إقــــــرار

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On Fourth Painlevé Hierarchie

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On Fourth Painlevé Hierarchies

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بناءً على موافقة شئون البحث العلمي والدراسات العليا بالجامعة الإسلامية بغزة على تشكيل لجنة الحكم على أطروحة الباحثة/ سجود عبد الهادى دياب الشيخ خليل لنيل درجة الماجستير في كلية العلوم قسم الرياضيات وموضوعها:

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On Fourth Painlevé Hierarchies

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واللجنة إذ تمنحها هذه الدرجة فإنها توصيها بتقوى الله ولزوم طاعته وأن تسخر علمها في خدمة دينها ووطنها.

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Abstract

In this thesis we will study a generalized fourth and second (P_{IV}-P_{II}) Painlevé hierarchy. Then we will obtain a fourth Painlevé hierarchy as special case of this generalized P_{IV}-P_{II} hierarchy. we will also find the first and second member for this fourth Painlevé hierarchy and some special solutions of this hierarchy will be considered.

And we will derive a second fourth Painlevé hierarchy by using scalar isomonodromy problem. The first and second member for this fourth Painlevé hierarchy and special solutions for the second member of this hierarchy will be considered.

Finally, we will derive a third fourth Painlevé hierarchy, we will also find the first and second member for this fourth Painlevé hierarchy and some special solutions of this hierarchy will be considered.

Contents

Chapter 1

Introduction

One century ago Painlevé and his school discovered six ordinary differential equations that define new transcendental functions with respect to constants of integrations. This was achieved by classifying second-order ordinary differential equations of a certian form having what is today referred to as the Painlevé property. A differential equation is said to have the Painlevé property if its solutions have no movable branch points ; that is, the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen and so are dependent only on the equation. Painlevé and his collaborators found fifty canonical classes of equations whose solutions have no movable critical points. Furthermore, they also showed that among these fifty equations there are exactly six differential equations that define new functions. These six equations are called the six Painlevé equations and denoted by P_I-P_{VI}.

The six Painlevé equations, P_I-P_{VI}, in the form $u'' = F(z, u, u')$, are listed below [1]

$$
\begin{aligned}
P_I \quad &: \; u'' = 6u^2 + z, \\
P_{II} \quad &: \; u'' = 2u^3 + zu + \alpha, \\
P_{III} \quad &: \; u'' = \frac{(u')^2}{u} - \frac{1}{z}u' + \alpha u^3 + \frac{1}{z}(\beta u^2 + \gamma) + \frac{\delta}{u}, \\
P_{IV} \quad &: \; u'' = \frac{(u')^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + (2z^2 - 2\alpha)u + \frac{\beta}{u}, \\
P_V \quad &: \; u'' = \frac{3u-1}{2u(u-1)}(u')^2 - \frac{1}{z}u' + \frac{1}{z^2}(u-1)^2(\alpha u + \frac{\delta}{u}) + \frac{\gamma u}{z} + \frac{\delta u(u+1)}{u-1}, \\
P_{VI} \quad &: \; u'' = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-z}\right)(u')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{u-z}\right)u' \\
&\qquad + \frac{u(u-1)(u-z)}{z^2(z-1)^2}\left(\frac{\alpha z(z-1)}{(u-z)^2} + \frac{\beta(z-1)}{(u-1)^2} + \frac{\gamma z}{u^2} + \delta\right),
\end{aligned}
$$

where α, β, γ and δ are constant parameters and primes denote differentiation with respect to z.

These six Painlevé equations, P_I-P_{VI}, have mathematical and physical significance.

Their mathematical importance originates from:

1. P_{II}-P_{VI} possess rational solutions and solutions expressible in terms of special functions for certain values of the parameters.

2. Painlevé equations arise as reductions of solutions of nonlinear partial differential equations (PDE's) solvable by the inverse scattering transformations.

3. P_I-P_V can be obtained from P_{VI} by the process of contraction and it is possible to derive transformations for P_{II}-P_{IV} from P_V.

4. The initial value problems of P_I-P_{VI} can be studied using the inverse monodromy transformation, which is an expansion of inverse spectral method to ordinary differential equations.

5. Painlevé equations appear as the compatibility conditions of linear system of equations (Lax-pairs, consists of two matrices functions on the phase space of the system, such the Hamiltonian system (2.2) may be written as
$$\frac{dL}{dt} \equiv \dot{L} = [L, M],$$
where $[M, L] = ML - LM$ denotes the commutator of matrices M, L). possessing regular and irregular singular points. By using Lax-pairs, one can find the general solution of a given Painlevé equation as the Fredholm integral equations.

6. Painlevé equations, P_I-P_{VI}, may be obtained from the discrete ones by a suitable limiting.

7. P_I-P_{VI} can be written as (non-autonomous) Hamiltonian system which is a function on phase space, denoted by $\mathcal{H}(t, l, s)$.
$$\frac{ds}{dt} = \frac{\partial \mathcal{H}}{\partial l} \quad , \quad \frac{dl}{dt} = -\frac{\partial \mathcal{H}}{\partial s}.$$

8. One of the important properties of the Painlevé equations is the existence of Schlesinger transformations, which transform the solutions of the associated linear system but preserve the monodromy data.

9. P_{II}-P_{VI} have gained more importance since they possess Bäcklund transformations, which map the solutions of a given Painlevé equation to the solutions of the same Painlevé equation, but with different values of the parameters.

Besides the mathematical significance of the Painlevé equations, they have arisen in a variety of important physical applications including statistical mechanics (correlation functions of the XY model and the Ising model), random matrix theory, topological field theory, plasma physics, nonlinear waves (resonant oscillations in shallow water, convective flows with viscous dissipation, Görtler vortices in boundary layers and Hele-shaw problems), quantum gravity, quantum field theory, general relativity, nonlinear and fiber optics, polyelectrolytes, Bose-Einstein condensation and stimulated Raman scattering

A Painlevé hierarchy is an infinite sequence of nonlinear ordinary differential equations whose first member is a Painlevé equation. The first higher order integrable system of Painlevé type was published by Garnier and is now known as the Garnier system. A first Painlevé hierarchy was given by Kudryashov [2] and Airault was the first to derive a second Painlevé hierarchy [3]. After that, Gordoa, Joshi and Pickering have used non-isospectral scattering problems to derive new second Painlevé hierarchies and new fourth Painlevé hierarchies [4].

Garnier found that five of the six Painlevé equations can be presented as isomonodromic linear problems in the form

$$\psi_{xx} = U\psi, \quad w(\lambda)\psi_\lambda = 2A\psi_x - A_x\psi, \tag{1.1}$$

where $A(x,\lambda) = \sum_{k=0}^{n} a_k(x)\lambda^{n-k}$ and $U(x,\lambda)$ has one of the following two forms $U(x,\lambda) = p(x) - \lambda$ or $U(x,\lambda) = p(x) - y(x)\lambda + \lambda^2$.
The compatibility condition of the linear system (1.1), $(\psi_{xx})_\lambda = (\psi_\lambda)_{xx}$, implies

$$wU_\lambda = 4UA_x + 2U_xA - A_{xxx}.$$

In addition to the scalar linear system (1.1) each Painlevé equation can be written as a compatibility condition of a matrix linear system

$$\phi_\lambda(x,\lambda) = A(x,\lambda)\phi(x,\lambda), \quad \phi_x(x,\lambda) = B(x,\lambda)\phi(x,\lambda), \tag{1.2}$$

where

$$A(x,\lambda) = \sum_{j=0}^{N+n} A_j\lambda^{N-j}, \quad B(x,\lambda) = \sum_{j=0}^{L+l} B_j\lambda^{L-j}, \tag{1.3}$$

and A_j and B_j are matrices with entries depending on the solution $y(x)$ of the Painlevé equation. The compatibility condition of the linear system (1.2), $\phi_{\lambda x} = \phi_{x\lambda}$, implies

$$A_x - B_\lambda + AB - BA = 0. \tag{1.4}$$

In this thesis we are interested in fourth Painlevé hierarchies. Fourth Painlevé equation is one of the six classical Painlevé equations which are regarded as completely integrable because they can be solved through an associated system of linear equation.

This thesis is organized as follows.
In chapter two we will study a generalized fourth and second (P_{IV}-P_{II}) Painlevé

hierarchy. Then we will obtain a fourth Painlevé hierarchy as special case of this generalized P_{IV}-P_{II} hierarchy, we will also find the first and second member for this fourth Painlevé hierarchy and some special solutions of this hierarchy will be considered.

In chapter three we will derive a second fourth Painlevé hierarchy by using scalar isomonodromy problem, and we will study the first and second member for this fourth Painlevé hierarchy and special solutions for the second member of this hierarchy will be considered.

In chapter four we will use matrix isomonodromy problem to derive a third fourth Painlevé hierarchy, we will also find the first and second member for this fourth Painlevé hierarchy and transformation between the second member for hierarchy in this chapter and the second member of the hierarchy in chapter two, and some special solutions of this hierarchy will be considered.

Chapter 2

The Fourth Painlevé Hierarchy P_{IV-1}

In this chapter we will study a generalized fourth and second (P_{IV}-P_{II}) hierarchy, and we will obtain a fourth Painlevé hierarchy as special case of this generalized P_{IV}-P_{II} hierarchy. We will study special solutions for this fourth Painlvé hierarchy.

2.1 Generalized P_{IV}-P_{II} Hierarchy

Consider the hierarchy of the partial differential equations [4]

$$\mathbf{u}_{t_n} = \mathbf{M}_n = R^n \mathbf{u}_y + \sum_{i=0}^{n+1} R^{n+1-i} \mathbf{G}_i, \tag{2.1}$$

where $\mathbf{u} = (u, v)^T$, $\mathbf{M}_n = (M_n, N_n)^T$, $R = \frac{1}{2} \begin{pmatrix} \partial_x u \partial_x^{-1} - \partial_x & 2 \\ 2v + v_x \partial_x^{-1} & u + \partial_x \end{pmatrix}$, $G_i = (g_i, 0)^T$. Assume $\partial_{t_n} = 0$ and $\partial_y = \partial_x$. Then hierarchy (2.1) reduces to the following hierarchy of ordinary differential equations

$$R^n u_x + \sum_{i=0}^{n+1} R^{n+1-i} G_i = (0, 0)^T, \tag{2.2}$$

where $G_i = (g_i, 0)^T$ with all g_i are now constant parameters. The operator R can be written as

$$R = B_2 B_1^{-1}, \tag{2.3}$$

where B_1 and B_2 are Hamiltonian operators given by

$$B_2 = \frac{1}{2}\begin{pmatrix} 2\partial_x & \partial_x u - \partial_x^2 \\ u\partial_x + \partial_x^2 & v\partial_x + \partial_x v \end{pmatrix}, B_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}. \tag{2.4}$$

We can use non-isospectral scattering problems to obtain linear problems for the hierarchy (2.2). In this way we obtain the linear problem

$$\psi_{xx} = [(\lambda - \frac{1}{2}u)^2 + \frac{1}{2}u_x - v]\psi, \tag{2.5}$$

$$(\frac{1}{2}\sum_{i=0}^{n+1}\lambda^{n+1-i}g_i)\psi_\lambda = (\lambda^n + \frac{1}{2}P_n)\psi_x - \frac{1}{4}P_{n,x}\psi, \tag{2.6}$$

where we assume that not all g_i are zero. Here P_n is given by

$$P_n = \partial_x^{-1}\sum_{i=-1}^{n-1}\lambda^{n-1-i}M_i, \tag{2.7}$$

where

$$\mathbf{M}_i = R^i\mathbf{u}_x + \sum_{j=0}^{i+1}R^{i+1-j}\mathbf{G}_j. \tag{2.8}$$

Calculating M_0 we find

$$\begin{aligned} \mathbf{M}_0 &= \mathbf{u}_x + \sum_{j=0}^{1}R^{1-j}G_j, \\ &= \begin{pmatrix} u_x \\ v_x \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} \partial_x u\partial_x^{-1} - \partial_x & 2 \\ 2v + v_x\partial_x^{-1} & u + \partial_x \end{pmatrix}\begin{pmatrix} g_0 \\ 0 \end{pmatrix} + \begin{pmatrix} g_1 \\ 0 \end{pmatrix}. \end{aligned} \tag{2.9}$$

Thus,

$$\mathbf{M}_0 = \begin{pmatrix} M_0 \\ N_0 \end{pmatrix} = \begin{pmatrix} u_x + \frac{1}{2}g_0(xu)_x + g_1 \\ v_x + \frac{1}{2}g_0(2v + xv_x) \end{pmatrix}. \tag{2.10}$$

Equation (2.8) gives

$$\mathbf{M}_{-1} \quad = R^0G_0 = \begin{pmatrix} g_0 \\ 0 \end{pmatrix}. \tag{2.11}$$

A matrix linear problem of the hierarchy (2.2) is given by

$$\psi_x = F\psi, \tag{2.12}$$

$$\left(\frac{1}{2}\sum_{i=0}^{n+1}\lambda^{n+1-i}g_i\right)\psi_\lambda = H\psi = (\lambda^n F + G)\psi, \qquad (2.13)$$

where the matrices F and G are given by

$$F = \begin{pmatrix} -\frac{1}{2}(2\lambda - u) & 1 \\ -v & \frac{1}{2}(2\lambda - u) \end{pmatrix},$$

$$G = \begin{pmatrix} -\frac{1}{4}((2\lambda - u)P_n + P_{n,x}) & \frac{1}{2}P_n \\ \frac{1}{2}\sum_{i=0}^{n+1}\lambda^{n+1-i}g_i + \frac{1}{2}\lambda^n u_x - \frac{1}{2}M_n & \\ -\frac{1}{4}((2\lambda - u)P_n + P_{n,x})_x - \frac{1}{2}vP_n & \frac{1}{4}((2\lambda - u)P_n + P_{n,x}) \end{pmatrix},$$

and where P_n and M_n are defined by (2.7) and (2.8) respectively.

In the case $g_i = 0$, $i = 0, 1, 2, \ldots, n-2$, the hierarchy (2.2) reads

$$R^n \mathbf{u}_x + R^2 G_{n-1} + RG_n + G_{n+1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (2.14)$$

Since $R = \frac{1}{2}\begin{pmatrix} \partial_x u \partial_x^{-1} - \partial_x & 2 \\ 2v + v_x \partial_x^{-1} & u + \partial_x \end{pmatrix}$, $G_i = g_i(1,0)^T$ with g_i are constants, we have $R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} xu_x + u \\ 2v + xv_x \end{pmatrix}$, and

$$R^2\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \partial_x(u^2 x - xu_x - u) + 4v + 2xv_x & 4u + 2xu_x \\ 2xvu_x + 2xuv_x + 4uv + 3v_x + xv_{xx} & 4v + 2xv_x u^2 + 2u_x \end{pmatrix}. \qquad (2.15)$$

As a result, (2.14) can be written as

$$R^n \mathbf{u}_x + \frac{1}{4}g_{n-1}\begin{pmatrix} (u^2 x - xu_x - u)_x + 4v + 2xv_x \\ 2xvu_x + 2xuv_x + 4uv + 3v_x + xv_{xx} \end{pmatrix} + \frac{1}{2}g_n\begin{pmatrix} xu_x + u \\ 2v + xv_x \end{pmatrix} + g_{n+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (2.16)$$

The hierarchy (2.16) is called the generalized $P_{IV} - P_{II}$ hierarchy [4].

Since a second and fourth Painlevé hierarchies can be obtained as special cases of this hierarchy (2.16). In this special case we obtain our P_{IV} and P_{II} hierarchies, according as to whether $g_n \neq 0$ or $g_n = 0$ respectively.

2.2 The Hierarchy P_{IV}

Assume that $g_{n-1} = 0$ and $g_n \neq 0$. We will show that the hierarchy (2.16) has a first integral

$$\begin{pmatrix} I_n \\ J_n \end{pmatrix} = \begin{pmatrix} L_{n,x} - 2K_n - (u + 2\frac{g_{n+1}}{g_n})L_n \\ L_nK_{n,x} + vL_n^2 + K_n^2 - L_{n,x}K_n + (u + 2\frac{g_{n+1}}{g_n})L_nK_n \end{pmatrix}, \quad (2.17)$$

where

$$\begin{pmatrix} K_n \\ L_n \end{pmatrix} = B_1^{-1}\left[R^{n-1}\mathbf{u}_x + g_n\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{i=0}^{n-2}\left(\frac{-g_{n+1}}{g_n}\right)^{n-i-1} R^i\mathbf{u}_x\right] + 2\left(\frac{-g_{n+1}}{g_n}\right)^n\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (2.18)$$

Let K_n, L_n be given by (2.18).

