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Ulam stability of Reciprocal, Reciprocal Difference and Reciprocal Adjoint Functional Equations

حول استقرار المعادلات الدالية المنعكسة و فرق المنعكسة و جور المنعكسة

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إقـــــــــــرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

حول استقرار المعادلات الدالية المنعكسة و فرق المنعكسة و جور المنعكسة

Ulam stability of Reciprocal, Reciprocal Difference and Reciprocal Adjoint Functional Equations

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الملخص

لقد قمنا في هذه الاطروحة بدراسة الحل العام و استقراره لبعض انواع المعادلات الدالية حيث درسنا ما يلي :

١. المعادلات الدالية المنعكسة و المعادلات الدالية فرق المنعكسة و المعادلات الدالية جوار المنعكسة.
٢. تعميم استقرار هايرز اولام للمعادلة الدالية المنعكسة.
٣. الحل العم للمعادلة الدالية المنعكسة في عدة متغيرات.
٤. تعميم استقرار هايرز اولام للمعادلة الدالية المنعكسة في عدة متغيرات.
٥. استقرار اولام للمعادلة الدالية فرق المنعكسة و المعادلة الدالية جوار المنعكسة .

الإهـداء

إلى من حملتني كرها ووضعتني كرها...إلى من أرضعتني من الإيمان ما يقويني...

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ABSTRACT

In this thesis, we study the general solution and the stability of some types of functional equations, namely reciprocal , difference reciprocal and reciprocal adjoint functional equation We study : The generalized Hyers-Ulam stability of reciprocal type functional equation $r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}$ where $r : \mathbb{R}-\{0\} \longrightarrow \mathbb{R}$ is a mapping $x+y \neq 0$ and $r(x) + r(y) \neq 0$ for all $x, y \in \mathbb{R} - \{0\}$. The general solution and the generalized Hyers-Ulam stability of a reciprocal type functional equation in several variables of the form
$\frac{\prod_{i=2}^{m} r(x_1+x_i)}{\sum_{i=2}^{m}[\prod_{j=2,j\neq i}^{m} r(x_1+x_i)]} = \frac{\prod_{i=1}^{m} r(x_i)}{\sum_{i=2}^{m} r(x_1)[\prod_{j=2,j\neq i}^{m} r(x_j)]+(m-1)\prod_{i=2}^{m} r(x_i)}$ where m is a positive integer with $m \geq 3$ is study.Moreover is analysed the Hyers-Ulam stability of reciprocal difference functional equation (or RDF equation)of the form $r(\frac{x+y}{2}) - r(x + y) = \dfrac{r(x)r(y)}{r(x) + r(y)}$, and the reciprocal adjoint functional equation (or RAF equation) of the form $r(\frac{x+y}{2}) + r(x + y) = \dfrac{3r(x)r(y)}{r(x) + r(y)}$.

Introduction

Functional equations form a modern branch of mathematics. To solve a functional equation means to find all functions that satisfy the functional equation. The field of functional equations includes differential equations, difference equations and integral equations. Functional equations appeared in the literature around the same time as the modern theory of function. A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam in 1940. In the next year, Hyers gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias proved a generalization of Hyers theorem for additive mappings. Furthermore, in 1994, a generalization of Rassias theorem was obtained by Gavruta by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$ [10].

In 1982-1994, a generalization of the Hyers result was established by J. M. Rassias with a weaker condition controlled by a product of different powers of norms. However, there was a singular case. Then for this singularity, a counter example was given by P. G-avruta . This stability is called Hyers-Ulam-Rassias stability involving a product of different powers of norms [14].

Very recently, S. M. Jung applied a fixed point method for proving the Hyers-Ulam stability for the reciprocal functional equation

$$r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \tag{0.1}$$

In 2008, K. Ravi and B. V. Senthil Kumar investigated some results on Ulam-Gavruta-Rassias stability of the functional equation (0.1). It was proved that the reciprocal function $r(x) = \frac{c}{x}$ is a solution of the functional equation (0.3). Later, J. M. Rassias and et.al., introduced the Reciprocal Difference Functional equation

$$r(\frac{x+y}{2}) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \tag{0.2}$$

and the Reciprocal Adjoint Functional equation

$$r(\frac{x+y}{2}) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)} \tag{0.3}$$

and investigated the Hyers-Ulam stability of the equations (0.2) and (0.3). Further in the same paper, it was proved that the functional equations (0.1), (0.2) and (0.3) are equivalent [16].

This thesis is organized as follows.

Chapter one This chapter consists of two sections. In the first section, we study reciprocal type functional equation in one variable and its solution . In the second section, we study generalized Hyars - Ulam stability of reciprocal type functional equation in one variable .

Chapter two This chapter consists of four sections.In the first section, we study reciprocal difference functional equation and its solution . In the second section, we study Hyars - Ulam stability of reciprocal difference functional equation . In the third section, we study generalized Ulam stability of reciprocal difference functional equation . In the fourth section, we study extended Ulam stability of reciprocal difference functional equation .

Chapter three This chapter consists of four sections. In the first section, we study reciprocal adjoint functional equation and its solution. In the second section, we study Hyars - Ulam stability of reciprocal adjoint functional equation. In the third section, we study generalized Hyers-Ulam stability of reciprocal adjoint functional equation. In the fourth section, we study extended Hyers-Ulam stability of reciprocal adjoint functional equation.

Chapter four This chapter consists of three sections. In the first section, we study reciprocal type functional equation in several variable. In the second section, we study generalized Hyars - Ulam stability of reciprocal type functional in several variable. In the third section, we study counter examples.

Chapter 1

Stability of Reciprocal Type Functional Equation

This chapter consists of two sections . In the first one reciprocal type functional equation in one variable and its solution are studied . The second one is generalized Hyars - Ulam stability of reciprocal type functional equation in one variable

1.1 Reciprocal Type Functional Equation in One Variable and its Solution

Theorem 1.1.1. [12] Let r be a continuous real-valued function of a non-zero real variable satisfying the reciprocal type functional equation

$$r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}, \tag{1.1}$$

for all $x, y \in \mathbb{R} - \{0\}$. If $r(x) \neq 0$, $2r(x) + r(y) \neq 0$ and $2r(x) + r(-y) \neq 0$ for all $x, y \in \mathbb{R} - \{0\}$ then $r(x)$ is of the form

$$r(x) = \frac{c}{x}, \tag{1.2}$$

for all $x \in \mathbb{R} - \{0\}$, where c is a constant

Proof .Replacing (x,y) by (x,x) in (1.1) we obtain

$$r(2x) = \frac{1}{2}r(x). \tag{1.3}$$

Similarly, replacing (x,y) by $(x,2x)$ in (1.1) and using (1.3) we obtain

$$r(3x) = \frac{r(x)r(2x)}{r(x)+r(2x)} = \frac{r(x)\frac{1}{2}r(x)}{r(x)+\frac{1}{2}r(x)} = \frac{r(x)}{3} \tag{1.4}$$

for all $x \in \mathbb{R} - \{0\}$.

Now, replacing (x,y) by $(x,2y)$ in (1.1) and using (1.3) we obtain

$$r(x+2y) = \frac{r(x)r(2y)}{r(x)+r(2y)} = \frac{r(x)r(y)}{2r(x)+r(y)} \tag{1.5}$$

for all $x,y \in \mathbb{R} - \{0\}$.

Replacing (x,y) by $(x,-2y)$ in (1.1) and using(1.3) we obtain

$$r(x-2y) = \frac{r(x)r(-2y)}{r(x)+r(-2y)} = \frac{r(x)r(-y)}{2r(x)+r(-y)} \tag{1.6}$$

for all $x,y \in \mathbb{R} - \{0\}$.

Dividing equation (1.5) by equation (1.6) and then replacing (x,y) by $(x,-x)$ in the resulting equation, we get

$$\begin{aligned}\frac{r(-x)}{r(3x)} &= \frac{r(x)r(-x)}{2r(x)+r(-x)}\frac{2r(x)+r(x)}{r(x)r(x)} \\ &= \frac{3r(-x)}{2r(x)+r(-x)}.\end{aligned}$$

Then $3r(-x)r(3x) = r(-x)[2r(x)+r(-x)]$

By using (1.4) we get

$$3r(-x)\frac{1}{3}r(x) = 2r(-x)r(x) + r(-x)r(-x)$$

$$r(x) = 2r(x) + r(-x)$$

$$r(-x) = -r(x)$$

for all $x, y \in \mathbb{R} - \{0\}$, which also shows that r is an odd function. From (1.1) and using induction on a positive integer n, we get

$$r(nx) = \frac{1}{n} r(x). \tag{1.7}$$

To see this , it is clear that (1.7) is true for $n = 1$.
Suppose that (1.7) is true for $n = k$, then
$r(kx) = \frac{1}{k} r(x)$.
Replacing y by kx in (1.1) to get

$$r((k+1)x) = \tfrac{r(x)r(kx)}{r(x)+r(kx)} = \tfrac{r(x)\frac{1}{k}r(x)}{r(x)+\frac{1}{k}r(x)} = \tfrac{1}{(k+1)} r(x).$$

Hence (1.7) is true for $n = k + 1$.
Therefore,(1.7) is true for all positive integers n.

Replacing x by $\frac{x}{n}$ in (1.7), to get
$r(x) = \frac{1}{n} r(\frac{x}{n})$ and hence

$$r(\frac{x}{n}) = nr(x) \tag{1.8}$$

for all $x \in \mathbb{R} - \{0\}$, where n is a positive integer. Now, replacing x by $-x$ in equations (1.7) and (1.8) and using oddness of r, we get

$$r(-nx) = -\frac{1}{n} r(x) \tag{1.9}$$

and

$$r(-\frac{x}{n}) = -nr(x) \tag{1.10}$$

respectively for $x \in \mathbb{R} - \{0\}$. When $x = 1$, the equations (1.7) and (1.8) become

$$r(n) = \frac{1}{n} r(1) = \frac{c}{n} \tag{1.11}$$

and

$$r(\frac{1}{n}) = nr(1) = nc \tag{1.12}$$

respectively for some constant $c = r(1)$. Similarly, When $x = 1$, the equations (1.9) and (1.10) become

$$r(-n) = -\frac{1}{n}r(1) = -\frac{c}{n} \tag{1.13}$$

and

$$r(-\frac{1}{n}) = -nr(1) = -nc. \tag{1.14}$$

Hence, we conclude that

$$r(m) = \tfrac{c}{m} \text{ and } r(\tfrac{1}{m}) = mc$$

as $m \in \mathbb{Z} - \{0\}$.

Next, let $k = \frac{m}{n}$ be any non-zero rational number, where $m, n \in \mathbb{Z} - \{0\}$. Then (1.7),(1.8), (1.9) and (1.10) yield

$$r(kx) = r(m.\frac{x}{n}) = \frac{1}{m}r(\frac{x}{n}) = \frac{n}{m}r(x) = \frac{1}{k}r(x). \tag{1.15}$$

for all $x \in \mathbb{R} - \{0\}, k \in \mathbb{Q} - \{0\}$. When $x = 1$ the equation (1.15) becomes $r(k) = \frac{1}{k}r(1) = \frac{c}{k}$. Let x be any fixed element in $\mathbb{R} - \{0\}$, then there is a sequence (k_n) of non-zero rational numbers such that $\lim_{n\to\infty} k_n = x$. Then by the continuity of r, where, we hava $r(x) = \lim_{n\to\infty} r(k_n) = \lim_{n\to\infty} \frac{c}{k_n} = \frac{c}{x}$. Hence, we conclude that $r(x) = \frac{c}{x}$ for all $x \in \mathbb{R} - \{0\}$, which completes the proof of Theorem .□

1.2 Generalized Hyars - Ulam Stability of Reciprocal Type Functional Equation in One Variable

Definition 1.2.1. [12] A mapping $r : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ defined as $r(x) = \frac{c}{x}$, c being a constant, is called reciprocal, if the reciprocal type functional equation

$$r(x + y) = \frac{r(x)r(y)}{r(x) + r(y)}$$

holds with $2r(x) + r(y) \neq 0$, $2r(x) + r(-y) \neq 0$, $r(x) + r(y) \neq 0$ and $r(x) \neq 0$ for all $x, y \in \mathbb{R} - \{0\}$.

Throughout this section X will denote the set of non zero real numbers

Notation: If $g : X \longrightarrow \mathbb{R}$ is a function, then we denoted

$$|Dg(x,y)| = |g(x+y) - \frac{g(x)g(y)}{g(x)+g(y)}|$$

for all $x, y \in X$.

Definition 1.2.2. [11] A metric space is a pair (Y, d), where Y is a set and d is a metric on Y (or distance function on Y), that is; a real valued function defined on $Y \times Y$ such that for all $x, y, z \in Y$ we have:

1. $d(x, y) \geq 0$.
2. $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) = d(y, x)$. (Symmetry)
4. $d(x, y) \leq d(x, z) + d(z, y)$. (Triangle inequality)

Lemma 1.2.3. [5] Assume that $(Y,d\ )$ is a complete metric space, K is a nonempty set and, $a : K \to K$, $\Psi : Y \to Y$, and $f : K \to Y$ be functions satisfying

$$d(\Psi\{f[a(x)]\}, f(x)) \leq h(x) \tag{1.16}$$

for all $x \in K$ and for some function $h : K \to \mathbb{R}^+$.

If Ψ satisfies

$$d(\Psi(x), \Psi(y)) \leq \phi d(x, y) \tag{1.17}$$

for all $x, y \in Y$, for a certain non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, then, for each integer n, we have

$$d(\Psi^{n+1}\{f[a^{n+1}(x)]\}, \Psi^n\{f[a^n(x)]\}) \leq (\phi^n(h[a^n(x)])) \tag{1.18}$$

where Ψ^i, a^i and ϕ^i denote the i-th iterate of Ψ, a and ϕ, respectively.

Proof .We prove (1.18) by induction on n. Setting in (1.16) $a(x)$ instead of x we get

$$d(\Psi\{f[a^2(x)]\}, f[a(x)]) \leq h[a(x)] \tag{1.19}$$

setting $x = \Psi(f[a^2(x)])$ and $y = f(a(x))$ in (1.17) and use (1.19) to get

$$d(\Psi^2\{f[a^2(x)]\}, \Psi\{f[a(x)]\}) \leq \phi(d(\Psi\{f[a^2(x)]\}, f[a(x)])) \leq \phi(h[a(x)]) \tag{1.20}$$

since ϕ is non-decreasing. Therefor (1.18) is true for $n = 1$.
Suppose that (1.18) is true for $n = k$, then

$$d(\Psi^{k+1}\{f[a^{k+1}(x)]\}, \Psi^k\{f[a^k(x)]\}) \leq (\phi^k(h[a^k(x)])) \tag{1.21}$$

Setting in (1.21) $a(x)$ instead of x to get

$$d(\Psi^{k+1}\{f[a^{k+2}(x)]\}, \Psi^k\{f[a^{k+1}(x)]\}) \leq (\phi^k(h[a^{k+1}(x)])) \tag{1.22}$$

setting $x = \Psi^{k+1}(f[a^{k+2}(x)])$ and $y = \Psi^k f(a^{k+1}(x))$ in (1.17) and use (1.22) to get $d(\Psi^{k+2}\{f[a^{k+2}(x)]\}, \Psi^{k+1}\{f[a^{k+1}(x)]\}) \leq \phi(d(\Psi^{k+1}\{f[a^{k+2}(x)]\}, \Psi^k(f[a^{k+1}(x)]))$

$$\leq \phi^{k+1}(h[a^{k+1}(x)]) \tag{1.23}$$

since ϕ is non-decreasing . Hence,(1.18) is true for $n = k + 1$.
Therefore (1.18) is true for all positive integer n.

Lemma 1.2.4. [5] Assume that $(Y, d\ )$ is a complete metric space, K is a nonempty set, and $a : K \to K$, $\Psi : Y \to Y$ and $f : K \to Y$ are functions satisfying $d(\Psi\{f[a(x)]\}, f(x)) \leq h(x)$ for all $x \in K$ and for some function $h : K \to \mathbb{R}^+$. Assume also that Ψ satisfies $d(\Psi(x), \Psi(y)) \leq\ \phi d(x, y)$ for all $x, y \in Y$ for a certain non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, and for each integer n, we have

$d(\Psi^{n+1}\{f[a^{n+1}(x)]\}, \Psi^n\{f[a^n(x)]\}) \leq (\phi^n(h[a^n(x)]))$,

where Ψ^i, a^i and ϕ^i denote the i-th iterate of Ψ, a and ϕ, respectively.
If the series $\sum_{i=0}^{\infty} \phi^i\{h[a^i(x)]\}$, is convergent for every $x \in K$ then $\{Q_n(x)\}$ is a Cauchy sequence, where $Q_n(x) = \Psi^n[f(a^n(x))]$.
Define $r(x) = \lim_{n\to\infty} Q_n(x)$ we have

$$d(r(x), f(x)) \leq \sum_{i=0}^{\infty} \phi^i\{h[a^i(x)]\} \tag{1.24}$$

Proof

Let $m > n$; then by using triangle inequality we have,

$d(Q_n(x), Q_m(x)) \leq d(Q_{n+1}(x), Q_n(x)) + d(Q_{n+2}(x), Q_{n+1}(x)) + ... + d(Q_{m-1}(x), Q_{m-2}(x)) + d(Q_m(x), Q_{m-1}(x))$

$= d(\Psi^{n+1}\{f[a^{n+1}(x)]\}, \Psi^n\{f[a^n(x)]\}) + ... + d(\Psi^m\{f[a^m(x)]\}, \Psi^{m-1}\{f[a^{m-1}(x)]\})$

$\leq (\phi^n(h[a^n(x)])) + ... + (\phi^{m-1}(h[a^{m-1}(x)]))$

$= \sum_{i=n}^{m-1} \phi^i\{h[a^i(x)]\}$. However $\sum_{i=0}^{\infty} \phi^i\{h(a^i(x)\}$ is convergent , then $\{Q_n(x)\}$ is Cauchy .

Using triangle inequality we get

$d(Q_n(x), f(x)) = d(\Psi^n\{f[a^n(x)]\}, f(x)) \leq$

$d(\Psi\{f(a(x\}), f(x)) + d(\Psi^2\{f(a^2(x\}), \Psi f(x)) + ... + d(\Psi^n\{f(a^n(x\}), \Psi^{n-1}\{f(a^{n-1}(x\}))$

$= \sum_{i=1}^{n} d(\Psi^i\{f[a^i(x)]\}, \Psi^{i-1}\{f[a^{i-1}(x)]\}) \leq \sum_{i=1}^{n} \phi^{i-1}(h[a^{i-1}(x)])$

Taking the limit as n goes to infinity we obtain (1.23).□

Lemma 1.2.5. [5] Assume the hypotheses of Lemmas 1.2.3 and 1.2.4. If the function Ψ is continuous, then the function r is a solution of the functional equation

$$\Psi\{r[a(x)]\} = r(x), \tag{1.25}$$

$x \in K$. Moreover, if ϕ is subadditive, then r is the only function satisfying (1.24) and (1.25)

Proof. By the continuity of Ψ we have the following chain of equalities:

$\Psi\{r[a(x)]\} = \Psi[\lim_{n\to\infty} Q_n\{a(x)\}] = \lim_{n\to\infty} \Psi[Q_n\{a(x)\}]$

$= \lim_{n\to\infty} \Psi^{n+1}[f\{a^{n+1}(x)\}] = r(x)$ since $Q_n(x) = \Psi^n[f(a^n(x))]$. Suppose that a function r_1 satisfies (1.24) and (1.25) and ϕ is subadditive. Thus

$d(r_1(x), Q_n(x)) = d(\Psi^n\{r_1[a^n(x)]\}, \Psi^n\{f[a^n(x)]\})$ by (1.24)

$\leq \phi^n(d(r_1[a^n(x)], f[a^n(x)]))$ by(1.18)

$\leq \phi^n(\sum_{i=0}^{\infty}\phi^i(h[a^{n+i}(x)]))$ by (1.24) and since ϕ is non-decreasing ,
$\leq (\sum_{i=0}^{\infty}\phi^{n+i}(h[a^{n+i}(x)]))$ since ϕ is subadditive

Taking the limit as n goes to infinity, since the last term goes to zero [because$\sum_{i=1}^{\infty}\phi^i(h[a^i(x)])$ converges] we obtain $d(r_1(x), r(x)) = \lim_{n\to\infty} d(r_1(x), Q_n(x)) = 0$. Therefor r is unique $\square$.

We may now summarize the previous results in the following theorem.

Theorem 1.2.6. [12]

Assume that $(Y,d\ )$ is a complete metric space, K is a nonempty set, $f : K \to Y, \Psi : Y \to Y$, $a : K \to K$, $h : K \to [0,\infty)$,$v \in [0,\infty)$, ,$d(\Psi \circ f \circ a(x), f(x)) \leq h(x)$ for $x \in K$, $d(\Psi(x), \Psi(y)) \leq vd(x, y)$ for all $x, y \in Y$ and $H(x) = \sum_{i=0}^{\infty} v^i h(a^i(x)) < \infty$ for $x \in K$. Then, for every $x \in K$, the limit $r(x) = \lim_{n\to\infty} \Psi^n \circ f \circ a^n(x)$ exists and $r : K \to Y$ is a unique function such that $\Psi \circ r \circ a = r$ and $d(f(x), r(x)) \leq H(x)$, for $x \in K$.

Theorem 1.2.7. [12] Suppose that the mapping $g : X \longrightarrow \mathbb{R}$ satisfying

$$|Dg(x, y)| \leq \phi(x, y) \tag{1.26}$$

for all $x, y \in X$ where $\phi : X \times X \longrightarrow \mathbb{R}$ be a given function. Suppose there exists $\beta \in (0,\infty)$ such that $\frac{\beta}{2} < 1$, and

$$\phi(\frac{x}{2}, \frac{y}{2}) \leq \beta\phi(x, y) \tag{1.27}$$

for all $x, y \in X$. Then there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ which satisfies reciprocal type functional equation(1.1) and the inequality

$$|g(x) - r(x)| \leq \frac{2\beta}{2-\beta}\phi(x, x) \tag{1.28}$$

for all $x \in X$.

