$$

and

$$k'(t) = [2(2n+1)g_{n-1}t]^{-1} = T^{-1}, \qquad h'(t) = 0, m'(t) = 0$$

Substituting k', g', m' and h' in equation (3.8) gives

$$\begin{aligned} T^{-2n-3/(2n+1)}u_\tau \quad &= \quad T^{-2n-3/(2n+1)}\mathrm{R}^n[u]u_\xi + \sum_{i=0}^{n-1} B_i k(t) T^{-2n-3/(2n+1)} R^i[u]u_\xi \\ &\quad +2T^{-2n-3/(2n+1)}u_\xi(2u + \xi u_\xi). \end{aligned}$$

Multiplying both side by $T^{2n+3/(2n+1)}$ gives

$$u_\tau = \mathrm{R}^n[u]u_\xi + \sum_{i=0}^{n-1} B_i k(t) R^i[u]u_\xi + 2u_\xi(2u + \xi u_\xi).$$

It follows that

$$u_\tau = \mathrm{R}^n[u]u_\xi + \sum_{i=0}^{n-1} B_i \mathrm{R}^i[u]u_\xi + g_{n-1}(4u + 2\xi u_\xi),$$

□

The stationary reduction of (3.16) yields (3.3) with $g_n = 0$,

$$\mathrm{R}^n[u]u_\xi + \sum_{i=0}^{n-1} B_i \mathrm{R}^i[u]u_\xi + g_{n-1}(4u + 2\xi u_\xi) = 0.$$

The scaling reduction of (3.4) yields a P_{XXXIV} hierarchy. That is, the change of variables of Corollary 3.2 explains why the stationary reduction of the nonisospectral hierarchy (3.16) can also be obtained as a scaling reduction of the corresponding case of the standard hierarchy (3.4). Both techniques then yield the same Painlevè hierarchy.

3.3 A nonisospectral dispersive water wave hierarchy

The DWW hierarchy has an energy-dependent scattering problem,

$$\psi_{xx} = \left[(\lambda - \frac{1}{2}U)^2 + \frac{1}{2}U_x - v\right]\psi, \tag{3.17}$$

where λ is the constant spectral parameter. Consider the nonisospectral DWW hierarchy in $\mathbf{u} = (u, v)^T$

$$\mathbf{u}_\tau = \mathrm{R}^n[\mathbf{u}]\mathbf{u}_\xi + \sum_{j=0}^{n-1} B_j(\tau)\mathrm{R}^j[\mathbf{u}]\mathbf{u}_\xi + \frac{1}{2}g_n(\tau)\begin{pmatrix}(\xi u)_\xi \\ 2v + \xi v_\xi\end{pmatrix} + g_{n+1}(\tau)\begin{pmatrix}1 \\ 0\end{pmatrix}, \tag{3.18}$$

where $\mathrm{R}[\mathbf{u}]$ is given in (2.28) with $\mathbf{U}$ replaced by $\mathbf{u}$ and ∂_x by ∂_ξ. Equation (3.18) when $g_{n+1}(\tau)$, $g_n(\tau)$ and all $B_j(\tau)$ are constants yields the equation

$$\mathrm{R}^n[\mathbf{u}]\mathbf{u}_\xi + \sum_{j=0}^{n-1} B_j\mathrm{R}^j[\mathbf{u}]\mathbf{u}_\xi + \frac{1}{2}g_n\begin{pmatrix}(\xi u)_\xi \\ 2v + \xi v_\xi\end{pmatrix} + g_{n+1}\begin{pmatrix}1 \\ 0\end{pmatrix} = 0, \tag{3.19}$$

which is a generalized fourth Painlevé hierarchy if $g_{n+1} = 0$. In this section we will show that nonisospectral DWW hierarchy can be mapped to an isospectral DWW hierarchy.

Proposition 3.3. *(Gordoa et al., 2014) The nonisospectral hierarchy (3.18) is equivalent to the isospectral DWW hierarchy*

$$\boldsymbol{U}_t = \mathrm{R}^n[\boldsymbol{U}]\boldsymbol{U}_x + \sum_{i=1}^{n-1}\gamma_i(t)\mathrm{R}^i[\boldsymbol{U}]\boldsymbol{U}_x, \tag{3.20}$$

under a change of variables of the form

$$U = g(t)u(\xi,\tau) + m(t), \qquad V = g(t)^2 v(\xi,\tau), \qquad \xi = g(t)x + h(t), \qquad \tau = k(t). \tag{3.21}$$