Then $B_1\begin{pmatrix} K_n \\ L_n \end{pmatrix} = R^{n-1}\mathbf{u}_x + g_n\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{i=0}^{n-2}\left(\frac{-g_{n+1}}{g_n}\right)^{n-i-1} R^i\mathbf{u}_x$

$$+ 2\left(\frac{-g_{n+1}}{g_n}\right)^n B_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ and } B_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

applying B_1 to both sides of (2.18) and multiplying by $\frac{g_{n+1}}{g_n}$, we get

$$\frac{g_{n+1}}{g_n}B_1\begin{pmatrix} K_n \\ L_n \end{pmatrix} = \frac{g_{n+1}}{g_n}R^{n-1}\mathbf{u}_x + g_{n+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{i=0}^{n-2}\left(\frac{-g_{n+1}}{g_n}\right)^{n-i} R^i\mathbf{u}_x.$$

Re-indexing the last sum we get

$$\frac{g_{n+1}}{g_n}B_1\begin{pmatrix} K_n \\ L_n \end{pmatrix} = g_{n+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{i=-1}^{n-2}(-1)^{n-i}\left(\frac{g_{n+1}}{g_n}\right)^{n-i-1} R^{i+1}\mathbf{u}_x. \quad (2.19)$$

Using (2.18) and $R = B_2B_1^{-1}$, we have

$$B_2\begin{pmatrix} K_n \\ L_n \end{pmatrix} = R^n\mathbf{u}_x + g_nR\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{i=0}^{n-2}\left(\frac{-g_{n+1}}{g_n}\right)^{n-i-1} R^{i+1}\mathbf{u}_x + 2\left(\frac{-g_{n+1}}{g_n}\right)^n B_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $B_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u_x \\ v_x \end{pmatrix}$, the last equation gives

$$\begin{aligned} B_2\begin{pmatrix} K_n \\ L_n \end{pmatrix} &= R^n\mathbf{u}_x + g_nR\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{i=0}^{n-2}\left(\frac{-g_{n+1}}{g_n}\right)^{n-i-1} R^{i+1}\mathbf{u}_x + \left(\frac{-g_{n+1}}{g_n}\right)^n\begin{pmatrix} u_x \\ v_x \end{pmatrix} \\ &= R^n\mathbf{u}_x + g_nR\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{i=-1}^{n-2}(-1)^{n-i}(\frac{-g_{n+1}}{g_n})^{n-i-1}R^{i+1}\mathbf{u}_x. \end{aligned} \quad (2.20)$$

Now equations (2.19) and (2.20) give

$$\left[B_2 + \tfrac{g_{n+1}}{g_n} B_1\right] \begin{pmatrix} K_n \\ L_n \end{pmatrix} = \left[R^n \mathbf{u}_x + g_n R \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g_{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right].$$

Since $g_{n-1} = 0$, (2.16) yields

$$R^n \mathbf{u}_x = \frac{-1}{2} g_n \begin{pmatrix} (xu)_x \\ 2v + xv_x \end{pmatrix} - g_{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and hence

$$\begin{aligned} &\begin{pmatrix} -2 & 0 \\ 2K_n & 2L_n \end{pmatrix} \left[B_2 + \tfrac{g_{n+1}}{g_n} B_1\right] \begin{pmatrix} K_n \\ L_n \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 \\ 2K_n & 2L_n \end{pmatrix} \left[R^n \mathbf{u}_x + g_n R \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g_{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \partial_x \begin{pmatrix} I_n \\ J_n \end{pmatrix}. \end{aligned} \tag{2.21}$$

This shows that (2.17) is a first integral of (2.16).

Define two constants of integration α_n and β_n by

$$\begin{pmatrix} I_n \\ J_n \end{pmatrix} = \begin{pmatrix} g_n - 2\alpha_n \\ (\frac{1}{2} g_n - \alpha_n)^2 - \frac{1}{4}\beta_n^2 \end{pmatrix}. \tag{2.22}$$

From equation (2.17), we have $g_n - 2\alpha_n = L_{n,x} - 2K_n - (u + 2\frac{g_{n+1}}{g_n})L_n$, and hence

$$L_{n,x} = 2K_n + (u + \frac{2g_{n+1}}{g_n})L_n + g_n - 2\alpha_n. \tag{2.23}$$

Similarly $(\frac{1}{2}g_n - \alpha_n)^2 - \frac{1}{4}\beta_n^2 = L_n K_{n,x} + vL_n^2 + K_n^2 - L_{n,x}K_n + (u + \frac{2g_{n+1}}{g_n})L_n K_n$, or

$$K_{n,x} = \frac{(K_n + \frac{1}{2}g_n - \alpha_n)^2 - \frac{1}{4}\beta_n^2}{L_n} - vL_n. \tag{2.24}$$

We will show that in the case $n = 1$, the system of equations (2.23), (2.24) is just the fourth Painlevé equation P_{IV}. Thus, the hierarchy (2.23), (2.24) is a fourth Painlevé hierarchy, and we will denote it by P_{IV-1}.

It should be noted that we can obtain a second Painlevé hierarchy from (2.16).

2.3 Examples

2.3.1 The first member of the P_{IV-1} hierarchy

When $n = 1$, equation (2.16) gives

$$R\mathbf{u}_x + g_0 R^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g_1 R \begin{pmatrix} 1 \\ 0 \end{pmatrix} + g_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.25}$$

We note that

$$\begin{aligned} R\mathbf{u}_x &= \tfrac{1}{2} \begin{pmatrix} \partial_x u \partial_x^{-1} - \partial_x & 2 \\ 2v + v_x \partial_x^{-1} & u + \partial_x \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} \\ &= \tfrac{1}{2} \begin{pmatrix} (u^2 - u_x + 2v)_x \\ (2uv + v_x)_x \end{pmatrix}. \end{aligned} \tag{2.26}$$

Substituting $R\mathbf{u}_x$ from (2.26) into (2.25) we get

$$\frac{1}{2}(u^2 - u_x + 2v)_x + \frac{1}{4} g_0((u^2 x - x u_x - u)_x + 4v + 2x v_x) + \frac{1}{2} g_1 (xu)_x + g_2 = 0, \tag{2.27}$$

$$\frac{1}{2}(2uv + v_x)_x + \frac{1}{4} g_0 (2xv u_x + 2xu v_x + 4uv + 3v_x + x v_{xx}) + \frac{1}{2} g_1 (2v + x v_x) = 0. \tag{2.28}$$

These two equations are indeed the first member of generalized $P_{IV} - P_{II}$ hierarchy.

In the case $g_0 = 0, g_1 \neq 0$ we obtain first integral as in section 1.2

$$\begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = B_1^{-1} \left[\mathbf{u}_x + g_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] - \frac{2g_2}{g_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

applying B_1 to both sides of the last equation, we get

$$B_1 \begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = \begin{pmatrix} u_x \\ v_x \end{pmatrix} + g_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{2g_2}{g_1} B_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

now since $\begin{pmatrix} L_{1,x} \\ K_{1,x} \end{pmatrix} = B_1 \begin{pmatrix} K_1 \\ L_1 \end{pmatrix}$, then

$$\begin{pmatrix} L_{1,x} \\ K_{1,x} \end{pmatrix} = \begin{pmatrix} u_x \\ v_x \end{pmatrix} + g_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.29}$$

From the last equatin we have

$L_{1,x} = u_x + g_1$. Integrate it with respect to x we have

$L_1 = u + g_1 x + c_1$,

where c_1 is a constant of integration. We will take $c_1 = -2\frac{g_2}{g_1}$.
Equation (2.29) gives $K_{1,x} = v_x$. Integrate it with respect to x we have

$$K_1 = v + c_2,$$

where c_2 is a constant of integration. We will take $c_2 = 0$.
We obtain the first integral as in Section 1.2 in equation (2.22) with K_1 and L_1 given by :

$$\begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = \begin{pmatrix} v \\ u + g_1 x - 2\frac{g_2}{g_1} \end{pmatrix}. \tag{2.30}$$

Corresponding to (2.23), (2.24) we thus obtain:

$$u_x = 2v + u^2 + g_1 x u + 2g_2 x - 2\alpha_1 - 4\left(\frac{g_2}{g_1}\right)^2, \tag{2.31}$$

and

$$v_x = \frac{(v - \alpha_1 + \frac{1}{2}g_1)^2 - \frac{1}{4}\beta_1^2}{u + g_1 x - 2\frac{g_2}{g_1}} - v\left(u + g_1 x - 2\frac{g_2}{g_1}\right), \tag{2.32}$$

where α_1 and β_1 are two independent constants of integration. Solving (2.31) for v, we obtain

$$v = \frac{1}{2}\left[u_x - u^2 - g_1 x u - 2g_2 x + 2\alpha_1 + 4\left(\frac{g_2}{g_1}\right)^2\right]. \tag{2.33}$$

Substituting v into the (2.32) then yields a scalar second-order ODE in u:

$$\begin{aligned} u_{xx} = & \frac{(u_x - u^2 - g_1 x u - 2g_2 x + \alpha_1 + 4(\frac{g_2}{g_1})^2 + \frac{1}{2}g_1)^2 - \frac{1}{4}\beta^2}{u + g_1 x - \dfrac{2g_2}{g_1}} \\ & - 2\left(\frac{1}{2}u_x - u^2 - g_1 x u - 2g_2 x + \alpha_1 + 4\left(\frac{g_2}{g_1}\right)^2\right)\left(u + g_1 x - \frac{2g_2}{g_1}\right) \\ & + 2(2uu_x + g_1 x u_x + g_1 u + 2g_2). \end{aligned} \tag{2.34}$$

Let

$$u(x) = y(z) - g_1 z - 2\frac{g_2}{g_1}, \tag{2.35}$$

$$x = z + 4\frac{g_2}{g_1^2}. \tag{2.36}$$

Then $u_x = y_z - g_1$ and using (2.33), we find

$v = \frac{1}{2}[y_z - g_1 - y^2 - g_1 z y + 2\alpha_1]$
or $v - \alpha_1 + \frac{1}{2}g_1 = \frac{1}{2}[y_z - y^2 + g_1 z y]$
Since $u = y - g_1 z - 2\frac{g_2}{g_1}$, and $x = z + 4\frac{g_2}{g_1^2}$, we have
$u + g_1 x - 2\frac{g_2}{g_1} = u + g_1 z + 4\frac{g_2}{g_1} - 2\frac{g_2}{g_1} = u + g_1 z + 2\frac{g_2}{g_1} = y.$
Substituting these expression into (2.32) yields
$\frac{1}{2}[y_{zz} - 2yy_z + g_1 y] = \frac{1}{4y}[(y_z - y^2 - g_1 z y)^2 - \frac{1}{4}\beta_1^2] - \frac{1}{2}y[y_z - y^2 + g_1 z y - g_1 + 2\alpha_1],$
As a result we obtain the equation

$$y_{zz} = \frac{1}{2y}y_z^2 + \frac{3}{2}y^3 - 2g_1 z y^2 + (\frac{1}{2}g_1^2 z^2 - 2\alpha_1)y - \frac{\beta_1^2}{2y}. \tag{2.37}$$

Setting $g_1 = -2$, which can be done without loss of generality for $g_1 \neq 0$, gives

$$y_{zz} = \frac{1}{2y}y_z^2 + \frac{3}{2}y^3 + 4zy^2 + 2(z^2 - \alpha_1)y - \frac{\beta_1^2}{2y}, \tag{2.38}$$

which is just the fourth Painlevé equation P_{IV}. Thus, we see that (2.22), or equivalently (2.23), (2.24), is indeed a P_{IV} hierarchy. Corresponding to (2.12), (2.13) we obtain for (2.38) the linear problem

$$F = \begin{pmatrix} \frac{1}{2}(y - zg_1 - 2\mu) & 1 \\ \frac{1}{2}(y^2 - y_z - zg_1 y + g_1 - 2\alpha_1) & -\frac{1}{2}(y - zg_1 - 2\mu) \end{pmatrix} \tag{2.39}$$

$$H = \begin{pmatrix} \frac{1}{4}(y^2 - zg_1 y - 2z\mu g_1 - 4\mu^2) & \frac{1}{2}(2\mu + y) \\ \frac{1}{8}(2yy_z - \frac{y_z^2}{y} - y^3 + 4\mu(y^2 - y_z) & \\ -z^2 g_1^2 y + 2g_1(zy^2 - zy_z + 2\mu & \\ -2z\mu y) + \frac{\beta_1^2}{y} - 8\alpha_1\mu) & -\frac{1}{4}(y^2 - y_z - zg_1 y - 2z\mu g_1 - 4\mu^2) \end{pmatrix} \tag{2.40}$$

where we set $\lambda = \mu - (\frac{g_2}{g_1})$.

2.3.2 The second member of P_{IV-1} hierarchy

In the case $g_1 = 0$ and $g_2 \neq 0$, from (2.18) we have
$$\begin{pmatrix} K_2 \\ L_2 \end{pmatrix} = B_1^{-1}\left(R\mathbf{u}_x + g_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{-g_3}{g_2}\right)\mathbf{u}_x\right) + 2\left(\frac{-g_3}{g_2}\right)^2\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
Applying B_1 to both side in the last equation, we get
$$B_1\begin{pmatrix} K_2 \\ L_2 \end{pmatrix} = R\mathbf{u}_x + g_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{-g_3}{g_2}\right)\mathbf{u}_x + 2B_1\left(\frac{-g_3}{g_2}\right)^2\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} L_{2,x} \\ K_{2,x} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (u^2-u_x+2v)_x \\ (2uv+v_x)_x \end{pmatrix} + g_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{-g_3}{g_2}\right)\mathbf{u}_x + 2\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \left(\frac{-g_3}{g_2}\right)^2 \end{pmatrix},$$

From this, we have

$L_{2,x} = \frac{1}{2}(u^2-u_x+2v)_x + g_2 + \left(\frac{-g_3}{g_2}\right)u_x + \left(2\left(\frac{-g_3}{g_2}\right)^2\right)_x,$

Integrate it with respect to x,

$$L_2 = v + \frac{1}{2}u^2 - \frac{1}{2}u_x + g_2x - \frac{g_3}{g_2}u + 2\left(\frac{-g_3}{g_2}\right)^2 + 2c_1, \tag{2.41}$$

where c_1 is the constant of integration. In the same way,

$K_{2,x} = \frac{1}{2}(2uv+v_x)_x - \frac{g_3}{g_2}v_x,$

Integrate it with respect to x,

$$K_2 = uv + \frac{1}{2}v_x - \frac{g_3}{g_2}v + c_2. \tag{2.42}$$

We will take $c_2 = 0$.