Proof Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (1.26) and use the notation in page 7 we get

$$|Dg(\frac{x}{2},\frac{x}{2})| = |g(\frac{x}{2}+\frac{x}{2}) - \frac{g(\frac{x}{2})g(\frac{x}{2})}{g(\frac{x}{2})+g(\frac{x}{2})})| = |g(x) - \frac{1}{2}g(\frac{x}{2})| \le \phi(\frac{x}{2},\frac{x}{2}) \tag{1.29}$$

for all $x \in X$. Considering $f = g$,$\Psi(z) = \frac{1}{2}z$,$v = \frac{1}{2}$, $h(x) = \phi(\frac{x}{2},\frac{x}{2})$,$a(x) = \frac{1}{2}x$ and $d(x,y) = |x-y|$ for all $x,y \in X$ in Theorem (1.2.6) we see that the limit $r(x) = \lim_{n\to\infty} 2^{-n}g(2^{-n}x)$ exists and $|r(x)-g(x)| \le H(x)$ where $H(x) = \sum_{i=0}^{\infty}\frac{1}{2^i}\phi(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}) = \sum_{i=0}^{\infty}\frac{1}{2^i}\beta^{i+1}\phi(x,x) = \beta\phi(x,x)\sum_{i=0}^{\infty}(\frac{\beta}{2})^i = \frac{\beta}{1-\frac{\beta}{2}}\phi(x,x)$
$= \frac{2\beta}{2-\beta}\phi(x,x)$,for all $x \in X$. Therefor, $|g(x)-r(x)| \le H(x) = \frac{2\beta}{2-\beta}\phi(x,x)$ which is exactly (1.28). Using (1.26) and(1.27) we obtain

$$2^{-n}|Dg(2^{-n}x,2^{-n}y)| \le (\frac{\beta}{2})^n\phi(x,y) \tag{1.30}$$

for all $x,y \in X$ and $n \in \mathbb{N}$. Allowing $n \to \infty$ in (1.30) ,

$$|r(x+y) - \frac{r(x)r(y)}{r(x)+(y)}|$$

$$= |\lim_{n\to\infty} 2^{-n}\{g(2^{-n}(x+y)) - \frac{g(2^{-n}x)g(2^{-n}y)}{g(2^{-n}x)+g(2^{-n}y)}\}|$$

$$= \lim_{n\to\infty} 2^{-n}|g(2^{-n}(x+y)) - \frac{g(2^{-n}x)g(2^{-n}y)}{g(2^{-n}x)+g(2^{-n}y)}|$$

$$\lim_{n\to\infty} 2^{-n}|Dg(2^{-n}x,2^{-n}y)| \le \lim_{n\to\infty}(\frac{\beta}{2})^n\phi(x,y) = 0.$$

Therefor, $r(x+y) = \frac{r(x)r(y)}{r(x)+(y)}$,hence r satisfies (1.1). Next, we show that r is the unique reciprocal mapping satisfying (1.1) and (1.28). Let $r_1 : X \to R$ be another reciprocal mapping satisfying (1.1) and (1.28), and $|r_1(x)-g(x)| \le H(x)$ for all $x \in X$, $\Psi \circ r_1 \circ a(x) = r_1(x)$ i.e. $\frac{1}{2}[r_1(\frac{x}{2})] = \frac{2}{2}r_1(x) = r_1(x)$ and hence by Theorem (1.2.6) $r = r_1$, which proves that r is unique □

Theorem 1.2.8. [12] Suppose that the mapping $g : X \longrightarrow \mathbb{R}$ satisfying $|Dg(x,y)| \le \phi(x,y)$ for all $x,y \in X$ where $\phi : X \times X \longrightarrow \mathbb{R}$ be a given function . Suppose there exists $\beta \in (0,\infty)$ such that $2\beta < 1$ and

$$\phi(2x,2y) \le \beta\phi(x,y) \tag{1.31}$$

for all $x, y \in X$. Then there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ which satisfies (1.1) and the inequality

$$|g(x) - r(x)| \leq \frac{2}{1-2\beta}\phi(x,x) \tag{1.32}$$

for all $x \in X$.

Proof . Replacing (x, y) by (x, x) in (1.26) and use the notation in page 7 we get

$$|Dg(x,x)| = |g(x+x) - \frac{g(x)g(x)}{g(x)+g(x)}| = |g(2x) - \frac{1}{2}g(x)| \leq \phi(x,x) \tag{1.33}$$

for all $x \in X$, that is

$$|2g(2x) - g(x)| \leq 2\phi(x,x) \tag{1.34}$$

for all $x \in X$. The rest of the proof is obtained by taking $f = \frac{1}{2}g$,$\Psi(z) = 2z$,$v = 2$,$h(x) = 2\phi(x, x)$,$a(x) = 2x$, and $d(x, y) = |x - y|$ for all $x, y \in X$ in Theorem (1.2.6) and we see that the limit $r(x) = \lim_{n\to\infty} 2^n g(2^n x)$ exists and $|r(x) - g(x)| \leq H(x)$ where $H(x) = \sum_{i=0}^{\infty} 2^{i+1}\phi(2^i x, 2^i x) = \sum_{i=0}^{\infty} 2^{i+1}\beta^i\phi(x, x) = 2\phi(x, x)\sum_{i=0}^{\infty}(2\beta)^i = \frac{2}{1-2\beta}\phi(x, x)$,for all $x \in X$. Therefor ,$|g(x) - r(x)| \leq H(x) = \frac{2}{1-2\beta}\phi(x, x)$ which is exactly (1.32). Using (1.26) and (1.31), we obtain

$$2^n|Dg(2^n x, 2^n y)| \leq (2\beta)^n\phi(x,y) \tag{1.35}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Allowing $n \to \infty$ in (1.35), we get

$$|r(x+y) - \frac{r(x)r(y)}{r(x)+(y)}|$$

$$= |\lim_{n\to\infty} 2^n\{g(2^n(x+y)) - \frac{g(2^n x)g(2^n y)}{g(2^n x)+g(2^n y)}\}|$$

$$= \lim_{n\to\infty} 2^n|g(2^n(x+y)) - \frac{g(2^n x)g(2^n y)}{g(2^n x)+g(2^n y)}|$$

$$\lim_{n\to\infty} 2^n|Dg(2^n x, 2^n y)| \leq \lim_{n\to\infty}(2\beta)^n\phi(x,y) = 0.$$

Therefor, $r(x + y) = \frac{r(x)r(y)}{r(x)+(y)}$ and hence r satisfies (1.1). Next, we show that r is the unique reciprocal mapping satisfying (1.1) and (1.32). Let $r_1 : X \to \mathbb{R}$ be another reciprocal mapping satisfying (1.1) and (1.32), and $|r_1(x)-g(x)| \leq H(x)$ for all $x \in X$, $\Psi \circ r_1 \circ a(x) = r_1(x)$ i.e. $2[r_1(2x)] = \frac{2}{2}r_1(x) = r_1(x)$ and hence by Theorem(1.2.6) $r = r_1$, which proves that r is unique $\square$

Theorem 1.2.9. [12] Let $g : X \longrightarrow \mathbb{R}$ be a mapping satisfying

$$|Dg(x, y)| \leq \phi(x, y) \tag{1.36}$$

for all $x, y \in X$ where $\phi : X \times X \longrightarrow \mathbb{R}$ be a given function such that

$$\Gamma(x) = \sum_{i=0}^{\infty} 2^{i+1}\phi(2^i x, 2^i x) \tag{1.37}$$

(or respectively

$$\Gamma(x) = \sum_{i=0}^{\infty} \frac{1}{2^i}\phi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}})) \tag{1.38}$$

with the conditions $x + y \neq 0$,$g(x) + g(y) \neq 0$ and $g(x) \neq 0$ for all $x, y \in X$ and

$$\lim_{n\to\infty} 2^{n+1}\phi(2^n x, 2^n x) = 0$$

$$(\text{or respectively } \lim_{n\to\infty} \tfrac{1}{2^n}\phi(\tfrac{x}{2^{n+1}}, \tfrac{x}{2^{n+1}}) = 0 \text{)}$$

holds for every $x \in X$. Then there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ which satisfies reciprocal type functional equation (1.1) for all $x \in X$ and the inequality $|g(x) - r(x)| \leq \Gamma(x)$.

Corollary 1.2.10. [12] Let $g : X \longrightarrow \mathbb{R}$ be a mapping such that $|Dg(x, y)| \leq c_1(|x|^p + |y|^p)$ for all $x, y \in X$,where $c_1 \geq 0$ and $p \neq -1$.

for all $x, y \in X$. Then there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ satisfying reciprocal type functional equation (1.1), for every $x \in X$ and for $p \neq -1$

$$|g(x) - r(x)| \leq \frac{4c_1}{|2^{p+1} - 1|}|x|^p \tag{1.39}$$

.

Proof Let $\phi(x,y) = c_1(|x|^p + |y|^p)$

Case 1. If $p < -1$ then (see 1.37)

$\Gamma(x) = \sum_{i=0}^{\infty} 2^{i+1} c_1(|2^i x|^p + |2^i x|^p) = \sum_{i=0}^{\infty} 2^{i+1} c_1(2|2^i x|^p) = 4c_1|x|^p \sum_{i=0}^{\infty} 2^{(p+1)i}$

$$= \tfrac{4c_1}{1-2^{p+1}}|x|^p.$$

This means that the series above converges and so $\lim_{n\to\infty} 2^{n+1}\phi(2^n x, 2^n x) = 0$.

Case 2. If $p > -1$,then (see 1.38)

$\Gamma(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} c_1(|\frac{x}{2^{i+1}}|^p + |\frac{x}{2^{i+1}}|^p) = 2c_1 \sum_{i=0}^{\infty} \frac{1}{2^i}|\frac{x}{2^{i+1}}|^p = \frac{2c_1}{2^p}\sum_{i=0}^{\infty} \frac{1}{2^{(p+1)i}}|x|^p$

$$= \tfrac{2c_1}{2^p}|x|^p \tfrac{1}{1-2^{-(p+1)}} = \tfrac{4c_1}{2^{p+1}-1}|x|^p.$$

Similarly as in case 1 the series converges , so that $\lim_{n\to\infty} \frac{1}{2^n}\phi(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}) = 0$. By using the two cases we get $\Gamma(x) = \frac{4c_1}{|2^{p+1}-1|}|x|^p$.

Therefor, by Theorem (1.2.9) there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ satisfying reciprocal type functional equation(1.1)

and $|g(x) - r(x)| \leq \frac{4c_1}{|2^{p+1}-1|}|x|^p$, for $p \neq -1$ □

Corollary 1.2.11. [12] Let $g : X \longrightarrow \mathbb{R}$ be a mapping such that

$$|Dg(x,y)| \leq c_2|x|^a|y|^b \text{ for all } x,y \in X \text{ , where } c_2 \geq 0 \text{ and } a+b \neq -1$$

. Then there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ satisfying reciprocal type functional equation (1.1), for every $x \in X$, $\rho \neq -1$ and

$$|g(x) - r(x)| \leq \frac{2c_2}{|2^{\rho+1}-1|}|x|^\rho \tag{1.40}$$

Proof Let $\phi(x,y) = c_2|x|^a|y|^b$

Case 1. If $\rho < -1$, then see (1.37)

$\Gamma(x) = \sum_{i=0}^{\infty} 2^{i+1} c_2(|2^i x|^a|2^i x|^b) = \sum_{i=0}^{\infty} 2^{i+1} c_2(|2^i x|^{a+b}) = 2c_2|x|^\rho \sum_{i=0}^{\infty} 2^{(\rho+1)i}$

$= \frac{2c_2}{1-2^{\rho+1}}|x|^\rho$, where $\rho = a+b$.

Therefor , as above, $\lim_{n\to\infty} 2^{n+1}\phi(2^n x, 2^n x) = 0$.

Case 2 .If $\rho > -1$, then see(1.38)

$\Gamma(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} c_2(|\frac{x}{2^{i+1}}|^a |\frac{x}{2^{i+1}}|^b) = c_2 \sum_{i=0}^{\infty} \frac{1}{2^i} |\frac{x}{2^{i+1}}|^{a+b} = \frac{c_2}{2^\rho} \sum_{i=0}^{\infty} \frac{1}{2^{(\rho+1)i}} |x|^\rho = \frac{c_2}{2^\rho} |x|^\rho \frac{1}{1-2^{-(\rho+1)}}$
$= \frac{2c_2}{2^{\rho+1}-1} |x|^\rho$.
Similarly as above, $\lim_{n\to\infty} \frac{1}{2^n} \phi(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}) = 0$.

By using the two cases we get $\Gamma(x) = \frac{2c_1}{|2^{\rho+1}-1|} |x|^\rho$.
Therefor, by Theorem (1.2.9) there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ satisfying reciprocal type functional equation (1.1) and $|g(x) - r(x)| \leq \frac{2c_2}{|2^{\rho+1}-1|} |x|^\rho$, for every $x \in X$ and $\rho \neq -1$. $\square$

Corollary 1.2.12. [12] Let $g : X \longrightarrow \mathbb{R}$ be a mapping , such that

$|Dg(x, y)| \leq c_3(|x|^q |y|^q + (|x|^{2q} + |y|^{2q}))$ for all $x, y \in X$,where $c_3 \geq 0$ and, $q \neq -\frac{1}{2}$.

Then there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ satisfying reciprocal type functional equation(1.1) , and for $q \neq -\frac{1}{2}$ and all $x \in X$,

$$|g(x) - r(x)| \leq \frac{6c_3}{|2^{2q+1} - 1|} |x|^{2q} \tag{1.41}$$

. **Proof** Let $\phi(x, y) = c_3(|x|^q |y|^q + (|x|^{2q} + |y|^{2q}))$

Case 1. If $2q < -1$, then see (1.37)

$\Gamma(x) = \sum_{i=0}^{\infty} 2^{i+1} c_3(|2^i x|^q |2^i x|^q + (|2^i x|^{2q}| + 2^i x|^{2q})) = \sum_{i=0}^{\infty} 2^{i+1} 3c_3(|2^i x|^{2q})$
$= 6c_3 |x|^{2q} \sum_{i=0}^{\infty} 2^{(2q+1)i} = \frac{6c_3}{1-2^{2q+1}} |x|^{2q}$.
Therefore, $\lim_{n\to\infty} 2^{n+1} \phi(2^n x, 2^n x) = 0$.

Case 2 .If $2q > -1$, then (see 1.38)

$\Gamma(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} c_3(|\frac{x}{2^{i+1}}|^q |\frac{x}{2^{i+1}}|^q + (|\frac{x}{2^{i+1}}|^{2q} + |\frac{x}{2^{i+1}}|^{2q})) = 3c_3 \sum_{i=0}^{\infty} \frac{1}{2^i} |\frac{x}{2^{i+1}}|^{2q} = \frac{3c_3}{2^{2q}} \sum_{i=0}^{\infty} \frac{1}{2^{(2q+1)i}} |x|^{2q}$
$= \frac{3c_3}{2^{2q}} |x|^{2q} \frac{1}{1-2^{-(2q+1)}} = \frac{6c_3}{2^{2q+1}-1} |x|^{2q}$
Similarly , $\lim_{n\to\infty} \frac{1}{2^n} \phi(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}) = 0$. By using the two cases we get $\Gamma(x) = \frac{6c_3}{|2^{2q+1}-1|} |x|^{2q}$.
Therefore by Theorem (1.2.9) there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ satisfying reciprocal type functional equation (1.1) and $|g(x) - r(x)| \leq \frac{6c_3}{|2^{2q+1}-1|} |x|^{2q}$, for $2q \neq -1$. $\square$

Chapter 2

Ulam Stability of Reciprocal Difference Functional Equation

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Introduction

This chapter consists of four sections:

In the first section, we study reciprocal difference functional equation and its solution. In the second section, we study Hyars - Ulam stability of reciprocal difference functional equation. In the third section, we study generalized Ulam stability of reciprocal difference functional equation. In the fourth section, we study extended Ulam stability of reciprocal difference functional equation.

Let X and Y be sets of nonzero real numbers. Let $r : X \to Y$ satisfies

$$r(\frac{x+y}{2}) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \tag{2.1}$$

for all $x, y \in X$.

Equation (2.1)is called the reciprocal difference functional equation

2.1 Reciprocal Difference Functional Equation and its Solution

Theorem 2.1.1. [15] Let X and Y be sets of non-zero real numbers. A function $r : X \longrightarrow Y$ satisfies the functional equation

$$r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}$$

if and only if $r : X \longrightarrow Y$ satisfies the functional equation

$$r(\frac{x+y}{2}) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}$$

Therefore, every solution of functional equations (2.1) is also a reciprocal function .

Proof. Let $r : X \longrightarrow Y$ satisfies the functional equation(1.1). Letting $y = x$ in (1.1), we get

$$r(2x) = \frac{1}{2}r(x). \tag{2.2}$$

Replacing x by $\frac{x}{2}$ in (2.2) , we obtain

$$r(\frac{x}{2}) = 2r(x). \tag{2.3}$$

Now, replacing (x, y) by $(\frac{x}{2}, \frac{y}{2})$ in (1.1) we get

$$r(\frac{x+y}{2}) = \frac{r(\frac{x}{2})r(\frac{y}{2})}{r(\frac{x}{2})+r(\frac{y}{2})} \tag{2.4}$$

Using (2.3)we obtain

$$r(\frac{x+y}{2}) = \frac{2r(x)r(y)}{r(x)+r(y)} \tag{2.5}$$

Subtracting (1.1) from (2.5) we get

$$r(\frac{x+y}{2}) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}$$

Conversely, let $r : X \longrightarrow Y$ satisfy the functional equation(2.1).Putting $y = x$ in (2.1) we obtain

$$r(x) - r(2x) = \frac{r(x)}{2} \tag{2.6}$$

Multiplying both side by 2 to get

$$2r(x) - 2r(2x) = r(x) \tag{2.7}$$

and hence

$$r(2x) = \frac{1}{2}r(x) \tag{2.8}$$

Replacing x by $\frac{x}{2}$ in (2.8), we obtain

$$2r(x) = r(\frac{x}{2}). \tag{2.9}$$

Using (2.9) in (2.1), we obtain (1.1)

$$2r(x+y) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \tag{2.10}$$

$$r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}.$$

This completes the proof of Theorem (2.1)□

2.2 Hyars - Ulam Stability of Reciprocal Difference Functional Equation

Theorem 2.2.1. [15] Let X and Y be sets of non-zero real numbers. Assume in addition that $f : X \longrightarrow Y$ is a mapping for which there exists a constant $c \geq 0$ independent of x, y such that the functional inequality

$$|f(\frac{x+y}{2}) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)}| \leq \frac{c}{2} \tag{2.11}$$

holds for all $x, y \in X$. Then the limit

$$r(x) = \lim_{n\to\infty} 2^{-n} f(2^{-n}x) \tag{2.12}$$

exists for all $x \in X$,$n \in N$ and $r : X \longrightarrow Y$ is the unique mapping satisfying the functional equation (2.1), such that

$$|f(x) - r(x)| \leq \epsilon \tag{2.13}$$

for all $x \in X$. Moreover, functional identity $r(x) = 2^{-n}r(2^{-n}x)$ holds for all $x \in X$ and all $n \in \mathbb{N}$.

Proof Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (2.11), we obtain

$$|f(\frac{x}{2}) - f(x) - \frac{1}{2}f(\frac{x}{2})| \leq \frac{c}{2}; \tag{2.14}$$

that is

$$|\frac{1}{2}f(\frac{x}{2}) - f(x)| \leq \frac{c}{2} \tag{2.15}$$

Replacing x by $\frac{x}{2^{k-1}}$ in (2.15) where k is a positive integer greater than or equal to 1, we obtain

$$|\frac{1}{2}f(\frac{x}{2^k}) - f(\frac{x}{2^{k-1}})| \leq \frac{c}{2} \tag{2.16}$$

for all $x \in X$.

Multiplying both sides of the above inequality by $\frac{1}{2^{k-1}}$ and adding the resulting n inequalities, we have

$$\sum_{k=1}^{n} |\frac{1}{2^k}f(\frac{x}{2^k}) - \frac{1}{2^{k-1}}f(\frac{x}{2^{k-1}})| \leq \sum_{k=1}^{n} \frac{1}{2^{k-1}}\frac{\epsilon}{2} = \epsilon\sum_{k=1}^{n}\frac{1}{2^k} = c(1 - \frac{1}{2^k}) \tag{2.17}$$

Using the triangle inequality and (2.17) we get

$$|\frac{1}{2^n}f(\frac{x}{2^n}) - f(x)| \leq \sum_{k=1}^{n} |\frac{1}{2^k}f(\frac{x}{2^k}) - \frac{1}{2^{k-1}}f(\frac{x}{2^{k-1}})| \leq c(1 - \frac{1}{2^n}). \tag{2.18}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now if $n > m > 0$, then $n - m$ is a natural number, and n can be replaced by $n - m$ in (2.18) to obtain

$$|\frac{1}{2^{n-m}}f(\frac{x}{2^{n-m}}) - f(x)| \leq \epsilon(1 - \frac{1}{2^{n-m}}). \tag{2.19}$$

Multiplying both sides by $\frac{1}{2^m}$ and simplifying , we get

$$|\frac{1}{2^n}f(\frac{x}{2^{n-m}}) - \frac{1}{2^m}f(x)| \leq \epsilon(\frac{1}{2^m} - \frac{1}{2^n}). \tag{2.20}$$

for all $x \in X$. Now we replace x by $\frac{x}{2^m}$ to have

$$|\frac{1}{2^n}f(\frac{x}{2^n}) - \frac{1}{2^m}f(\frac{x}{2^m})| \leq \epsilon(\frac{1}{2^m} - \frac{1}{2^n}). \tag{2.21}$$

If $m \to \infty$, then $(\frac{1}{2^m} - \frac{1}{2^n}) \to 0$ and therefore,

$$\lim_{m\to\infty} |\frac{1}{2^n}f(\frac{x}{2^n}) - \frac{1}{2^m}f(\frac{x}{2^m})| = 0. \tag{2.22}$$

Hence $\{2^{-n}f(2^{-n}x)\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $r : X \to \mathbb{R}$ by

$$r(x) = \lim_{n\to\infty} 2^{-n}f(2^{-n}x) \tag{2.23}$$

Now we show that $r : X \to \mathbb{R}$ defined by (2.23) is satisfying (2.1). Then by using (2.1) and (2.11) we get

$$|r(\frac{x+y}{2}) - r(x+y) - \frac{r(x)r(y)}{r(x)+(y)}|$$

$$= |\lim_{n\to\infty} 2^{-n}\{f(\frac{2^{-n}(x+y)}{2}) - f(2^{-n}(x+y)) - \frac{f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}\}|$$

$$= \lim_{n\to\infty} 2^{-n}|f(\frac{2^{-n}(x+y)}{2}) - f(2^{-n}(x+y)) - \frac{f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}|$$

$$\leq \lim_{n\to\infty} 2^{-n}\frac{c}{2} = 0.$$

Therefore $r(\frac{x+y}{2}) - r(x+y) = \frac{r(x)r(y)}{r(x)+(y)}$ for all $x, y \in X$.
Our next goal is to show that $|f(x) - r(x)| \leq c$. By using (2.18) we get

$$|f(x) - r(x)| = |f(x) - \lim_{n\to\infty} 2^{-n}f(2^{-n}x)|$$

$$= |\lim_{n\to\infty} 2^{-n}f(2^{-n}x) - f(x)| \leq \lim_{n\to\infty} \epsilon(1 - \frac{1}{2^n}) = c.$$

Hence we obtain $|f(x) - r(x)| \leq c$ for all $x \in X$

Finally we prove that r is unique. Suppose r is not unique, then there exists another reciprocal function $g : X \longrightarrow \mathbb{R}$ such that $|g(x) - f(x)| \leq c$, for all $x \in X$. Note that

$$|g(x) - r(x)| = |g(x) - f(x) + f(x) - r(x)| \leq |g(x) - f(x)| + |f(x) - r(x)| = c + c$$

Therefore,

$$|g(x) - r(x)| \leq 2c \tag{2.24}$$

Further, since r and g are reciprocal function , we have for any $n \in \mathbb{N}$,

$$|g(x) - r(x)| = |\frac{2^n g(x)}{2^n} - \frac{2^n r(x)}{2^n}| = |2^{-n} g(2^{-n}x) - 2^{-n} r(2^{-n}x)|$$
$$= 2^{-n}|g(2^{-n}x) - r(2^{-n}x)| \leq 2^{-n}(2c).$$

Taking the limit on both sides as $n \to \infty$, we get which is

$$|g(x) - r(x)| = 0.$$

Hence $g(x) = r(x)$ $\forall x \in X$. Therefore, the reciprocal map r is unique and the proof of the theorem is now complete.□

2.3 Generalized Ulam Stability of Reciprocal Difference Functional Equation

The generalized Ulam (or Ulam-Gavruta-Rassias) stability introduced by J. M. Rassias, concerns functional equations controlled by the product of powers of norms.

Theorem 2.3.1. [15] Let $f : X \longrightarrow Y$ be a mapping on the sets of non-zero real numbers. If there exist $a, b : \rho = a + b > -1$ and $c_1 \geq 0$ such that

$$|f(\frac{x+y}{2}) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)}| \leq c_1|x|^a|y|^b \tag{2.25}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \longrightarrow Y$ such that

$$|r(x) - f(x)| \leq c|x|^\rho \tag{2.26}$$

hold and r satisfies (2.1) for all $x, y \in X$ where $c = \frac{2c_1}{2^{\rho+1}-1}$.

