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Hamiltonian Structure of First and Second Painlevé Hierarchies

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Contents

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Abstract

In this thesis, the derivation of first and second Painlevé hierarchies and their Hamiltonian structures are studied. The first Painlevé hierarchy is derived from the Kadomtsev-Petviashvili hierarchy and the linear problem of that first Painlevé hierarchy is used to derive the Hamiltonian structure of that first Painlevé hierarchy.

In addition, The Modified Korteweg-de Vries hierarchy is derived from the Korteweg-de Vries hierarchy and the Modified Korteweg-de Vries hierarchy is used to derive a second Painlevé hierarchy. The linear problem for the second Painlevé hierarchy are found and the canonical coordinates for it are built. Moreover, the linear problem is used to drive the Hamiltonian structure of that second Painlevé hierarchy.

Chapter 1

Introduction

Around the beginning of the twentieth century, the six Painlevé equations (P_I-P_{VI}) were discovered by Painlevé , Gambier and their school [1] , [2] in an investigation of nonlinear second-order ordinary differential equations whose general solutions can not be expressed in terms of elementary and classical special functions; thus they define new transcendental functions.
The six Painlevé equations, P_I-P_{VI}, are listed below [1] , [2]

$$
\begin{aligned}
P_I \quad &: \quad W'' = 6W' + z, \\
P_{II} \quad &: \quad W'' = 2W^3 + zW + \alpha, \\
P_{III} \quad &: \quad W'' = \frac{(W')^2}{W} - \frac{1}{z}W' + \alpha W^3 + \frac{1}{z}(\beta W^2 + \gamma) + \frac{\delta}{W}, \\
P_{IV} \quad &: \quad W'' = \frac{(W')^2}{2W} + \frac{3}{2}W^3 + 4zW^2 + (2z^2 - 2\alpha)W + \frac{\beta}{W}, \\
P_V \quad &: \quad W'' = \frac{3W-1}{2W(W-1)}(W')^2 - \frac{1}{z}W' + \frac{1}{z^2}(W-1)^2(\alpha W + \frac{\delta}{W}) + \frac{\gamma W}{z} + \frac{\delta W(W+1)}{W-1}, \\
P_{VI} \quad &: \quad W'' = \frac{1}{2}\left(\frac{1}{W} + \frac{1}{W-1} + \frac{1}{W-z}\right)(W')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{W-z}\right)W' \\
&\qquad + \frac{W(W-1)(W-z)}{z^2(z^2-1)}\left(\frac{\alpha z(z-1)}{(W-z)^2} + \frac{\beta(z-1)}{(W-1)^2} + \frac{\gamma z}{W^2} + \delta\right),
\end{aligned}
$$

where α, β, γ and δ are constant parameters and primes denote differentiation with respect to z.

Painlevé transcendental functions appear in many areas of modern mathematics and physics and they play the same role in nonlinear problems as the classical special functions play in linear problems [2] , [3].

Ablowitz and Segur [4] demonstrated a close connection between completely integrable parial differential equations solvable by inverse scattering method. Flaschka and Newell [5] introduced the isomonodromy deformation method, which expresses the Painlevé equation as the compatibility condition of two linear systems of equations and are studied using Riemann-Hilbert methods [2].

In recent years there is a considerable interest in studying hierarchies of Painlevé equations. This interest is due to the connection between these hierarchies of Painlevé equations and completely integrable partial differential equations. A Painlevé hierarchy is an infinite sequence of nonlinear ordinary differential equations whose first member is a Painlevé equation. A first Painlevé hierarchy was given by Kudryashov through the reduction of the Korteweg-de Vries hierarchy [6], and Airault was the first to derive a second Painlevé hierarchy which obtained by similarity reduction of the modified Korteweg-de Vries hierarchy [7]. After that, Gordoa, Joshi and Pickering [8] have used non-isospectral scattering problems to derive new second Painlevé hierarchy and new fourth Painlevé hierarchy [2] , [9].

The Hamiltonian structure of the classical six Painlevé equations was discovered by Okamoto [10], Jimbo and Miwa [11]. Okamoto use the Hamiltonian structure of the second Painlevé equations to characterize the action of the Bäcklund transformations found by Gromak [12] and Lukashevich [13] in terms of affine Weyl groups and to produce immediately the so called Riccati-type classical solutions of the second Painelvé equation [12], [14]. Also several properties of the Yablonskii-Vorobév polynomials describing the rational solutions were proved using the Hamiltonian formulation [15] . Umemura and Watanabe used the Hamiltonian structure in to prove the irreducibility of P_{II} [16]. The Hamiltonians for $P_I - P_{VI}$ are rederived within the framework of the Adler-Kostant-Symes theorem and the classical R-matrix approach on a rational coadjoint orbit in the dual $L\mathfrak{g}^*$ of a loop algebra $L\mathfrak{g}$ [17].

In this thesis, our focus will be on the Hamiltonian structure of the first and second Painlevé hierarchies. This thesis is organized as follows:

In chapter two we give the basic definition and theorem related to our study.