Corresponding to (2.23), (2.24) we thus obtain

$\frac{1}{2}(u^2-u_x+2v)_x+g_2+\left(\frac{-g_3}{g_2}\right)u_x+(2(\frac{-g_3}{g_2})^2)_x = 2K_2+(u+\frac{2g_3}{g_2})L_2+(g_2-2\alpha_2),$

By substituting the values of L_2 and K_2 from (2.41),(2.42) we have,

$$u_{xx} = 3uu_x - u^3 - 6uv - 2g_2xu - 4g_3x + 4\alpha_2 + 8\left(\frac{g_3}{g_2}\right)^3 - 4c(u+\frac{2g_3}{g_2}), \tag{2.43}$$

and

$$\frac{1}{2}(2uv+v_x)_x - \frac{g_3}{g_2}v_x = \frac{(uv+\frac{1}{2}v_x-(\frac{g_3}{g_2}v)-\frac{1}{2}g_2-\alpha_2)^2-\frac{1}{4}\beta_2^2}{(v+\frac{1}{2}u^2-\frac{1}{2}u_x+g_2x-(\frac{g_3}{g_2})u+2(\frac{g_3}{g_2})^2+2c)} - v(v+\frac{1}{2}u^2-\frac{1}{2}u_x+g_2x-\frac{g_3}{g_2}u+2(\frac{-g_3}{g_2})^2+2c),$$

$$v_{xx} = 2(\frac{(uv+\frac{1}{2}v_x-(\frac{g_3}{g_2}v)-\frac{1}{2}g_2-\alpha_2)^2-\frac{1}{4}\beta_2^2}{(v+\frac{1}{2}u^2-\frac{1}{2}u_x+g_2x-(\frac{g_3}{g_2})u+2(\frac{g_3}{g_2})^2+2c)} - v(v+\frac{1}{2}u^2-\frac{1}{2}u_x+g_2x - \frac{g_3}{g_2}u+2(\frac{-g_3}{g_2})^2+2c)) - 2(uv)_x + 2(\frac{g_3}{g_2})v_x, \tag{2.44}$$

where α_2 and β_2 are two independent constants of integration. Solving (2.43) for v, we obtain

$$v = \frac{-1}{6u}\left[u_{xx} - 3uu_x + u^3 + 2g_2xu + 4g_3x - 4\alpha_2 + 8\left(\frac{g_3}{g_2}\right)^3 + 4c\left(u+\frac{2g_3}{g_2}\right)\right]. \tag{2.45}$$

Now substituting into (2.44) then yields a scalar fourth order ODE for u. The system (2.43), (2.44), is the second member of the P_{IV-1} hierarchy (2.2). Corresponding to which we have a matrix problem (2.12), (2.13) with

$$F = \begin{pmatrix} -\lambda + \frac{1}{2}u & 1 \\ 2c + g_2 x + 2(\frac{g_3}{g_2})^2 - (\frac{g_3}{g_2})u & \\ +\frac{1}{2}(u^2 - u') + \frac{w}{6u} & \lambda - \frac{1}{2}u \end{pmatrix}, \tag{2.46}$$

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & -H_{11} \end{pmatrix}, \tag{2.47}$$

where

$$\begin{aligned} H_{11} = \left(\tfrac{1}{2(24g_2^2u^2)}\right) & [24\lambda g_3^2 u^2 - 24\lambda^3 g_2^2 u^2 - 12 g_3^2 u^3 \\ & + 6\lambda g_2^2 u^4 + 6 g_2 g_3 u^4 + 2\lambda g_2^2 u w - g_2^2 u^2 u_x \\ & - 6 g_2 g_3 u^2 u_x - g_2^2 w u_x + g_2^2 u w_x], \end{aligned} \tag{2.48}$$

$$H_{12} = \left(\tfrac{1}{(12g_2^2u)}\right)[12\lambda^2 g_2^2 u - 12 g_3^2 u + 6\lambda g_2^2 u^2 + 6 g_2 g_3 u^2 - g_2^2 w], \tag{2.49}$$

$$\begin{aligned} H_{21} = \left(\tfrac{1}{(144u^2)}\right) & [16\gamma_2^2 - 32\gamma_2 g_3^2 x^2 - 32c\gamma_2 u - 96\lambda^2\gamma_2 u \\ & + 48\lambda g_3 u + 32c g_3 x u + 96\lambda^2 g_3 x u + 16c^2u^2 - 144\delta_2 u^2 \\ & - 48\lambda\gamma_2 u^2 + 96\lambda^2 c u^2 - 24 g_3 u^2 + 48\lambda g_3 x u^2 + 40\gamma_2 u^3 + 48\lambda c u^3 \\ & - 40 g_3 x u^3 - 40 c u^4 + 24\lambda^2 u^4 + 12\lambda u^5 - 11u^6 + 48\lambda\gamma_2 u_x \\ & - 48\lambda g_3 x u_x - 24\gamma_2 u u_x + 24 g_3 x u u_x - 72\lambda^2 u^2 u_x - 12\lambda u^3 u_x \\ & + 24\lambda u^2 u_{xx} - 9u^2 u_x^2 - 8\gamma_2 u_{xx} + 8 g_3 x u_{xx} + 8 c u u_{xx} + 24\lambda^2 u u_{xx} \\ & - 24\lambda u^2 u_{xx} + 8u^3 u_{xx} - 12\lambda u_x u_{xx} + 6 u u_x u_{xx} + u_{xx}^2 \\ & + 12\lambda u u_{xxx} - 6u^2 u_{xxx}]. \end{aligned} \tag{2.50}$$

2.4 Special solution of P_{IV-1} hierarchy and Relation between the second and fourth Painlevé hierarchy

As a special case of equation (2.23), (2.24) with $g_{n+1} = 0$ we consider the following Painlevé hierarchy [5]

$$\begin{aligned} L_{n,x} &= 2K_n + u l_n + g_n - 2\alpha_n, \\ K_{n,x} &= \tfrac{1}{L_n}\left[\left(K_n + \tfrac{1}{2}g_n - \alpha_n\right)^2 - \tfrac{1}{4}\beta_n^2\right] - vL_n. \end{aligned} \tag{2.51}$$

where $\mathbf{K}_n = (K_n, L_n)^T$ is defined recursively as follows:

$$\begin{aligned}
&\mathbf{K}_n[\mathbf{u}] = \mathbf{L}_n[\mathbf{u}] + \textstyle\sum_{j=1}^{n-1} \gamma_j L_j[u] + g_n x \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
&\mathbf{u} = (u, v)^T, \qquad \mathbf{L}_1[\mathbf{u}] = (v, u)^T, \\
&B_1 \mathbf{L}_{j+1}[\mathbf{u}] = B_2 \mathbf{L}_j[\mathbf{u}], \\
&B_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \qquad B_2 = \tfrac{1}{2} \begin{pmatrix} 2\partial_x & \partial_x u - \partial_x^2 \\ u\partial_x + \partial_x^2 & v\partial_x + \partial_x v \end{pmatrix}.
\end{aligned} \tag{2.52}$$

The fourth Painlevé hierarchy (2.51) can be obtained as the compatibility condition of the following linear system of equations

$$\frac{\partial \phi}{\partial \lambda} = A(x, \lambda)\phi(x, \lambda), \qquad \frac{\partial \phi}{\partial x} = B(x, \lambda)\phi(x, \lambda), \tag{2.53}$$

where

$$\begin{aligned}
&B = \begin{pmatrix} -\lambda & w \\ \frac{-v}{w} & \lambda \end{pmatrix}, \\
&A = \tfrac{1}{g_{n+1}} \sum_{j=0}^{n+1} A_j \lambda^{n-j}, \\
&A_0 = -2\sigma_3, \qquad u = -\tfrac{w_x}{w}.
\end{aligned} \tag{2.54}$$

Now we will use the linear system (2.53)-(2.54) to rewrite the hierarchy (2.51) in another form. Let $A_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}$, $0 \leq j \leq n+1$. Then the compatibility condition $\phi_{x\lambda} = \phi_{\lambda x}$ of the linear system (2.53) implies

$$A_x - B_\lambda + AB - BA = 0. \tag{2.55}$$

Now we will substitute (2.54) into (2.55) as follow:

$$\frac{1}{g_{n+1}} \sum_{j=0}^{n+1} A_{j,x} \lambda^{n-j} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{g_{n+1}} \sum_{j=0}^{n+1} A_j \lambda^{n-j} \begin{pmatrix} -\lambda & w \\ \frac{-v}{w} & \lambda \end{pmatrix}$$
$$- \begin{pmatrix} -\lambda & w \\ \frac{-v}{w} & \lambda \end{pmatrix} \frac{1}{g_{n+1}} \sum_{j=0}^{n+1} A_j \lambda^{n-j} = 0.$$

After multiplying the matrix and reindexing the last equation we obtain

$$\sum_{j=0}^{n+1} \begin{pmatrix} a_{j,x} & b_{j,x} \\ c_{j,x} & -a_{j,x} \end{pmatrix} \lambda^{n-j} + \sum_{j=0}^{n+1} \begin{pmatrix} -a_{j+1}\lambda^{n-j} - \frac{v}{w} b_j \lambda^{n-j} & w a_j \lambda^{n-j} + b_{j+1}\lambda^{n-j} \\ -c_{j+1}\lambda^{n-j} + \frac{v}{w} a_j \lambda^{n-j} & w c_j \lambda^{n-j} - a_{j+1}\lambda^{n-j} \end{pmatrix}$$
$$- \sum_{j=0}^{n+1} \begin{pmatrix} -a_{j+1}\lambda^{n-j} + w c_j \lambda^{n-j} & -b_{j+1}\lambda^{n-j} - w a_j \lambda^{n-j} \\ -\frac{v}{w} a_j \lambda^{n-j} + c_{j+1}\lambda^{n-j} & -\frac{v}{w} b_j \lambda^{n-j} - a_{j+1}\lambda^{n-j} \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$+\begin{pmatrix} 2\lambda^n & -2w\lambda^n \\ \frac{-2v}{w}\lambda^n & 2\lambda^n \end{pmatrix} - \begin{pmatrix} 2\lambda^n & 2w\lambda^n \\ \frac{2v}{w}\lambda^n & 2\lambda^n \end{pmatrix} = 0.$$

Next we will set the coefficient of each power of λ to zero.

Equating the coefficient of λ^{n-j} to zero we get

$$\begin{aligned} a_{j,x} &= \tfrac{v}{w}b_j + wc_j, && 0 \le j \le n-1, \\ b_{j,x} &= -2b_{j+1} - 2wa_j, && 0 \le j \le n, \\ c_{j,x} &= 2c_{j+1} - \tfrac{2v}{w}a_j, && 0 \le j \le n, \end{aligned} \tag{2.56}$$

Equating the constant term to zero we get

$$a_{n,x} = wc_n + \frac{v}{w}b_n - g_n, \tag{2.57}$$

Equating the coefficient of λ^{-1} to zero we get

$$\begin{aligned} a_{n+1,x} &= wc_{n+1} + \tfrac{v}{w}b_{n+1}, \\ b_{n+1,x} &= -2wa_{n+1}, \\ c_{n+1,x} &= -\tfrac{2v}{w}a_{n+1}. \end{aligned} \tag{2.58}$$

If $n \ge 2$, then equation (2.56)-(2.58) determines a_j, b_j, and c_j recursively as follows:

Using $A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$, we obtain

$$\begin{aligned} a_0 &= -2, \quad b_0 = c_0 = 0, \\ b_j &= \tfrac{-1}{2}b_{j-1,x} - wa_{j-1}, \quad 1 \le j \le n+1, \\ c_j &= \tfrac{1}{2}c_{j-1,x} + \tfrac{v}{w}a_{j-1}, \quad 1 \le j \le n+1, \\ a_j &= \frac{1}{4}\sum_{i=1}^{j-1}(b_i c_{j-i} + a_i a_{j-i}) + K_j, \quad 1 \le j \le n-1, \\ a_n &= \frac{1}{4}\sum_{i=1}^{n-1}(b_i c_{n-i} + a_i a_{n-i}) - g_n x + K_n, \\ a_{n+1} &= \frac{1}{4}\sum_{i=1}^{n}(b_i c_{n+1-i} + a_i a_{n+1-i}) - \frac{1}{2}g_n a_1 x + \alpha_n - \frac{1}{2}g_n. \end{aligned} \tag{2.59}$$

We will prove the third equation from (2.59) as follow:

since $a_0 = -2$, $a_1 = K_1$, $b_1 = 2w$, $c_1 = \frac{-2v}{w}$, then if $j = 2$

$a_{2,x} = wc_2 + \frac{v}{w}b_2 = w[\frac{1}{2}c_{1,x} + \frac{v}{w}a_1] - \frac{v}{w}[\frac{1}{2}b_{1,x} + wa_1]$

$= \frac{1}{4}b_1 c_{1,x} + \frac{1}{4}c_1 b_{1,x} \qquad = \frac{1}{4}(b_1 c_1)_x,$

Integrate the last equation with respect to x we have

$a_2 = \frac{1}{4}b_1c_1 + K_2.$

Now when $j = 3$ we get

$$\begin{aligned}
a_{3,x} &= \tfrac{1}{2}b_1c_3 - \tfrac{1}{2}c_1b_3 = \tfrac{1}{4}b_1[c_{2,x} - c_1a_2] + \tfrac{1}{4}c_1[b_{2,x} + b_1a_2] \\
&= \tfrac{1}{4}(b_1c_2 + c_1b_2)_x - \tfrac{1}{4}c_2b_{1,x} - \tfrac{1}{4}b_2c_{1,x} \\
&= \tfrac{1}{4}(b_1c_2 + c_1b_2)_x - \tfrac{1}{8}b_{1,x}[c_{1,x} - c_1a_1] + \tfrac{1}{8}c_{1,x}[b_{1,x} + b_1a_1] \\
&= \tfrac{1}{4}(b_1c_2 + c_1b_2)_x + \tfrac{1}{8}a_1(b_1c_1)_x = \tfrac{1}{4}(b_1c_2 + c_1b_2)_x + \tfrac{1}{2}a_1a_{2,x} \\
&= \tfrac{1}{4}(b_1c_2 + c_1b_2 + 2a_1a_2)_x.
\end{aligned}$$

Integrate $a_{3,x}$ with respect to x we have

$a_3 = \frac{1}{4}[b_1c_2 + c_1b_2 + 2a_1a_2] + K_3.$

Now since $4a_j + g_n\delta_{n,j} = \dfrac{1}{4}\displaystyle\sum_{i=1}^{j-1}(b_ic_{j-i} + a_ia_{j-i}) + K_j$, then

$$a_{j,x} + g_n\delta_{n,j} = \frac{1}{4}\sum_{i=1}^{j-1}(b_ic_{j-i,x} + b_{i,x}c_{j-i} + a_{i,x}a_{j-i} + a_ia_{j-i,x}),$$

After substituting the value of $c_{j-i,x}$, $b_{i,x}$, $a_{i,x}$ and $a_{j-i,x}$ from (2.56) in the last equation and reindexing it , we get

$$\begin{aligned}
4a_j + g_n\delta_{n,j} &= 2\sum_{i=1}^{j-1} b_ic_{j-i+1} - 2\sum_{k=2}^{j} b_kc_{j-k+1} + c_1\sum_{i=1}^{j-1} b_ia_{j-i} - b_1\sum_{i=1}^{j-1} a_ic_{j-i} \\
&+ \frac{1}{2}b_1\sum_{i=1}^{j-1} a_ic_{j-i} + \frac{1}{2}b_1\sum_{k=j-i}^{1} a_kc_{j-k} - \frac{1}{2}\sum_{i=1}^{j-1} b_ia_{j-i} - \frac{1}{2}\sum_{k=j-1}^{1} b_ka_{j-k} \\
&- g_n\sum_{i=1}^{j-1} \delta_{n,i}a_{j-i} - g_n\sum_{i=1}^{j-1} \delta_{n,j-i}a_i,
\end{aligned}$$

simplifing the last equation we obtain

$$4a_j + g_n\delta_{n,j} = 2b_1c_j - 2b_jc_1 - 2g_n\delta_{n,j-1}a_1 = 2wc_j + \frac{v}{w}b_j - 2g_n\delta_{n,j-1}K_1.$$

Thus we proved equation (2.59).

If $n = 1$, then $a_1 = -g_1x$, $a_2 = -v + \alpha_1 - \frac{1}{2}g_1$, and b_j, c_j are given by (2.59). Moreover the last equation in (2.58) can be integrated to obtain the first integral

$$b_{n+1}c_{n+1} + a_{n+1}^2 = K, \tag{2.60}$$

where K is a constant of integration.