Proof Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (2.25), we obtain

$$|f(\frac{\frac{x}{2}+\frac{x}{2}}{2}) - f(\frac{x}{2}+\frac{x}{2}) - \frac{f(\frac{x}{2})f(\frac{x}{2})}{f(\frac{x}{2})+f(\frac{x}{2})}| \leq c_1|\frac{x}{2}|^a|\frac{x}{2}|^b. \tag{2.27}$$

That is,

$$|f(\frac{x}{2}) - f(x) - \frac{1}{2}f(\frac{x}{2})| \leq \frac{c_1}{2^\rho}|x|^\rho \tag{2.28}$$

$$|\frac{1}{2}f(\frac{x}{2}) - f(x)| \leq \frac{c_1}{2^\rho}|x|^\rho. \qquad (2.29)$$

Replacing x by $\frac{x}{2}$ in (2.29), and dividing by 2 we get

$$|\frac{1}{2^2}f(\frac{x}{2^2}) - \frac{1}{2}f(x)| \leq \frac{c_1}{2^{\rho+1}}|\frac{x}{2}|^\rho. \qquad (2.30)$$

Summing (2.30) with (2.29) and use the triangle inequality to get

$|\frac{1}{2^2}f(\frac{x}{2^2}) - f(x)| \leq |\frac{1}{2^2}f(\frac{x}{2^2}) - \frac{1}{2}f(\frac{x}{2})| + |\frac{1}{2}f(\frac{x}{2}) - f(x)|$

$\leq \frac{c_1}{2^{\rho+1}}|\frac{x}{2}|^\rho + \frac{c_1}{2^\rho}|x|^\rho = \frac{c_1}{2^\rho}\sum_{i=0}^{1}\frac{1}{2^{i(\rho+1)}}|x|^\rho.$

Proceeding further and using induction on a positive integer n, we get

$$|\frac{1}{2^n}f(\frac{x}{2^n}) - f(x)| \leq \frac{c_1}{2^\rho}|x|^\rho\sum_{i=0}^{n-1}\frac{1}{2^{i(\rho+1)}}. \qquad (2.31)$$

Hence

$$|\frac{1}{2^n}f(\frac{x}{2^n}) - f(x)| \leq \frac{c_1}{2^\rho}|x|^\rho\sum_{i=0}^{\infty}\frac{1}{2^{i(\rho+1)}} = \frac{2c_1}{2^{\rho+1}-1}|x|^\rho. \qquad (2.32)$$

Setting $c = \frac{2c_1}{2^{\rho+1}-1}$ to get

$$|\frac{1}{2^n}f(\frac{x}{2^n}) - f(x)| \leq c|x|^\rho. \qquad (2.33)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Now if $n > m > 0$, then $n-m$ is a natural number, and n can be replaced by $n-m$ in (2.33) to obtain

$$|\frac{1}{2^{n-m}}f(\frac{x}{2^{n-m}}) - f(x)| \leq c|x|^\rho. \qquad (2.34)$$

Multiplying both sides by $\frac{1}{2^m}$ and simplifying , we get

$$|\frac{1}{2^n}f(\frac{x}{2^{n-m}}) - \frac{1}{2^m}f(x)| \leq \frac{c}{2^m}|x|^\rho. \qquad (2.35)$$

for all $x \in X$. Now we replace x by $\frac{x}{2^m}$ to have

$$|\frac{1}{2^n}f(\frac{x}{2^n}) - \frac{1}{2^m}f(\frac{x}{2^m})| \leq \frac{c}{2^m}|\frac{x}{2^m}|^\rho. \qquad (2.36)$$

If $m \to \infty$ and $\rho > -1$, then $\frac{c|x|^\rho}{2^{m(\rho+1)}} \to 0$, and therefore,

$$\lim_{m\to\infty}|\frac{1}{2^n}f(\frac{x}{2^n}) - \frac{1}{2^m}f(\frac{x}{2^m})| = 0. \qquad (2.37)$$

Hence

$$\{2^{-n} f(2^{-n} x)\}_{n=0}^{\infty} \tag{2.38}$$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $r : X \to \mathbb{R}$ by

$$r(x) = \lim_{n \to \infty} 2^{-n} f(2^{-n} x) \tag{2.39}$$

Now we show that $r : X \to \mathbb{R}$ defined by (2.39) is satisfying (2.1). Consider

$|r(\frac{x+y}{2}) - r(x+y) - \frac{r(x)r(y)}{r(x)+(y)}| = |\lim_{n\to\infty} 2^{-n}\{f(\frac{2^{-n}(x+y)}{2}) - f(2^{-n}(x+y)) - \frac{f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}\}|$

$= \lim_{n\to\infty} 2^{-n}|f(\frac{2^{-n}(x+y)}{2}) - f(2^{-n}(x+y)) - \frac{f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}|$

$\leq \lim_{n\to\infty} 2^{-n(a+b)} c_1 |x|^a |y|^b = 0$ by (2.25)

Therefore, $r(\frac{x+y}{2}) - r(x+y) = \frac{r(x)r(y)}{r(x)+(y)}$ for all $x, y \in X$.

Our next goal is to show that $|f(x) - r(x)| \leq c|x|^{\rho}$

$$|f(x) - r(x)| = |f(x) - \lim_{n\to\infty} 2^{-n} f(2^{-n}x)|$$

$= \lim_{n\to\infty} |2^{-n} f(2^{-n}x) - f(x)| \leq \lim_{n\to\infty} c|x|^{\rho} = c|x|^{\rho}$ by (2.33).

Hence we obtain $|f(x) - r(x)| \leq c|x|^{\rho}$ for all $x \in X$.

Finally we prove that r is unique. Suppose r is not unique, then there exists another reciprocal function $g : X \longrightarrow \mathbb{R}$ such that

$$|g(x) - f(x)| \leq c|x|^{\rho} \tag{2.40}$$

for all $x \in X$. Note that

$$|g(x) - r(x)| = |g(x) - f(x) + f(x) - r(x)|$$

$$\leq |g(x) - f(x)| + |f(x) - r(x)|$$

$$= c|x|^{\rho} + c|x|^{\rho}.$$

Therefore,

$$|g(x) - r(x)| \leq 2c|x|^\rho \tag{2.41}$$

Further, since r and g are reciprocal function , we have
$|g(x) - r(x)| = |\frac{2^n g(x)}{2^n} - \frac{2^n r(x)}{2^n}| = |2^{-n} g(2^{-n}x) - 2^{-n} r(2^{-n}x)|$
$= 2^{-n}|g(2^{-n}x) - r(2^{-n}x)| \leq 2^{-n}(2c|2^{-n}x|^\rho) = 2^{-n(\rho+1)+1}c|x|^\rho$, where $n \in \mathbb{N}$.
Taking the limit on both sides as $n \to \infty$, we get

$$|g(x) - r(x)| \leq \lim_{n\to\infty} 2^{-n(\rho+1)+1}c|x|^\rho = 0$$

Hence $g(x) = r(x)$ $\forall x \in X$. Therefore, the reciprocal map r is unique and the proof of the Theorem is now complete.□

Theorem 2.3.2. [15] Let $f : X \longrightarrow Y$ be a mapping on the sets of non-zero real numbers. If there exist a, b where $\rho = a + b < -1$ and $c_1 \geq 0$ such that

$$|f(\frac{x+y}{2}) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)}| \leq c_1|x|^a|y|^b \tag{2.42}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \longrightarrow Y$ such that

$$|r(x) - f(x)| \leq c|x|^\rho \tag{2.43}$$

hold and r satisfies (2.1) for all $x, y \in X$ where $c = \frac{2c_1}{1-2^{\rho+1}}$.

Proof Replacing(x, y) by (x, x)in (2.42), we have

$$|f(\frac{x+x}{2}) - f(x+x) - \frac{f(x)f(x)}{f(x)+f(x)}| \leq c_1|x|^a|x|^b \tag{2.44}$$

$$|f(x) - f(2x) - \frac{1}{2}f(x)| \leq c_1|x|^\rho \tag{2.45}$$

$$|\frac{1}{2}f(x) - f(2x)| \leq c_1|x|^\rho \tag{2.46}$$

$$|f(x) - 2f(2x)| \leq 2c_1|x|^\rho. \tag{2.47}$$

Replacing x by $2x$ in (2.47) and multiplying by 2 we get

$$|2f(2x) - 2^2 f(2^2x)| \leq 2^2 c_1|2x|^\rho. \tag{2.48}$$

Summing (2.48) with (2.47) and use the triangle inequality to get $|f(x)-2^2f(2^2x)| \leq |f(x)-2f(2x)| + |2f(2x)-2^2f(2^2x)|$

$$\leq 2c_1|x|^\rho + 2^2c_1|2x|^\rho = 2c_1\sum_{i=0}^{1}2^{i(\rho+1)}|x|^\rho. \tag{2.49}$$

Proceeding further and using induction on a positive integer n, we get

$$|f(x)-2^nf(2^nx)| \leq 2c_1\sum_{i=0}^{n-1}2^{i(\rho+1)}|x|^\rho. \tag{2.50}$$

Hence

$$|f(x)-2^nf(2^nx)| \leq 2c_1\sum_{i=0}^{\infty}2^{i(\rho+1)}|x|^\rho = \frac{2c_1}{1-2^{(\rho+1)}}|x|^\rho \tag{2.51}$$

Setting $c = \frac{2c_1}{1-2^{\rho+1}}$ to get

$$|f(x)-2^nf(2^nx)| \leq c|x|^\rho \tag{2.52}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now if $n > m > 0$, then $n-m$ is a natural number, and n can be replaced by $n-m$ in (2.52) to obtain

$$|f(x)-2^{n-m}f(2^{n-m}x)| \leq c|x|^\rho \tag{2.53}$$

Multiplying both sides by 2^m and simplifying , we get

$$|2^mf(x)-2^nf(2^{n-m}x)| \leq 2^mc|x|^\rho. \tag{2.54}$$

for all $x \in X$. Now we replace x by 2^mx to have

$$|2^mf(2^mx)-2^nf(2^nx)| \leq 2^mc|2^mx|^\rho \tag{2.55}$$

$$|2^mf(2^mx)-2^nf(2^nx)| \leq 2^{m(\rho+1)}c|x|^\rho. \tag{2.56}$$

If $m \to \infty$ and $\rho < -1$ then $2^{m(\rho+1)}c|x|^\rho \to 0$, and therefore,

$$\lim_{m\to\infty}|2^mf(2^mx)-2^nf(2^nx)| = 0. \tag{2.57}$$

Hence

$$\{2^nf(2^nx)\}_{n=0}^{\infty} \tag{2.58}$$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $r : X \to \mathbb{R}$ by

$$r(x) = \lim_{n\to\infty} 2^n f(2^n x) \tag{2.59}$$

Now we show that $r : X \to \mathbb{R}$ defined by (2.58) is satisfying (2.1). Consider

$$|r(\frac{x+y}{2}) - r(x+y) - \frac{r(x)r(y)}{r(x)+(y)}|$$

$$= |\lim_{n\to\infty} 2^n\{f(\frac{2^n(x+y)}{2}) - f(2^n(x+y)) - \frac{f(2^n x)f(2^n y)}{f(2^n x)+f(2^n y)}\}|$$

$$= \lim_{n\to\infty} 2^n|f(\frac{2^n(x+y)}{2}) - f(2^n(x+y)) - \frac{f(2^n x)f(2^n y)}{f(2^n x)+f(2^n y)}|$$

$$\leq \lim_{n\to\infty} 2^n c_1|2^n x|^a|2^n y|^b \text{ by } (2.42)$$

$$= \lim_{n\to\infty} 2^n c_1 2^{n(a+b)}|x|^a|y|^b$$

$$= \lim_{n\to\infty} 2^{n(\rho+1)} c_1|x|^a|y|^b = 0.$$

Therefore, $r(\frac{x+y}{2}) - r(x+y) = \frac{r(x)r(y)}{r(x)+(y)}$, for all $x, y \in X$.

Our next goal is to show that

$$|f(x) - r(x)| \leq c|x|^\rho$$

$$|f(x) - r(x)| = |f(x) - \lim_{n\to\infty} 2^n f(2^n x)|$$

$= |\lim_{n\to\infty} 2^n f(2^n x) - f(x)| \leq \lim_{n\to\infty} c|x|^\rho = c|x|^\rho$ by (2.52)

Hence we obtain

$$|f(x) - r(x)| \leq c|x|^\rho, \text{ for all } x \in X.$$

Finally we prove that r is unique. Suppose r is not unique, then there exists another reciprocal function $g : X \longrightarrow \mathbb{R}$ such that

$$|g(x) - f(x)| \leq c|x|^{\rho} \tag{2.60}$$

for all $x \in X$. Note that

$$|g(x) - r(x)| = |g(x) - f(x) + f(x) - r(x)|$$

$$\leq |g(x) - f(x)| + |f(x) - r(x)| = c|x|^{\rho} + c|x|^{\rho}$$

Therefore,

$$|g(x) - r(x)| \leq 2c|x|^{\rho} \tag{2.61}$$

Further, since r and g are reciprocal function , we have

$$|g(x) - r(x)| = |\frac{2^n g(x)}{2^n} - \frac{2^n r(x)}{2^n}|$$

$$= |2^n g(2^n x) - 2^n r(2^n x)|$$

$$= 2^n |g(2^n x) - r(2^n x)|$$

$$\leq 2^{n(\rho+1)+1} c|x|^{\rho},$$

where $n \in \mathbb{N}$. Hence

$$|g(x) - r(x)| \leq 2^{n(\rho+1)+1} c|x|^{\rho}.$$

Taking the limit on both sides, we get

$$0 \leq |g(x) - r(x)| = \lim_{n\to\infty} |g(x) - r(x)| \leq \lim_{n\to\infty} 2^{n(\rho+1)+1} c|x|^{\rho} = 0$$

Hence $g(x) = r(x)$ $\forall x \in X$. Therefore, the reciprocal map r is unique and the proof of the Theorem is now complete.□

Theorem 2.3.3. [15] Let $f : X \longrightarrow Y$ be a mapping on the sets of anon zero real number for which there exists a constant $\theta > 0$ and f satisfies

$$|f(\frac{x+y}{2}) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)}| \leq \theta H(x,y) \tag{2.62}$$

where $H : X^2 \longrightarrow Y$ is a function such that

$$\phi(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} H(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) \tag{2.63}$$

with the condition

$$\lim_{n\to\infty} \frac{1}{2^n} H(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}) = 0 \tag{2.64}$$

holds. Then there exists a unique reciprocal mapping $A : X \longrightarrow Y$ which satisfies (2.1) and the inequality

$$|f(x) - A(x)| \leq \theta\phi(x)$$

for all $x \in X$.

Proof Replacing(x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (2.62), we have

$$|f(\frac{\frac{x}{2}+\frac{x}{2}}{2}) - f(\frac{x}{2}+\frac{x}{2}) - \frac{f(\frac{x}{2})f(\frac{x}{2})}{f(\frac{x}{2})+f(\frac{x}{2})}| \leq \theta H(\frac{x}{2}, \frac{x}{2}). \tag{2.65}$$

That is,

$$|\frac{1}{2}f(\frac{x}{2}) - f(x)| \leq \theta H(\frac{x}{2}, \frac{x}{2}). \tag{2.66}$$

Replacing x by $\frac{x}{2}$ in(2.66) ,and dividing by 2 to get

$$|\frac{1}{2^2}f(\frac{x}{2^2}) - \frac{1}{2}f(x)| \leq \frac{\theta}{2}H(\frac{x}{2^2}, \frac{x}{2^2}) \tag{2.67}$$

Summing by (2.67) to (2.66) and using triangle inequality to get $|\frac{1}{2^2}f(\frac{x}{2^2}) - f(x)| \leq |\frac{1}{2^2}f(\frac{x}{2^2}) - \frac{1}{2}f(\frac{x}{2})| + |\frac{1}{2}f(\frac{x}{2}) - f(x)|$

$$\leq \frac{\theta}{2}H(\frac{x}{2^2}, \frac{x}{2^2}) + \theta H(\frac{x}{2}, \frac{x}{2}) = \theta \sum_{i=0}^{1} \frac{1}{2^i} H(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) \tag{2.68}$$

Proceeding further and using induction on a positive integer n, we get

$$|\frac{1}{2^n}f(\frac{x}{2^n})-f(x)|\leq\theta\sum_{i=0}^{n}\frac{1}{2^i}H(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}) \tag{2.69}$$

for all $x\in X$ and $n\in\mathbb{N}$. Now if $n>m>0$, then $n-m$ is a natural number, and n can be replaced by $n-m$ in (2.69) to obtain

$$|\frac{1}{2^{n-m}}f(\frac{x}{2^{n-m}})-f(x)|\leq\theta\sum_{i=1}^{n-m}\frac{1}{2^i}H(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}). \tag{2.70}$$

Multiplying both sides by $\frac{1}{2^m}$ and simplifying , we get

$$|\frac{1}{2^n}f(\frac{x}{2^{n-m}})-\frac{1}{2^m}f(x)|\leq\theta\sum_{i=1}^{n-m}\frac{1}{2^{i+m}}H(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}). \tag{2.71}$$

for all $x\in X$. Now we replace x by $\frac{x}{2^m}$ to have

$$|\frac{1}{2^n}f(\frac{x}{2^n})-\frac{1}{2^m}f(\frac{x}{2^m})|\leq\theta\sum_{i=1}^{n-m}\frac{1}{2^{i+m}}H(\frac{x}{2^{i+m+1}},\frac{x}{2^{i+m+1}}). \tag{2.72}$$

If $m\to\infty$, then $\frac{1}{2^{i+m}}H(\frac{x}{2^{i+m+1}},\frac{x}{2^{i+m+1}})\to 0\ \forall i$ by (2.63)

Therefore, $\lim_{m\to\infty}|\frac{1}{2^n}f(\frac{x}{2^n})-\frac{1}{2^m}f(\frac{x}{2^m})|=0$.

Hence

$$\{2^{-n}f(2^{-n}x)\}_{n=0}^{\infty} \tag{2.73}$$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $A:X\to\mathbb{R}$ by

$$A(x)=\lim_{n\to\infty}2^{-n}f(2^{-n}x) \tag{2.74}$$

Now we show that $A:X\to\mathbb{R}$ defined by (2.74) is satisfying(2.1). Consider

$$|A(\frac{x+y}{2})-A(x+y)-\frac{A(x)A(y)}{A(x)+A(y)}|$$

$$= |\lim_{n\to\infty} 2^{-n}\{f(\frac{2^{-n}(x+y)}{2}) - f(2^{-n}(x+y)) - \frac{f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}\}|$$

$$= \lim_{n\to\infty} 2^{-n}|f(\frac{2^{-n}(x+y)}{2}) - f(2^{-n}(x+y)) - \frac{f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}|$$

$$\leq \theta \lim_{n\to\infty} 2^{-n} H(2^{-n}x, 2^{-n}y) = 0 \text{ by (2.62) and (2.64).}$$

Therefore, $A(\frac{x+y}{2}) - A(x+y) = \frac{A(x)A(y)}{A(x)+A(y)}$ for all $x, y \in X$.

Our next goal is to show that

$$|f(x) - A(x)| \leq \theta\phi(x)$$

$$|f(x) - A(x)| = |f(x) - \lim_{n\to\infty} 2^{-n} f(2^{-n}x)|$$

$= |\lim_{n\to\infty} 2^{-n} f(2^{-n}x) - f(x)| \leq \lim_{n\to\infty} \theta\phi(x) = \theta\phi(x)$ by (2.69).

Hence we obtain

$$|f(x) - A(x)| \leq \theta\phi(x)$$

for all $x \in X$

Finally we prove that A is unique. Suppose that there exists another reciprocal function $G : X \longrightarrow \mathbb{R}$ such that

$$|G(x) - f(x)| \leq \theta\phi(x) \tag{2.75}$$

for all $x \in X$. Note that

$$|G(x) - A(x)| = |G(x) - f(x) + f(x) - A(x)|$$

$$\leq |G(x) - f(x)| + |f(x) - A(x)|$$

$$= \theta\phi(x) + \theta\phi(x)$$

Therefore

$$|G(x) - A(x)| \leq 2\theta\phi(x) \tag{2.76}$$

Further, since A and G are reciprocal function , we have

$$|G(x) - A(x)| = |\frac{2^n G(x)}{2^n} - \frac{2^n A(x)}{2^n}|$$

$$= |2^{-n}G(2^{-n}x) - 2^{-n}A(2^{-n}x)|$$

$$= 2^{-n}|G(2^{-n}x) - A(2^{-n}x)|$$

$$\leq 2^{-n}\theta\phi(x),$$

where $n \in \mathbb{N}$. Hence

$$|G(x) - A(x)| \leq 2^{-n}\theta\phi(x).$$

Taking the limit on both sides, we get

$$0 \leq |G(x) - A(x)| = \lim_{n\to\infty} |G(x) - A(x)| \leq \lim_{n\to\infty} 2^{-n}\theta\phi(x) = 0.$$

Hence $G(x) = A(x)\ \forall x \in X$. Therefore, the reciprocal map A is unique and the proof of the theorem is now complete.□

Theorem 2.3.4. [15] Let $f : X \longrightarrow Y$ be a mapping on the sets of non-zero real numbers for which there exists a constant $\theta > 0$ and f satisfies

$$|f(\frac{x+y}{2}) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)}| \leq \theta H(x,y) \tag{2.77}$$

where $H : X^2 \longrightarrow Y$ be a function such that

$$\phi(x) = \sum_{i=0}^{\infty} 2^i H(2^{i+1}x, 2^{i+1}x) \tag{2.78}$$

with the condition

$$\lim_{n\to\infty} 2^n H(2^{n+1}x, 2^{n+1}x) = 0 \tag{2.79}$$

holds. Then there exists a unique reciprocal mapping $A : X \longrightarrow Y$ which satisfies (2.1) and the inequality

$$|f(x) - A(x)| \leq \theta\phi(x)$$

for all $x \in X$

Proof Replacing(x, y) by (x, x) in (2.77), we have

$$|f(\frac{x+x}{2}) - f(x+x) - \frac{f(x)f(x)}{f(x)+f(x)}| \leq \theta H(x, x) \quad (2.80)$$

$$|f(x) - f(2x) - \frac{1}{2}f(x)| \leq \theta H(x, x) \quad (2.81)$$

$$|\frac{1}{2}f(x) - f(2x)| \leq \theta H(x, x) \quad (2.82)$$

$$|f(x) - 2f(2x)| \leq \theta H(x, x) \quad (2.83)$$

Replacing x by $2x$ in ,(2.83),and multiplying by 2 we get

$$|2f(x) - 2^2 f(2^2 x)| \leq 2\theta H(2x, 2x) \quad (2.84)$$

Summing (2.84) and (2.83) and use triangle inequality to get $|f(x) - 2^2 f(2^2 x)| \leq$ $|f(x) - 2f(2x)| + |2f(2x) - 2^2 f(2^2 x)|$

$\theta H(x, x) + 2\theta H(2x, 2x)$

$$= \theta \sum_{i=0}^{1} 2^i H(2^{i+1}x, 2^{i+1}x) \quad (2.85)$$

Proceeding further and using induction on a positive integer n, we get

$$|f(x) - 2^n f(2^n x)| \leq \theta \sum_{i=0}^{n} 2^i H(2^{i+1}x, 2^{i+1}x) \quad (2.86)$$

for all $x \in X$ and $n \in \mathbb{N}$. Now if $n > m > 0$, then $n - m$ is a natural number, and n can be replaced by $n - m$ in (2.86) to obtain

$$|f(x) - 2^{n-m} f(2^{n-m} x)| \leq \theta \sum_{i=0}^{n-m} 2^i H(2^{i+1}x, 2^{i+1}x) \quad (2.87)$$

Multiplying both sides by 2^m and simplifying , we get

$$|2^m f(x) - 2^n f(2^{n-m} x)| \leq \theta \sum_{i=0}^{n-m} 2^{i+m} H(2^{i+1}x, 2^{i+1}x) \quad (2.88)$$

for all $x \in X$. Now we replace x by $2^m x$ to have

$$|2^m f(2^m x) - 2^n f(2^n x)| \leq \theta \sum_{i=0}^{n-m} 2^{i+m} H(2^{i+m+1}x, 2^{i+m+1}x) \tag{2.89}$$

If $m \to \infty$,then $\theta 2^{i+m} H(2^{i+m+1}x, 2^{i+m+1}x) \to 0 \ \forall i$.