Proof. By differentiating $\mathbf{U}$ with respect to t we obtain

$$\mathbf{U}_t = \begin{pmatrix} g(t)[u_\xi(g'x+h')+u_\tau k']+g'u+m' \\ g^2(t)[v_\xi(g'x+h')+v_\tau k']+2gg'v \end{pmatrix}$$

That is

$$\begin{aligned} \mathbf{U}_t &= \begin{pmatrix} g'(t)(\xi u)_\xi \\ g(t)g'(t)(2v+\xi v_\xi) \end{pmatrix} + \begin{pmatrix} [g(t)h\prime(t)-g\prime(t)h(t)]u_\xi \\ g(t)[g(t)h\prime(t)-g\prime(t)h(t)]v_\xi \end{pmatrix} + \begin{pmatrix} g(t)k'(t)u_\tau \\ g^2(t)k'(t)v_\tau \end{pmatrix} \\ &\quad + m'(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

Differentiating $\mathbf{U}$ with respect to x we get

$$\mathbf{U}_x = \begin{pmatrix} g^2(t)u_\xi \\ g^3(t)v_\xi \end{pmatrix}$$

Thus $\mathbf{U}_x$ can be written as

$$\mathbf{U}_x = \begin{pmatrix} g^2(t) & 0 \\ 0 & g^3(t) \end{pmatrix}\mathbf{u}_\xi.$$

Let

$$\mathbf{w}(x,t) = (w_1, w_2)^T,$$

where

$$w_1 = g(t)u(\xi,\tau), \quad w_2 = g^2(t)v(\xi,\tau), \quad \xi = g(t)x+h(t).$$

Then

$$\partial_x = g\partial_\xi, \quad \partial_x^{-1} = g^{-1}\partial_\xi^{-1},$$

and

$$R[\mathbf{w}]\mathbf{w}_x = \frac{1}{2}\begin{pmatrix} \partial_x w_1\partial_x^{-1}-\partial_x & 2 \\ 2w_2+(w_2)_x\partial_x^{-1} & w_1+\partial_x \end{pmatrix}\begin{pmatrix} g^2(t) & 0 \\ 0 & g^3(t) \end{pmatrix}\mathbf{u}_\xi.$$

Substituting $w_1 = g(t)u(\xi,\tau)$ and $w_2 = g^2(t)v(\xi,\tau)$ we have

$$\begin{aligned} R[\mathbf{w}]\mathbf{w}_x &= \frac{1}{2}\begin{pmatrix} g\partial_\xi gug^{-1}\partial_\xi^{-1}-g\partial_\xi & 2 \\ 2g^2v+g^3v_\xi g^{-1}\partial_\xi^{-1} & gu+g\partial_\xi \end{pmatrix}\begin{pmatrix} g^2(t) & 0 \\ 0 & g^3(t) \end{pmatrix}\mathbf{u}_\xi. \\ &= \frac{1}{2}\begin{pmatrix} g[\partial_\xi u\partial_\xi^{-1}-\partial_\xi] & 2 \\ g^2[2v+v_\xi\partial_\xi^{-1}] & g[u+\partial_\xi] \end{pmatrix}\begin{pmatrix} g^2(t) & 0 \\ 0 & g^3(t) \end{pmatrix}\mathbf{u}_\xi. \end{aligned}$$

It follows that

$$R[\mathbf{w}]\mathbf{w}_x = \begin{pmatrix} g^3(t) & 0 \\ 0 & g^4(t) \end{pmatrix} R[\mathbf{u}]\mathbf{u}_\xi,$$

where

$$R[\mathbf{w}] = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} R[\mathbf{u}],$$

$$\mathbf{U} = (w_1 + m, w_2)^T$$

By (3.21) we have $\mathbf{U} = (w_1 + m, w_2)$. Now using Lemma 2.2, we obtain

$$\begin{aligned} R^n[\mathbf{U}]\mathbf{U}_x &= \sum_{j=0}^{n} \alpha_{n,j} m^{n-j} R^j[\mathbf{w}]\mathbf{w}_x \\ &= \sum_{j=0}^{n} \alpha_{n,j} m^{n-j} \left[\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}^j R^j[\mathbf{u}] \right] \begin{pmatrix} g^2(t) & 0 \\ 0 & g^3(t) \end{pmatrix} \mathbf{u}_\xi. \end{aligned}$$