We will prove equation (2.60). We will multiply the last two equations in (2.58) by c_{n+1}, b_{n+1}, respectively, and adding them, so we get

$b_{n+1}c_{n+1,x} + c_{n+1}b_{n+1,x} = \dfrac{-2v}{w}b_{n+1}a_{n+1} - 2wa_{n+1}c_{n+1}$. And from (2.58), the

last equation becomes
$(b_{n+1}c_{n+1})_x = -a_{n+1}\left[\frac{2vb_{n+1}}{w} + 2wc_{n+1}\right] = -2a_{n+1}a_{n+1,x}.$
Now integrating the last equation with respect to x we get equation (2.60). Let $K = \frac{1}{4}\beta_n^2$. Then the fourth Painlevé hierarchy (2.51) can be written as

$$\begin{aligned} &b_{n+1,x} + 2wa_{n+1} = 0, \\ &b_{n+1}c_{n+1} = \tfrac{1}{4}\beta_n^2 - a_{n+1}^2. \end{aligned} \tag{2.61}$$

2.4.1 Special solutions of the fourth Painlevé hierarchy

The first member of the fourth Painlevé hierarchy (2.61) with $g_2 = 1$ reads

$$\begin{aligned} u_x &= 2v + u^2 + g_1xu - 2\alpha_1, \\ v_x &= \frac{1}{(u+g_1x)}\left[\left(v - \alpha_1 - \frac{1}{2}g_1\right)^2 - \frac{1}{4}\beta_1^2\right] - v(u+g_1x). \end{aligned} \tag{2.62}$$

Equation (2.62) has the special solution $(2\alpha_1 + g_1)^2 = \beta_1^2$, $v = u_x$, and u solves the Riccati equation

$$u_x + u^2 + g_1xu - 2\alpha_1 = 0. \tag{2.63}$$

Setting $g_1 = -2$, which can be done without loss of generality, and using the linearizing transformation $u = \frac{U_x}{U}$, then (2.63) is transformed to the Weber-Hermite equation

$$U_{xx} - 2xU_x - 2\alpha_1U = 0. \tag{2.64}$$

In [6] the second member of the hierarchy (2.59), (2.61), with $g_2 = 1$ reads

$$u_{xx} = 3uu_x - u^3 - 6uv - 2xu + 4\alpha_2 - 4cu, \tag{2.65}$$

$$v_{xx} = 2\left(\frac{(uv + \frac{1}{2}v_x - \frac{1}{2} - \alpha_2)^2 - \frac{1}{4}\beta_2^2}{(v + \frac{1}{2}u^2 - \frac{1}{2}u_x + x + 2c)} - v(v + \frac{1}{2}u^2 - \frac{1}{2}u_x + x + 2c)\right) - 2(uv)_x. \tag{2.66}$$

Equation (2.65), (2.66) has the special solution $2\alpha + \beta + 1 = 0$, $v = u_x$ and u is a solution of the linearizable equation

$$u_{xx} = -(3u - K_1)u_x - u^3 + k_1u^2 - 2xu - \beta + 2\alpha - 1. \tag{2.67}$$

As a generalization of the above results, we will prove that the fourth Painlevé hierarchy (2.59), (2.61) has the special solutions

$$(2\alpha_n + g_n)^2 = \beta_n^2, \qquad v = u_x, \tag{2.68}$$

and u solves the nth order differential equation

$$b_{n+1,x} + 2wa_{n+1} = 0. \tag{2.69}$$

In order to prove that (2.68)-(2.69) is a solution of the hierarchy (2.61), we only have to prove that

$$b_{n+1}c_{n+1} = \frac{1}{4}\beta_n^2 - a_{n+1}^2. \tag{2.70}$$

As a first step, we will use induction to prove that

$$a_j - K_j = -\frac{d}{dx}\left(\frac{b_j}{w}\right), \qquad 2 \le j \le n-1. \tag{2.71}$$

Using (2.59), we find $a_1 = K_1$, $b_1 = 2w$, $c_1 = -\frac{2u_x}{w}$, $a_2 = \frac{1}{4}(b_1c_1 + a_1^2) + K_2$ so $a_2 = -u_x + K_2$, $b_2 = \frac{1}{2}b_{1,x} - wa_1 = w(u - K_1)$. Since $\frac{d}{dx}\left(\frac{b_2}{w}\right) = u_x$, we have $a_2 - K_2 = -\frac{d}{dx}\left(\frac{b_2}{w}\right)$.

Assume it is true for $j = m$, $2 \le m \le n-1$; that is, assume that

$$a_m - k_m = -\frac{d}{dx}\left(\frac{b_m}{w}\right) = -\frac{1}{w}(b_{m,x} + ub_m). \tag{2.72}$$

Then the first equation in (2.56) at $j = m+1$ implies that $a_{m+1,x} = wc_{m+1} + \frac{u_x}{w}b_{m,x}$. Using (2.59) to substitute b_{m+1} and c_{m+1} we get
$a_{m+1,x} = w(\frac{1}{2}c_{m,x} + \frac{u_x}{w}a_m) + \frac{u_x}{w}(\frac{-1}{2}b_{m,x} - wa_m)$.
Hence

$$2a_{m+1,x} = wc_{m,x} - \frac{u_x}{w}b_{m,x}. \tag{2.73}$$

Now using $w_x = -uw$ and

$$\frac{d}{dx}(wc_m) = wc_{m,x} - uwc_m, \tag{2.74}$$

then from the first equation in (2.56) at $j = m$ we have $a_{m,x} = wc_m + \frac{u_x}{w}b_m$, this yields

$$\frac{d}{dx}[u(a_m - K_m)] = u_x(a_m - K_m) + ua_{m,x} = u(a_m - K_m) + uwc_m + \frac{uu_x}{w}b_m. \tag{2.75}$$

Thus (2.73)-(2.75) give
$\frac{d}{dx}[2a_{m+1} - wc_m - u(a_m - K_m)] = 2a_{m+1,x} - wc_{m,x} - w_xc_m - u_x(a_m - K_m) - ua_{m,x}.$

After substituting the value of $2a_{m+1,x}$ from (2.73) we have

$$\frac{d}{dx}\left[2a_{m+1} - wc_m - u(a_m - K_m)\right] = -u_x\left[a_m - K_m + \frac{1}{w}(b_{m,x} + ub_m)\right]. \tag{2.76}$$

The induction hypothesis (2.72) implies that the right-hand side of (2.76) is zero and hence we get $\frac{d}{dx}\left[2a_{m+1} - wc_m - u(a_m - K_m)\right] = 0$. After integrate it with respect to x we have

$$a_{m+1} - K_{m+1} = \frac{1}{2}[u(a_m - k_m) + wc_m]. \tag{2.77}$$

On the other hand (2.59) at $j = m+1$ we get $\frac{1}{w}b_{m+1} = \frac{-1}{2w}b_{m,x} - a_m$. Using (2.72) to substitute $b_{m,x}$, we get

$$\frac{1}{w}b_{m+1} = -\frac{1}{2}\left[a_m - \frac{u}{w}b_m\right]. \tag{2.78}$$

Now differentiating (2.78) we get $\frac{d}{dx}\left(\frac{b_{m+1}}{w}\right) = \frac{-1}{2}\left(a_{m,x} - \frac{u}{w}b_{m,x} - \frac{wu_x - uw_x}{w^2}b_m\right)$ Then using the first equation in (2.56) to substitute $a_{m,x}$ and (2.72) to substitute $b_{m,x}$, we obtain

$$\frac{d}{dx}\left(\frac{b_{m+1}}{w}\right) = -\frac{1}{2}[u(a_m - K_m) + wc_m]. \tag{2.79}$$

Therefore, (2.77) and (2.79) yield the result.

Using similar arguments we can prove that

$$a_n + g_n x - k_n = -\frac{d}{dx}\left(\frac{b_n}{w}\right), \tag{2.80}$$

and

$$a_{n+1} - k_{n+1} - g_n = -\frac{d}{dx}\left(\frac{b_{n+1}}{w}\right). \tag{2.81}$$

Since $b_{n+1,x} = -2wa_{n+1}$, (2.81) gives

$$\begin{aligned} a_{n+1} - k_{n+1} - g_n &= \tfrac{-b_{n+1,x}}{w} + \tfrac{w_x}{w^2}b_{n+1}, \\ &= 2a_{n+1} - \tfrac{u}{w}b_{n+1}, \\ a_{n+1} + k_{n+1} + g_n &= \tfrac{u}{w}b_{n+1}. \end{aligned} \tag{2.82}$$

The first equation in (2.58) implies $c_{n+1} = \frac{1}{w}\left[a_{n+1,x} - \frac{u_x}{w}b_{n+1}\right] = \frac{1}{w^2}\left[wa_{n+1,x} - u_x b_{n+1}\right]$.

Using (2.82) to substitute a_{n+1} we obtain

$$\begin{aligned} c_{n+1} &= \frac{1}{w^2}\left[w(\frac{u}{w}b_{n+1} - K_{n+1} - g_n)_x - u_x b_{n+1}\right], \\ &= \frac{1}{w^2}\left[w(\frac{u}{w}b_{n+1,x} + \frac{wu_x - uw_x}{w^2}b_{n+1}) - u_x b_{n+1}\right], \\ &= \frac{u}{w}\left[\frac{1}{w}b_{n+1,x} - \frac{w_x}{w^2}b_{n+1}\right], \\ &= \frac{u}{w}\frac{d}{dx}\left(\frac{b_{n+1}}{w}\right). \end{aligned}$$

Hence (2.81) gives

$$a_{n+1} - K_{n+1} - g_n = -\frac{w}{u}c_{n+1}. \tag{2.83}$$

Therefore, using (2.82) and (2.83) to substituting b_{n+1} and c_{n+1} into (2.70) we get

$$\begin{aligned} (a_{n+1} + k_{n+1} + g_n)(a_{n+1} - k_{n+1} - g_n) &= \frac{1}{4}\beta_n^2 - a_{n+1}^2, \\ -(a_{n+1}^2 - 2a_{n+1}K_{n+1} - K_{n+1}^2 - g_n^2) &= \frac{1}{4}\beta_n^2 - a_{n+1}^2, \\ g_n^2 + K_{n+1}^2 + 2g_nK_{n+1} &= \frac{1}{4}\beta_n^2, \end{aligned}$$

Let $K_{n+1} = \alpha_n - \frac{1}{2}g_n$. Then (2.70) is satisfied provided that $(2\alpha_n + g_n)^2 = \beta_n^2$; and this complete the proof.

Now we will show that equation (2.69) is linearizable. Using $w_x = -uw$, we find

$$b_{j,x} = w(\partial_x - u)\left(\frac{b_j}{w}\right). \tag{2.84}$$

Now, from (2.59) we have
$\frac{b_{j+1}}{w} = \frac{1}{w}\left[\frac{-1}{2}b_{j,x} - wa_j\right]$. Using (2.84), the last equation becomes
$\frac{b_{j+1}}{w} = \frac{1}{w}\left[\frac{-1}{2}\left(w(\partial_x - u)\left(\frac{b_j}{w}\right)\right) - wa_j\right]$. Since $a_j - K_j = -\frac{d}{dx}\left(\frac{b_j}{w}\right)$, we get

$$\frac{b_{j+1}}{w} = \frac{1}{2}(\partial_x + u)\left(\frac{b_j}{w}\right) - K_j, \quad 2 \le j \le n-1, \tag{2.85}$$

And from (2.59)

$$\frac{b_{n+1}}{w} = \frac{1}{w}\left[\frac{-1}{2}b_{n,x} - wa_n\right], \text{ and from (2.84), the last equation becomes}$$
$$\frac{b_{n+1}}{w} = \frac{1}{w}\left[\frac{-1}{2}\left(w(\partial_x - u)\left(\frac{b_n}{w}\right)\right) - wa_n\right], \text{ since } a_n + g_n x - K_n = -\frac{d}{dx}\left(\frac{b_j}{w}\right),$$

$$\frac{b_{n+1}}{w} = \frac{1}{2}(\partial_x + u)\left(\frac{b_n}{w}\right) - K_n + g_n x. \tag{2.86}$$

Since $b_2 = w(u - K_1)$, (2.84) and (2.86) imply that

$$\begin{aligned}\frac{b_{n+1}}{w} &= \frac{1}{2}(\partial_x + u)\left(\frac{b_n}{w}\right) - K_n + g_n x,\\ &= \frac{1}{2}(\partial_x + u)\left[\frac{1}{2}(\partial_x + u)\left(\frac{b_{n-1}}{w}\right) - K_{n-1} + g_{n-1}x\right] - K_n + g_n x,\\ &= \frac{1}{2}(\partial_x+u)\left[\frac{1}{2}(\partial_x + u)\left(\frac{1}{2}(\partial_x + u)\left(\frac{b_{n-2}}{w}\right) - K_{n-2} + g_{n-2}x\right) - K_{n-1} + g_{n-1}x\right]\\ &\quad - K_n + g_n x,\end{aligned}$$

Follow at the same method we obtain

$$\frac{b_{n+1}}{w} = \frac{1}{2^{n-1}}(\partial_x + u)^{n-1}u - \sum_{i=1}^{n-1}\frac{K_{n-i}}{2^i}(\partial_x + u)^{i-1}u + g_n x - K_n. \tag{2.87}$$

Now using (2.81) and (2.84), (2.69) can be written in the explicit form

$$(\partial_x + u)\left(\frac{b_{n+1}}{w}\right) - 2K_{n+1} - 2g_n = 0. \tag{2.88}$$

Hence (2.87) and (2.88) imply that (2.69) can be written in the form

$$(\partial_x+u)\left(\frac{1}{2^{n-1}}(\partial_x + u)^{n-1}u - \sum_{i=1}^{n-1}\frac{K_{n-i}}{2^i}(\partial_x + u)^{i-1}u + g_n x - K_n\right) - 2K_{n+1} - 2g_n = 0,$$

and multiplying it by $\frac{1}{2}$ we get

$$\frac{1}{2^n}(\partial_x + u)^n u - \sum_{i=1}^{n-1}\frac{K_{n-i}}{2^{i+1}}(\partial_x + u)^i u + \frac{1}{2}g_n(xu - 1) - \frac{1}{2}K_n u - K_{n+1} = 0. \tag{2.89}$$

Equation (2.89) can be linearized by the substitution $u = \frac{U_x}{U}$.

2.4.2 Relation between the second and fourth Painlevé hierarchy

In this subsection, we will show that the fourth Painlevé hierarchy (2.61) have special solutions in terms of the second Painlevé hierarchy that is infinite sequence of nonlinear ordinary differential equations whose first member is a second Painlevé equation

$$\begin{aligned} &b_{n+1,x} + \tilde{w} a_{n+1} = 0, \\ &\sum_{i=1}^{n+1} (b_i c_{n+2-i} + a_i a_{n+2-i}) + 4\delta_n = 0. \end{aligned} \tag{2.90}$$

Thus, we will derive a relation between the second Painlevé hierarchy (2.90) and the fourth Painlevé hierarchy (2.61).

Let u and v be solutions of the hierarchy (2.61) of order $n = m+1$ such that $a_1 = 0$, $\beta_{m+1} = 0$ and $b_{m+2} = 0$. Setting $2w = \tilde{w}$, then we have $u = -\frac{\tilde{w_x}}{\tilde{w}}$ Then equations (2.59) and (2.61) become yields

$$\begin{aligned} &a_0 = -2, \quad a_1 = 0, \quad b_0 = c_0 = 0, \\ &b_j = -\frac{1}{2}(b_{j-1,x} + \tilde{w} a_{j-1}), \qquad 1 \leq j \leq m+1, \\ &c_j = \frac{1}{2} c_{j-1,x} + \frac{2v}{\tilde{w}} a_{j-1}, \qquad 1 \leq j \leq m+2, \\ &a_j = \frac{1}{4} \sum_{i=1}^{j-1} (b_i c_{j-i} + a_i a_{j-i}) + K_j, \quad 2 \leq j \leq m, \\ &a_{m+1} = \frac{1}{4} \sum_{i=1}^{m} (b_i c_{m+1-i} + a_i a_{m+1-i}) - g_{m+1} x + K_{m+1}, \\ &a_{m+2} = \frac{1}{4} \sum_{i=1}^{m+1} (b_i c_{m+2-i} + a_i a_{m+2-i}) + \alpha_{m+1} - \frac{1}{2} g_{m+1}, \\ &a_{m+2} = 0. \end{aligned} \tag{2.91}$$

Moreover, since $a_{m+2} = b_{m+2} = 0$, (2.91) gives

$$\sum_{i=1}^{m+1} (b_i c_{m+2-i} + a_i a_{m+2-i}) + 4(\alpha_{m+1} - \frac{1}{2} g_{m+1}) = 0, \tag{2.92}$$

$$b_{m+1,x} = -\tilde{w} a_{m+1}. \tag{2.93}$$

Thus, u and v are solutions of the second Painleví hierarchy (2.90) of order $n = m$ with parameter $\delta_m = \alpha_{m+1} - \frac{1}{2} g_{m+1}$.

Therefore we have shown that any n^{th} member of the fourth Painlevé hierarchy (2.61), $n \geq 2$, has a special solution $a_1 = \beta_n = 0$ in terms of the $(n-1)^{st}$ member of the second Painlevé hierarchy (2.90).