Therefore,

$$\lim_{m\to\infty} |2^m f(2^m x) - 2^n f(2^n x)| = 0. \tag{2.90}$$

Hence

$$\{2^n f(2^n x)\}_{n=1}^{\infty} \tag{2.91}$$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $A : X \to \mathbb{R}$ by

$$A(x) = \lim_{n\to\infty} 2^n f(2^n x) \tag{2.92}$$

Now we show that $A : X \to \mathbb{R}$ defined by (2.92) is satisfying (2.1). Consider

$$|A(\frac{x+y}{2}) - A(x+y) - \frac{A(x)A(y)}{A(x)+A(y)}|$$

$$= |\lim_{n\to\infty} 2^n \{f(\frac{2^n(x+y)}{2}) - f(2^n(x+y)) - \frac{f(2^n x)f(2^n y)}{f(2^n x)+f(2^n y)}\}|$$

$$= \lim_{n\to\infty} 2^n |f(\frac{2^n(x+y)}{2}) - f(2^n(x+y)) - \frac{f(2^n x)f(2^n y)}{f(2^n x)+f(2^n y)}|$$

$$\leq \lim_{n\to\infty} 2^n \theta H(2^n x, 2^n y) = 0 \text{ by } (2.77)$$

Therefore

$$A(\frac{x+y}{2}) - A(x+y) = \frac{A(x)A(y)}{A(x)+(y)}$$

for all $x, y \in X$.

Our next goal is to show that

$$|f(x) - A(x)| \leq \theta\phi(x)$$

$$|f(x) - A(x)| = |f(x) - \lim_{n\to\infty} 2^n f(2^n x)|$$

$= |\lim_{n\to\infty} 2^n f(2^n x) - f(x)| \leq \theta\phi(x)$ by (2.86)

Hence we obtain

$$|f(x) - A(x)| \leq \theta\phi(x)$$

for all $x \in X$

Finally we prove that A is unique. Suppose that there exists another reciprocal function $G : X \longrightarrow \mathbb{R}$ such that

$$|G(x) - f(x)| \leq \theta\phi(x) \tag{2.93}$$

for all $x \in X$. Note that

$$|G(x) - A(x)| = |G(x) - f(x) + f(x) - A(x)|$$

$$\leq |G(x) - f(x)| + |f(x) - A(x)| = \theta\phi(x) + \theta\phi(x)$$

Therefore,

$$|G(x) - A(x)| \leq 2\theta\phi(x) \tag{2.94}$$

Further, since A and G are reciprocal function , we have

$$|G(x) - A(x)| = |\frac{2^n G(x)}{2^n} - \frac{2^n A(x)}{2^n}|$$

$$= |\frac{1}{2^n} G(\frac{x}{2^n}) - \frac{1}{2^n} A(\frac{x}{2^n}) A|$$

$$= \frac{1}{2^n} |G(\frac{x}{2^n}) - A(\frac{x}{2^n})|$$

$$\leq \frac{1}{2^n} \theta \phi(\frac{x}{2^n}),$$

where $n \in \mathbb{N}$. Hence

$$|G(x) - A(x)| \leq \frac{1}{2^n} \theta \phi(\frac{x}{2^n})$$

Taking the limit on both sides, we get

$$0 \leq |G(x) - A(x)| = \lim_{n \to \infty} |G(x) - A(x)| \leq \theta \lim_{n \to \infty} \frac{1}{2^n} \phi(\frac{x}{2^n}) = 0$$

Hence $G(x) = A(x)\ \forall x \in X$. Therefore, the reciprocal map A is unique and the proof of the theorem is now complete.□

2.4 Extended Ulam Stability of Reciprocal Difference Functional Equation

The extended Ulam(or Rassias) stability introduced by J. M. Rassias, concerns functional equations controlled by the mixed product-sum of powers of norms.

Theorem 2.4.1. [15] Let $f : X \longrightarrow Y$ be a mapping on the sets of non-zero real numbers. If there exist k and α with $k > 0$ and $\alpha > -\frac{1}{2}$ such that

$$|f(\frac{x+y}{2}) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)}| \leq k(|x|^\alpha |y|^\alpha + (|x|^{2\alpha} + |y|^{2\alpha})) \tag{2.95}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \longrightarrow Y$ such that

$$|r(x) - f(x)| \leq c|x|^{2\alpha} \tag{2.96}$$

and r satisfies (2.1)for all $x, y \in X$ where $c = \frac{6k}{2^{2\alpha+1}-1}$.

Proof Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (2.95), we obtain

$$|f(\frac{\frac{x}{2}+\frac{x}{2}}{2}) - f(\frac{x}{2}+\frac{x}{2}) - \frac{f(\frac{x}{2})f(\frac{x}{2})}{f(\frac{x}{2})+f(\frac{x}{2})}| \leq k(|\frac{x}{2}|^{\alpha}|\frac{x}{2}|^{\alpha} + (|\frac{x}{2}|^{2\alpha} + |\frac{x}{2}|^{2\alpha})) \tag{2.97}$$

$$|f(\frac{x}{2}) - f(x) - \frac{1}{2}f(\frac{x}{2})| \leq \frac{3k}{2^{2\alpha}}|x|^{2\alpha} \tag{2.98}$$

$$|\frac{1}{2}f(\frac{x}{2}) - f(x)| \leq \frac{3k}{2^{2\alpha}}|x|^{2\alpha} \tag{2.99}$$

Replacing x by $\frac{x}{2}$ in (2.99) ,and dividing by 2 to get

$$|\frac{1}{2^2}f(\frac{x}{2^2}) - \frac{1}{2}f(\frac{x}{2})| \leq \frac{3k}{2^{2\alpha}}\frac{1}{2}|\frac{x}{2}|^{2\alpha} \tag{2.100}$$

Summing(2.99)and(2.100) and use the triangle inequality to get $|\frac{1}{2^2}f(\frac{x}{2^2}) - f(x)| \leq |\frac{1}{2^2}f(\frac{x}{2^2}) - \frac{1}{2}f(\frac{x}{2})| + |\frac{1}{2}f(\frac{x}{2}) - f(x)| \leq \frac{3k}{2^{2\alpha}}\frac{1}{2}|\frac{x}{2}|^{2\alpha} + \frac{3k}{2^{2\alpha}}|x|^{2\alpha}$

$$= \sum_{i=0}^{1}\frac{1}{2^{i(2\alpha+1)}}|x|^{2\alpha} \tag{2.101}$$

Then by induction on n we get

$$|\frac{1}{2^n}f(\frac{x}{2^n}) - f(x)| \leq \frac{3k}{2^{2\alpha}}\sum_{i=0}^{n-1}\frac{1}{2^{i(2\alpha+1)}}|x|^{2\alpha} \tag{2.102}$$

Hence

$$|\frac{1}{2^n}f(\frac{x}{2^n}) - f(x)| \leq \frac{3k}{2^{2\alpha}}\sum_{i=0}^{\infty}\frac{1}{2^{i(2\alpha+1)}}|x|^{2\alpha} = \frac{6k}{2^{2\alpha+1}-1}|x|^{2\alpha}. \tag{2.103}$$

Setting $c = \frac{6k}{2^{2\alpha+1}-1}$

$$|\frac{1}{2^n}f(\frac{x}{2^n}) - f(x)| \leq c|x|^{2\alpha}. \tag{2.104}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now if $n > m > 0$, then $n - m$ is a natural number, and n can be replaced by $n - m$ in (2.104) to obtain

$$|\frac{1}{2^{n-m}}f(\frac{x}{2^{n-m}}) - f(x)| \leq c|x|^{2\alpha}. \tag{2.105}$$

Multiplying both sides by $\frac{1}{2^m}$ and simplifying , we get

$$|\frac{1}{2^n}f(\frac{x}{2^{n-m}}) - \frac{1}{2^m}f(x)| \leq \frac{c}{2^m}|x|^{2\alpha}. \tag{2.106}$$

for all $x \in X$. Now we replace x by $\frac{x}{2^m}$ to have

$$|\frac{1}{2^n}f(\frac{x}{2^n}) - \frac{1}{2^m}f(\frac{x}{2^m})| \leq \frac{c}{2^m}|\frac{x}{2^m}|^{2\alpha}. \tag{2.107}$$

If $m \to \infty$ and $\alpha > -\frac{1}{2}$, then $\frac{c|x|^{2\alpha}}{2^{m(2\alpha+1)}} \to 0$, and therefore,

$$\lim_{m\to\infty} |\frac{1}{2^n}f(\frac{x}{2^n}) - \frac{1}{2^m}f(\frac{x}{2^m})| = 0. \tag{2.108}$$

Hence

$$\{2^{-n}f(2^{-n}x)\}_{n=0}^{\infty} \tag{2.109}$$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $r : X \to \mathbb{R}$ by

$$r(x) = \lim_{n\to\infty} 2^{-n}f(2^{-n}x) \tag{2.110}$$

Now we show that $r : X \to \mathbb{R}$ defined by (2.110) is satisfying (2.1). Consider

$$|r(\frac{x+y}{2}) - r(x+y) - \frac{r(x)r(y)}{r(x)+(y)}|$$

$$= |\lim_{n\to\infty} 2^{-n}\{f(\frac{2^{-n}(x+y)}{2}) - f(2^{-n}(x+y)) - \frac{f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}\}|$$

$$= \lim_{n\to\infty} 2^{-n}|f(\frac{2^{-n}(x+y)}{2}) - f(2^{-n}(x+y)) - \frac{f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}|$$

$$\leq \lim_{n\to\infty} 2^{-n(2\alpha+1)}k(|x|^{\alpha}|y|^{\alpha} + (|x|^{2\alpha} + |y|^{2\alpha})) = 0 \text{ by } (2.95)$$

Therefore, $r(\frac{x+y}{2}) - r(x+y) = \frac{r(x)r(y)}{r(x)+(y)}$, for all $x, y \in X$.

Our next goal is to show that

$$|f(x) - r(x)| \leq c|x|^{2\alpha}$$

By (2.110), we hava

$$|f(x) - r(x)| = |f(x) - \lim_{n\to\infty} 2^{-n}f(2^{-n}x)|$$

$$= |\lim_{n\to\infty} 2^{-n}f(2^{-n}x) - f(x)| \leq c|x|^{2\alpha}$$

Hence we obtain $|f(x) - r(x)| \leq c|x|^{2\alpha}$ for all $x \in X$

Finally we prove that r is unique. Suppose that there exists another reciprocal function $g : X \longrightarrow \mathbb{R}$ such that

$$|g(x) - f(x)| \leq c|x|^{2\alpha} \tag{2.111}$$

for all $x \in X$. Note that

$$|g(x) - r(x)| = |g(x) - f(x) + f(x) - r(x)|$$

$$\leq |g(x) - f(x)| + |f(x) - r(x)| = c|x|^{2\alpha} + c|x|^{2\alpha}$$

Therefore,

$$|g(x) - r(x)| \leq 2c|x|^{2\alpha} \tag{2.112}$$

Further, since r and g are reciprocal function , we have

$$|g(x) - r(x)| = |\frac{2^n g(x)}{2^n} - \frac{2^n r(x)}{2^n}|$$

$$= |2^{-n} g(2^{-n}x) - 2^{-n} r(2^{-n}x)|$$

$= 2^{-n}|g(2^{-n}x) - r(2^{-n}x)| \leq 2^{-n(2\alpha+1)+1}c|x|^{2\alpha}$, where $n \in \mathbb{N}$. Hence

$$|g(x) - r(x)| \leq 2^{-n(2\alpha+1)+1}c|x|^{2\alpha}$$

Taking the limit on both sides, we get

$$0 \leq |g(x) - r(x)| = \lim_{n\to\infty} |g(x) - r(x)| \leq \lim_{n\to\infty} 2^{-n(2\alpha+1)+1}c|x|^{2\alpha}$$

Hence $g(x) = r(x)$ $\forall x \in X$. Therefore, the reciprocal map r is unique and the proof of the theorem is now complete.□

Theorem 2.4.2. [15] Let $f : X \longrightarrow Y$ be a mapping on the sets of non-zero real numbers. If there exist k and α with $k > 0$ and $\alpha < -\frac{1}{2}$ such that

$$|f(\frac{x+y}{2}) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)}| \leq k(|x|^{\alpha}|y|^{\alpha} + (|x|^{2\alpha} + |y|^{2\alpha})) \tag{2.113}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \longrightarrow Y$ such that

$$|f(x) - r(x)| \leq c|x|^{2\alpha}$$

and r satisfies (2.1) for all $x, y \in X$ where $c = \frac{6k}{1-2^{2\alpha+1}}$

Proof Replacing (x, y) by (x, x) in (2.113), we get

$$|f(\frac{x+x}{2}) - f(x+x) - \frac{f(x)f(x)}{f(x)+f(x)}| \leq k(|x|^{\alpha}|x|^{\alpha} + (|x|^{2\alpha} + |x|^{2\alpha})) \tag{2.114}$$

$$|f(x) - f(2x) - \frac{1}{2}f(x)| \leq 3k|x|^{2\alpha} \tag{2.115}$$

$$|\frac{1}{2}f(x) - f(2x)| \leq 3k|x|^{2\alpha} \tag{2.116}$$

$$|f(x) - 2f(2x)| \leq 6k|x|^{2\alpha} \tag{2.117}$$

Replacing x by $2x$ in (2.117) and multiplying by 2 to get

$$|2f(2x) - 2^2f(2^2x)| \leq 12k|2x|^{2\alpha}. \tag{2.118}$$

Summing (2.17) and (2.118) and use the triangle inequality to get $|f(x)-2^2f(2^2x)| \leq$
$|f(x) - 2f(2x)| + |2f(2x) - 2^2f(2^2x)|$
$\leq 6k|x|^{2\alpha} + 12k|x|^{2\alpha}$

$$= 6k\sum_{i=0}^{1} 2^{i(2\alpha+1)}|x|^{2\alpha}. \tag{2.119}$$

Using induction on a positive integer n, we get

$$|f(x) - 2^nf(2^nx)| \leq 6k\sum_{i=0}^{n-1} 2^{i(2\alpha+1)}|x|^{2\alpha}. \tag{2.120}$$

Hence

$$|f(x) - 2^nf(2^nx)| \leq 6k\sum_{i=0}^{\infty} 2^{i(2\alpha+1)}|x|^{2\alpha} = \frac{6k}{1-2^{2\alpha+1}}|x|^{2\alpha} \tag{2.121}$$

Setting $c = \frac{6k}{1-2^{2\alpha+1}}$, (2.121) becomes

$$|f(x) - 2^n f(2^n x)| \leq c|x|^{2\alpha} \tag{2.122}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now if $n > m > 0$, then $n - m$ is a natural number and n can be replaced by $n - m$ in (2.122) to obtain

$$|f(x) - 2^{n-m} f(2^{n-m} x)| \leq c|x|^{2\alpha} \tag{2.123}$$

Multiplying both sides by 2^m and simplifying , we get

$$|2^m f(x) - 2^n f(2^{n-m} x)| \leq 2^m c|x|^{2\alpha} \tag{2.124}$$

for all $x \in X$. Now we replace x by $2^m x$ to have

$$|2^m f(2^m x) - 2^n f(2^n x)| \leq 2^m c|2^m x|^{2\alpha} \tag{2.125}$$

$$|2^m f(2^m x) - 2^n f(2^n x)| \leq 2^{m(2\alpha+1)} c|x|^{2\alpha} \tag{2.126}$$

If $m \to \infty$ and $\alpha < -\frac{1}{2}$ then $2^{m(2\alpha+1)} c|x|^{2\alpha} \to 0$, and therefore,

$$\lim_{m\to\infty} |2^m f(2^m x) - 2^n f(2^n x)| = 0. \tag{2.127}$$

Hence

$$\{2^n f(2^n x)\}_{n=0}^{\infty} \tag{2.128}$$

is a Cauchy sequence in X. Hence the limit of this sequence exists.
Define $r : X \to \mathbb{R}$ by

$$r(x) = \lim_{n\to\infty} 2^n f(2^n x) \tag{2.129}$$

Now we show that $r : X \to \mathbb{R}$ defined by (2.129) is satisfying (2.1). Consider

$$|r(\frac{x+y}{2}) - r(x+y) - \frac{r(x)r(y)}{r(x)+(y)}|$$

$$= |\lim_{n\to\infty} 2^n \{f(\frac{2^n(x+y)}{2}) - f(2^n(x+y)) - \frac{f(2^n x)f(2^n y)}{f(2^n x)+f(2^n y)}\}|$$

$$= \lim_{n\to\infty} 2^n |f(\frac{2^n(x+y)}{2}) - f(2^n(x+y)) - \frac{f(2^n x)f(2^n y)}{f(2^n x)+f(2^n y)}|$$

$$\leq \lim_{n\to\infty} 2^n c_1 |2^n x|^a |2^n y|^b \text{ by (2.113)}$$

$$= \lim_{n\to\infty} 2^n c_1 2^{n(a+b)} |x|^a |y|^b$$

$$\leq \lim_{n\to\infty} 2^{n(2\alpha+1)} k(|x|^\alpha |y|^\alpha + (|x|^{2\alpha} + |y|^{2\alpha})) = 0$$

Therefore, $r(\frac{x+y}{2}) - r(x+y) = \frac{r(x)r(y)}{r(x)+(y)}$, for all $x, y \in X$.

Our next goal is to show that

$|f(x) - r(x)| \leq c|x|^{2\alpha}$ Using $r(x) = \lim_{n\to\infty} 2^n f(2^n x)$ we have

$$|f(x) - r(x)| = |f(x) - \lim_{n\to\infty} 2^n f(2^n x)|$$

$= |\lim_{n\to\infty} 2^n f(2^n x) - f(x)| \leq c|x|^{2\alpha}$

Hence we obtain $|f(x) - r(x)| \leq c|x|^{2\alpha}$,for all $x \in X$

Finally we prove that r is unique. Suppose that there exists another reciprocal function $g : X \longrightarrow \mathbb{R}$ such that

$$|g(x) - f(x)| \leq c|x|^{2\alpha} \tag{2.130}$$

for all $x \in X$. Note that

$$|g(x) - r(x)| = |g(x) - f(x) + f(x) - r(x)|$$

$$\leq |g(x) - f(x)| + |f(x) - r(x)| = c|x|^{2\alpha} + c|x|^{2\alpha}$$

Therefore

$$|g(x) - r(x)| \leq 2c|x|^{2\alpha} \tag{2.131}$$

Further, since r and g are reciprocal function , we have

$$|g(x) - r(x)| = |\frac{2^n g(x)}{2^n} - \frac{2^n r(x)}{2^n}|$$

$$= |2^n g(2^n x) - 2^n r(2^n x)|$$

$= 2^n |g(2^n x) - r(2^n x)| \leq 2^{n(2\alpha+1)+1} c|x|^{2\alpha}$,

where $n \in \mathbb{N}$. Hence

$$|g(x) - r(x)| \leq 2^{n(2\alpha+1)+1} c|x|^{2\alpha}$$

Taking the limit of both sides, we get

$$0 \leq |g(x) - r(x)| = \lim_{n\to\infty} |g(x) - r(x)| \leq \lim_{n\to\infty} 2^{n(2\alpha+1)+1} c|x|^{2\alpha} = 0.$$

Hence $g(x) = r(x)$ $\forall x \in X$. Therefore, the reciprocal map r is unique and the proof of the theorem is now complete.□

Chapter 3

Ulam Stability of Reciprocal Adjoint Functional Equation

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Introduction

This chapter consists of four sections:

In the first section, we study reciprocal adjoint functional equation and its solution. In the second section, we study Hyars - Ulam stability of reciprocal adjoint functional equation. In the third section, we study generalized Ulam stability of reciprocal adjoint functional equation. In the fourth section, we study extended Ulam stability of reciprocal adjoint functional equation.

Definition 3.0.3. Let X and Y be sets of non-zero real numbers and $r : X \to Y$ satisfies

$$r(\frac{x+y}{2}) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)} \tag{3.1}$$

for all $x, y \in X$.

Equation (3.1) is called the reciprocal adjoint functional equation.

3.1 Reciprocal Adjoint Functional Equation and its Solution

Theorem 3.1.1. [15] Let X and Y be sets of non-zero real numbers. A function $r : X \longrightarrow Y$ satisfies the functional equation

$$r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}$$

if and only if $r : X \longrightarrow Y$ satisfies the functional equation

$$r(\frac{x+y}{2}) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)}$$

Therefore, every solution of functional equations (3.1)is also a reciprocal function .