Then

$$R^n[\mathbf{U}]\mathbf{U}_x = \sum_{j=0}^{n} \alpha_{n,j} m^{n-j} \mathbf{G}_{j+2} R^j[\mathbf{u}]\mathbf{u}_\xi,$$

where

$$\mathbf{G}_j = \begin{pmatrix} g^j(t) & 0 \\ 0 & g^{j+1}(t) \end{pmatrix}.$$

Thus

$$\mathbf{U}_t = \sum_{j=0}^{n} \alpha_{n,j} m^{n-j} \mathbf{G}_{j+2} R^j[\mathbf{u}]\mathbf{u}_\xi + \sum_{i=1}^{n-1} \gamma_i \sum_{j=0}^{i} \alpha_{i,j} m^{i-j} \mathbf{G}_{j+2} R^j[\mathbf{u}]\mathbf{u}_\xi.$$

Using $\mathbf{U}_t$ as in equation (**??**) we obtain

$$\begin{aligned} \begin{pmatrix} g(t)k'(t)u_\tau \\ g^2(t)k'(t)v_\tau \end{pmatrix} &= \sum_{j=0}^{n} \alpha_{n,j} m^{n-j} \mathbf{G}_{j+2} R^j[\mathbf{u}]\mathbf{u}_\xi + \sum_{i=1}^{n-1} \gamma_i \sum_{j=0}^{i} \alpha_{i,j} m^{i-j} \mathbf{G}_{j+2} R^j[\mathbf{u}]\mathbf{u}_\xi \\ &\quad - \begin{pmatrix} g'(t)(\xi u)_\xi \\ g(t)g'(t)(2v + \xi v_\xi) \end{pmatrix} - \begin{pmatrix} [g(t)h\prime(t) - g\prime(t)h(t)]u_\xi \\ g(t)[g(t)h\prime(t) - g\prime(t)h(t)]v_\xi \end{pmatrix} \\ &\quad - m'(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

Which can be written as

$$\begin{pmatrix} g(t)k'(t)u_\tau \\ g^2(t)k'(t)v_\tau \end{pmatrix} = G_{n+2}R^n[\mathbf{u}]\mathbf{u}_\xi + \sum_{i=1}^{n-1} B_i k(t) G_{n+2}R^i[\mathbf{u}]\mathbf{u}_\xi - \begin{pmatrix} g'(t)(\xi u)_\xi \\ g(t)g'(t)(2v+\xi v_\xi) \end{pmatrix} - m'(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.22}$$

Let

$$k'(t) = g^{n+1}(t), \quad g'(t) = \frac{-1}{2} g_n(k(t))g^{n+2}(t), \quad m'(t) = -g_{n+1}(k(t))g^{n+2}(t) \tag{3.23}$$

Substituting k', m', and g' in equation (3.22), and multiplying both side by $\mathbf{G}_{n+2}^{-1}$ gives

$$\begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} = R^n[\mathbf{u}]\mathbf{u}_\xi + \sum_{i=1}^{n-1} B_i k(t) R^i[\mathbf{u}]\mathbf{u}_\xi + \frac{1}{2} g_n k(t) \begin{pmatrix} (\xi u)_\xi \\ 2v+\xi v_\xi \end{pmatrix} + g_{n+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Witting the coefficient as a function of τ gives

$$\mathbf{u}_\tau = \mathrm{R}^n[\mathbf{u}]\mathbf{u}_\xi + \sum_{i=0}^{n-1} B_i(\tau)\mathrm{R}^i[\mathbf{u}]\mathbf{u}_\xi + \frac{1}{2} g_n(\tau) \begin{pmatrix} (\xi u)_\xi \\ 2v+\xi v_\xi \end{pmatrix} + g_{n+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

□

Proposition 3.4. *(Gordoa et al., 2014) Extending the change of variables (3.21) with*

$$\psi(x,t) = \varphi(\xi,\tau), \qquad \mu(k(t)) = \frac{2\lambda - m(t)}{2g(t)}, \tag{3.24}$$

the scattering problem (3.17) is transformed into the nonisospectral scattering problem

$$\varphi_{\xi\xi} = \left[\left(\mu(\tau) - \frac{1}{2}u\right)^2 + \frac{1}{2}u_\xi - v\right]\varphi \tag{3.25}$$

where $\mu(\tau)$ satisfies the nonisospectral condition

$$\frac{d\mu}{d\tau} = \frac{1}{2} g_n(\tau)\mu(\tau) + \frac{1}{2} g_{n+1}(\tau). \tag{3.26}$$

Proof. Since

$$\psi(x,t) = \varphi(\xi,\tau), \qquad \lambda = \frac{1}{2}[2g(t)\mu(k(t)) + m(t)],$$

we have

$$\psi_x = \varphi_\xi \frac{d\xi}{dx} = g(t)\varphi_\xi,$$

and

$$\psi_{xx} = g(t)^2\varphi_{\xi\xi}.$$

Substituting ψ_{xx} in equation (3.17) we have

$$g(t)^2\varphi_{\xi\xi} = \left[\left(\frac{2g(t)\mu(\tau) + m(t) - g(t)u(\xi,\tau)}{2} - m(t)\right)^2 + \frac{1}{2}g(t)^2u_\xi - g(t)^2v(\xi,\tau)\right]\varphi = 0.$$