Chapter 3

The Fourth Painlevé Hierarchy P_{IV-2}

3.1 Scalar isomonodromy problems and the fourth Painlevé hierarchy

In this section, we will derive a fourth Painlevé hierarchy from scalar isomonodromy problems [7]. We start from the linear system of equations

$$\psi_{xx} = (\lambda - y)\psi_x - P_n(x)\psi, \tag{3.1}$$

$$\lambda\psi_\lambda = A(x,\lambda)\psi_x + B(x,\lambda)\psi, \tag{3.2}$$

where $y(x)$ is a solution of the fourth Painlevé hierarchy, λ is a spectral parameter, and $A(x,\lambda)$, $B(x,\lambda)$, and $P_n(x)$ are functions that can be found for every equation of the fourth Painlevé hierarchy.
The compatibility condition of equations (3.1), (3.2) is given by

$$(\psi_{xx})_\lambda = (\psi_\lambda)_{xx}. \tag{3.3}$$

Differentiates (3.1) with respect to λ we obtain

$$\psi_{xx\lambda} = \psi_x + (\lambda - y)\psi_{x\lambda} - \frac{1}{\lambda}P_n(A\psi_x + B\psi). \tag{3.4}$$

Differentiating (3.2) twice with respect to x, we get

$$\lambda\psi_{\lambda xx} = (A(\lambda-y)^2 + 2A_x(\lambda-y) + B(\lambda-y) - Ay_x + A_{xx} + 2B_x - AP_n)\psi_x \\ + (B_{xx} - 2A_xP_n - AP_{n,x} - A(\lambda-y)P_n - BP_n)\psi. \tag{3.5}$$

It follows that

$$\psi_{xx\lambda} = \psi_x + ((\lambda-y)\frac{1}{\lambda}(\lambda-y)A + A_x + B)\psi_x + (B_x - AP_n)\psi - P_n(\frac{1}{\lambda}(A\psi_x + B\psi)). \tag{3.6}$$

By comparing the coefficients of ψ_x and ψ in the two equations (3.5) and (3.6) we find

$\frac{1}{\lambda}(A(\lambda-y)^2 + 2A_x(\lambda-y) + B(\lambda-y) - Ay_x + A_{xx} + 2B_x - AP_n)$
$= 1 + \frac{1}{\lambda}((\lambda-y)^2A + (\lambda-y)A_x + (\lambda-y)B) - \frac{1}{\lambda}P_nA,$

and

$\frac{1}{\lambda}(B_{xx} - 2A_xP_n - AP_{n,x} - A(\lambda-y)P_n - BP_n)$
$= \frac{1}{\lambda}((\lambda-y)B_x - AP_n(\lambda-y)) - \frac{1}{\lambda}BP_n,$

Simplifying these two equation we get

$$A_xy + Ay_x + \lambda - A_{xx} - \lambda A_x - 2B_x = 0. \tag{3.7}$$

$$2A_xP_n + AP_{n,x} + \lambda B_x - B_{xx} - yB_x = 0. \tag{3.8}$$

From (3.7) we obtain
$B_x = \frac{1}{2}(A_xy + Ay_x + \lambda - A_{xx} - \lambda A_x).$
After integration with respect to x, we find

$$B = \frac{1}{2}(\lambda x + yA - A_x - \lambda A). \tag{3.9}$$

Substituting B from (3.9) into (3.8) we get

$$4A_xP_n + 2AP_{n,x} + 2\lambda yA_x + \lambda Ay_x - yy_xA + A_{xxx} + \lambda^2 - 2A_xy_x - Ay_{xx} - \lambda^2A_x - y^2A_x - \lambda y = 0. \tag{3.10}$$

We seek a solution $A(x,\lambda)$ of this equation in the form:

$$A(x,\lambda) = \sum_{k=0}^{n} a_k(x)\lambda^{n-k}, \quad n = 1, 2, \ldots \tag{3.11}$$

Substituting equation (3.11) into (3.10) yields

$$\begin{aligned}
&4P_n\sum_{k=0}^{n}a_{k,x}(x)\lambda^{n-k}+2P_{n,x}\sum_{k=0}^{n}a_k(x)\lambda^{n-k}+2\lambda y\sum_{k=0}^{n}a_{k,x}(x)\lambda^{n-k}\\
&+\lambda y_x\sum_{k=0}^{n}a_k(x)\lambda^{n-k}-yy_x\sum_{k=0}^{n}a_k(x)\lambda^{n-k}+\sum_{k=0}^{n}a_{k,xxx}(x)\lambda^{n-k}+\lambda^2\\
&-2y_x\sum_{k=0}^{n}a_k(x)\lambda^{n-k}-y_{xx}\sum_{k=0}^{n}a_k(x)\lambda^{n-k}-\lambda^2\sum_{k=0}^{n}a_{k,x}(x)\lambda^{n-k}\\
&-y^2\sum_{k=0}^{n}a_{k,x}(x)\lambda^{n-k}-\lambda y=0,
\end{aligned}\tag{3.12}$$

Insert λ in the summation and reindexing we get

$$\begin{aligned}
&4P_n\sum_{k=0}^{n}a_{k,x}(x)\lambda^{n-k}+2P_{n,x}\sum_{k=0}^{n}a_k(x)\lambda^{n-k}+2y\sum_{k=-1}^{n-1}a_{k+1,x}(x)\lambda^{n-k}\\
&+y_x\sum_{k=-1}^{n-1}a_{k+1}(x)\lambda^{n-k}-yy_x\sum_{k=0}^{n}a_k(x)\lambda^{n-k}+\sum_{k=0}^{n}a_{k,xxx}(x)\lambda^{n-k}+\lambda^2\\
&-2y_x\sum_{k=0}^{n}a_k(x)\lambda^{n-k}-y_{xx}\sum_{k=0}^{n}a_k(x)\lambda^{n-k}-\sum_{k=-2}^{n-2}a_{k+2,x}(x)\lambda^{n-k}\\
&-y^2\sum_{k=0}^{n}a_{k,x}(x)\lambda^{n-k}-\lambda y=0.
\end{aligned}\tag{3.13}$$

Equating the coefficients of λ^{n+2} to zero, we get $-a_{0,x}=0$ and hence $a_0(x)$=constant. As a special case, we set

$$a_0(x)=1.\tag{3.14}$$

Equating the coefficients of λ^{n+1} to zero, we obtain $2ya_{0,x}+y_xa_0-a_{1,x}=0$ or $a_{1,x}=y_xa_0$, provided that $n\geq 2$. When $n=1$ we have $a_{1,x}=y_xa_0+1$ and hence $a_1=y+x+c_0$, where c_0 is a constant of integration. For simplicity we take $c_0=0$.

At $n\geq 2$, $a_1=y+c_1$, where c_1 is a constant of integration. Thus,

$$a_1(x)=\begin{cases} y(x)+x, & n=1;\\ y(x), & n\geq 2.\end{cases}\tag{3.15}$$

We introduce the notation

$$E_k=4P_na_{k,x}+2P_{n,x}a_k+2ya_{k,x}+y_xa_{k+1}-yy_xa_k+a_{k,xxx}-2y_xa_k-y_{xx}a_k-a_{k+2,x}-y^2a_{k,x}.\tag{3.16}$$

Then, for $k < n-2$ equation (3.10) gives the recursion relation

$$E_k + 2ya_{k,x} + y_x a_{k+1} - a_{k+2,x} = 0. \tag{3.17}$$

Equating the coefficients of λ to zero, we get
$4P_n a_{n-1,x} + 2P_{n,x}a_{n-1} + 2ya_{n,x} + y_x a_n - yy_x a_{n-1} + a_{n-1,xxx} - 2y_x a_{n-1,x}$
$- y_{xx}a_{n-1} - y^2 a_{n-1,x} - y = 0.$
That is

$$E_{n-1} + 2ya_{n,x} + y_x a_n - y = 0. \tag{3.18}$$

Equating the coefficients of λ^2 to zero, we obtain
$4P_n a_{n-2,x} + 2P_{n,x}a_{n-2} + 2ya_{n-1,x} + y_x a_{n-1} - yy_x a_{n-2} + 1 + a_{n-2,xxx} - 2y_x a_{n-2,x} -$
$y_{xx}a_{n-2} - a_{n,x} - y^2 a_{n-2,x} = 0.$
That is

$$E_{n-2} + 2ya_{n-1,x} + y_x a_{n-1} + 1 - a_{n,x} = 0. \tag{3.19}$$

Equating the constant term to zero we get
$4P_n a_{n,x} + 2P_{n,x}a_n - yy_x a_n + a_{n,xxx} - 2y_x a_{n,x} - y_{xx}a_n - y^2 a_{n,x} = 0,$
That is

$$E_n = 0. \tag{3.20}$$

We set $V(x) = a_n(x)$, $V_x = a_{n,x}$. It follows from equation (3.20) that the function $V(x)$ satisfies $E_n = 0$:

$$4P_n V_x + 2P_{n,x}V - yy_x V + V_{xxx} - 2y_x V_x - y_{xx}V - y^2 V_x = 0. \tag{3.21}$$

We will show that this equation has a first integral given by

$$VV_{xx} - \frac{1}{2}V_x^2 + 2V^2 P_n - (y_x + \frac{1}{2}y^2)V^2 + \delta = 0, \tag{3.22}$$

where δ is a constant of integration.
Since $\int VV_{xxx}dx = VV_{xx} - \int V_x V_{xx}dx$ and $\int V_x V_{xx}dx$ by let $y = V_x$ then $dy = V_{xx}dx$ so $\int ydy = \frac{y^2}{2}$ so $\int V_x V_{xx}dx = \frac{V_x^2}{2}$ so we have $\int VV_{xxx}dx = VV_{xx} - \frac{V_x^2}{2}$, and knowing that $4P_n VV_x + 2P_{n,x}V^2 = (2V^2 P_n)_x$ and $-yy_x V^2 - y^2 VV_x = (-\frac{1}{2}V^2 y^2)_x$ and $-2y_x VV_x - y_{xx}V^2 = (-V^2 y_x)_x$.
Multiply equation (3.21) by V and integrate it with respect to x we have
$4P_n VV_x + 2P_{n,x}V^2 - yy_x V^2 + VV_{xxx} - 2y_x VV_x - y_{xx}V^2 - y^2 VV_x = 0.$
$2V^2 P_n - \frac{1}{2}V^2 y^2 - V^2 y_x + VV_{xx} - \frac{1}{2}V_x^2 = 0.$

This shows that (3.22) is a first integral of (3.21).

Equations (3.14)-(3.19) and (3.22) determine a fourth Painlevé hierarchy. In order to show this we consider the case $n = 1$.
In this case we have $V(x) = a_1(x) = y(x) + x$. We use equation (3.16) to obtain:
$E_0 = 4P_1 a_{0,x} + 2a_0 P_{1,x} - yy_x a_0 + a_{0,xxx} - 2y_x a_{0,x} - a_0 y_{xx} - y^2 a_{0,x}$.
Since $a_0 = 1$, we have $E_0 = 2P_{1,x} - yy_x - y_{xx}$.
Then (3.18) gives

$$E_0 + 2ya_{1,x} + y_x a_1 - y = 0. \quad (3.23)$$

Using $a_1 = y + x$ and $a_{1,x} = y_x + 1$, equation (3.23) becomes
$E_0 + 2y(y_x + 1) + y_x(y + x) - y = 0$ or
$E_0 + 3yy_x + y + xy_x = 0$.
By Substituting the last equation into $E_0 = 2P_{1,x} - yy_x - y_{xx}$ we have
$2P_{1,x} - yy_x - y_{xx} = -3yy_x - y - xy_x$,
and hence $2P_{1,x} + 2yy_x - y_{xx} + xy_x + y = 0$.
Integrate the last equation with respect to x yields
$2P_1 + y^2 - y_x + xy + \beta = 0$, where β is a constant of integration.
As a result we obtain

$$P_1(x) = \frac{1}{2}(y_x - xy - y^2 - \beta). \quad (3.24)$$

Assume that $y(x) = Y(x) - x$
Then from (3.24) we have
$P_1(x) = \frac{1}{2}(Y_x + xY - Y^2 - 1 - \beta)$.
Substituting the value of P_1 into equation (3.22) we get
$YY_{xx} - \frac{1}{2}Y_x^2 - Y^2 Y_x + Y^2 - \frac{1}{2}Y^4 + xY^3 - \frac{1}{2}x^2Y^2 + Y^2 Y_x + xY^3 - Y^4 - Y^2$
$- \beta Y^2 + \delta = 0$.
Thus

$$YY_{xx} - \frac{1}{2}Y_x^2 - (\beta Y^2 + \frac{x^2}{2})Y^2 + 2xY^3 - \frac{3}{2}Y^4 + \delta = 0. \quad (3.25)$$

This is the fourth Painlevé equation and hence the hierarchy is a fourth Painlevé hierarchy and we will denote it by P_{IV-2}.

3.2 The second member of the fourth Painlevé hierarchy P_{IV-2}

The second member of the fourth Painlevé hierarchy P_{IV-2} can be found from (3.14)-(3.19) and (3.22) when $n = 2$. From equation (3.18) and (3.19) where $V = a_n$ and $V_x = a_{n,x}$ we have

$$E_0 + 2ya_{1,x} + y_x a_1 + 1 - V_x = 0, \tag{3.26}$$

$$E_1 + 2yV_x + y_x V - y = 0. \tag{3.27}$$

From (3.17) we get

$$E_0 = 4P_n a_{0,x} + 2P_{2,x} - yy_x - y_{xx} = 2P_{2,x} - yy_x - y_{xx}.$$

Substituting $a_0(x) = 1$ and $a_1(x) = y(x)$ into equation (3.26) we find
$2P_{2,x} - y_{xx} + 2yy_x + 1 - V_x = 0$
Integrating the last equation with respect to x gives

$$2P_2 - V + y^2 - y_x + x + \epsilon = 0, \tag{3.28}$$

where ϵ is a constant of integration. We will take $\varepsilon = 0$.
Solving equation (3.28) for P_2, we obtain $P_2 = \frac{1}{2}(V - y^2 + y_x - x)$.
Now from (3.16)
$E_1 = 4P_2 a_{1,x} + 2a_1 P_{2,x} - y^2 y_x + y_{xxx} - 2y_x^2 - yy_{xx} - y^2 y_x$.
Then substituting the value of a_1, $a_{1,x}$ we get
$E_1 = 4y_x P_2 + 2yP_{2,x} - y^2 y_x + y_{xxx} - 2y_x^2 - yy_{xx} - y^2 y_x$
Substituting the value of E_1 from (3.27) we have
$2y_x(V - y^2 + y_x - x) + y(V_x - 2yy_x + y_{xx} - 1) - y^2 y_x + y_{xxx} - 2y_x^2 - yy_{xx} - y^2 y_x + 2yV_x + y_x V - y = 0$.
Thus,
$y_{xxx} - 6y^2 y_x + 3Vy_x + 3yV_x - 2xy_x - 2y = 0$.
Integrate the last equation with respect to x gives

$$y_{xx} - 2y^3 + 3Vy - 2xy - \beta = 0, \tag{3.29}$$

where β is a constant of integration.
Substituting $P_2(x)$ into equation (3.22) gives
$VV_{xx} - \frac{1}{2}V_x^2 - (y_x + \frac{1}{2}y^2)V^2 + (V - y^2 + y_x - x)V^2 + \delta = 0$, or

$$VV_{xx} - \frac{1}{2}V_x^2 - \frac{3}{2}y^2V^2 - xV^2 + V^3 + \delta = 0. \qquad (3.30)$$

The second member of P_{IV-2} hierarchy is determined by (3.29) and (3.30).