Proof. Let $r : X \longrightarrow Y$ satisfies the functional equation (1.1). Letting $y = x$ in (1.1), we get

$$r(2x) = \frac{1}{2}r(x). \tag{3.2}$$

Replacing x by $\frac{x}{2}$ in (3.2), we obtain

$$r(\frac{x}{2}) = 2r(x). \tag{3.3}$$

Now, replacing (x, y) by $(\frac{x}{2}, \frac{y}{2})$ in (1.1) we get

$$r(\frac{x+y}{2}) = \frac{r(\frac{x}{2})r(\frac{y}{2})}{r(\frac{x}{2}) + r(\frac{y}{2})}. \tag{3.4}$$

Using (3.3) we obtain

$$r(\frac{x+y}{2}) = \frac{2r(x)r(y)}{r(x)+r(y)} \tag{3.5}$$

Adding (1.1) to (3.5) we get

$$r(\frac{x+y}{2}) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)} \tag{3.6}$$

Conversely let $r : X \longrightarrow Y$ satisfies the functional equation (3.1). Putting $y = x$ in (3.1) we obtain

$$r(\frac{x+x}{2}) + r(x+x) = \frac{3r(x)r(x)}{r(x)+r(x)}$$

$$r(x) + r(2x) = \frac{3r^2(x)}{2r(x)} \tag{3.7}$$

$$r(x) + r(2x) = \frac{3r(x)}{2} \tag{3.8}$$

Multiplying both side by 2 to get

$$2r(x) + 2r(2x) = 3r(x) \tag{3.9}$$

$$2r(2x) = r(x) \tag{3.10}$$

Replacing x by $\frac{x}{2}$ in (3.10) we get

$$2r(x) = r(\frac{x}{2}) \tag{3.11}$$

Using (3.11) in (3.1), we obtain

$$2r(x+y) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)} \tag{3.12}$$

$$3r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)} \tag{3.13}$$

Dividing both by 3 to get

$$r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}$$

This completes the proof of Theorem (3.1.1).□

3.2 Hyars - Ulam Stability of Reciprocal Adjoint Functional Equation

Theorem 3.2.1. [15] Let X and Y be sets of non-zero real numbers. Assume in addition that $f : X \longrightarrow Y$ is a mapping for which there exists a constant $c \geq 0$ (independent of x, y) such that the functional inequality

$$|f(\frac{x+y}{2}) + f(x+y) - \frac{3f(x)f(y)}{f(x)+f(y)}| \leq \frac{c}{2} \tag{3.14}$$

holds for all $x, y \in X$. Then the limit

$$r(x) = \lim_{n\to\infty} 2^{-n} f(2^{-n}x)$$

exists for all $x \in X$, $n \in \mathbb{N}$ and $r : X \longrightarrow Y$ is the unique mapping satisfying the functional equation (3.1), such that

$$|f(x) - r(x)| \leq c$$

for all $x \in X$. Moreover, functional identity $r(x) = 2^{-n} r(2^{-n}x)$ holds for all $x \in X$ and $n \in N$

Proof Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (3.14), we obtain

$$|f(\frac{\frac{x}{2}+\frac{x}{2}}{2}) + f(\frac{x}{2}+\frac{x}{2}) - \frac{3f(\frac{x}{2})f(\frac{x}{2})}{f(\frac{x}{2})+f(\frac{x}{2})}| \leq \frac{c}{2} \tag{3.15}$$

$$|f(\frac{x}{2}) + f(x) - \frac{3f^2(\frac{x}{2})}{2f(\frac{x}{2})}| \leq \frac{c}{2} \tag{3.16}$$

$$|f(\frac{x}{2}) + f(x) - \frac{3}{2}f(\frac{x}{2})| \leq \frac{c}{2} \tag{3.17}$$

$$|f(x) - \frac{1}{2}f(\frac{x}{2})| \leq \frac{c}{2} \tag{3.18}$$

Replacing x by $\frac{x}{2^{k-1}}$ in (3.18) where k is a positive integer greater than or equal to 1, we obtain

$$|f(\frac{x}{2^{k-1}}) - \frac{1}{2}f(\frac{x}{2^k})| \leq \frac{c}{2} \tag{3.19}$$

for all $x \in X$. Multiplying both sides of the above inequality by $\frac{1}{2^{k-1}}$ and adding the resulting n inequalities, we have

$$\sum_{k=1}^{n} |\frac{1}{2^{k-1}} f(\frac{x}{2^{k-1}}) - \frac{1}{2^k} f(\frac{x}{2^k})| \leq \sum_{k=1}^{n} \frac{1}{2^{k-1}} \frac{c}{2} = c \sum_{k=1}^{n} \frac{1}{2^k} = c(1 - \frac{1}{2^n}) \tag{3.20}$$

Using the triangle inequality and (3.20) to get

$$|f(x) - \frac{1}{2^n} f(\frac{x}{2^n})| \leq c(1 - \frac{1}{2^n}). \tag{3.21}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now if $n > m > 0$, then $n - m$ is a natural number, and n can be replaced by $n - m$ in (3.21) to obtain

$$|f(x) - \frac{1}{2^{n-m}} f(\frac{x}{2^{n-m}})| \leq c(1 - \frac{1}{2^{n-m}}). \tag{3.22}$$

Multiplying both sides by $\frac{1}{2^m}$ and simplifying , we get

$$|\frac{1}{2^m} f(x) - \frac{1}{2^n} f(\frac{x}{2^{n-m}})| \leq c(\frac{1}{2^m} - \frac{1}{2^n}). \tag{3.23}$$

for all $x \in X$. Now we replace x by $\frac{x}{2^m}$ to have

$$|\frac{1}{2^m} f(\frac{x}{2^m}) - \frac{1}{2^n} f(\frac{x}{2^n})| \leq c(\frac{1}{2^m} - \frac{1}{2^n}). \tag{3.24}$$

If $m \to \infty$, then $(\frac{1}{2^m} - \frac{1}{2^n}) \to 0$ and therefore,

$$\lim_{m\to\infty} |\frac{1}{2^m} f(\frac{x}{2^m}) - \frac{1}{2^n} f(\frac{x}{2^n})| = 0. \tag{3.25}$$

Hence $\{2^{-n}f(2^{-n}x)\}_{n=1}^{\infty}$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $r : X \to \mathbb{R}$ by

$$r(x) = \lim_{n\to\infty} 2^{-n}f(2^{-n}x) \tag{3.26}$$

Now we show that $r : X \to \mathbb{R}$ defined by (3.26) is satisfies (3.1). Then by using (3.14) we get $|r(\frac{x+y}{2}) + r(x+y) - \frac{3r(x)r(y)}{r(x)+(y)}|$

$$= |\lim_{n\to\infty} 2^{-n}\{f(\frac{2^{-n}(x+y)}{2}) + f(2^{-n}(x+y)) - \frac{3f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}\}|$$

$$= \lim_{n\to\infty} 2^{-n}|f(\frac{2^{-n}(x+y)}{2}) + f(2^{-n}(x+y)) - \frac{3f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}|$$

$$\leq \lim_{n\to\infty} 2^{-n}\frac{c}{2} = 0$$

Therefore

$$r(\frac{x+y}{2}) + r(x+y) = \frac{3r(x)r(y)}{r(x)+(y)}$$

for all $x, y \in X$.

Our next goal is to show that $|f(x) - r(x)| \leq c$ .By using (3.21) we get

$$|f(x) - r(x)| = |f(x) - \lim_{n\to\infty} 2^{-n}f(2^{-n}x)|$$

$= |\lim_{n\to\infty} 2^{-n}f(2^{-n}x) - f(x)| \leq \lim_{n\to\infty} c(1 - \frac{1}{2^n}) = c.$

Hence we obtain

$$|f(x) - r(x)| \leq c$$

for all $x \in X$.

Finally we prove that r is unique. Suppose r is not unique, then there exists another reciprocal function $g : X \longrightarrow \mathbb{R}$ such that

$$|g(x) - f(x)| \leq c \tag{3.27}$$

for all $x \in X$. Note that

$$\begin{aligned} |g(x) - r(x)| &= |g(x) - f(x) + f(x) - r(x)| \\ &\leq |g(x) - f(x)| + |f(x) - r(x)| \\ &= c + c \end{aligned}$$

Therefore

$$|g(x) - r(x)| \leq 2c \tag{3.28}$$

Further, since r and g are reciprocal function , we have for any $n \in N$,

$$\begin{aligned} |g(x) - r(x)| &= |\frac{2^n g(x)}{2^n} - \frac{2^n r(x)}{2^n}| \\ &= |2^{-n} g(2^{-n} x) - 2^{-k} r(2^{-n} x)| \\ &= 2^{-n} |g(2^{-n} x) - r(2^{-n} x)| \\ &\leq 2^{-n} 2c \end{aligned}$$

Taking the limit on both sides as $n \to \infty$, we get

$$\lim_{n \to \infty} |g(x) - r(x)| \leq 2c \lim_{n \to \infty} 2^{-n}$$

which gives

$$|g(x) - r(x)| = 0.$$

Hence $g(x) = r(x)$ $\forall x \in X$. Therefore, the reciprocal map r is unique and the proof of the theorem is now complete.□

3.3 Generalized Ulam Stability of Reciprocal Adjoint Functional Equation

Theorem 3.3.1. [15] Let X and Y be sets of non zero real numbers and $f : X \longrightarrow Y$ be a mapping . If there exist a, b where $\rho = a + b > -1$ and $c_1 \geq 0$ such that

$$|f(\frac{x+y}{2}) + f(x+y) - \frac{3f(x)f(y)}{f(x)+f(y)}| \leq c_1|x|^a|y|^b \tag{3.29}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \longrightarrow Y$ such that

$$|f(x) - r(x)| \leq c|x|^\rho$$

holds and r satisfies (3.1) for all $x, y \in X$ where $c = \frac{2c_1}{2^{\rho+1}-1}$.

Proof Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (3.29), we obtain

$$|f(\frac{\frac{x}{2}+\frac{x}{2}}{2}) + f(\frac{x}{2}+\frac{x}{2}) - \frac{3f(\frac{x}{2})f(\frac{x}{2})}{f(\frac{x}{2})+f(\frac{x}{2})}| \leq c_1|\frac{x}{2}|^a|\frac{x}{2}|^b \tag{3.30}$$

that is,

$$|f(x) - \frac{1}{2}f(\frac{x}{2})| \leq \frac{c_1}{2^\rho}|x|^\rho \tag{3.31}$$

Replacing x by $\frac{x}{2}$ in (3.31) and dividing by 2 we get

$$|\frac{1}{2}f(\frac{x}{2}) - \frac{1}{2^2}f(\frac{x}{2^2})| \leq \frac{c_1}{2^{\rho+1}}|\frac{x}{2}|^\rho \tag{3.32}$$

Summing (3.31) with (3.32) and use the triangle inequality to get $|f(x) - \frac{1}{2^2}f(\frac{x}{2^2})| \leq |f(x) - \frac{1}{2}f(\frac{x}{2})| + |\frac{1}{2}f(\frac{x}{2}) - \frac{1}{2^2}f(\frac{x}{2^2})| \leq \frac{c_1}{2^\rho}|x|^\rho + \frac{c_1}{2^{\rho+1}}|\frac{x}{2}|^\rho$

$$= \frac{c_1}{2^\rho}\sum_{i=0}^{1}\frac{1}{2^{i(\rho+1)}}|x|^\rho. \tag{3.33}$$

Proceeding further and using induction on a positive integer n, we get

$$|f(x) - \frac{1}{2^n}f(\frac{x}{2^n})| \leq \frac{c_1|x|^\rho}{2^\rho}\sum_{i=1}^{n-1}\frac{1}{2^{i(\rho+1)}}. \tag{3.34}$$

Then

$$|f(x) - \frac{1}{2^n} f(\frac{x}{2^n})| \le \frac{c_1|x|^\rho}{2^\rho} \sum_{i=1}^{\infty} \frac{1}{2^{i(\rho+1)}} = \frac{2c_1}{2^{\rho+1}-1}|x|^\rho. \tag{3.35}$$

Setting $c = \frac{2c_1}{2^{\rho+1}-1}$ we get

$$|f(x) - \frac{1}{2^n} f(\frac{x}{2^n})| \le c|x|^\rho. \tag{3.36}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Now if $n > m > 0$, then $n - m$ is a natural number, and n can be replaced by $n - m$ in (3.36) to obtain

$$|f(x) - \frac{1}{2^{n-m}} f(\frac{x}{2^{n-m}})| \le c|x|^\rho. \tag{3.37}$$

Multiplying both sides by $\frac{1}{2^m}$ and simplifying , we get

$$|\frac{1}{2^m} f(x) - \frac{1}{2^n} f(\frac{x}{2^{n-m}})| \le \frac{c}{2^m}|x|^\rho. \tag{3.38}$$

for all $x \in X$. Now we replace x by $\frac{x}{2^m}$ to have

$$|\frac{1}{2^m} f(\frac{x}{2^m}) - \frac{1}{2^n} f(\frac{x}{2^n})| \le \frac{c}{2^m}|\frac{x}{2^m}|^\rho. \tag{3.39}$$

If $m \to \infty$ and $\rho > -1$, then $\frac{c|x|^\rho}{2^{m(\rho+1)}} \to 0$ and therefore,

$$\lim_{m\to\infty} |\frac{1}{2^m} f(\frac{x}{2^m}) - \frac{1}{2^n} f(\frac{x}{2^n})| = 0. \tag{3.40}$$

Hence

$$\{2^{-n} f(2^{-n}x)\}_{n=0}^{\infty} \tag{3.41}$$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $r : X \to \mathbb{R}$ by

$$r(x) = \lim_{n\to\infty} 2^{-n} f(2^{-n}x) \tag{3.42}$$

We will show that $r : X \to \mathbb{R}$ defined by (3.42) is satisfying (3.1). By using (3.29) we have,

$$|r(\frac{x+y}{2}) + r(x+y) - \frac{3r(x)r(y)}{r(x)+(y)}|$$

$$= |\lim_{n\to\infty} 2^{-n}\{f(\frac{2^{-n}(x+y)}{2}) + f(2^{-n}(x+y)) - \frac{3f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}\}|$$

$$= \lim_{n\to\infty} 2^{-n}|f(\frac{2^{-n}(x+y)}{2}) + f(2^{-n}(x+y)) - \frac{3f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}|$$

$$\leq \lim_{n\to\infty} 2^{-n}c_1|2^{-n}x|^a|2^{-n}y|^b$$

$$= c_1 \lim_{n\to\infty} 2^{-n}2^{-n(a+b)}|x|^a|y|^b$$

$$= c_1 \lim_{n\to\infty} 2^{-n(\rho+1)}|x|^a|y|^b = 0$$

Therefore,

$$r(\frac{x+y}{2}) + r(x+y) = \frac{3r(x)r(y)}{r(x)+(y)}$$

for all $x, y \in X$.

Our next goal is to show that

$$|f(x) - r(x)| \leq c|x|^\rho$$

$$|f(x) - r(x)| = |f(x) - \lim_{n\to\infty} 2^{-n}f(2^{-n}x)|$$

$$= |\lim_{n\to\infty} 2^{-n}f(2^{-n}x) - f(x)|$$

$\leq \lim_{n\to\infty} c|x|^\rho = c|x|^\rho$. (By (3.34).)

Hence we obtain

$$|f(x) - r(x)| \leq c|x|^{\rho}$$

for all $x \in X$.

Finally we prove that r is unique. Suppose r is not unique, then there exists another reciprocal function $g : X \longrightarrow \mathbb{R}$ such that

$$|g(x) - f(x)| \leq c|x|^{\rho} \tag{3.43}$$

for all $x \in X$. Note that

$$\begin{aligned}|g(x) - r(x)| &= |g(x) - f(x) + f(x) - r(x)| \\ &\leq |g(x) - f(x)| + |f(x) - r(x)| \\ &= c|x|^{\rho} + c|x|^{\rho}\end{aligned}$$

Therefore,

$$|g(x) - r(x)| \leq 2c|x|^{\rho} \tag{3.44}$$

Further, since r and g are reciprocal function , we have

$|g(x) - r(x)| = |\frac{2^n g(x)}{2^n} - \frac{2^n r(x)}{2^n}| = |2^{-n}g(2^{-n}x) - 2^{-n}r(2^{-n}x)|$
$= 2^{-n}|g(2^{-n}x) - r(2^{-n}x)| \leq 2^{-n}2c|2^{-n}x|^{\rho} = 2^{-n(\rho+1)+1}c|x|^{\rho}$,

Taking the limit on both sides as $n \to \infty$, we get

$$|g(x) - r(x)| \leq \lim_{n\to\infty} 2^{-n(\rho+1)+1}c|x|^{\rho} = 0$$

Hence $g(x) = r(x)$ $\forall x \in X$. Therefore, the reciprocal map r is unique and the proof of the theorem is now complete.□

Theorem 3.3.2. [15] let X and Y be sets of non zero real numbers and $f : X \longrightarrow Y$ be a mapping . If there exist a, b where $\rho = a + b < -1$ and $c_1 \geq 0$ such that

$$|f(\frac{x+y}{2}) + f(x+y) - \frac{3f(x)f(y)}{f(x)+f(y)}| \leq c_1|x|^a|y|^b \tag{3.45}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \longrightarrow Y$ such that

$$|r(x) - f(x)| \leq c|x|^\rho$$

holds and r satisfies (3.1)for all $x, y \in X$ where $c = \frac{2c_1}{1-2^{\rho+1}}$.

Proof

Replacing (x, y) by (x, x)in (3.45), we have

$$|f(\frac{x+x}{2}) + f(x+x) - \frac{3f(x)f(x)}{f(x)+f(x)}| \leq c_1|x|^a|x|^b \tag{3.46}$$

$$|f(x) + f(2x) - \frac{3}{2}f(x)| \leq c_1|x|^\rho \tag{3.47}$$

$$|f(2x) - \frac{1}{2}f(x)| \leq c_1|x|^\rho \tag{3.48}$$

$$|2f(2x) - f(x)| \leq 2c_1|x|^\rho \tag{3.49}$$

Replacing x by $2x$ in (3.49) and multiplying by 2 we get

$$|2^2f(2^2x) - 2f(2x)| \leq 2^2c_1|2x|^\rho \tag{3.50}$$

Summing (3.50) with (3.49) and use the triangle inequality to get $|2^2f(2^2x) - f(x)| \leq |2^2f(2^2x) - 2f(2x)| + |2f(2x) - f(x)| \leq 2^2c_1|2x|^\rho + 2c_1|x|^\rho$

$$= 2c_1 \sum_{i=1}^{1} 2^{i(\rho+1)}|x|^\rho \tag{3.51}$$

Proceeding further and using induction on a positive integer n, we get

$$|2^n f(2^n x) - f(x)| \leq 2c_1 \sum_{i=1}^{n-1} 2^{i(\rho+1)}|x|^\rho \tag{3.52}$$

Hence

$$|2^n f(2^n x) - f(x)| \leq 2c_1 \sum_{i=1}^{\infty} 2^{i(\rho+1)}|x|^\rho = \frac{2c_1}{1 - 2^{(\rho+1)}}|x|^\rho \tag{3.53}$$

Setting $c = \frac{2c_1}{1-2^{\rho+1}}$ to get

$$|2^n f(2^n x) - f(x)| \leq c|x|^\rho \tag{3.54}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now if $n > m > 0$, then $n - m$ is a natural number, and n can be replaced by $n - m$ in (3.54) to obtain

$$|2^{n-m} f(2^{n-m} x) - f(x)| \leq c|x|^\rho \tag{3.55}$$

Multiplying both sides by 2^m and simplifying , we get

$$|2^n f(2^{n-m} x) - 2^m f(x)| \leq 2^m c|x|^\rho \tag{3.56}$$

for all $x \in X$. Now we replace x by $2^m x$ to have

$$|2^n f(2^n x) - 2^m f(2^m x)| \leq 2^m c|2^m x|^\rho \tag{3.57}$$

$$|2^n f(2^n x) - 2^m f(2^m x)| \leq 2^{m(\rho+1)} c|x|^\rho \tag{3.58}$$

If $m \to \infty$ and $\rho < -1$ then $2^{m(\rho+1)} c|x|^\rho \to 0$ and therefore,

$$\lim_{m\to\infty} |2^n f(2^n x) - 2^m f(2^m x)| = 0. \tag{3.59}$$

Hence

$$\{2^n f(2^n x)\}_{n=0}^{\infty} \tag{3.60}$$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $r : X \to \mathbb{R}$ by

$$r(x) = \lim_{n\to\infty} 2^n f(2^n x) \tag{3.61}$$

Now we show that $r : X \to \mathbb{R}$ defined by (3.61) is satisfying (3.1). Consider

$$|r(\frac{x+y}{2}) + r(x+y) - \frac{3r(x)r(y)}{r(x)+(y)}|$$

$$= |\lim_{n\to\infty} 2^n \{f(\frac{2^n(x+y)}{2}) + f(2^n(x+y)) - \frac{3f(2^n x)f(2^n y)}{f(2^n x) + f(2^n y)}\}|$$

$$= \lim_{n\to\infty} 2^n |f(\frac{2^n(x+y)}{2}) + f(2^n(x+y)) - \frac{3f(2^n x)f(2^n y)}{f(2^n x) + f(2^n y)}|$$

$\leq \lim_{n\to\infty} 2^n c_1 |2^n x|^a |2^n y|^b$. By (3.45)

$$= \lim_{n\to\infty} 2^n c_1 2^{n(a+b)} |x|^a |y|^b$$

$$= \lim_{n\to\infty} 2^{n(\rho+1)} c_1 |x|^a |y|^b = 0$$

Therefore,

$$r(\frac{x+y}{2}) + r(x+y) = \frac{3r(x)r(y)}{r(x)+(y)}$$

for all $x, y \in X$.

Our next goal is to show that

$$|f(x) - r(x)| \leq c|x|^\rho$$

$$|f(x) - r(x)| = |f(x) - \lim_{n\to\infty} 2^n f(2^n x)|$$

$$= |\lim_{n\to\infty} 2^n f(2^n x) - f(x)|$$

$$\leq \lim_{n\to\infty} c|x|^\rho = c|x|^\rho. \text{ By } (3.54)$$

Hence we obtain

$$|f(x) - r(x)| \leq c|x|^\rho$$

for all $x \in X$

Finally we prove that r is unique. Suppose that there exists another reciprocal function $g : X \longrightarrow \mathbb{R}$ such that

$$|g(x) - f(x)| \leq c|x|^\rho \tag{3.62}$$

for all $x \in X$. Note that

$$|g(x) - r(x)| = |g(x) - f(x) + f(x) - r(x)|$$

$$\leq |g(x) - f(x)| + |f(x) - r(x)|$$

$$= c|x|^\rho + c|x|^\rho$$

Therefore

$$|g(x) - r(x)| \leq 2c|x|^\rho \tag{3.63}$$

Further, since r and g are reciprocal function , we have

$|g(x) - r(x)| = |\frac{2^n g(x)}{2^n} - \frac{2^n r(x)}{2^n}| = |2^n g(2^n x) - 2^n r(2^n x)| = 2^n |g(2^n x) - r(2^n x)|$
$\leq 2^{n(\rho+1)+1} c|x|^\rho$, where $n \in \mathbb{N}$. Hence

$$|g(x) - r(x)| \leq 2^{n(\rho+1)+1} c|x|^\rho$$

Taking the limit on both sides, we get

$$0 \leq |g(x) - r(x)| = \lim_{n\to\infty} |g(x) - r(x)| \leq \lim_{n\to\infty} 2^{n(\rho+1)+1} c|x|^\rho = 0$$

Hence $g(x) = r(x)$ $\forall x \in X$. Therefore, the reciprocal map r is unique and the proof of the theorem is now complete.$\Box$

Theorem 3.3.3. [15] Let X and Y be sets of non zero real numbers and $f : X \longrightarrow Y$ be a mapping for which there exists a constant $\theta > 0$ and satisfies

$$|f(\frac{x+y}{2}) + f(x+y) - \frac{3f(x)f(y)}{f(x)+f(y)}| \leq \theta H(x, y) \tag{3.64}$$

where $H : X^2 \longrightarrow Y$ is a function such that

$$\phi(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} H(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) \tag{3.65}$$

with the condition

$$\lim_{n\to\infty} \tfrac{1}{2^n} H(\tfrac{x}{2^{n+1}}, \tfrac{x}{2^{n+1}}) = 0$$

holds. Then there exists a unique reciprocal mapping $A : X \longrightarrow Y$ which satisfies (3.1) and the inequality

$$|f(x) - A(x)| \leq \theta\phi(x)$$

for all $x \in X$.

Proof Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (3.64), we have

$$|f(\frac{\frac{x}{2}+\frac{x}{2}}{2}) + f(\frac{x}{2}+\frac{x}{2}) - \frac{3f(\frac{x}{2})f(\frac{x}{2}}{f(\frac{x}{2})+f(\frac{x}{2})}| \leq \theta H(\frac{x}{2}, \frac{x}{2}) \tag{3.66}$$

$$|f(x) - \frac{1}{2}f(\frac{x}{2})| \leq \theta H(\frac{x}{2}, \frac{x}{2}) \tag{3.67}$$

Replacing x by $\frac{x}{2}$ in (3.67) and dividing by 2 to get

$$|\frac{1}{2}f(\frac{x}{2}) - \frac{1}{2^2}f(\frac{x}{2^2})| \leq \frac{\theta}{2}H(\frac{x}{2^2}, \frac{x}{2^2}) \tag{3.68}$$

Summing (3.68) with (3.67) and using triangle inequality and multiplying the right side by $\frac{2}{2}$to get $|f(x) - \frac{1}{2^2}f(\frac{x}{2^2})| \leq |f(x) - \frac{1}{2}f(\frac{x}{2})| + |\frac{1}{2}f(\frac{x}{2}) - \frac{1}{2^2}f(\frac{x}{2^2})|$

$$\leq \theta H(\frac{x}{2}, \frac{x}{2}) + \frac{\theta}{2}H(\frac{x}{2^2}, \frac{x}{2^2}) = \theta\sum_{i=0}^{1}\frac{1}{2^i}H(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) \tag{3.69}$$

Proceeding further and using induction on a positive integer n , we get

$$|f(x) - \frac{1}{2^n}f(\frac{x}{2^n})| \leq \theta\sum_{i=0}^{n-1}\frac{1}{2^i}H(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}). \tag{3.70}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now if $n > m > 0$, then $n - m$ is a natural number, and n can be replaced by $n - m$ in (3.70) to obtain

$$|f(x) - \frac{1}{2^{n-m}}f(\frac{x}{2^{n-m}})| \leq \theta\sum_{i=0}^{n-m-1}\frac{1}{2^i}H(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}). \tag{3.71}$$

Multiplying both sides by $\frac{1}{2^m}$ and simplifying , we get

$$|\frac{1}{2^m}f(x) - \frac{1}{2^n}f(\frac{x}{2^{n-m}})| \leq \theta \sum_{i=0}^{n-m-1} \frac{1}{2^{i+m}} H(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}). \tag{3.72}$$

for all $x \in X$. Now we replace x by $\frac{x}{2^m}$ to have

$$|\frac{1}{2^m}f(\frac{x}{2^m}) - \frac{1}{2^n}f(\frac{x}{2^n})| \leq \theta \sum_{i=0}^{n-m-1} \frac{1}{2^{i+m}} H(\frac{x}{2^{i+m+1}}, \frac{x}{2^{i+m+1}}). \tag{3.73}$$

If $m \to \infty$,then $\frac{1}{2^{i+m}}H(\frac{x}{2^{i+m+1}}, \frac{x}{2^{i+m+1}}) \to 0 \ \forall i$ by $(\lim_{n\to\infty} \frac{1}{2^n}H(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}) = 0)$

Therefore,

$$\lim_{m\to\infty} |\frac{1}{2^m}f(\frac{x}{2^m}) - \frac{1}{2^n}f(\frac{x}{2^n})| = 0. \tag{3.74}$$

Hence

$$\{2^{-n}f(2^{-n}x)\}_{n=0}^{\infty} \tag{3.75}$$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $A : X \to \mathbb{R}$ by

$$A(x) = \lim_{n\to\infty} 2^{-n}f(2^{-n}x) \tag{3.76}$$

Now we show that $A : X \to \mathbb{R}$ defined by (3.75) is satisfying (3.1). Consider

$$|A(\frac{x+y}{2}) + A(x+y) - \frac{3A(x)A(y)}{A(x)+A(y)}|$$

$$= |\lim_{n\to\infty} 2^{-n}\{f(\frac{2^{-n}(x+y)}{2}) + f(2^{-n}(x+y)) - \frac{3f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}\}|$$

$$= \lim_{n\to\infty} 2^{-n}|f(\frac{2^{-n}(x+y)}{2}) + f(2^{-n}(x+y)) - \frac{3f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}|$$

$\leq \lim_{n\to\infty} 2^{-n}\theta H(2^{-n}x, 2^{-n}y) = 0$ by (3.64)and (3.65)

Therefore,

$$A(\frac{x+y}{2}) + A(x+y) = \frac{3A(x)A(y)}{A(x)+A(y)}$$

for all $x, y \in X$.