Then

$$g(t)^2\varphi_{\xi\xi} = g(t)^2\left[(\mu(\tau) - \frac{1}{2}u)^2 + \frac{1}{2}u_\xi - v(\xi,\tau)\right]\varphi.$$

That is

$$\varphi_{\xi\xi} = \left[(\mu(\tau) - \frac{1}{2}u)^2 + \frac{1}{2}u_\xi - v(\xi,\tau)\right]\varphi.$$

To proof that $\mu(\tau)$ satisfies the nonisospectral condition, we calculate

$$\begin{aligned} \frac{d}{dt}\mu &= \frac{\partial\mu}{\partial\tau}\frac{\partial\tau}{dt} \\ &= \frac{-2m'(t)g(t) - 2g'(t)(2\lambda - m(t))}{4g(t)^2}. \end{aligned}$$

Then

$$\frac{\partial\mu}{\partial\tau}k'(t) = -\frac{2g'(t)(2\lambda - m(t))}{4g(t)^2} - \frac{2m'(t)g(t)}{4g(t)^2}$$

That is

$$\frac{\partial\mu}{\partial\tau}k'(t) = -\left(\frac{2\lambda - m(t)}{2g(t)}\right)\left(\frac{g'(t)}{g(t)}\right) - \frac{1}{2}\frac{m'(t)}{g(t)}. \tag{3.27}$$

Substituting $g'(t)$, $m'(t)$ and $k'(t)$ from (3.23) into equation (3.27) we have

$$\begin{aligned} \frac{\partial\mu}{\partial\tau}g^{n+1}(t) &= -\frac{2\lambda - m(t)}{2g(t)}\frac{-\frac{1}{2}g_n(k(t))g(t)^{n+2}}{g(t)} + \frac{1}{2}\frac{g_{n+1}(k(t))g(t)^{n+2}}{g(t)} \\ &= \frac{1}{4}g_n(k(t))g(t)^n(2\lambda - m(t)) + \frac{1}{2}g_{n+1}(k(t))g(t)^{n+1}. \end{aligned}$$

Substituting $\lambda = \frac{1}{2}\left(2g(t)\mu(k(t)) + m(t)\right)$ gives

$$\frac{\partial \mu}{\partial \tau} g^{n+1}(t) = \frac{1}{2} g(t)^{n+1} \left[\mu(k(t)) g_n(k(t)) + g_{n+1}(k(t))\right].$$

Cancelling $g^{n+1}(t)$ and writing the coefficients as functions of τ yields

$$\frac{\partial \mu}{\partial \tau} = \frac{1}{2} g_n(\tau)\mu(\tau) + \frac{1}{2} g_{n+1}(\tau).$$

□

3.4 Scaling reductions of the dispersive water wave hierarchy

We have studied in Chapter 2 the scaling reduction of DWW. In this section we will explain the relationship between the derivation of Painlevè hierarchies using nonisospectral scattering problems and scaling reductions and show that the same Painlevè hierarchies are found using the two method.

Corollary 3.3. *(Gordoa et al., 2014) For the special case of equation (3.20) and transformation (3.21) having $g(t) = 1$, $m(t) = -g_{n+1}t$ (g_{n+1} constant), $k(t) = t$ and $h(t)$ and all $\gamma_k(t)$ such that all $B_k(t)$, $k = 0, 1, \ldots n-1$ are constants, the corresponding equivalent nonisospectral equation is*

$$u_t = \mathrm{R}^n[u]u_\xi + \sum_{j=0}^{n-1} B_j(t)\mathrm{R}^j[u]u_\xi + g_{n+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.28}$$

Proof. If $g(t) = 1$, $m(t) = -g_{n+1}t$ and $k(t) = t$ transformation (3.21) becomes

$$U = u(\xi, \tau) - g_{n+1}t, \qquad V = v(\xi, \tau), \qquad \xi = x + h(t), \qquad \tau = t. \tag{3.29}$$

Then

$$\begin{aligned} U_t &= u_\xi h'(t) + u_\tau - g_{n+1}, \\ V_t &= v_\xi h'(t) + v_\tau, \\ U_x &= u_\xi, \\ V_x &= v_\xi. \end{aligned}$$

That is

$$\mathbf{U_t} = \mathbf{u}_\xi h'(t) + \mathbf{u}_\tau - g_{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{and} \quad \mathbf{U}_x = \mathbf{u}_\xi$$