Equation (3.29), (3.30) has a Lax pair of the form

$\psi_{xx} = (\lambda - y)\psi_x + \frac{1}{2}(y^2 - V - y_x + x)\psi,$

$\lambda\psi_x = (V + \lambda y + \lambda^2)\psi_x + \frac{1}{2}(yV - V_x + \lambda x + \lambda y^2 - \lambda y_x - \lambda V - \lambda^3)\psi.$

Taking equation (3.28) in account and differentiating it with respect to x, we have
$2P_{2,x} - V_x + 2yy_x - y_{xx} + 1 = 0.$
Then substituting the value of y_{xx} from (3.29), we obtain
$2P_{2,x} - V_x + 2yy_x - 2y^3 - 2xy + 3Vy - \beta + 1 = 0.$
But $P_2 = \frac{1}{2}(V + y_x - y^2 - x)$.
So $4yP_2 + yV = 2Vy + 2yy_x - 2y^3 - 2xy + Vy = 3Vy + 2yy_x - 2y^3 - 2xy.$
Thus, we can write equation (3.29) as:

$$2P_{2,x} - V_x + y(4P_2 + V) + 1 - \beta = 0 \qquad (3.31)$$

From equation (3.31) we can solve for $y(x)$ as :

$$y = \frac{V_x - 2P_{2,x} + \beta - 1}{4P_2 + V}. \qquad (3.32)$$

We note that we have special solutions of equations (3.28),(3.30) and (3.31) at $V(x) = P_2 = \delta = 0$ and $\beta = 1$, (3.28) becomes

$$y_x - y^2 - x = 0. \qquad (3.33)$$

This is a Ricatti equation and has a solution in the form

$$y = -\frac{\varphi_x}{\varphi}, \qquad (3.34)$$

where φ satisfies the Airy equation

$$\varphi_{xx} + x\varphi_x = 0. \tag{3.35}$$

3.3 Special and rational solutions of the second member of the P_{IV-2} hierarchy

To find special and rational solutions of the second member of the P_{IV-2} hierarchy, we consider equations (3.29),(3.30).

From equation (3.30), we can see that there is special solution $V(x) = 0$ at $\delta = 0$. We then have the second-order ODE from equation (3.29) in the form

$$y_{xx} = 2y^3 + 2xy + \beta. \tag{3.36}$$

Equation (3.36) is the second Painlevé equation, and we can see that this equation give special and rational solutions of the second member of the P_{IV-2} hierarchy.

Equation (3.36) has special functions and rational solutions at integer and half integer values of the parameter β.

The second member of the P_{IV-2} hierarchy, therefore, has special functions and rational solutions expressed by means of solutions of the second Painlevé equation.

Chapter 4

The Fourth Painlevé hierarchy P_{IV-3}

In this chapter we will use a matrix isomonodromy linear problem to derive a fourth Painlevé hierarchy of ordinary differential equations [8].

Painlevé equations can be written as a compatibility condition of a matrix linear system

$$\phi_\lambda(x,\lambda) = A(x,\lambda)\phi(x,\lambda), \quad \phi_x(x,\lambda) = B(x,\lambda)\phi(x,\lambda), \tag{4.1}$$

where

$$A(x,\lambda) = \sum_{j=0}^{N+n} A_j \lambda^{N-j}, \quad B(x,\lambda) = \sum_{j=0}^{L+l} B_j \lambda^{L-j}, \tag{4.2}$$

and A_j and B_j are matrices with entries depending on the solution $u(x)$ of the Painlevé equation. The compatibility condition of the linear system (4.1), $\phi_{\lambda x} = \phi_{x\lambda}$, implies

$$A_x - B_\lambda + AB - BA = 0. \tag{4.3}$$

4.1 Derivation of P_{IV-3} hierarchy

The fourth Painlevé equation:

$$u_{xx} = \frac{u_x^2}{2u} + \frac{3}{2}u^3 + 4xu^2 + 2(x^2 - \alpha) + \frac{\beta}{u}, \tag{4.4}$$

can be written as the compatibility condition of the matrix linear problem (4.1) with

$$
\begin{array}{llll}
B = B_0\lambda^2 + B_1\lambda + B_2, & A = \sum_{j=0}^{4} A_j\lambda^{3-j}, & & \\
B_0 = \frac{1}{2}\sigma_3, & B_1 = \begin{pmatrix} 0 & iw \\ iv & 0 \end{pmatrix}, & B_2 = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}, & \\
A_0 = \frac{1}{2}\sigma_3, & A_1 = B_1, & A_2 = \begin{pmatrix} x+u & 0 \\ 0 & -x-u \end{pmatrix}, & \\
A_3 = \begin{pmatrix} 0 & i(w_x + 2xw) \\ i(2xv - v_x) & 0 \end{pmatrix}, & A_4 = \gamma_0\sigma_3, & u = vw. &
\end{array}
\tag{4.5}
$$

As a generalization of this linear problem we will use

$$
A = \sum_{j=0}^{2m+2} A_j\lambda^{2m+1-j}, \qquad B = B_0\lambda^2 + B_1\lambda + B_2, \tag{4.6}
$$

where m is a positive integer. Furthermore we set

$$
\begin{array}{llll}
B_0 = \frac{1}{2}\sigma_3, & B_1 = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}, & B_2 = \begin{pmatrix} -pq & 0 \\ 0 & pq \end{pmatrix}, & \\
A_0 = \frac{1}{2}\sigma_3, & A_{2j} = a_{2j}\sigma 3, & j = 1, \dots, m, & \\
A_{2j+1} = \begin{pmatrix} 0 & b_j \\ c_j & 0 \end{pmatrix}, & j = 0, 1, \dots, m, & A_{2m+2} = \gamma_0\sigma_3. &
\end{array}
\tag{4.7}
$$

Now we substitute A and B from (4.6) into (4.3) and we get

$$
\begin{array}{l}
\left(\sum_{j=0}^{2m+2} A_j\lambda^{2m+1-j}\right)(B_0\lambda^2 + B_1\lambda + B_2) - (B_0\lambda^2 + B_1\lambda + B_2)\left(\sum_{j=0}^{2m+2} A_j\lambda^{2m+1-j}\right) \\
+ \sum_{j=0}^{2m+2} A_{j,x}\lambda^{2m+1-j} - (2B_0\lambda + B_1) = 0 \\
(A_0\lambda^{2m+1} + A_1\lambda^{2m} + \cdots + A_{2m}\lambda + A_{2m+1} + A_{2m+2}\lambda^{-1})(B_0\lambda^2 + B_1\lambda + B_2) - \\
(B_0\lambda^2 + B_1\lambda + B_2)(A_0\lambda^{2m+1} + A_1\lambda^{2m} + \cdots + A_{2m}\lambda + A_{2m+1} + A_{2m+2}\lambda^{-1}) \\
+(A_{0,x}\lambda^{2m+1} + A_{1,x}\lambda^{2m} + A_{2,x}\lambda^{2m-1} + \cdots + A_{2m-1,x}\lambda^2 + A_{2m,x}\lambda + A_{2m+1,x} + A_{2m+2,x}\lambda^{-1}) \\
-(2B_0\lambda + B_1) = 0,
\end{array}
\tag{4.8}
$$

Next we will set the coefficient of each power of λ to zero. Equating the coefficient of λ^{-1} to zero, we get
$A_{2m+2}B_2 - B_2A_{2m+2} + A_{2m+2,x} = 0$ or

$$A_{2m+2,x} = [B_2, A_{2m+2}]. \tag{4.9}$$

Equating the constant term to zero, we get
$A_{2m+1}B_2 - B_2A_{2m+1} + A_{2m+1,x} - B_1 + A_{2m+2}B_1 - B_1A_{2m+2} = 0.$
It follows that

$$A_{2m+1,x} = B_1 + [B_1, A_{2m+2}] + [B_2, A_{2m+1}]. \tag{4.10}$$

Equating the coefficient of λ to zero, we obtain
$A_{2m}B_2 + A_{2m+1}B_1 + A_{2m+2}B_0 - B_2A_{2m} - B_1A_{2m+1} - B_0A_{2m+2} + A_{2m,x}$
$- 2B_0 = 0$ and hence

$$A_{2m,x} = 2B_0 + [B_0, A_{2m+2}] + [B_1, A_{2m+2}] + [B_2, A_{2m}]. \tag{4.11}$$

Equating the coefficient of λ^{2m-1} to zero gives
$A_2B_2 - B_2A_2 + A_3B_1 - B_1A_3 + A_{2,x} = 0$ and so

$$A_{2,x} = [B_2, A_2] + [B_1, A_3]. \tag{4.12}$$

Equating the coefficient of λ^{2m+3} to zero, we have
$A_0B_0 - B_0A_0 = 0$ or

$$[B_0, A_0] = 0. \tag{4.13}$$

Equating the coefficient of λ^{2m+2} to zero, we get

$$[B_0, A_1] + [B_1, A_0] = 0. \tag{4.14}$$

Equating the coefficient of λ^j to zero, we obtain

$$A_{j,x} = [B_0, A_{j+2}] + [B_1, A_{j+1}] + [B_2, A_j], \qquad j = 0, 1, \ldots, 2m-1. \tag{4.15}$$

Substituting A_j and B_j from (4.7) into (4.15) yields

$$\begin{pmatrix} a_{j,x} & b_{j,x} \\ c_{j,x} & -a_{j,x} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} a_{j,2} & b_{j,2} \\ c_{j,2} & -a_{j,2} \end{pmatrix} - \begin{pmatrix} a_{j,2} & b_{j,2} \\ c_{j,2} & -a_{j,2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}\begin{pmatrix} a_{j+1} & b_{j+1} \\ c_{j+1} & -a_{j+1} \end{pmatrix} - \begin{pmatrix} a_{j+1} & b_{j+1} \\ c_{j+1} & -a_{j+1} \end{pmatrix}\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} -pq & 0 \\ 0 & pq \end{pmatrix}\begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} - \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}\begin{pmatrix} -pq & 0 \\ 0 & pq \end{pmatrix}. \tag{4.16}$$

Equation (4.16) gives

$a_{j,x} = \frac{1}{2}a_{j+2} - \frac{1}{2}a_{j+2} + pc_{j+1} - qb_{j+1} - pqa_j + pqa_j$

Replacing j by $2j$ gives

$$a_{2j,x} = pc_{2j+1} - qb_{2j+1}, \qquad j = 1, 2, \ldots, m. \tag{4.17}$$

Equation (4.16) also gives $b_{j,x} = \frac{1}{2}b_{j+2} + \frac{1}{2}b_{j+2} - pa_{j+1} - pa_{j+1} - pqb_j - pqb_j$

$c_{j,x} = -c_{j+2} + 2qa_{j+1} + 2pqc_j$

Replacing j by $2j-1$ gives

$$b_{2j-1,x} = b_{2j+1} - 2pa_{2j} - 2pqb_{2j-1}, \qquad j = 1, 2, \ldots, m. \tag{4.18}$$

$$c_{2j-1,x} = -c_{2j+1} + 2qa_{2j} + 2pqc_{2j-1}, \qquad j = 1, 2, \ldots, m. \tag{4.19}$$

From (4.11) we have

$a_{2m,x} = 1 + \frac{1}{2}a_{2m+2} - \frac{1}{2}a_{2m+2} + pc_{2m+1} - qb_{2m+1} - pqa_{2m} + pqa_{2m}$.

Thus,

$$a_{2m,x} = 1 + pc_{2m+1} - qb_{2m+1}, \tag{4.20}$$

From (4.10) we obtain

$$\begin{pmatrix} a_{2m+1,x} & b_{2m+1,x} \\ c_{2m+1,x} & -a_{2m+1,x} \end{pmatrix} = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} + \begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix}\begin{pmatrix} \gamma_0 & 0 \\ 0 & -\gamma_0 \end{pmatrix}$$
$$- \begin{pmatrix} \gamma_0 & 0 \\ 0 & -\gamma_0 \end{pmatrix}\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} + \begin{pmatrix} -pq & 0 \\ 0 & pq \end{pmatrix}\begin{pmatrix} a_{2m+1} & b_{2m+1} \\ c_{2m+1} & -a_{2m+1} \end{pmatrix}$$
$$- \begin{pmatrix} a_{2m+1} & b_{2m+1} \\ c_{2m+1} & -a_{2m+1} \end{pmatrix}\begin{pmatrix} -pq & 0 \\ 0 & pq \end{pmatrix}. \tag{4.21}$$

As a result we get
$c_{2m+1,x} = q + q\gamma_0 + q\gamma_0 + pqc_{2m+1} + pqc_{2m+1} = q(2pc_{2m+1} + 2\gamma_0 + 1)$ so

$$c_{2m+1,x} - q(2pc_{2m+1} + 2\gamma_0 + 1) = 0, \tag{4.22}$$

and
$b_{2m+1,x} = p - p\gamma_0 - p\gamma_0 - pqb_{2m+1} - pqb_{2m+1} = p(-2qb_{2m+1} - 2\gamma_0 + 1)$
Thus,

$$b_{2m+1,x} + p(2qb_{2m+1} + 2\gamma_0 - 1) = 0. \tag{4.23}$$

We will show that the system (4.22)-(4.23) has the following first integral

$$\sum_{j=1}^{m+1} b_{2j-1}c_{2m+3-2j} + \sum_{j=1}^{m} a_{2j}a_{2m-2j+2} = 2x(a_2 + pq) + \gamma_1, \tag{4.24}$$

where γ_1 is a constant of integration.
First of all when $m = 1$ we have

$$\begin{aligned} c_{3,x} - q(2pc_3 + 2\gamma_0 + 1) = 0, \\ b_{3,x} + p(2qb_3 + 2\gamma_0 - 1) = 0. \end{aligned} \tag{4.25}$$

and from equations (4.18), (4.19), (4.20) we have

$$\begin{aligned} b_{1,x} &= b_3 - 2pa_2 - 2pqb_1, \\ c_{1,x} &= -c_3 + 2qa_2 + 2pqc_1, \\ a_{2,x} &= 1 + pc_3 - qb_3, \end{aligned} \tag{4.26}$$

Then from (4.7) we have $a_0 = \frac{1}{2}$, $b_0 = 0$, $c_0 = 0$, and $a_1 = 0$, $b_1 = p$, $c_1 = q$.
From (4.26) we get

$$\begin{aligned} b_3 &= p_x + 2pa_2 + 2p^2q, \\ c_3 &= -q_x + 2qa_2 + 2pq^2. \end{aligned} \tag{4.27}$$

Substituting (4.26) into (4.27) we get $a_{2,x} = 1 - pq_x - qp_x$ and hence

$$a_2 = -pq + x + \alpha_2, \tag{4.28}$$

where α_2 is a constant of integration. Let $\alpha_2 = 0$. Then multiply equation (4.25) with b_1 and c_1 respectively we get
$b_1c_{3,x} = pq(2pc_3 + 2\gamma_0 + 1)$, and $c_1b_{3,x} = -pq(2qb_3 + 2\gamma_0 - 1)$.
So $b_1c_{3,x} + c_1b_{3,x} = 2pq(pc_3 - qp_3 + 1) = 2pqa_2$.