Our next goal is to show that

$$|f(x) - A(x)| \leq \theta\phi(x)$$

$$|f(x) - A(x)| = |f(x) - \lim_{n\to\infty} 2^{-n} f(2^{-n}x)|$$

$$= |\lim_{n\to\infty} 2^{-n} f(2^{-n}x) - f(x)|$$

$\leq \lim_{n\to\infty} \theta \sum_{i=0}^{n-1} H(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}) = \theta\phi(x)$. (By (3.70))

Hence we obtain

$$|f(x) - A(x)| \leq \theta\phi(x)$$

for all $x \in X$.

Finally we prove that A is unique. Suppose that there exists another reciprocal function $G : X \longrightarrow \mathbb{R}$ such that

$$|G(x) - f(x)| \leq \theta\phi(x) \tag{3.77}$$

for all $x \in X$. Note that

$$|G(x) - A(x)| = |G(x) - f(x) + f(x) - A(x)|$$

$$\leq |G(x) - f(x)| + |f(x) - A(x)| = \theta\phi(x) + \theta\phi(x)$$

Therefore,

$$|G(x) - A(x)| \leq 2\theta\phi(x) \tag{3.78}$$

Further, since G and f are reciprocal function , we have
$|G(x) - A(x)| = |\frac{2^n G(x)}{2^n} - \frac{2^n A(x)}{2^n}|$
$= |2^{-n}G(2^{-n}x) - 2^{-n}A(2^{-n}x)| = 2^{-n}|G(2^{-n}x) - A(2^{-n}x)| \leq 2^{-n}\theta\phi(x),$

where $n \in \mathbb{N}$. Hence

$$|G(x) - A(x)| \leq 2^{-n}\theta\phi(x)$$

Taking the limit on both sides, we get

$$0 \leq |G(x) - A(x)| = \lim_{n\to\infty} |G(x) - A(x)| \leq \lim_{n\to\infty} 2^{-n}\theta\phi(x) = 0$$

Hence $G(x) = A(x)$ $\forall x \in X$. Therefore the reciprocal map A is unique and the proof of the theorem is now complete.□

Theorem 3.3.4. [15] Let X and Y be sets of non-zero real numbers and $f : X \longrightarrow Y$ be a mapping for which there exists a constant $\theta > 0$ and f satisfies

$$|f(\frac{x+y}{2}) + f(x+y) - \frac{3f(x)f(y)}{f(x)+f(y)}| \leq \theta H(x,y) \tag{3.79}$$

where $H : X^2 \longrightarrow Y$ be a function such that

$$\phi(x) = \textstyle\sum_{i=0}^{\infty} 2^i H(2^{i+1}x, 2^{i+1}x)$$

with the condition

$$\lim_{n\to\infty} 2^n H(2^{n+1}x, 2^{n+1}x) = 0$$

holds. Then there exists a unique reciprocal mapping $A : X \longrightarrow Y$ which satisfies (3.1) and the inequality

$$|A(x) - f(x)| \leq \theta\phi(x)$$

for all $x \in X$

Proof

Replacing (x, y) by (x, x) in (3.79), we have

$$|f(\frac{x+x}{2}) + f(x+x) - \frac{3f(x)f(x)}{f(x)+f(x)}| \leq \theta H(x, x) \tag{3.80}$$

$$|f(x) + f(2x) - \frac{3}{2}f(x)| \leq \theta H(x, x) \tag{3.81}$$

$$|f(2x) - \frac{1}{2}f(x)| \leq \theta H(x, x) \tag{3.82}$$

$$|2f(2x) - f(x)| \leq 2\theta H(x, x) \tag{3.83}$$

Replacing x by $2x$ in (3.86) and multiplying by 2 we get

$$|2^2 f(2^2 x) - 2f(2x)| \leq 2^2\theta H(2x, 2x) \tag{3.84}$$

Summing (3.87) with (3.86) and use triangle inequality to get

$$|2^2 f(2^2 x) - f(x)| \leq |2^2 f(2^2 x) - 2f(2x)| + |2f(2x) - f(x)|$$

$$\leq 2\theta H(x, x) + 2^2\theta H(2x, 2x) = 2\theta \sum_{i=0}^{1} 2^i H(2^{i+1}x, 2^{i+1}x) \tag{3.85}$$

Proceeding further and using induction on a positive integer n, we get

$$|2^n f(2^n x) - f(x)| \leq 2\theta \sum_{i=0}^{n-1} 2^i H(2^{i+1}x, 2^{i+1}x) \tag{3.86}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now if $n > m > 0$, then $n - m$ is a natural number, and n can be replaced by $n - m$ in (3.89) to obtain

$$|2^{n-m} f(2^{n-m} x) - f(x)| \leq 2\theta \sum_{i=0}^{n-m-1} 2^i H(2^{i+1}x, 2^{i+1}x) \tag{3.87}$$

Multiplying both sides by 2^m and simplifying , we get

$$|2^n f(2^{n-m}x) - 2^m f(x)| \leq \theta \sum_{i=0}^{n-m-1} 2^{i+m} H(2^{i+1}x, 2^{i+1}x) \tag{3.88}$$

for all $x \in X$. Now we replace x by $2^m x$ to have

$$|2^n f(2^n x) - 2^m f(2^m x)| \leq \theta 2^{i+m+1} \sum_{i=0}^{n-m-1} H(2^{i+m+1}x, 2^{i+m+1}x) \tag{3.89}$$

If $m \to \infty$ then $\theta 2^{i+m+1} H(2^{i+m+1}x, 2^{i+m+1}x) \to 0 \ \forall i$.

Therefore,

$$\lim_{m\to\infty} |2^n f(2^n x) - 2^m f(2^m x)| = 0. \tag{3.90}$$

Hence $\{2^n f(2^n x)\}_{n=0}^{\infty}$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $r : X \to \mathbb{R}$ by

$$A(x) = \lim_{n\to\infty} 2^n f(2^n x) \tag{3.91}$$

Now we show that $A : X \to \mathbb{R}$ defined by (3.91) is satisfying (3.1). Consider

$$|A(\frac{x+y}{2}) - A(x+y) + \frac{3A(x)A(y)}{A(x)+A(y)}|$$

$$= |\lim_{n\to\infty} 2^n \{f(\frac{2^n(x+y)}{2}) + f(2^n(x+y)) - \frac{3f(2^n x)f(2^n y)}{f(2^n x)+f(2^n y)}\}|$$
$$= \lim_{n\to\infty} 2^n |f(\frac{2^n(x+y)}{2}) + f(2^n(x+y)) - \frac{3f(2^n x)f(2^n y)}{f(2^n x)+f(2^n y)}|$$
$$\leq \lim_{n\to\infty} 2^n \theta H(2^n x, 2^n y) = 0 \text{ by } (3.79)$$

Therefore

$$A(\frac{x+y}{2}) + A(x+y) = \frac{3A(x)A(y)}{A(x)+A(y)}$$

for all $x, y \in X$.

Our next goal is to show that

$$|f(x) - A(x)| \leq \theta\phi(x)$$

$$|f(x) - A(x)| = |f(x) - \lim_{n\to\infty} 2^n f(2^n x)|$$

$$= |\lim_{n\to\infty} 2^n f(2^n x) - f(x)| \leq \theta\phi(x) \text{ by } (3.86)$$

Hence we obtain

$$|f(x) - A(x)| \leq \theta\phi(x)$$

for all $x \in X$

Finally we prove that A is unique. Suppose that there exists another reciprocal function $G : X \longrightarrow \mathbb{R}$ such that

$$|G(x) - f(x)| \leq \theta\phi(x) \tag{3.92}$$

for all $x \in X$. Note that

$$|G(x) - A(x)| = |G(x) - f(x) + f(x) - A(x)|$$

$$\leq |G(x) - f(x)| + |f(x) - A(x)| = \theta\phi(x) + \theta\phi(x)$$

Therefore,

$$|G(x) - A(x)| \leq 2\theta\phi(x) \tag{3.93}$$

Further, since A and G are reciprocal function , we have

$$|f(x) - A(x)| = |\tfrac{2^n G(x)}{2^n} - \tfrac{2^n A(x)}{2^n}| = |\tfrac{1}{2^n} G(\tfrac{x}{2^n}) - \tfrac{1}{2^n} A(\tfrac{x}{2^n})| = \tfrac{1}{2^n}|G(\tfrac{x}{2^n}) - A(\tfrac{x}{2^n})|$$
$$\leq \tfrac{1}{2^n}\theta\phi(\tfrac{x}{2^n}),$$

where $n \in \mathbb{N}$. Hence

$$|g(x) - A(x)| \leq \frac{1}{2^n}\theta\phi(\frac{x}{2^n}).$$

Taking the limit on both sides, we get

$$0 \leq |G(x) - A(x)| \lim_{n\to\infty} |G(x) - A(x)| \leq \theta \lim_{n\to\infty} \frac{1}{2^n}\phi(\frac{x}{2^n}) = 0$$

Hence $g(x) = r(x)\ \forall x \in X$. Therefore, the reciprocal map A is unique and the proof of the theorem is now complete.□

3.4 Extended Ulam Stability of Reciprocal adjoint Functional Equation

Theorem 3.4.1. [15] Let X and Y be sets of non zero real numbers and $f : X \longrightarrow Y$ be a mapping . If there exist k and α with $k > 0$ and $\alpha > -\frac{1}{2}$ such that

$$|f(\frac{x+y}{2}) + f(x+y) - \frac{3f(x)f(y)}{f(x)+f(y)}| \leq k(|x|^{\alpha}|y|^{\alpha} + |x|^{2\alpha} + |y|^{2\alpha}) \tag{3.94}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \longrightarrow Y$ such that

$$|f(x) - r(x)| \leq c|x|^{2\alpha}$$

and r satisfies (3.1) for all $x, y \in X$ where $c = \frac{6k}{2^{2\alpha+1}-1}$.

Proof

Replacing (x, y) by $(\frac{x}{2}, \frac{x}{2})$ in (3.94), we obtain

$$|f(\frac{\frac{x}{2}+\frac{x}{2}}{2}) + f(\frac{x}{2}+\frac{x}{2}) - \frac{3f(\frac{x}{2})f(\frac{x}{2})}{f(\frac{x}{2})+f(\frac{x}{2})}| \leq k(|\frac{x}{2}|^{\alpha}|\frac{x}{2}|^{\alpha} + |\frac{x}{2}|^{2\alpha} + |\frac{x}{2}|^{2\alpha}) \tag{3.95}$$

$$|f(x) - \frac{1}{2}f(\frac{x}{2})| \leq \frac{3k}{2^{2\alpha}}|x|^{2\alpha} \tag{3.96}$$

Replacing x by $\frac{x}{2}$ in (3.96) and dividing by 2 to get

$$|\frac{1}{2}f(\frac{x}{2}) - \frac{1}{2^2}f(\frac{x}{2^2})| \leq \frac{3k}{2^{2\alpha}}\frac{1}{2}|\frac{x}{2}|^{2\alpha} \tag{3.97}$$

Summing (3.97) with (3.96) and use triangle inequality to get $|f(x) - \frac{1}{2^2}f(\frac{x}{2^2})| \leq |f(x) - \frac{1}{2}f(\frac{x}{2})| + |\frac{1}{2}f(\frac{x}{2}) - \frac{1}{2^2}f(\frac{x}{2^2})| \leq \frac{3k}{2^{2\alpha}}|x|^{2\alpha} + \frac{3k}{2^{2\alpha}}\frac{1}{2}|\frac{x}{2}|^{2\alpha}$

$$= \frac{3k}{2^{2\alpha}}\sum_{i=0}^{1}\frac{1}{2^{i(2\alpha+1)}}|x|^{2\alpha} \tag{3.98}$$

Then by induction on n we get

$$|f(x) - \frac{1}{2^n}f(\frac{x}{2^n})| \leq \frac{3k}{2^{2\alpha}}\sum_{i=0}^{n-1}\frac{1}{2^{i(2\alpha+1)}}|x|^{2\alpha} \tag{3.99}$$

Hence

$$|f(x) - \frac{1}{2^n}f(\frac{x}{2^n})| \leq \frac{3k}{2^{2\alpha}}\sum_{i=0}^{\infty}\frac{1}{2^{i(2\alpha+1)}}|x|^{2\alpha} = \frac{6k}{2^{2\alpha+1}-1}|x|^{2\alpha} \tag{3.100}$$

$$|f(x) - \frac{1}{2^n}f(\frac{x}{2^n})| \leq c|x|^{2\alpha} \tag{3.101}$$

Where $c = \frac{6k}{2^{2\alpha+1}-1}$ for all $x \in X$ and $n \in \mathbb{N}$. Now if $n > m > 0$, then $n - m$ is a natural number, and n can be replaced by $n - m$ in (3.101) to obtain

$$|f(x) - \frac{1}{2^{n-m}}f(\frac{x}{2^{n-m}})| \leq c|x|^{2\alpha}. \tag{3.102}$$

Multiplying both sides by $\frac{1}{2^m}$ and simplifying , we get

$$|\frac{1}{2^m}f(x) - \frac{1}{2^n}f(\frac{x}{2^{n-m}})| \leq \frac{c}{2^m}|x|^{2\alpha}. \tag{3.103}$$

for all $x \in X$. Now we replace x by $\frac{x}{2^m}$ to have

$$|\frac{1}{2^m}f(\frac{x}{2^m}) - \frac{1}{2^n}f(\frac{x}{2^n})| \leq \frac{c}{2^m}|\frac{x}{2^m}|^{2\alpha}. \tag{3.104}$$

If $m \to \infty$ and $\alpha > -\frac{1}{2}$, then $\frac{c}{2^{m(2\alpha+1)}}|x|^{2\alpha} \to 0$, and therefore,

$$\lim_{m\to\infty} |\frac{1}{2^m}f(\frac{x}{2^m}) - \frac{1}{2^n}f(\frac{x}{2^n})| = 0. \tag{3.105}$$

Hence

$$\{2^{-n}f(2^{-n}x)\}_{n=0}^{\infty} \tag{3.106}$$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $r : X \to \mathbb{R}$ by

$$r(x) = \lim_{n\to\infty} 2^{-n}f(2^{-n}x) \tag{3.107}$$

Now we show that $r : X \to \mathbb{R}$ defined by (3.107) is satisfying(3.1). Consider

$$|r(\frac{x+y}{2}) + r(x+y) - \frac{3r(x)r(y)}{r(x)+(y)}|$$

$$= |\lim_{n\to\infty} 2^{-n}\{f(\frac{2^{-n}(x+y)}{2}) + f(2^{-n}(x+y)) - \frac{3f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}\}|$$

$$= \lim_{n\to\infty} 2^{-n}|f(\frac{2^{-n}(x+y)}{2}) + f(2^{-n}(x+y)) - \frac{3f(2^{-n}x)f(2^{-n}y)}{f(2^{-n}x)+f(2^{-n}y)}|$$

$$\leq \lim_{n\to\infty} 2^{-n(2\alpha+1)}k(|x|^{\alpha}|y|^{\alpha} + (|x|^{2\alpha} + |y|^{2\alpha})) = 0 \text{ by } (3.94)$$

Therefore,

$$r(\frac{x+y}{2}) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)}$$

for all $x, y \in X$.

Our next goal is to show that

$$|f(x) - r(x)| \leq c|x|^{2\alpha}$$

$$|f(x) - r(x)| = |f(x) - \lim_{n\to\infty} 2^{-n} f(2^{-n}x)|$$

$$= |\lim_{n\to\infty} 2^{-n} f(2^{-n}x) - f(x)| \leq c|x|^{2\alpha} \text{ by } (3.101)$$

Hence we obtain

$$|f(x) - r(x)| \leq c|x|^{2\alpha}$$

for all $x \in X$

Finally we prove that r is unique. Suppose that there exists another reciprocal function $g : X \longrightarrow \mathbb{R}$ such that

$$|g(x) - f(x)| \leq c|x|^{2\alpha} \tag{3.108}$$

for all $x \in X$. Note that

$$|g(x) - r(x)| = |g(x) - f(x) + f(x) - r(x)|$$

$$\leq |g(x) - f(x)| + |f(x) - r(x)| = c|x|^{2\alpha} + c|x|^{2\alpha}$$

Therefore,

$$|g(x) - r(x)| \leq 2c|x|^{2\alpha} \tag{3.109}$$

Further, since r and g are reciprocal function , we have

$$|g(x) - r(x)| = |\tfrac{2^n g(x)}{2^n} - \tfrac{2^n r(x)}{2^n}| = |2^{-n} g(2^{-n}x) - 2^{-n} r(2^{-n}x)|$$
$$= 2^{-n}|g(2^{-n}x) - r(2^{-n}x)| \leq 2^{-n(2\alpha+1)+1} c|x|^{2\alpha},$$

where $n \in \mathbb{N}$. Hence

$$|g(x) - r(x)| \leq 2^{-n(2\alpha+1)+1} c|x|^{2\alpha}$$

Taking the limit on both sides, we get

$$0 \leq |g(x) - r(x)| = \lim_{n\to\infty} |g(x) - r(x)| \leq \lim_{n\to\infty} 2^{-n(2\alpha+1)+1} c|x|^{2\alpha} = 0$$

Hence $g(x) = r(x)$ $\forall x \in X$. Therefore, the reciprocal map r is unique and the proof of the theorem is now complete.□

Theorem 3.4.2. [15] Let X and Y be sets of non zero real numbers and $f : X \longrightarrow Y$ be a mapping. If there exist k and α with $k > 0$ and $\alpha < -\frac{1}{2}$ such that

$$|f(\frac{x+y}{2}) + f(x+y) - \frac{3f(x)f(y)}{f(x)+f(y)}| \leq k(|x|^{\alpha}|y|^{\alpha} + |x|^{2\alpha} + |y|^{2\alpha}) \tag{3.110}$$

for all $x, y \in X$, then there exists a unique reciprocal mapping $r : X \longrightarrow Y$ such that

$$|r(x) - f(x)| \leq c|x|^{2\alpha}$$

and r satisfies (3.1) for all $x, y \in X$ where $c = \frac{6k}{1-2^{2\alpha+1}}$.

Proof

Replacing (x, y) by (x, x) in (3.110), we have

$$|f(\frac{x+x}{2}) + f(x+x) - \frac{3f(x)f(x)}{f(x)+f(x)}| \leq k(|x|^{\alpha}|x|^{\alpha} + (|x|^{2\alpha} + |x|^{2\alpha})) \tag{3.111}$$

$$|f(x) + f(2x) - \frac{3}{2}f(x)| \leq 3k|x|^{2\alpha} \tag{3.112}$$

$$|f(2x) - \frac{1}{2}f(x)| \leq 3k|x|^{2\alpha} \tag{3.113}$$

$$|2f(2x) - f(x)| \leq 6k|x|^{2\alpha} \tag{3.114}$$

Replacing x by $2x$ in (3.114) and multiplying by 2 to get

$$|2^2 f(2^2 x) - 2f(2x)| \leq 12k|2x|^{2\alpha} \tag{3.115}$$

Summing (3.115) with (3.114) and use triangle inequality to get $|2^2 f(2^2 x) - f(x)| \leq |2^2 f(2^2 x) - 2f(2x)| + |2f(2x) - f(x)| \leq 12k|2x|^{2\alpha} + 6k|x|^{2\alpha}$

$$= 6k \sum_{i=1}^{1} 2^{i(2\alpha+1)} |x|^{2\alpha} \tag{3.116}$$

Then by induction one can see that for all natural number n,

$$|2^n f(2^n x) - f(x)| \leq 6k \sum_{i=0}^{n-1} 2^{i(2\alpha+1)} |x|^{2\alpha} \tag{3.117}$$

$$|2^n f(2^n x) - f(x)| \leq 6k \sum_{i=0}^{\infty} 2^{i(2\alpha+1)} |x|^{2\alpha} = \frac{6k}{1 - 2^{(2\alpha+1)}} |x|^{2\alpha} \tag{3.118}$$

Setting $c = \frac{6k}{1-2^{(2\alpha+1)}}$

$$|2^n f(2^n x) - f(x)| \leq c|x|^{2\alpha} \tag{3.119}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now if $n > m > 0$, then $n - m$ is a natural number, and n can be replaced by $n - m$ in (3.119) to obtain

$$|2^{n-m} f(2^{n-m} x) - f(x)| \leq c|x|^{2\alpha} \tag{3.120}$$

Multiplying both sides by 2^m and simplifying , we get

$$|2^n f(2^{n-m} x) - 2^m f(x)| \leq 2^m c|x|^{2\alpha} \tag{3.121}$$

for all $x \in X$. Now we replace x by $2^m x$ to have

$$|2^n f(2^n x) - 2^m f(2^m x)| \leq 2^m c|2^m x|^{2\alpha} \tag{3.122}$$

$$|2^n f(2^n x) - 2^m f(2^m x)| \leq 2^{m(2\alpha+1)} c|x|^{2\alpha} \tag{3.123}$$

If $m \to \infty$ and $\alpha < -\frac{1}{2}$ $2^{m(2\alpha+1)} c|x|^{2\alpha} \to 0$, and therefore,

$$\lim_{m\to\infty} |2^n f(2^n x) - 2^m f(2^m x)| = 0. \tag{3.124}$$

Hence

$$\{2^n f(2^n x)\}_{n=0}^{\infty} \tag{3.125}$$

is a Cauchy sequence in X. Hence the limit of this sequence exists.

Define $r : X \to \mathbb{R}$ by

$$r(x) = \lim_{n\to\infty} 2^n f(2^n x) \tag{3.126}$$

Now we show that $r : X \to \mathbb{R}$ defined by (3.126) is satisfying (3.1). Consider

$$|r(\frac{x+y}{2}) + r(x+y) - \frac{3r(x)r(y)}{r(x)+r(y)}|$$

$$= |\lim_{n\to\infty} 2^n \{f(\frac{2^n(x+y)}{2}) + f(2^n(x+y)) - \frac{3f(2^n x)f(2^n y)}{f(2^n x)+f(2^n y)}\}|$$

$$= \lim_{n\to\infty} 2^n |f(\frac{2^n(x+y)}{2}) + f(2^n(x+y)) - \frac{3f(2^n x)f(2^n y)}{f(2^n x)+f(2^n y)}|$$

$$\leq \lim\nolimits_{n\to\infty} 2^n k(|2^n x|^\alpha |2^n y|^\alpha + |2^n x|^{2\alpha} + |2^n y|^{2\alpha} \text{ by (3.110)}$$

$$= \lim_{n\to\infty} 2^{n(2\alpha+1)} k(|x|^{\alpha}|y|^{\alpha} + |x|^{2\alpha} + |y|^{2\alpha}) = 0$$

Therefore,

$$r(\frac{x+y}{2}) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)}$$

for all $x, y \in X$.