By using Proposition 3.3, substituting $\mathbf{U}_t$ and $\mathbf{U}_x$ in equation (3.20) we have

$$\mathbf{u}_\tau - g_{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{R}^n[\mathbf{u}]\mathbf{u}_\xi + \sum_{i=1}^{n-1} \beta_i(t)\mathrm{R}^i[\mathbf{u}]\mathbf{u}_\xi.$$

Since $\tau = k(t) = t$. As a result we obtain

$$\mathbf{u}_t = \mathrm{R}^n[u]\mathbf{u}_\xi + \sum_{i=1}^{n-1} \beta_i(t)\mathrm{R}^i[\mathbf{u}]\mathbf{u}_\xi + g_{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Which is the nonisospectral DWW hierarchy (3.18) with $g_n = 0$.

□

Corollary 3.4. *(Gordoa et al., 2014) For the special case of equation (3.20) and transformation (3.21) having $g(t) = 1/[\frac{1}{2}(n+1)g_n t]^{1/(n+1)}$ (g_n constant), $m(t) = d$ (constant), $k(t) = \log(t)/[\frac{1}{2}(n+1)g_n]$ and $h(t)$ and all $\gamma_k(t)$ such that all $B_k(t)$, $k = 0, 1, \ldots n-1$ are constants, the corresponding equivalent nonisospectral equation is*

$$\boldsymbol{u}_\tau = \mathrm{R}^n[\boldsymbol{u}]\boldsymbol{u}_\xi + \sum_{j=0}^{n-1} B_j(t)\mathrm{R}^j[\boldsymbol{u}]\boldsymbol{u}_\xi + \frac{1}{2}g_n \begin{pmatrix} (\xi u)_\xi \\ 2v + \xi v_\xi \end{pmatrix}. \tag{3.30}$$

Proof. Let

$$T = \frac{1}{2}(n+1)g_n t$$

By differentiating $g(t), k(t), m(t)$ and $h(t)$ with respect to t we obtain

$$g'(t) = \frac{-1}{2}g_n[\frac{1}{2}(n+1)g_n t]^{-n-2/(n+1)} = \frac{-1}{2}g_n T^{-n-2/(n+1)},$$

and

$$k'(t) = [\frac{1}{2}t(n+1)g_n]^{-1} = T^{-1}, \qquad h'(t) = 0, m'(t) = 0$$

Substituting $g'(t)$, $m'(t)$ and $k'(t)$ into equation (3.22) we have

$$\begin{pmatrix} T^{-n-2/(n+1)}u_\tau \\ T^{-n-3/(n+1)}v_\tau \end{pmatrix} = G_{n+2}R^n[\mathbf{u}]\mathbf{u}_\xi + \sum_{i=1}^{n-1} B_i k(t) G_{n+2} R^i[\mathbf{u}]\mathbf{u}_\xi$$
$$+\frac{1}{2}g_n \begin{pmatrix} T^{-n-2/(n+1)}(\xi u)_\xi \\ T^{-n-3/(n+1)}(2v + \xi v_\xi) \end{pmatrix}.$$

Multiplying both side by $\mathbf{G}_{n+2}^{-1}$ where

$$\mathbf{G}_{n+2}^{-1} = \begin{pmatrix} T^{n+2/(n+1)} & 0 \\ 0 & T^{n+3/(n+1)} \end{pmatrix},$$

gives

$$\mathbf{u}_\tau = \mathrm{R}^n[\mathbf{u}]\mathbf{u}_\xi + \sum_{j=0}^{n-1} B_j(t)\mathrm{R}^j[\mathbf{u}]\mathbf{u}_\xi + \frac{1}{2}g_n \begin{pmatrix} (\xi u)_\xi \\ 2v + \xi v_\xi \end{pmatrix}.$$

□

The stationary reduction of (3.30) yields (3.19) with $g_{n+1} = 0$,

$$\mathrm{R}^n[\mathbf{u}]\mathbf{u}_\xi + \sum_{j=0}^{n-1} B_j(t)\mathrm{R}^j[\mathbf{u}]\mathbf{u}_\xi + \frac{1}{2}g_n \begin{pmatrix} (\xi u)_\xi \\ 2v + \xi v_\xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The scaling reduction of (3.20) yields a P_{IV} hierarchy. That is, the change of variables of Corollary 3.4 explains why the stationary reduction of the nonisospectral hierarchy (3.30) can also be obtained as a scaling reduction of the corresponding case of the standard hierarchy (3.20). Both techniques then yield the same Painlevè hierarchy.

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