By integration by parts we have

$$(b_1c_3)_x - b_{1,x}c_3 + (c_1b_3)_x - c_{1,x}b_3 = 2pqa_2.(b_1c_3 + c_1b_3)_x = p_xc_3 + q_xb_3 + 2pqa_2$$

Substituting the value of b_3, c_3 from (4.27) into the last equation, we obtain $(b_1c_3 + c_1b_3)_x = 2(pqa_2)_x + (p^2q^2)_x$.
Now integrating with respect to x gives

$$b_1c_3 + c_1b_3 = 2pqa_2 + p^2q^2 + \gamma_1, \tag{4.29}$$

where γ_1 is the constant of integration.
Next we consider the case $m = 2$.
$c_{5,x} = q(2pc_5 + 2\gamma_0 + 1)$, $b_{5,x} = -p(2qb_5 + 2\gamma_0 - 1)$.
and from equation (4.18), (4.19), (4.20) we have

$$\begin{aligned} b_{2j-1,x} &= b_{2j+1} - 2pa_{2j} - 2pqb_{2j-1}, \quad j = 1, 2 \\ c_{2j-1,x} &= -c_{2j+1} + 2qa_{2j} + 2pqc_{2j-1}, \quad j = 1, 2 \\ a_{4,x} &= 1 + pc_5 - qb_5, \end{aligned} \tag{4.30}$$

thus, $b_{3,x} = b_5 - 2pa_4 - 2pqb_3$ and $c_{3,x} = -c_5 + 2qa_4 + 2pqc_3$

$$\begin{aligned} b_5 &= b_{3,x} + 2pa_4 + 2pqb_3, \\ c_5 &= -qc_{3,x} + 2qa_4 + 2pqc_3, \end{aligned} \tag{4.31}$$

Then multiply $c_{5,x}$ and $b_{5,x}$ with b_1 and c_1 respectively we get $b_1c_{5,x} = pq(2pc_5 + 2\gamma_0 + 1)$, $c_1b_{5,x} = -pq(2qb_5 + 2\gamma_0 - 1)$, so

$$b_1c_{5,x} + c_1b_{5,x} = 2pq(a_{4,x} - 1) + 2pq = 2pqa_{4,x}. \tag{4.32}$$

Now since $b_1c_{5,x} = (b_1c_5)_x - b_{1,x}c_5$, and $c_1b_{5,x} = (c_1b_5)_x - c_{1,x}b_5$, then equation (4.32) becomes

$$(b_1c_5 + c_1b_5)_x = 2pqa_{4,x} + b_{1,x}c_5 + c_{1,x}b_5. \tag{4.33}$$

Substituting the value of b_5, c_5 from equation (4.31) into equation (4.33), we get

$$(b_1c_5+c_1b_5)_x = 2pqa_{4,x} - b_{1,x}c_{3,x} + c_{1,x}b_{3,x} + 2p_xqa_4 + 2pq_xa_4 + 2pp_xqc_3 + 2pqq_xb_3$$

Substituting the value of $b_{1,x}$, $c_{1,x}$ from equation (4.26) into the last equation, and since $(2pqa_4)_x = 2pqa_{4,x} + 2p_xqa_4 + 2pq_xa_4$ we get

$$(b_1c_5 + c_1b_5)_x = (2pqa_4)_x - (c_3b_3)_x + (pc_{3,x} + qb_{3,x})(2pq + 2a_2) + 2pq(p_xc_3 + q_xb_3)$$

since $a_2 = -pq$, and substituting the value of b_3, c_3 from equation (4.27) into the last equation, we get
$(b_1c_5 + c_1b_5)_x - (2pqa_4)_x + (c_3b_3)_x = 0$. Now integrate it with respect to x, we have

$$b_1c_5 + c_1b_5 + 2a_2a_4 + c_3b_3 = \gamma_2 \tag{4.34}$$

where γ_2 is the constant of integration.
This shows that (4.24) is the first integral of the system (4.22) and (4.23).

In order to derive a hierarchy of the ordinary differential equation, we proceed as follows.

Define $u = -pq$, $v = \frac{p_x}{p}$, and introduce the notation U_j, V_j, $j = 0, 1, \ldots, m$ as follows:

$$\begin{aligned}
&U_j = a_{2j+2} - K_{2j+2}, \qquad j = 0, 1, \ldots, m-2,\\
&U_{m-1} = a_{2m} - x,\\
&U_m = 2x(a_2 - 2u) - \textstyle\sum_{j=1}^{m+1} b_{2j} - c_{2m+3-2j} - \sum_{j=1}^{m} a_{2j}a_{2m-2j-2},\\
&V_j = \tfrac{1}{p}b_{2j+3} - 2K_{2j+2}, \qquad j = 0, 1, \ldots, m-2,\\
&V_{m-1} = \tfrac{1}{p}b_{2m+1} - 2x,\\
&V_m = \tfrac{1}{p}b_{2m+1,x} + 2qb_{2m+1} + 2U_m + 2x(2u - v) - 2,
\end{aligned} \tag{4.35}$$

where K_j are constants.
Equation (4.24) can be written in the form
$$\sum_{j=1}^{m+1} b_{2j-1}c_{2m+3-2j} + \sum_{j=1}^{m} a_{2j}a_{2m-2j+2} = 2x(a_2 + pq) + \gamma_1.$$
Then from (4.35), we have $2x(a_2 - 2u) - U_m = 2x(a_2 + pq) + \gamma_1$
Thus,

$$U_m + 2xu + \gamma_1 = 0. \tag{4.36}$$

Equation (4.23) can be written in the form $b_{2m+1,x} + p(2qb_{2m+1} + 2\gamma_0 - 1) = 0$
Multiply it by $\frac{1}{p}$ yields

$\frac{1}{p}b_{2m+1,x} + (2qb_{2m+1} + 2\gamma_0 - 1) = 0$. Then from (4.35), we have
$V_m - 2qb_{2m+1} - 2U_m - 2x(2u - v) + 2 + 2\gamma_0 - 1 + 2qb_{2m+1} + 1 = 0$
and hence $V_m - 2qb_{2m+1} - 2(a_{2m} - x) - 2x(2u - v) + 2 + 2\gamma_0 - 1$. Thus,

$$V_m + 2xv + 2\gamma_0 + 2\gamma_1 + 1 = 0. \tag{4.37}$$

We will show that U_j, V_j, $j = 1.2. \ldots, m$, satisfy

$$\begin{pmatrix} U_j \\ V_j \end{pmatrix} = R_{IV} \begin{pmatrix} U_{j-1} \\ V_{j-1} \end{pmatrix} + 2K_j \begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.38}$$

where R_{IV} is the recursion operator

$$R_{IV} = \begin{pmatrix} D_x - 2u + v + D_x^{-1}(2u_x - v_x) & 2u - D_x^{-1}u_x \\ -2D_x - 4u + 2v + 2D_x^{-1}(2u_x - v_x) & D_x + 2u + v - 2D_x^{-1}u_x \end{pmatrix}. \tag{4.39}$$

Substituting a_{2j+2} from equation (4.17) into equation (4.35), we have

$$U_j = D_x^{-1}(pc_{2j+3} - qb_{2j+3}) - k_{2j+2}.$$

Substituting c_{2j+3} from equation (4.19) and b_{2j+3} from equation (4.18) into the last equation, we have

$$U_j = D_x^{-1}[-pc_{2j+1,x}] + D_x^{-1}[-2upc_{2j+1}] - D_x^{-1}[qb_{2j+1,x}] + D_x^{-1}[2uqb_{2j+1}].$$

Using $b_{2j+1} = p(V_{j-1} + 2k_{2j+2})$ and $c_{2j+1} = \frac{1}{p}[U_{j-1,x} + u(V_{j-1} + 2k_{2j}]$ we get

$$\begin{aligned} U_j = {} & D_x^{-1}[-p(\tfrac{-p_x}{p^2}(D_xU_{j-1} - u(V_{j-1} + 2k_{2j}))) \\ & -\tfrac{p}{p}(U_{j-1,xx} - u_xV_{j-1} - 2k_{2j}u_x - uV_{j-1,x})] + D_x^{-1}[\tfrac{-2up}{p}(U_{j-1,x} - u(V_{j-1} + 2k_{2j}))] \\ & -D_x^{-1}[q(p_xV_{j-1} + 2p_xk_{2j} + pV_{j-1,x})] + D_x^{-1}[2uq(p(V_{j-1} + 2k_{2j}))]. \end{aligned}$$

Using $v = \frac{p_x}{p}$ and $u = -pq$, we obtain

$$\begin{aligned} U_j = {} & D_x^{-1}[vU_{j-1,x} - vuV_{j-1} - 2uvk_{2j} - U_{j-1,xx} + u_xV_{j-1} + 2u_xk_{2j} + uV_{j-1,x}] \\ & +D_x^{-1}[-2uU_{j-1,x} + 2u^2V_{j-1} + 4u^2k_{2j}] - D_x^{-1}[-uvV_{j-1} - 2uvk_{2j} - uV_{j-1,x}] \\ & +D_x^{-1}[-2u^2V_{j-1} - 4u^2k_{2j}]. \end{aligned}$$

Rearanging the terms we have

$$U_j = -U_{j-1,x} - 2uU_{j-1} + vU_{j-1} + D_x^{-1}(2u_x - v_x)U_{j-1} + 2uV_{j-1} - D_x^{-1}u_xV_{j-1} + 2k_{2j}u.$$

Similarly, we can prove that $V_j = \frac{1}{p} b_{2j+3} - 2k_{2j+2}$. Substituting b_{2j+3} from equation (4.18) into equation (4.35), we get

$$V_j = \frac{1}{p}[b_{2j+1,x} + 2pa_{2j+2} + 2pqb_{2j+1}] - 2k_{2j+2}.$$

Using $b_{2j+1} = p(V_{j-1} - 2k_{2j})$, we obtain

$$V_j = \frac{1}{p}[p_x(V_{j-1} + 2k_{2j}) + pV_{j-1,x}] + 2a_{2j+2} + 2qb_{2j+1} - 2k_{2j+2}.$$

Substituting a_{2j+2} from equation (4.17) into the last equation, we have

$$V_j = \frac{p_x}{p}(V_{j-1} + 2k_{2j}) + V_{j-1,x} + 2D_x^{-1}(pc_{2j+3} - qb_{2j+3}) + 2q(p(V_{j-1} + 2k_{2j})).$$

Using $v = \frac{p_x}{p}$ and $u = -pq$, we obtain

$$V_j = vV_{j-1} + 2vk_{2j} + V_{j-1,x} - 2uV_{j-1} - 4uk_{2j} + 2D_x^{-1}(pc_{2j+3} - qb_{2j+3}).$$

Since $D_x^{-1}(pc_{2j+3} - qb_{2j+3}) = -U_{j-1,x} - 2uU_{j-1} + vU_{j-1} + D_x^{-1}(2u_x - v_x)U_{j-1} + 2uV_{j-1} - D_x^{-1}u_xV_{j-1} + 2k_ju = U_j + k_{2j+2}$, we get

$$\begin{aligned} V_j &= vV_{j-1} + 2vk_{2j} + V_{j-1,x} - 2uV_{j-1} - 4uk_{2j} - 2U_{j-1,x} - 4uU_{j-1} + 2vU_{j-1} \\ &+2D_x^{-1}(2u_x - v_x)U_{j-1} + 4uV_{j-1} - 2D_x^{-1}u_xV_{j-1} + 4k_{2j}u + 2k_{2j+2} - 2k_{2j+2}. \end{aligned}$$

Rearanging the terms we have

$$\begin{aligned} V_j &= -2U_{j-1,x} - 4uU_{j-1} + 2vU_{j-1} + 2D_x^{-1}(2u_x - v_x)U_{j-1} \\ &+V_{j-1,x} + 2uV_{j-1} + vV_{j-1} - 2D_x^{-1}u_xV_{j-1} + 2k_{2j}v. \end{aligned}$$

Using equation (4.38) and hence $U_0 = u$ and $V_0 = v$, we find

$$\begin{pmatrix} U_j \\ V_j \end{pmatrix} = R_{IV}^j \begin{pmatrix} u \\ v \end{pmatrix} + 2\sum_{i=1}^{j-1} K_{2i} R_{IV}^{j-i} \begin{pmatrix} u \\ v \end{pmatrix} + 2K_j \begin{pmatrix} u \\ v \end{pmatrix}, j = 0, 1, ..., m. \tag{4.40}$$

Therefore, equation (4.36)-(4.37) can be written as

$$R_{IV}^m \begin{pmatrix} u \\ v \end{pmatrix} + 2\sum_{i=1}^{m-1} k_{2i} R_{IV}^{m-i} \begin{pmatrix} u \\ v \end{pmatrix} + 2x \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \gamma_1 \\ \gamma_0 + \gamma_1 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.41}$$

The coefficients A and B in the linear problem (4.1) has the form (4.6), where

$$\begin{aligned}
&B_0 = \tfrac{1}{2}\sigma_3, && B_1 = \begin{pmatrix} 0 & p \\ \frac{-u}{p} & 0 \end{pmatrix}, && B_2 = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}, \\
&A_0 = \tfrac{1}{2}\sigma_3, && A_1 = B_1, && A_{2j} = a_{2j}\sigma 3, \qquad j = 1, \ldots, m, \\
&A_{2j+1} = \begin{pmatrix} 0 & b_j \\ c_j & 0 \end{pmatrix}, && j = 0, 1, \ldots, m, && A_{2m+2} = \gamma_0\sigma_3.
\end{aligned} \tag{4.42}$$

p satisfies $p_x = pv$, a_{2j}, b_{2j+1}, c_{2j+1}, $j+1, 2, \ldots, m$ are given by

$$\begin{aligned}
&a_{2j} = U_{j-1} + K_{2j}, \qquad j = 1, \ldots, m-1, \\
&a_{2m} = U_{m-1} + x, \\
&b_{2j+1} = p(V_{j-1} + 2K_{2j}), \qquad j = 1, \ldots, m-1, \\
&b_{2m+1} = p(V_{jm-1} + 2x), \\
&c_{2j+1} = \tfrac{1}{p}[D_x U_{j-1} - u(V_{j-1} + 2K_{2j})], \qquad j = 1, \ldots, m-1, \\
&c_{2m+1} = \tfrac{1}{p}[D_x U_{m-1} - u(V_{m-1} + 2x)].
\end{aligned} \tag{4.43}$$

When $m = 1$, u satisfies the fourth Painlevé equation. Thus, The hierarchy (4.41) is called a fourth Painlevé hierarchy and we denote it by P_{IV-3}.

4.1.1 Example

When $m = 2$, (4.41) becomes

$$R_{IV}^2 + 2\sum_{i=1}^{1} K_{2i} R_{IV}^{2-i} \begin{pmatrix} u \\ v \end{pmatrix} + 2x \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \gamma_1 \\ \gamma_0 + \gamma_1 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$R_{IV}^2 + 2K_2 R_{IV} \begin{pmatrix} u \\ v \end{pmatrix} + 2x \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \gamma_1 \\ \gamma_0 + \gamma_1 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.44}$$

Calculation shows that

$$\begin{aligned}
R_{IV} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} -D_x - 2u + v + D_x^{-1}(2u_x - v_x) & 2u - D_x^{-1}u_x \\ -2D_x - 4u + 2v + 2D_x^{-1}(2u_x - v_x) & D_x + 2u + v - 2D_x^{-1}u_x \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \begin{pmatrix} -u_x - u^2 + 2uv \\ v_x - 2u_x - 2u^2 + v^2 + 2uv \end{pmatrix}.
\end{aligned} \tag{4.45}$$

Using $R_{IV}^2\begin{pmatrix} u \\ v \end{pmatrix} = R_{IV}\big(R_{IV}\begin{pmatrix} u \\ v \end{pmatrix}\big)$, we find

$$\begin{aligned} R_{IV}^2\begin{pmatrix} u \\ v \end{pmatrix} &= R_{IV}\begin{pmatrix} -u_x - u^2 + 2uv \\ v_x - 2u_x - 2u^2 + v^2 + 2uv \end{pmatrix} \\ &= \begin{pmatrix} u_{xx} - 3vu_x + 4uv^2 - 2u^3 \\ v_{xx} + 2uu_x + 3vv_x - 6u_xv - 6u^2v + 6v^2u + v^3 \end{pmatrix}. \end{aligned} \tag{4.46}$$

So from (4.45),(4.46),(4.44) we have

$$u_{xx} = (3v + 2K_2)u_x - 3uv^2 - 4K_2uv + 2u^3 + 2K_2u^2 + -2xu - \gamma_1, \tag{4.47}$$

$$\begin{aligned} v_{xx} &= -(3v + 2K_2)v_x + 2(3v + 2K_2)u_x - v^3 - 6uv^2 - 2K_2v^2 - 2xv + 6u^2v - 4K_2uv \\ &+4K_2u^2 - (\gamma_0 + \gamma_1 + 1). \end{aligned} \tag{4.48}$$

The elimination of v between (4.47) and (4.48) gives a fourth-order equation of u.

linear system for (4.47) and (4.48) is given by (4.1) with $B = B_0\lambda^2 + B_1\lambda + B_2$, $A = \sum_{j=0}^{6} A_j\lambda^{5-j}$, where $B_j, j = 0, 1, 2$, and $A_j, j = 0, 1$ are given by (4.42) and

$$\begin{aligned} &A_2 = (u + K_2)\sigma_3, \qquad A_3 = \begin{pmatrix} 0 & p(v + 2K_2) \\ \frac{1}{p}(u_x - uv - 2K_2u) & 0 \end{pmatrix}, \\ &A_4 = -(u_x + u^2 - 2uv - 2K_2u - x)\sigma_3, \qquad A_5 = \begin{pmatrix} 0 & b_5 \\ c_5 & 0 \end{pmatrix}, \qquad A_6 = \gamma_0\sigma_3, \\ &b_5 = p(v_x - 2u_x + 2uv - 2u^2 + 2K_2v + 2x), \\ &c_5 = \tfrac{-1}{p}[u_{xx} - 2(v + K_2)u_x - uv_x - 2u^3 + 2u^2v + 2uv^2 + 2K_2uv + 2xu]. \end{aligned} \tag{4.49}$$