Our next goal is to show that $|f(x) - r(x)| \leq c|x|^{2\alpha}$

$$|f(x) - r(x)| = |f(x) - \lim_{n\to\infty} 2^n f(2^n x)|$$

$$= |\lim_{n\to\infty} 2^n f(2^n x) - f(x)| \leq c|x|^{2\alpha} \text{ by } (3.123)$$

Hence we obtain

$$|f(x) - r(x)| \leq c|x|^{2\alpha}$$

for all $x \in X$.

Finally we prove that r is unique. Suppose that there exists another reciprocal function $g : X \longrightarrow \mathbb{R}$ such that

$$|g(x) - f(x)| \leq c|x|^{2\alpha} \tag{3.127}$$

for all $x \in X$. Note that

$$|g(x) - r(x)| = |g(x) - f(x) + f(x) - r(x)|$$

$$\leq |g(x) - f(x)| + |f(x) - r(x)| = c|x|^{2\alpha} + c|x|^{2\alpha}$$

Therefore

$$|g(x) - r(x)| \leq 2c|x|^{2\alpha} \tag{3.128}$$

Further, since r and g are reciprocal function , we have

$$|g(x) - r(x)| = |\frac{2^n g(x)}{2^n} - \frac{2^n r(x)}{2^n}|$$

$$= |2^n g(2^n x) - 2^n r(2^n x)|$$

$$= 2^n |g(2^n x) - r(2^n x)|$$

$$\leq 2^n 2c|2^n x|^{2\alpha} = 2^{n(2\alpha+1)+1} c|x|^{2\alpha},$$

where $n \in \mathbb{N}$. Hence

$$|g(x) - r(x)| \leq 2^{n(2\alpha+1)+1} c|x|^{2\alpha}$$

Taking the limit on both sides, we get

$$0 \leq |g(x) - r(x)| = \lim_{n\to\infty} |g(x) - r(x)| \leq \lim_{n\to\infty} 2^{n(\rho+1)+1} c|x|^{2\alpha} = 0$$

which is

$$|g(x) - r(x)| \leq 0.$$

Hence $g(x) = r(x)$ $\forall x \in X$. Therefore the reciprocal map r is unique and the proof of the theorem is now complete.$\square$

Chapter 4

Stability of Reciprocal Type Functional Equation in Several Variables

Introduction

This chapter consists of three sections:

In the first section, we study reciprocal type functional equation in several variables. In the second section, we study generalized Hyars - Ulam stability of reciprocal type functional equation in several variables. In the third section,we study counter examples .

4.1 Reciprocal Type Functional Equation in Several Variables

The Reciprocal Type Functional Equation in Several variables is:

$$\frac{\prod_{i=2}^{m} r(x_1 + x_i)}{\sum_{i=2}^{m}[\prod_{j=2,j\neq i}^{m} r(x_1 + x_i)]} = \frac{\prod_{i=1}^{m} r(x_i)}{\sum_{i=2}^{m} r(x_1)[\prod_{j=2,j\neq i}^{m} r(x_j)] + (m-1)\prod_{i=2}^{m} r(x_i)} \tag{4.1}$$

where m is a positive integer with $m \geq 3$.

Notation: We define the difference operator $D_m r : X^m \to \mathbb{R}$ by

$D_m r(x_1, x_2, ...x_m) = \frac{\prod_{i=2}^{m} r(x_1+x_i)}{\sum_{i=2}^{m}[\prod_{j=2,j\neq i}^{m} r(x_1+x_j)]} - \frac{\prod_{i=1}^{m} r(x_i)}{\sum_{i=2}^{m} r(x_1)[\prod_{j=2,j\neq i}^{m} r(x_j)]+(m-1)\prod_{i=2}^{m} r(x_i)}$

for $x_1, x_2, ..., x_m \in X$.

In the following results, we will set $\frac{0^{m-1}}{0^{m-2}} = 0$ for $m \geq 3$ and assume $\sum_{i=2}^{m}[\prod_{j=2,j\neq i}^{m} r(x_1 + x_j)] \neq 0, \sum_{i=2}^{m}[\prod_{j=2,j\neq i}^{m} r(x_j)] + (m-1)\prod_{i=2}^{m} r(x_i) \neq 0$ for all $x_i \in X; i = 1, 2, ..., m$; $m \geq 3$ and $x_1 \neq -x_i$ for all $i; 2 \leq i \leq m; m \geq 3$.

hugGeneral solution of functional equation (4.1)

Theorem 4.1.1. [16] A mapping $r : X \to \mathbb{R}$ satisfies the functional equation (4.1) for all $x_1, x_2, ..., x_m \in X$ if and only if there exists a reciprocal mapping $r : X \to \mathbb{R}$ satisfying the reciprocal functional equation (1.1) for all $x, y \in X$.

Proof. Let the mapping $r : X \to \mathbb{R}$ satisfies the functional equation (4.1). Replacing x_1 by x and x_i by y for $i = 1, 2, ..., m$ in (4.1)

$$\frac{\prod_{i=2}^{m} r(x+y)}{\sum_{i=2}^{m}[\prod_{j=2,j\neq i}^{m} r(x+y)]} = \frac{r(x)\prod_{i=2}^{m} r(y)}{\sum_{i=2}^{m} r(x)[\prod_{j=2,j\neq i}^{m} r(y)] + (m-1)\prod_{i=2}^{m} r(y)}$$

$$\frac{r^{m-1}(x+y)}{(m-1)[r^{m-2}(x+y)]} = \frac{r(x)r^{m-1}(y)}{(m-1)r(x)r^{m-2}(y) + (m-1)r^{m-1}(y)}$$

$$\frac{r(x+y)}{(m-1)} = \frac{r(x)r^{m-1}(y)}{(m-1)r^{m-2}(y)[r(x) + r(y)]}$$

Multiplying by $(m-1)$

$$r(x+y) = \frac{r(x)r(y)}{r(x) + r(y)},$$

which is (1.1).

Conversely, let the mapping $r : X \to \mathbb{R}$ satisfies the functional equation (1.1). Replacing (x, y) by $(x_1, x_2 + x_3)$ in (1.1), we obtain

$$r((x_1 + x_2 + x_3) = \frac{r(x_1)r(x_2 + x_3)}{r(x_1) + r(x_2 + x_3)}$$

$$= \frac{r(x_1)\frac{r(x_2)r(x_3)}{r(x_2)+r(x_3)}}{r(x_1) + \frac{r(x_2)r(x_3)}{r(x_2)+r(x_3)}}$$

$$= \frac{r(x_1)r(x_2)r(x_3)}{r(x_1)r(x_2)+r(x_1)r(x_3)+r(x_2)r(x_3)}$$

$$= \frac{\prod_{i=1}^{3} r(x_i)}{\sum_{i=2}^{3} r(x_1)[\prod_{j=2,j\neq i}^{3} r(x_j)] + \prod_{i=2}^{3} r(x_i)}$$

for all $x_1, x_2, x_3 \in X$. Using induction on a positive integer $m-1$, we have

$$r(x_1+x_2+...+x_{m-1}) = \frac{\prod_{i=2}^{m-1} r(x_i)}{\sum_{i=2}^{m-1} r(x_1)[\prod_{j=2,j\neq i}^{m-1} r(x_j)] + \prod_{i=2}^{m-1} r(x_i)} \tag{4.2}$$

for all $x_1, x_2, ..., x_{m-1} \in X$. Now, replacing x_i by x for $i = 1, 2, ..., m-1$ in (4.2) , we get $r((m-1)x) = \frac{1}{m-1} r(x)$ for all $x \in X$. Replacing x_i by x_{i+1} for $i = 1, 2, ..., m-1$ in (4.2), we obtain

$$r(x_2+x_3+,...+x_m) = \frac{\prod_{i=2}^{m} r(x_i)}{\sum_{i=3}^{m} r(x_2)[\prod_{j=3,j\neq i}^{m} r(x_j)] + \prod_{i=3}^{m} r(x_i)}$$

for all $x_1, x_2, ..., x_m \in X$. Now, replacing x_i by $x_1 + x_{i+1}$ for $i = 1, 2, ..., m$ in (2.3) , we get

$$\frac{\prod_{i=2}^{m} r(x_1+x_i)}{\sum_{i=3}^{m} r(x_1+x_2)[\prod_{j=3,j\neq i}^{m} r(x_1+x_j)] + \prod_{i=3}^{m} r(x_1+x_i)}$$

$$= r((m-1)x_1+x_2+...+x_m)$$

$$= \frac{\frac{1}{m-1} r(x_1) \frac{\prod_{i=2}^{m} r(x_i)}{\sum_{i=3}^{m} r(x_2)[\prod_{j=3,j\neq i}^{m} r(x_j)]+\prod_{i=3}^{m} r(x_i)}}{\frac{1}{m-1} r(x_1) + \frac{\prod_{i=2}^{m} r(x_i)}{\sum_{i=3}^{m} r(x_2)[\prod_{j=3,j\neq i}^{m} r(x_j)]+\prod_{i=3}^{m} r(x_i)}}$$

$$= \frac{\prod_{i=1}^{m} r(x_i)}{\sum_{i=2}^{m} r(x_1)[\prod_{j=2,j\neq i}^{m} r(x_j)] + (m-1)\prod_{i=2}^{m} r(x_i)} \tag{4.3}$$

$$\frac{\prod_{i=2}^{m} r(x_1+x_i)}{\sum_{i=2}^{m}[\prod_{j=2,j\neq i}^{m} r(x_1+x_j)]} = \frac{\prod_{i=1}^{m} r(x_i)}{\sum_{i=2}^{m} r(x_1)[\prod_{j=2,j\neq i}^{m} r(x_j)] + (m-1)\prod_{i=2}^{m} r(x_i)}$$

for all $x_1, x_2, ..., x_m \in X$. This completes the proof of the Theorem

4.2 Generalized Hyars - Ulam Stability of Reciprocal Type Functional Equation in Several Variables

Through this chapter$X$ will denoted the set of non-zero real numbers.

Theorem 4.2.1. [16] Let $\varphi : X^m \longrightarrow \mathbb{R}$ be a function satisfying

$$\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi(\frac{x_1}{2^{i+1}}, \frac{x_2}{2^{i+1}}, ..., \frac{x_m}{2^{i+1}}) < \infty \tag{4.4}$$

for all $x_1, x_2, ..., x_m \in X$. If a function $f : X \to \mathbb{R}$ satisfies the functional inequality

$$|D_m f(x_1, x_2, ...x_m)| \leq \varphi(x_1, x_2, ..., x_m) \tag{4.5}$$

for all $x_1, x_2, ..., x_m \in X$, then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ which satisfies (4.1) and the inequality

$$|r(x) - f(x)| \leq 2(m-1) \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi(\frac{x_1}{2^{i+1}}, \frac{x_2}{2^{i+1}}, ..., \frac{x_m}{2^{i+1}}) \tag{4.6}$$

for all $x \in X$.

Proof. Replacing x_i by $\frac{x}{2}$ for $i = 1, 2, ..., m$ in (4.5)

$$|D_m f(\frac{x}{2}, \frac{x}{2}, ..., \frac{x}{2})|$$

$$= |\frac{\prod_{i=2}^{m} f(\frac{x}{2} + \frac{x}{2})}{\sum_{i=2}^{m} [\prod_{j=2, j\neq i}^{m} f(\frac{x}{2} + \frac{x}{2})]} - \frac{\prod_{i=1}^{m} f(\frac{x}{2})}{\sum_{i=2}^{m} f(\frac{x}{2}) [\prod_{j=2, j\neq i}^{m} f(\frac{x}{2})] + (m-1) \prod_{i=2}^{m} f(\frac{x}{2})}| \leq \varphi(\frac{x}{2}, \frac{x}{2}, ..., \frac{x}{2}) \tag{4.7}$$

$$|\frac{f^{m-1}(x)}{\sum_{i=2}^{m} f^{m-2}(x)} - \frac{f^m(\frac{x}{2})}{\sum_{i=2}^{m} f(\frac{x}{2}) f^{m-2}(\frac{x}{2}) + (m-1) f^{m-1}(\frac{x}{2})}| \leq \varphi(\frac{x}{2}, \frac{x}{2}, ..., \frac{x}{2}). \tag{4.8}$$

$$|\frac{f^{m-1}(x)}{(m-1) f^{m-2}(x)} - \frac{f^m(\frac{x}{2})}{(m-1) f^{m-1}(\frac{x}{2}) + (m-1) f^{m-1}(\frac{x}{2})}| \leq \varphi(\frac{x}{2}, \frac{x}{2}, ..., \frac{x}{2}). \tag{4.9}$$

$$|\frac{f(x)}{(m-1)} - \frac{f^m(\frac{x}{2})}{2(m-1) f^{m-1}(\frac{x}{2})}| \leq \varphi(\frac{x}{2}, \frac{x}{2}, ..., \frac{x}{2}). \tag{4.10}$$

$$|\frac{f(x)}{(m-1)} - \frac{f(\frac{x}{2})}{2(m-1)}| \leq \varphi(\frac{x}{2}, \frac{x}{2}, ..., \frac{x}{2}). \tag{4.11}$$

and multiplying by $(m-1)$, we get

$$|f(x) - \frac{1}{2}f(\frac{x}{2})| \leq (m-1)\varphi(\frac{x}{2}, \frac{x}{2}, ..., \frac{x}{2}). \tag{4.12}$$

Now, replacing x by $\frac{x}{2}$ in (4.12), and dividing by 2 to get

$$|\frac{1}{2}f(\frac{x}{2}) - \frac{1}{2^2}f(\frac{x}{2^2})| \leq 2(m-1)\frac{1}{2^2}\varphi(\frac{x}{2^2}, \frac{x}{2^2}, ..., \frac{x}{2^2}). \tag{4.13}$$

Summing (4.12) and (4.13) and using triangle inequality to get

$$\begin{aligned} |f(x) - \frac{1}{2^2}f(\frac{x}{2^2})| &\leq (m-1)\varphi(\frac{x}{2}, \frac{x}{2}, ..., \frac{x}{2} + 2(m-1)\frac{1}{2^2}\varphi(\frac{x}{2^2}, \frac{x}{2^2}, ..., \frac{x}{2^2}) \\ &= 2(m-1)\sum_{i=0}^{1}\frac{1}{2^{i+1}}\varphi(\frac{x}{2}, \frac{x}{2}, ..., \frac{x}{2}) \end{aligned} \tag{4.14}$$

for all $x \in X$. Proceeding further and using induction arguments on a positive integer n, we arrive

$$|f(x) - \frac{1}{2^n}f(\frac{x}{2^n})| \leq 2(m-1)\sum_{i=0}^{n-1}\frac{1}{2^{i+1}}\varphi(\frac{x}{2}, \frac{x}{2}, ..., \frac{x}{2}) \tag{4.15}$$

for all $x \in X$. Hence for any integers l, k with $l > k > 0$, we obtain by using the triangle inequality

$$\begin{aligned} &|\frac{1}{2^l}f(\frac{x}{2^l}) - \frac{1}{2^k}f(\frac{x}{2^k})| \\ &= |\frac{1}{2^l}f(\frac{x}{2^l}) - \frac{1}{2^{l-1}}f(\frac{x}{2^{l-1}}) + \frac{1}{2^{l-1}}f(\frac{x}{2^{l-1}}) - ... + \frac{1}{2^k}f(\frac{x}{2^k})| \\ &\leq 2(m-1)\frac{1}{2^l}\varphi(\frac{x}{2^l}, \frac{x}{2^l}, ..., \frac{x}{2^l}) + ... + 2(m-1)\frac{1}{2^k}\varphi(\frac{x}{2^k}, \frac{x}{2^k}, ..., \frac{x}{2^k}) \\ &\leq 2(m-1)\sum_{i=k}^{l}\frac{1}{2^i}\varphi(\frac{x}{2^i}, \frac{x}{2^i}, ..., \frac{x}{2^i}) \\ &\leq 2(m-1)\sum_{i=k}^{l}\frac{1}{2^{i+1}}\varphi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, ..., \frac{x}{2^{i+1}}) \end{aligned} \tag{4.16}$$

for all $x \in X$.

Taking the limit as $l \to \infty$ in (4.16) and considering (4.4), it follows that the sequence

$$\{\frac{1}{2^n}f(\frac{x}{2^n})\}_{n=0}^{\infty} \tag{4.17}$$

is a Cauchy sequence for each $x \in X$. Since $\mathbb{R}$ is complete, we can define $r : X \to \mathbb{R}$ by

$$r(x) = \lim_{n\to\infty} \frac{1}{2^n} f(\frac{x}{2^n}). \quad (4.18)$$

To show that r satisfies (4.1), replacing $(x_1, x_2, ...x_m)$ by $(2^{-n}x_1, 2^{-n}x_2, ...2^{-n}x_m)$ in (4.5) and dividing by 2^n, we obtain

$$|2^{-n}D_m f(2^{-n}x_1, 2^{-n}x_2, ...2^{-n}x_m)| \le 2^{-n}\varphi(2^{-n}x_1, 2^{-n}x_2, ...2^{-n}x_m) \quad (4.19)$$

$= |\lim_{n\to\infty} 2^{-n}\{\frac{\prod_{i=2}^m f(2^{-n}x_1+2^{-n}x_i)}{\sum_{i=2}^m f(x_1)[\prod_{j=2,j\neq i}^m f(2^{-n}x_1+2^{-n}x_j)]} -$
$\frac{\prod_{i=1}^m f(2^{-n}x_i)}{2^{-n}\sum_{i=2}^m f(2^{-n}x_1)[\prod_{j=2,j\neq i}^m f(2^{-n}x_j)]+(m-1)\prod_{i=2}^m f(2^{-n}x_i)}\}|$
$= \lim_{n\to\infty} 2^{-n}|\{\frac{\prod_{i=2}^m f(2^{-n}(x_1+x_i))}{\sum_{i=2}^m f(2^{-n}x_1)[\prod_{j=2,j\neq i}^m f(2^{-n}(x_1+x_j))]} -$
$\frac{\prod_{i=1}^m f(2^{-n}x_i)}{2^{-n}\sum_{i=2}^m f(2^{-n}x_1)[\prod_{j=2,j\neq i}^m f(2^{-n}x_j)]+(m-1)\prod_{i=2}^m f(2^{-n}x_i)}\}|$
$\le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^{-n}x_1, 2^{-n}x_2, ...2^{-n}x_m) = 0$ by (4.5) and (4.4)

Therefore,

$\frac{\prod_{i=2}^m r(x_1+x_i)}{\sum_{i=2}^m[\prod_{j=2,j\neq i}^m r(x_1+x_i)]} = \frac{\prod_{i=1}^m r(x_i)}{\sum_{i=2}^m r(x_1)[\prod_{j=2,j\neq i}^m r(x_j)]+(m-1)\prod_{i=2}^m r(x_i)}$

for all $x_1, x_2, ..., x_m \in X$ and for all positive integer n.

Taking limit $n \to \infty$ in (4.15), we arrive (4.6). Now, it remains to show that r is uniquely defined. Let $r_1 : X \to \mathbb{R}$ be another reciprocal mapping which satisfies (4.1) and the inequality (4.6) , since $r(2^{-n}x) = 2^n r(x)$ then $r_1(2^{-n}x) = 2^n r_1(x)$ and using (4.6), we arrive $|r_1(x)-r(x)| = |r_1(x)-f(x)+f(x)-r(x)| = 2^{-n}|r_1(2^{-n}x)-r(2^{-n}x)|$ $\le 4(m-1)\sum_{i=0}^{\infty} \frac{1}{2^{i+n+1}}\varphi(\frac{x_1}{2^{i+n+1}}, \frac{x_2}{2^{i+n+1}}, ..., \frac{x_m}{2^{i+n+1}})$

$$\le 4(m-1)\sum_{i=n}^{\infty} \frac{1}{2^{i+1}}\varphi(\frac{x_1}{2^{i+1}}, \frac{x_2}{2^{i+1}}, ..., \frac{x_m}{2^{i+1}}) \quad (4.20)$$

for all $x \in X$. Allowing $n \to \infty$ in (4.20), we find that r is unique. This completes the proof of the Theorem. □

Theorem 4.2.2. [16] Let $\varphi : X^m \longrightarrow \mathbb{R}$ be a function satisfying

$$\sum_{i=0}^{\infty} 2^i\varphi(2^i x_1, 2^i x_2, ..., 2^i x_m) < \infty \quad (4.21)$$

for all $x_1, x_2, ..., x_m \in X$. If a function $f : X \to \mathbb{R}$ satisfies the functional inequality

$$|D_m f(x_1, x_2, ..., x_m)| \leq \varphi(x_1, x_2, ..., x_m) \tag{4.22}$$

for all $x_1, x_2, ..., x_m \in X$, then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ which satisfies (4.1) and the inequality

$$|r(x) - f(x)| \leq 2(m-1)\sum_{i=0}^{\infty} 2^i \varphi(2^i x_1, 2^i x_2, ..., 2^i x_m) \tag{4.23}$$

for all $x \in X$.