In chapter three we derive the Korteweg-de Vries hierarchy from the Kadomtsev-Petviashvili hierarchy. Then we derive the first Painlevé hierarchy from the Korteweg-de Vries hierarchy.

In chapter four we derive the second Painlevé hierarchy from the Modified Korteweg-de Vries hierarchy.

In chapter five we derive the Hamiltonian structure for first Painlevé hierarchy.

In chapter six we derive the Hamiltonian structure for the second Painlevé hierarchy.

Chapter 2

Preliminaries

Our aim in this chapter is to review the basic definitions and theorems related to our study.

Definition 2.1. [18]

(1) Let $F(X, U)$ be a smooth function (infinitely differentiable) of $X = (x_1, ..., x_p)$, $U = (u_1, ..., u_q)$. The total derivative of F is the unique smooth function dF given by

$$dF = d_{x_1}F + ... + d_{x_p}F,$$

where $d_{x_i}F$ is the x_i total derivative of F defined by

$$d_{x_i}F = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_i} + ... + \frac{\partial F}{\partial u_q}\frac{\partial u_q}{\partial x_i}.$$

(2) The derivations with respect to x is denoted by ∂_x and ∂_x^{-1} denotes the inverse of ∂_x ; that is

$$\partial_x^{-1}\partial_x = 1 = \partial_x\partial_x^{-1}.$$

Definition 2.2. [18] A pseudo-differential operator is a formal infinite series

$$D = \sum_{i=-\infty}^{n} P_i[u]\partial_x^i,$$

whose coefficients P_i are differential functions. We say that D has order n provided its leading coefficient $P_n[u]$ is not identically zero.

Definition 2.3. [19] Let D be pseudo-differential operator. The projection $(D)_{\geq 0}$ of D is the part of positive powers of ∂_x. Similarly, we use the notation $(D)_{<0}$ for the projection of D to the part of negative powers of ∂_x.

Definition 2.4. [20] An m-dimensional smooth manifold is a set M, together with a countable collection of subsets $U_\alpha \subset M$, called coordinate charts, and one-to-one functions $\chi_\alpha : U_\alpha \to V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbb{R}^m$ which satisfy the following properties:

1. The coordinate charts cover M: $\bigcup_\alpha U_\alpha = M$.
2. On the overlap of any pair of coordinate charts $U_\alpha \cap U_\beta$ the composite map $\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha(U_\alpha \cap U_\beta) \to \chi_\beta(U_\alpha \cap U_\beta)$ is a smooth function.
3. If $x \in U_\alpha,\ \tilde{x} \in U_\beta$ are distinct points of M, then there exist open subsets $W \subset U_\alpha,\ \tilde{W} \subset U_\beta$, with $\chi_\alpha(x) \in W,\ \chi_\beta(\tilde{x}) \in \tilde{W}$, satisfying

$$\chi_\alpha^{-1}(W)\ \cap\ \chi_\beta^{-1}(\tilde{W}) = \phi.$$

Example 2.5. [18] The simplest m-dimensional smooth manifold is just the Euclidean space $\mathbb{R}^m$ itself. There is a single coordinate chart $U = \mathbb{R}^m$, with local coordinate map given by the identity: $\chi = I : \mathbb{R}^m \to \mathbb{R}^m$. More generally, any open subset $U \subset \mathbb{R}^m$ is an m-dimensional manifold with a single coordinate chart given by U itself, with local coordinate map the identity again .

Definition 2.6. [21] A Lie group is a group G which is at the same time a differentiable manifold, and such that the group operation $(g, h) \to gh^{-1}$ is differentiable.

Definition 2.7. [22] A Lie algebra $\mathfrak{g}$ is a vector space with a bilinear map

$$[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which is skew-symmetric

$$[X, Y] = -[Y, X], \qquad \forall X, Y \in \mathfrak{g}$$

and satisfies Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0, \quad \forall X, Y, Z \in \mathfrak{g}.$$

Definition 2.8. [23] A symplectic structure ω on a smooth manifold M is given by a bilinear form $\omega : M \times M \longrightarrow \mathbb{R}$ which satisfy the following properties

1. Skew-symmetric: $\omega(u, v) = -\omega(v, u)$ for all $u, v \in M$.
2. Totally isotropic: $\omega(v, v) = 0$ for all $v \in M$.
3. Nondegenerate: if $\omega(u, v) = 0$ for all $v \in M$ then $u = 0$.

Definition 2.9. [23] A symplectic manifold is a pair (M, ω) consisting of a smooth manifold M of even dimension, $\dim M = 2n$, and a symplectic structure ω.