The hierarchy was given before (2.2). Indeed the transformation $y = -u_x + uv - u^2$, $w = -v$ transforms the system (4.47) and (4.48) into the system

$$\begin{aligned} u_{xx} &= (3v + 2K_2)u_x - 3uv^2 - 4K_2uv + 2u^3 + 2K_2u^2 - 2xu - \gamma_1 \\ &= 3(v + \tfrac{2}{3}K_2)u_x - 3u(v^2 + \tfrac{4}{3}K_2v) + 2u^3 + 2K_2u^2 - 2xu - \gamma_1, \end{aligned} \tag{4.50}$$

Add and delete the term $\frac{4}{9}k_2^2$ to the term $v^2 + \frac{4}{3}K_2v$ we have

$$u_{xx} = 3(v + \frac{2}{3}K_2)u_x - 3u(v + \frac{2}{3}K_2)^2 + \frac{4}{3}K_2^2u + 2u^3 + 2K_2u^2 - 2xu - \gamma_1, \tag{4.51}$$

and

$$
\begin{aligned}
v_{xx} &= -(3v+2K_2)v_x + 2(3v+2K_2)u_x - v^3 - 6uv^2 - 2K_2v^2 - 2xv + 6u^2v - 4K_2uv \\
&\quad + 4K_2u^2 - (2\gamma_0 + 2\gamma_1 + 1), \\
&= -3(v+\tfrac{2}{3}K_2)v_x + 6(v+\tfrac{2}{3}K_2)u_x - 6uv(v+\tfrac{2}{3}K_2) + 6u^2(v+\tfrac{2}{3}K_2) \\
&\quad - v^3 - 2K_2v^2 - 2xv - (2\gamma_0 + 2\gamma_1 + 1),
\end{aligned} \tag{4.52}
$$

Take $-6(v + \frac{2}{3}K_2)$ as a common factor we have

$$
\begin{aligned}
v_{xx} &= -3(v+\tfrac{2}{3}K_2)v_x - 6(v+\tfrac{2}{3}K_2)[-u_x + uv - u^2] - v^3 - 2K_2v^2 - 2xv - (2\gamma_0 + 2\gamma_1 + 1), \\
v_{xx} &= -3(v+\tfrac{2}{3}K_2)v_x - 6y(v+\tfrac{2}{3}K_2) - v^3 - 2K_2v^2 - 2xv - (2\gamma_0 + 2\gamma_1 + 1).
\end{aligned} \tag{4.53}
$$

Now

$$
y_x = -u_{xx} + u_xv + v_xu - 2uu_x \tag{4.54}
$$

Then substituting u_{xx} from equation (4.51) into equation (4.54), we obtain

$$
\begin{aligned}
-y_x &= 3(v+\tfrac{2}{3}K_2)u_x - u(3v^2 + 4K_2v) + 2u^3 - 2K_2u^2 - 2xu + \gamma_1 - u_xv - uv_x + 2uu_x, \\
&= 2(v+\tfrac{2}{3}K_2)u_x - uv(3v + 4k_2) + 2u^3 - 2u(K_2u) - 2xu + \gamma_1 - uv_x, \\
&= 2(v+\tfrac{2}{3}K_2)(-y + uv - u^2) - uv(3v + 4k_2) + 2u^3 - 2u(K_2u) - 2xu + \gamma_1 - uv_x, \\
&= -2uy - 2vy - 2K_2y - uv^2 - uv_x - 2k_2uv + \gamma_1 - 2xu,
\end{aligned} \tag{4.55}
$$

Then

$-y_x - \gamma_1 + 2K_2y + 2vy = -u(v_x + 2y + 2K_2v + 2x + v^2)$

Thus

$$
u = \frac{y_x + 2y(K_2 + v) + \gamma_1}{(v_x + 2y + 2K_2v + 2x + v^2)} \tag{4.56}
$$

Substituting u into $y = -u_x + uv - u^2$, we obtain

$$y=-\frac{(v_x+2y+v^2+2K_2v+2x)(y_{xx}-2y_x(v+K_2)-2yv_x)}{(v_x+2y+2K_2v+2x+v^2)^2}$$

$$-\frac{(y_x+2y(K_2+v)+\gamma_1)(v_{xx}+2y_x+2vv_x+2v_x+2)}{(v_x+2y+2K_2v+2x+v^2)^2}$$

$$+v\frac{y_x+2y(K_2+v)+\gamma_1}{(v_x+2y+2K_2v+2x+v^2)}-\left(\frac{y_x+2y(K_2+v)+\gamma_1}{(v_x+2y+2K_2v+2x+v^2)}\right)^2,$$

or

$$y=\frac{(y_x+2y(K_2+v)+\gamma_1)(v_{xx}+2y_x+2vv_x+2v_x+2)}{(v_x+2y+2K_2v+2x+v^2)^2}+v\frac{y_x+2y(K_2+v)+\gamma_1}{(v_x+2y+2K_2v+2x+v^2)}$$

$$-(\frac{y_x+2y(K_2+v)+\gamma_1}{(v_x+2y+2K_2v+2x+v^2)})^2+\frac{(-y_{xx}-2y_x(v+K_2)(v_x+2y+v^2+2K_2v+2x)}{(v_x+2y+v^2+2K_2v+2x)^2}$$

$$-\frac{(-2y_x(v+K_2)-2yv_x)(v_x+2y+v^2+2K_2v+2x)}{(v_x+2y+v^2+2K_2v+2x)^2},$$

Multiply it by $(v_x+2y+v^2+2K_2v+2x)$ we get

$$y(v_x+2y+v^2+2K_2v+2x)=\frac{(y_x+2y(K_2+v)+\gamma_1)(v_{xx}+2y_x+2vv_x+2v_x+2)}{(v_x+2y+2K_2v+2x+v^2)}$$

$$+v(y_x+2y(K_2+v)+\gamma_1)-\frac{(y_x+2y(K_2+v)+\gamma_1)^2}{(v_x+2y+2K_2v+2x+v^2)}-y_{xx}+2y_x(v+K_2)+2yv_x,$$

so we have

$$y_{xx}=2y_x(v+K_2)+2yv_x+\frac{(y_x+2y(K_2+v)+\gamma_1)(v_{xx}+2y_x+2vv_x+2v_x+2)}{(v_x+2y+2K_2v+2x+v^2)}+$$
$$v(y_x+2y(K_2+v)+\gamma_1)-\frac{(y_x+2y(K_2+v)+\gamma_1)^2}{(v_x+2y+2K_2v+2x+v^2)}-y(v_x+2y+v^2+2K_2v+2x),$$

Thus

$$y_{xx}=2(yv)_x+2y_xK_2+\frac{(y_x+2y(K_2+v)+\gamma_1)}{(v_x+2y+2K_2v+2x+v^2)}[v_{xx}+2y_x+2vv_x+2v_x+$$
$$2+vv_x+2vy+v^3+2K_2v^2+2xv-y_x+2y(v+K_2)-\gamma_1],$$

After substituting by v_{xx} and y_x and $w = -v$ we have

$$y_{xx} = \frac{[y_x + 2y(w - K_2) - \gamma_1 - \gamma_0 + \frac{1}{2}]^2 - (\gamma_0 - \frac{1}{2})^2}{[2y - w_x + w^2 - 2K_2 w + 2x]} - 2(yw)_x + 2K_2 y_x - y[2y - w_x + w^2 - 2K_2 w + 2x]. \quad (4.57)$$

From (4.53) and $w = -v$ we have

$$w_{xx} = (3w-2K_2)w_x - 2y(3w-2K_2) - w^3 + 2K_2 w^2 - 2xw + (2\gamma_0 + 2\gamma_1 + 1). \quad (4.58)$$

The two equation (4.57) and (4.58) is the second member of the fourth Painlevé hierarchy P_{IV-1}.

4.1.2 Special solutions of P_{IV-3} hierarchy

In this subsection, we will show that the fourth Painlevé hierarchy P_{IV-3}(4.41) admits special solutions in terms of second Painlevé hierarchy

$$R_{II}^m u - \sum_{i=1}^{m-1} K_{2i} R_{II}^{m-i} u - 4^{m-1}(xu + \alpha) = 0, \quad (4.59)$$

where $R_{II} = D_x^2 - 4u^2 + 4uD_x^{-1}u_x$.

Suppose that $m = 2n$, $p = 1$, $2\gamma_1 + 2\gamma_0 + 1 = 0$, $K_{4j-2} = 0$, $\quad j = 1, 2, .., n$. Then $u = -q$, $v = 0$, and the hierarchy (4.36)-(4.37) reduces to the following hierarchy

$$U_{2n} + 2xu + \gamma_1 = 0, \qquad V_{2n} = 0. \quad (4.60)$$

The operator (4.39) becomes

$$R = \begin{pmatrix} -D_x - 2u + 2D_x^{-1}u_x & 2u - D_x^{-1}u_x \\ -2D_x - 4u + 4D_x^{-1}u_x & D_x + 2u - 2D_x^{-1}u_x \end{pmatrix}. \quad (4.61)$$

Now we will use induction to prove that

$$\begin{aligned} &U_{2j} = (D_x - 2u)(D_x + 2u - 2D_x^{-1}u_x)U_{2j-2} + 2K_{4j}u, && j = 1, 2, .., n, \\ &V_{2j} = 0, && j = 1, 2, .., n, \\ &U_{2j+1} = -(D_x + 2u - 2D_x^{-1}u_x)U_{2j}, && j = 1, 2, .., n-1, \\ &V_{2j+1} = 2U_{2j+1}, && j = 1, 2, ..., n-1. \end{aligned} \quad (4.62)$$

First, we note that $U_0 = u$, $V_0 = 0$. Thus (4.38) gives
$U_1 = -(D_x + 2u - 2D_x^{-1}u_x)U_0$, $V_1 = 2U_1$,
$V_2 = (D_x + 2u - 2D_x^{-1}u_x)(V_1 - 2U_1) = 0$, and

$$\begin{aligned} U_2 &= -(D_x - 2u)U_1 + 2K_4 u \\ &= (D_x - 2u)(D_x + 2u - 2D_x^{-1}u_x)U_0 + 2K_4 u. \end{aligned} \tag{4.63}$$

Hence the formula (4.62) are true when $j = 1$.

Assume that (4.62) is true for $j = k$, $1 \leq k \leq n-1$. Then substituting $V_{2k} = 0$ into (4.38) implies that $U_{2k+1} = -(D_x + 2u - 2D_x^{-1}u_x)U_{2k}$ and
$V_{2k+1} = -2(D_x + 2u - 2D_x^{-1}u_x)U_{2k} = 2U_{2k+1}$.
Since $V_{2k+2} = (D_x + 2u - 2D_x^{-1}u_x)(V_{2k+1} - 2U_{2k+1})$, we get $V_{2k+2} = 0$. Using (4.38), $V_{2k+1} = 2U_{2k+1}$, and $U_{2k+1} = -(D_x + 2u - 2D_x^{-1}u_x)U_{2k}$, we get

$$\begin{aligned} U_{2k+2} &= -(D_x + 2u - 2D_x^{-1}u_x)U_{2k+1} + (2u - D_x^{-1}u_x)V_{2k+1} + 2K_{4k+1}u \\ &= -(D_x - 2u)U_{2k+1} + 2K_{4k+4}u \\ &= (D_x - 2u)(D_x + 2u - 2D_x^{-1}u_x)U_{2k} + 2K_{4k+4}u. \end{aligned} \tag{4.64}$$

This ends the proof.

Now using $(D_x - 2u)(D_x + 2u - 2D_x^{-1}u_x) = D_x^2 - 4u^2 + 4uD_x^{-1}u_x = R_{II}$, $U_0 = u$, we obtain

$$U_{2j} = R_{II}U_{2j-2} + 2K_{4j}u, \tag{4.65}$$

and hence

$$U_{2j} = R_{II}^j u + 2\sum_{i=1}^{j-1} K_{4i}R_{II}^{j-i}u + 2K_{4j}u. \tag{4.66}$$

Thus, equation (4.60) yields

$$R_{II}^n u + 2\sum_{i=1}^{n-1} K_{4i}R_{II}^{n-i}u + 2xu + \gamma_1 = 0. \tag{4.67}$$

The hierarchy (4.67) is equivalent to the second Painlevé hierarchy (4.59). Therefore, if u is a solution of the $2n$ th member of the fourth Painlevé hierarchy (4.41) with $v = 0$, $2\gamma_1 + 2\gamma_0 + 1 = 0$, $K_{4j-2} = 0, \quad j = 1, 2, .., n$, then u satisfies the nth member of the second Painlevé hierarchy (4.67). The linear problem for the second Painlevé hierarchy (4.67) is given by

$$
\begin{aligned}
&B = B_0\lambda^2 + B_1\lambda + B_2, \quad A = \sum_{j=0}^{4n+2} A_j \lambda^{4n+1-j},\\
&B_0 = A_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix}, \quad A_1 = B_1 = \begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix},\\
&B_2 = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}, \quad A_{2j} = \begin{pmatrix} a_{2j} & 0 \\ 0 & -a_{2j} \end{pmatrix}, \quad j = 1, ..., 2n,\\
&A_{2j+1} = \begin{pmatrix} 0 & b_j \\ c_j & 0 \end{pmatrix}, j = 1, 2, ..., n, \quad A_{4n+2} = \begin{pmatrix} \gamma_0 & 0 \\ 0 & -\gamma_0 \end{pmatrix},
\end{aligned}
\tag{4.68}
$$

where a_j, b_j and c_j are given by

$$
\begin{aligned}
&a_{4j} = -(D_x + 2u - 2D_x^{-1}u_x)U_{2j-2} + K_{4j}, \quad j = 1, ..., n-1,\\
&a_{4j-2} = U_{2j-2}, \quad j = 1, ..., n,\\
&a_{4n} = -(D_x + 2u - 2D_x^{-1}u_x)U_{2n-2} + x,\\
&c_{4j-1} = D_x U_{2j-2}, \quad j = 1, ..., n,\\
&c_{4j+1} = -[R_{II}U_{2j-2} + 2K_{4j}u], \quad j = 1, 2, ..., n-1,\\
&c_{4n+1} = -[R_{II}U_{2n-2} + 2xu], \quad j = 1, 2, .., n,\\
&b_{4j-1} = 0, \quad b_{4j+1} = 2a_{4j}, \quad j = 1, 2, .., n.
\end{aligned}
\tag{4.69}
$$

Therefore, the above relation between the fourth Painlevé hierarchy (4.41) and the second Painlevé hierarchy (4.67) gives rise to the linear problem (4.68) for the second Painlevé hierarchy.

For example, the second member of the fourth Painlevé hierarchy (4.36) and (4.37), that is equation (4.47) and (4.48) has the special solution $K_2 = 0$, $2\gamma_0 + 2\gamma_1 + 1 = 0$, $v = 0$ and u satisfies the second Painlevé equation

$$
u_{xx} = 2u^3 - 2xu + \gamma_0 + \frac{1}{2}. \tag{4.70}
$$

Bibliography

[1] P. A. Clarkson, "Painlevé Equations Nonlinear Special Functions", http://math.nist.gov/Dlozier/SF21/SF21slides/Clarkson.pdf.

[2] N.A. Kudryashov, "The first and second Painlevé equations of higher order and some relations between them", Phys. Lett. A **224** (1997), 353-360.

[3] H. Airault, "Rational solutions of Painlevé equations", Stud. Appl. Math. **61** (1979), 31-53.

[4] P. R. Gordoa, N. Joshi and A. Pickering, "On a Generalized 2 + 1 Dispersive Water Wave Hierarchy", Publ. RIMS, Kyoto Univ. **37** (2001), 327-347

[5] A. Sakka, "On special solutions of second and fourth Painlevé hierarchies", A. Math. Phys., 174 (2008), 1-5.

[6] A. Sakka, "Linear problems and hierarchies of Painlevé equations", J. Phys. A:Theor. 40(2007)7687 .

[7] N.A. Kudryashov, "On The Fourth Painlevé Hierarchies", Theor. Math. Phys., 134 (2003), 86-93 .

[8] A.H. Sakka, "Linear problems and hierarchies of Painlevé equations", J. Phys. (2008) AM2/025210.