Proof. Replacing x_i by x for $i = 1, 2, ..., m$ in (4.22) we get

$$|D_m f(x_1, x_2, ...x_m)| = |\frac{\prod_{i=2}^{m} f(x+x)}{\sum_{i=2}^{m}[\prod_{j=2,j\neq i}^{m} f(x+x)]} - \frac{\prod_{i=1}^{m} f(x)}{\sum_{i=2}^{m} f(x)[\prod_{j=2,j\neq i}^{m} f(x)] + (m-1)\prod_{i=2}^{m} f(x)}|$$

$$\leq \varphi(x, x, ..., x) \tag{4.24}$$

$$|\frac{f^{m-1}(2x)}{\sum_{i=2}^{m} f^{m-2}(2x)} - \frac{f^m(x)}{\sum_{i=2}^{m} f(x) f^{m-2}(x) + (m-1) f^{m-1}(x)}| \leq \varphi(x, x, ..., x) \tag{4.25}$$

$$|\frac{f^{m-1}(2x)}{(m-1) f^{m-2}(2x)} - \frac{f^m(x)}{(m-1) f^{m-1}(x) + (m-1) f^{m-1}(x)}| \leq \varphi(x, x, ..., x) \tag{4.26}$$

$$|\frac{f(2x)}{(m-1)} - \frac{f^m(x)}{2(m-1) f^{m-1}(x)}| \leq \varphi(x, x, ..., x) \tag{4.27}$$

$$|\frac{f(2x)}{(m-1)} - \frac{f(x)}{2(m-1)}| \leq \varphi(x, x, ..., x) \tag{4.28}$$

Multiplying by $(m-1)$, we get

$$|f(2x) - \frac{1}{2} f(x)| \leq (m-1)\varphi(x, x, ..., x) \tag{4.29}$$

Multiplying by 2 we get

$$|2f(2x) - f(x)| \leq 2(m-1)\varphi(x, x, ..., x) \tag{4.30}$$

Replacing x by $2x$ in (2.30), we obtain

$$|2f(2^2x) - f(2x)| \leq 2(m-1)\varphi(2x, 2x, ..., 2x) \tag{4.31}$$

Multiplying by 2 we get

$$|2^2f(2^2x) - 2f(2x)| \leq 2(m-1)2\varphi(2x, 2x, ..., 2x) \tag{4.32}$$

Using triangle inequality and both (4.31) and (4.32) to get

$$|2^2f(2^2x) - 2f(2x)| \leq |2^2f(2^2x) - 2f(2x)| + |2f(2x) - f(x)|$$

$$\leq 2(m-1)[\varphi(2x, 2x, ..., 2x) + \varphi(x, x, ..., x)] = 2(m-1)\sum_{i=0}^{1} 2^i\varphi(2^ix, 2^ix, ..., 2^ix) \tag{4.33}$$

Proceeding further and using induction arguments on a positive integer n, we arrive

$$|2^nf(2^nx) - f(x)| \leq 2(m-1)\sum_{i=0}^{n-1} 2^i\varphi(2^ix, 2^ix, ..., 2^ix) \tag{4.34}$$

for all $x \in X$. Hence for any integers l, k with $l > k > 0$, we obtain by using the triangular inequality

$$|2^lf(2^lx) - 2^kf(2^kx)|$$
$$= |2^lf(2^lx) - 2^{l-1}f(2^{l-1}x) + 2^{l-1}f(2^{l-1}x) - ... + 2^kf(2^kx)|$$
$$\leq 2(m-1)2^l\varphi(2^lx, 2^lx, ..., 2^lx) + ... + 2(m-1)2^k\varphi(2^kx, 2^kx, ..., 2^kx)$$

$$\leq 2(m-1)\sum_{i=k}^{l} 2^i\varphi(2^ix, 2^ix, ..., 2^ix) \tag{4.35}$$

for all $x \in X$ .Taking the limit as $l \to \infty$ in (4.35) and considering (4.21), it follows that the sequence $\{2^nf(2^nx)\}$ is a Cauchy sequence for each $x \in X$. Since $\mathbb{R}$ is complete, we can define $r : X \to \mathbb{R}$ by $r(x) = \lim_{n\to\infty} 2^nf(2^nx)$ .To show that r satisfies

(4.1), replacing $(x_1, x_2, ...x_m)$ by $(2^n x_1, 2^n x_2, ...2^n x_m)$ in (4.22) and multiplying by 2^n, we obtain

$$|2^n D_m f(2^n x_1, 2^n x_2, ...2^n x_m)| \leq 2^n \varphi(2^n x_1, 2^n x_2, ...2^n x_m) \tag{4.36}$$

$= |\lim_{n\to\infty} \frac{2^n \prod_{i=2}^{m} f(2^n x_1 + 2^n x_i)}{2^n \sum_{i=2}^{m}[\prod_{j=2, j\neq i}^{m} f(2^n x_1 + 2^n x_j)]} - \frac{2^n \prod_{i=1}^{m} f(2^n x_i)}{2^n \sum_{i=2}^{m} f(x_1)[\prod_{j=2,j\neq i}^{m} f(2^n x_j)] + 2^n (m-1) \prod_{i=2}^{m} f(2^n x_i)}|$
$= |\lim_{n\to\infty} \frac{2^n \prod_{i=2}^{m} f(2^n (x_1 + x_i))}{2^n \sum_{i=2}^{m}[\prod_{j=2, j\neq i}^{m} f(2^n (x_1 + x_j))]} - \frac{2^n \prod_{i=1}^{m} f(2^n x_i)}{2^n \sum_{i=2}^{m} f(x_1)[\prod_{j=2,j\neq i}^{m} f(2^n x_j)] + 2^n (m-1) \prod_{i=2}^{m} f(2^n x_i)}|$
$\leq \lim_{n\to\infty} 2^n \varphi(2^n x_1, 2^n x_2, ..., 2^n x_m) = 0$ by (4.21) and (4.22)
Therefore, $\frac{\prod_{i=2}^{m} r(x_1+x_i)}{\sum_{i=2}^{m}[\prod_{j=2,j\neq i}^{m} r(x_1+x_i)]} = \frac{\prod_{i=1}^{m} r(x_i)}{\sum_{i=2}^{m} r(x_1)[\prod_{j=2,j\neq i}^{m} r(x_j)] + (m-1)\prod_{i=2}^{m} r(x_i)}$ for all $(x_1, x_2, ...x_m) \in X$ and for all positive integer n. Taking limit $n \to \infty$ in (4.34), we arrive (4.23). Now, it remains to show that r is uniquely defined.

Let $r_1 : X \to \mathbb{R}$ be another reciprocal mapping which satisfies (4.1) and the inequality (4.23) , since $r(2^{-n}x) = 2^n r(x)$ then $r_1(2^n x) = 2^{-n} r_1(x)$ and using (4.23), we arrive $|r_1(x) - r(x)| = |r_1(x) - f(x) + f(x) - r(x)| = 2^n |r_1(2^n x) - r(2^n x)|$

$$\leq 4(m-1)\sum_{i=0}^{\infty} 2^{i+n}\varphi(2^{i+n}x_1, 2^{i+n}x_2, ..., 2^{i+n}x_m) \leq 4(m-1)\sum_{i=n}^{\infty} 2^i \varphi(2^i x_1, 2^i x_2, ..., 2^i x_m) \tag{4.37}$$

for all $x \in X$. Allowing $n \to \infty$ in (4.37), we find that r is unique. This completes the proof of Theorem □

Corollary 4.2.3. [16] For any fixed $c_1 \geq 0$ and $p > -1$ or $p < -1$, if $f : X \longrightarrow \mathbb{R}$ satisfies

$$|D_m f(x_1, x_2, ...x_m)| \leq c_1 (\sum_{i=1}^{m} |x_i|^p) \tag{4.38}$$

for all $x_1, x_2, ...x_m \in X$,then there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ such that

$$|r(x) - f(x)| \leq \begin{cases} \frac{2m(m-1)c_1}{2^{p+1}-1}|x|^p, & \text{for } p > -1 \\ \frac{2m(m-1)c_1}{1-2^{p+1}}|x|^p, & \text{for } p < -1 \end{cases}$$

for all $x \in X$.

Proof. If we choose $\varphi(x_1, x_2, ..., x_m) = c_1(\sum_{i=1}^{m} |x_i|^p)$, for all $x_1, x_2, ..., x_m \in X$, then by Theorem 4.2.1, we arrive $\Gamma(x) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} c_1(\sum_{i=1}^{m} |\frac{x}{2^{i+1}}|^p) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \frac{mc_1}{2^{ip+p}} |x|^p = |x|^p \frac{mc_1}{2^{p+1}} \sum_{i=0}^{\infty} \frac{1}{2^{(p+1)i}} = \frac{mc_1}{2^{p+1}-1}$ then $|r(x) - f(x)| \leq \frac{2m(m-1)c_1}{2^{p+1}-1} |x|^p$, for all $x \in X$ and $p > -1$.

And using Theorem 4.2.2, we arrive $\Gamma(x) = \sum_{i=0}^{\infty} 2^i c_1(\sum_{i=1}^{m} |2^i x|^p) = \sum_{i=0}^{\infty} 2^i mc_1 |x|^p 2^{ip} = mc_1 |x|^p \sum_{i=0}^{\infty} 2^{(p+1)i} = \frac{mc_1}{1-2^{p+1}} |x|^p$ then $|r(x) - f(x)| \leq \frac{2m(m-1)c_1}{1-2^{p+1}} |x|^p$, for all $x \in X$ and $p < -1$.□

Corollary 4.2.4. [16] For any fixed $c_2 \geq 0$ and $p > -1$ or $p < -1$, if $f : X \longrightarrow \mathbb{R}$ satisfies

$$|D_m f(x_1, x_2, ..., x_m)| \leq c_2(\prod_{i=1}^{m} |x_i|^{\frac{p}{m}}) \tag{4.39}$$

for all $x_1, x_2, ..., x_m \in X$,then there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ such that

$$|r(x) - f(x)| \leq \begin{cases} \frac{2(m-1)c_2}{2^{p+1}-1} |x|^p, & \text{for } p > -1 \\ \frac{2(m-1)c_2}{1-2^{p+1}} |x|^p, & \text{for } p < -1 \end{cases}$$

for all $x \in X$

Proof. Considering $\varphi(x_1, x_2, ..., x_m) = c_2(\prod_{i=1}^{m} |x_i|^{\frac{p}{m}})$ for all $x_1, x_2, ..., x_m \in X$, then by Theorem 4.2.1, we arrive $\Gamma(x) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} c_2(\prod_{i=1}^{m} |\frac{x}{2^{i+1}}|^{\frac{p}{m}}) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \frac{c_2}{2^{ip+p}} |x|^p = |x|^p \frac{c_2}{2^{p+1}} \sum_{i=0}^{\infty} \frac{1}{2^{(p+1)i}} = \frac{c_2}{2^{p+1}-1}$ then $|r(x) - f(x)| \leq \frac{2(m-1)c_2}{2^{p+1}-1} |x|^p$ for all $x \in X$ and $p > -1$ and using Theorem 4.2.2, we arrive $\Gamma(x) = \sum_{i=0}^{\infty} 2^i c_2(\prod_{i=1}^{m} |2^i x|^p) = \sum_{i=0}^{\infty} 2^i c_2 |x|^p 2^{ip} = c_2 |x|^p \sum_{i=0}^{\infty} 2^{(p+1)i} = \frac{c_2}{1-2^{p+1}} |x|^p$ then $|r(x) - f(x)| \leq \frac{2(m-1)c_2}{1-2^{p+1}} |x|^p$ for all $x \in X$ and $p < -1$.□

Corollary 4.2.5. [16] For any fixed $c_3 \geq 0$ and $\alpha > \frac{-1}{m}$ or $\alpha < \frac{-1}{m}$ be real numbers, if $f : X \longrightarrow \mathbb{R}$ satisfying the functional inequality

$$|D_m f(x_1, x_2, ..., x_m)| \leq c_3((\sum_{i=1}^{m} |x_i|^{m\alpha}) + \prod_{i=1}^{m} |x_i|^{\alpha}) \tag{4.40}$$

for all $x_1, x_2, ..., x_m \in X$, then there exists a unique reciprocal mapping $r : X \longrightarrow \mathbb{R}$ satisfying the functional equation (4.1) and

$$|r(x) - f(x)| \leq \begin{cases} \frac{2(m-1)(m+1)c_3}{2^{m\alpha+1}-1}|x|^{m\alpha}, & \text{for } \alpha > \frac{-1}{m} \\ \frac{2(m-1)(m+1)c_3}{1-2^{m\alpha+1}}|x|^{m\alpha}, & \text{for } \alpha < \frac{-1}{m} \end{cases}$$

for all $x \in X$.

Proof. Considering $\varphi(x_1, x_2, ..., x_m) = c_3((\sum_{i=1}^{m} |x_i|^{m\alpha}) + \prod_{i=1}^{m} |x_i|^{\alpha})$ for all $x_1, x_2, ..., x_m \in X$, then by Theorem 4.2.1, we arrive $\Gamma(x) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} c_3\{\sum_{i=1}^{m} |\frac{x}{2^{i+1}}|^{m\alpha}(\prod_{i=1}^{m} |\frac{x}{2^{i+1}}|^{\alpha})\} = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} c_3\{\frac{m|x|^{m\alpha}}{2^{im\alpha+m\alpha}} + \frac{|x|^{m\alpha}}{2^{im\alpha+m\alpha}}\} = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} c_3\{(m+1)\frac{|x|^{m\alpha}}{2^{im\alpha+m\alpha}}\} = (m+1)|x|^{m\alpha} c_3 \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \frac{1}{2^{im\alpha+m\alpha}} = \frac{(m+1)c_3}{2^{m\alpha+1+}}|x|^{m\alpha} \sum_{i=0}^{\infty} \frac{1}{2^{(m\alpha+1)i}} = \frac{(m+1)c_3}{2^{m\alpha+1}-1}|x|^{m\alpha}$ then $|r(x) - f(x)| \leq \frac{2(m-1)(m+1)c_3}{2^{m\alpha+1}-1}|x|^{m\alpha}$, for all $x \in X$ and $\alpha > -\frac{1}{m}$.

And using Theorem 4.2.2, we arrive $\Gamma(x) = c_3((\sum_{i=1}^{m} |x_i|^{m\alpha}) + \prod_{i=1}^{m} |x_i|^{\alpha}) = \sum_{i=0}^{\infty} c_3 2^i(\sum_{i=1}^{m} |2^i x|^{m\alpha}) + \prod_{i=1}^{m} |2^i x|^{\alpha}) = \sum_{i=0}^{\infty} c_3 2^i\{m|2^i x|^{m\alpha}) + |2^i x|^{m\alpha})\} = \sum_{i=0}^{\infty} c_3 2^i\{(m+1)|2^i x|^{m\alpha}\} = c_3(m+1)|x|^{m\alpha} \sum_{i=0}^{\infty} 2^{(m\alpha+1)i} = \frac{c_3(m+1)}{1-2^{m\alpha+1}}|x|^{m\alpha}$ then $|r(x) - f(x)| \leq \frac{2(m-1)(m+1)c_3}{1-2^{m\alpha+1}}|x|^{m\alpha}$, for all $x \in X$ and $\alpha < -\frac{1}{m}$. □

4.3 Counter Examples

The following example illustrates the fact that the functional equation (4.1) is not stable for $p = -1$ in Corollary 4.2.3.

Example 4.3.1. [16] Let $\varphi : X \longrightarrow \mathbb{R}$ be a mapping defined by

$$\varphi(x) = \begin{cases} \frac{\mu}{x}, & x \in (1, \infty) \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a mapping $f : X \longrightarrow \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} \frac{\varphi(2^{-n}x)}{2^n}$ for all $x \in X$. Then the mapping f satisfies the inequality

$$|D_m f(x_1, x_2, ..., x_m)| \leq \frac{6\mu}{m-1}(\sum_{i=1}^{m} |x|^{-1}) \tag{4.41}$$

for all $x_1, x_2, ..., x_m \in X$.

Therefore, there do not exist a reciprocal mapping $r : X \longrightarrow \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - r(x)| \leq \beta |x|^{-1} \tag{4.42}$$

for all $x \in X$

Proof. $|f(x)| = \sum_{n=0}^{\infty} \frac{|\varphi(2^{-n}x)|}{|2^n|} \leq \sum_{n=0}^{\infty} \frac{\mu}{2^n} = \mu(1-\frac{1}{2})^{-1} = 2\mu$. Hence f is bounded by 2μ. If

$$(\sum_{i=1}^{m} |x_i|^{-1}) \geq 1$$

then the left hand side of (4.41) is less than $\frac{6\mu}{m-1}$.

Now, suppose that $0 < (\sum_{i=1}^{m} |x_i|^{-1}) < 1$. Then there exists a positive integer k such that

$$\frac{1}{2^{k+1}} < (\sum_{i=1}^{m} |x_i|^{-1}) < \frac{1}{2^k} \tag{4.43}$$

Hence $\sum_{i=1}^{m} |x_i|^{-1} < \frac{1}{2^k}$ implies

$2^k \sum_{i=1}^{m} |x_i|^{-1} < 1$ or$\frac{x_i}{2^k} > 1 > \frac{1}{2}$ for $i = 1, 2, ..., m$

or $\frac{x_i}{2^{k-1}} > 2 > 1$ for $i = 1, 2, ..., m$ and consequently

$\frac{1}{2^{k-1}}(x_1), \frac{1}{2^{k-1}}(x_i), \frac{1}{2^{k-1}}(x_1 + x_i) > 1$ for $i = 2, 3, ..., m$. Therefore, for each value of $n = 0, 1, 2, ..., k-1$, we obtain

$\frac{1}{2^n}(x_1), \frac{1}{2^n}(x_i), \frac{1}{2^n}(x_1 + x_i) > 1$ for i=2,3,...,m.

And $D_m\varphi(2^{-n}x_1, 2^{-n}x_2, ..., 2^{-n}x_m) = 0$ for $n = 0, 1, 2, ..., k-1$. Using (4.43) and the definition of f, we obtain $|D_m f(x_1, x_2, ...x_m)|$

$$= |\frac{\prod_{i=2}^{m} f(x_1 + x_i)}{\sum_{i=2}^{m}[\prod_{j=2, j\neq i}^{m} f(x_1 + x_j)]} - \frac{\prod_{i=1}^{m} f(x_i)}{\sum_{i=2}^{m} f(x_1)[\prod_{j=2,j\neq i}^{m} f(x_j)] + (m-1)\prod_{i=2}^{m} f(x_i)}|$$

$$\leq \frac{\prod_{i=2}^{m} \sum_{n=k}^{\infty} \frac{\mu}{2^n}}{(m-1)\prod_{i=2}^{m-1}(\sum_{n=k}^{\infty} \frac{\mu}{2^n})} + \frac{\prod_{i=1}^{m} \sum_{n=k}^{\infty} \frac{\mu}{2^n}}{2(m-1)\prod_{i=2}^{m}(\sum_{n=k}^{\infty} \frac{\mu}{2^n})}$$

$$\leq \frac{3}{2(m-1)} \frac{\mu}{2^k}(1 - \frac{1}{2})^{-1}$$

$$\leq \frac{6\mu}{m-1}\frac{1}{2^{k+1}}$$

$$\leq \frac{6\mu}{m-1}(\sum_{i=1}^{m}|x_i|^{-1})$$

for all $x_1, x_2, ..., x_m \in X$.Therefore, the inequality (4.41) holds true. We claim that the reciprocal functional equation (4.1) is not stable for $p = 1$ in Corollary 4.2.3. Assume that there exists a reciprocal mapping $r : X \longrightarrow \mathbb{R}$ satisfying (4.42). Therefore, we have

$$|f(x)| \leq (\beta + 1)|x|^{-1} \tag{4.44}$$

However, we can choose a positive integer m with $m\mu > \beta + 1$. If $x \in (1, 2^{m-1})$ then $2^{-n} \in (1, \infty)$ for all $n = 0, 1, 2, ..., m-1$ and therefore $|f(x)| = \sum_{n=0}^{\infty} \frac{|\varphi(2^{-n}x)|}{|2^n|} \geq \sum_{n=0}^{m-1} \frac{\frac{2^n\mu}{x}}{2^n} = \frac{m\mu}{x} > (\beta + 1)|x|^{-1}$ which contradicts (2.51). Therefore, the reciprocal type functional equation (4.1) is not stable for $p = -1$ in Corollary 4.2.3.□

The following example illustrates the fact that the functional equation (4.1) is not stable for $\alpha = \frac{-1}{m}$ in Corollary 4.2.5

Example 4.3.2. [16] Let $\varphi : X \longrightarrow \mathbb{R}$ be a mapping defined by

$$\varphi(x) = \begin{cases} \frac{\delta}{x}, & x \in (1, \infty) \\ \delta, & \text{otherwise} \end{cases}$$

where $\delta > 0$ is a constant, and define a mapping $f : X \longrightarrow \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\varphi(2^{-n}x)}{2^n} \text{for all} x \in X. \tag{4.45}$$

Then the mapping f satisfies the inequality

$$|D_m f(x_1, x_2, ..., x_m)| \leq \frac{6\delta}{m-1}(\sum_{i=1}^{m}|x_i|^{-1} + \prod_{i=1}^{m}|x_i|^{\frac{-1}{m}}) \tag{4.46}$$

for all $x_1, x_2, ..., x_m \in X$.

Therefore, there do not exist a reciprocal mapping $r : X \longrightarrow \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - r(x)| \leq \beta|x|^{-1} \tag{4.47}$$

for all $x \in X$.

Proof. $|f(x)| = \sum_{n=0}^{\infty} \frac{|\varphi(2^{-n}x)|}{|2^n|} \leq \sum_{n=0}^{\infty} \frac{\delta}{2^n} = \delta(1-\frac{1}{2})^{-1} = 2\delta$. Hence f is bounded by 2δ. If

$$\{\sum_{i=1}^{m} |x_i|^{-1} + (\prod_{i=1}^{m} |x_i|^{-\frac{1}{m}})\} \geq 1,$$

then the left hand side of (4.46) is less than $\frac{6\delta}{m-1}$. Now, suppose that

$$0 < \{\sum_{i=1}^{m} |x_i|^{-1} + (\prod_{i=1}^{m} |x_i|^{-\frac{1}{m}})\} < 1,$$

Then there exists a positive integer k such that

$$\frac{1}{2^{k+1}} \leq \{\sum_{i=1}^{m} |x_i|^{-1} + (\prod_{i=1}^{m} |x_i|^{-\frac{1}{m}})\} < \frac{1}{2^k}. \tag{4.48}$$

Hence $\{\sum_{i=1}^{m} |x_i|^{-1} + (\prod_{i=1}^{m} |x_i|^{-\frac{1}{m}})\} < \frac{1}{2^k}$ implies

$$\{\sum_{i=1}^{m} |x_i|^{-1} + (\prod_{i=1}^{m} |x_i|^{-\frac{1}{m}})\} < 1$$

$$or 2^k x_i^{-1}$$

for $i = 1, 2, ..., m$

$$or \frac{x_i}{2^k} > 1$$

for $i = 1, 2, ..., m$

$$or \frac{x_i}{2^k} > 1 > \frac{1}{2}$$

for $i = 1, 2, ..., m$

$$or \frac{x_i}{2^{k-1}} > 2 > 1$$

for $i = 1, 2, ..., m$ and consequently

$$\frac{1}{2^{k-1}}(x_1), \frac{1}{2^{k-1}}(x_i), \frac{1}{2^{k-1}}(x_1 + x_i) > 1$$

for $i = 2, 3, ..., m$. Therefore, for each value of $n = 0, 1, 2, ..., k-1$ we obtain

$$\frac{1}{2^n}(x_1), \frac{1}{2^n}(x_i), \frac{1}{2^n}(x_1 + x_i) > 1$$

for $i = 2, 3, ..., m$ and $D_m\varphi(2^{-n}x_1, 2^{-n}x_2, ..., 2^{-n}x_m) = 0$ for $n = 0, 1, 2, ..., k-1,$. Using (4.48) and the definition of f, we obtain

$|D_m f(x_1, x_2, ..., x_m)|$

$$= |\frac{\prod_{i=2}^m f(x_1 + x_i)}{\sum_{i=2}^m [\prod_{j=2, j\neq i}^m f(x_1 + x_j)]} - \frac{\prod_{i=1}^m f(x_i)}{\sum_{i=2}^m [\prod_{j=2, j\neq i}^m f(x_j)] + (m-1)\prod_{i=2}^m f(x_i)}|$$

$$\leq \frac{\prod_{i=2}^m \sum_{n=k}^\infty \frac{\delta}{2^n}}{(m-1)\prod_{i=2}^{m-1}(\sum_{n=k}^\infty \frac{\delta}{2^n})} + \frac{\prod_{i=1}^m \sum_{n=k}^\infty \frac{\delta}{2^n}}{2(m-1)\prod_{i=2}^m(\sum_{n=k}^\infty \frac{\delta}{2^n})}$$

$$\leq \frac{3}{2(m-1)}\frac{\delta}{2^k}(1 - \frac{1}{2})^{-1}$$

$$\leq \frac{6\delta}{m-1}\frac{1}{2^{k+1}}$$

$$\leq \frac{6\delta}{m-1}\{\sum_{i=1}^m |x_i|^{-1} + (\prod_{i=1}^m |x_i|^{-\frac{1}{m}})\}$$

for all $x_1, x_2, ..., x_m \in X$. Therefore, the inequality (4.1) holds true. We claim that the reciprocal functional equation (4.1) is not stable for $\alpha = -\frac{1}{m}$ in Corollary 4.2.5. Assume that there exists a reciprocal mapping $r : X \longrightarrow \mathbb{R}$ satisfying (4.47). Therefore, we have

$$|f(x)| \leq (\beta + 1)|x|^{-1} \tag{4.49}$$

However, we can choose a positive integer m with $m\delta > \beta + 1$. If $x \in (1, 2^{m-1})$ then $2^{-n} \in (1, \infty)$ for all $n = 0, 1, 2, ..., m-1$ and therefore, $|f(x)| = \sum_{n=0}^\infty \frac{|\varphi(2^{-n}x)|}{|2^n|} \geq \sum_{n=0}^{m-1} \frac{\frac{2^n\delta}{x}}{2^n} = \frac{m\delta}{x} > (\beta + 1)|x|^{-\frac{1}{m}}$ which contradicts (4.49). Therefore, the reciprocal type functional equation (4.1) is not stable for $\alpha = -\frac{1}{m}$ in Corollary 4.2.5.□

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