Example 2.10. [19] Consider the smooth manifold $\mathbb{R}^{2n}$. For any coordinate $(p_1, ..., p_n, q_1, ..., q_n) \in \mathbb{R}^{2n}$, we defined the symplectic structure ω by $\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i$, which is called the standard symplectic structure on $\mathbb{R}^{2n}$, where $(\mathrm{d}p_i \wedge \mathrm{d}q_i)(\zeta_1, \zeta_2) = \begin{vmatrix} \mathrm{d}p_i(\zeta_1) & \mathrm{d}q_i(\zeta_1) \\ \mathrm{d}p_i(\zeta_2) & \mathrm{d}q_i(\zeta_2) \end{vmatrix}$, for all $\zeta_1, \zeta_2 \in \mathbb{R}^{2n}$.

Definition 2.11. [24] , [20] Let M be a smooth manifold.

1. The set of all tangent vector at $x \in M$ is a vector space called the tangent space to M at x, and it is denoted by T_xM.
2. The dual space to tangent space T_xM, the vector space of all linear functions $f : T_xM \to \mathbb{R}$ is called the cotangent space at x, and it is denoted by T_x^*M.
3. The collection (union) of all tangent spaces at all points of M is called the tangent bundle of M and it denoted by TM; that is
$$TM = \bigcup_{x \in M} T_xM.$$
4. The collection (union) of all cotangent spaces at all points of M is called the cotangent bundle of M and it denoted by T^*M; that is
$$T^*M = \bigcup_{x \in M} T_x^*M.$$

Definition 2.12. [25] Assume M is an n-dimensional manifold. A vector field on M is a function v that associates with every point $x \in M$ a vector $v(x) \in T_xM$.

Definition 2.13. [25] Suppose that (M, ω) is a symplectic manifold. Then there is a bundle isomorphism $\omega : TM \to T^*M$, between the tangent bundle TM and the cotangent bundle T^*M, with the inverse $\Omega : T^*M \to TM, \quad \Omega = \omega^{-1}$.

Definition 2.14. [23] Let $H \in C^\infty(M)$, the vector space of smooth functions from M to $\mathbb{R}$, the Hamiltonian vector field of H is the smooth vector field X_H defined by $\mathrm{d}H(Y) = \omega(X_H, Y)$.

Definition 2.15. [25]

1. An open cover of a given subset of a topological space X is a collection of open sets of X whose union contains a given subset.

2. A topological space X is compact if every open cover of X has a finite subcover.

3. A compact manifold is a manifold that is compact as a topological space.

Definition 2.16. [23] For any finite dimensional compact smooth manifold M, let $\mathrm{Map}(M; G)$ denote the group of smooth maps $M \to G$, where G is a finite dimensional Lie group. Then $\mathrm{Map}(M; G)$ is an infinite dimensional Lie group. If $M = S^1$ then $\mathrm{Map}(S^1; G)$ is called the loop group of G and denoted by LG. The Lie algebra of $\mathrm{Map}(M; G)$ is clearly $\mathrm{Map}(X; \mathfrak{g})$. If $M = S^1$ then $\mathrm{Map}(S^1; \mathfrak{g})$ is called the loop algebra and denoted by $L\mathfrak{g}$.

Definition 2.17. [19] Let G be a lie group and M be a manifold . We say that G is an action group on M if there exists a map

$$\bullet : G \times M \to M,$$

$$(g, p) \to g \bullet p$$

such that

1. $e \bullet p = p$ for all $p \in M$, where e is the identity of G.

2. $g_2 \bullet (g_1 \bullet p) = (g_2 g_1) \bullet$p for all $g_2, g_1 \in G$ and $p \in M$.

Definition 2.18. [19] Let G be a Lie group that act on a manifold M. The orbit of a point $x \in M$ is $\varphi_x = \{g \bullet x | g \in M\}$.

Definition 2.19. [21] Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The group G acts on $\mathfrak{g}$ by the adjoint action denoted by Ad:

$$(\mathrm{Ad}g)(X) = gXg^{-1}, \quad g \in G, \ X \in \mathfrak{g}.$$

Similarly, the coadjoint action of G on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is Ad^* defined by:

$$(\mathrm{Ad}^*g \ \Xi)(X) = \Xi(\mathrm{Ad}g^{-1}(X)), \quad g \in G, \ X \in \mathfrak{g}, \ \Xi \in \mathfrak{g}^*.$$

Definition 2.20. [27] Let $\mathfrak{g}^*$ be the dual space of lie algebra $\mathfrak{g}$ and $\Xi \in \mathfrak{g}^*$. The coadjoint orbits of Ξ is given by $\varphi_\Xi := \{\mathrm{ad}^*_X \Xi | X \in \mathfrak{g}\}$.

Theorem 2.21. *(Kirillov-Kostant-Souriau) [18] Let $\mathfrak{g}^*$ be the dual space of Lie algebra $\mathfrak{g}$. Suppose that $\Xi \in \mathfrak{g}^*$ and φ_Ξ is the coadjoint orbit of Ξ. Then φ_Ξ carries a symplectic structure .*

Theorem 2.22. *[18] The orbits of the co-adjoint representation of G are even-dimensional submanifolds of $\mathfrak{g}^*$.*

Definition 2.23. [25] The Poisson bracket of two functions f, g of the variables $(p_1, p_2, ..., p_n, q_1, q_2, ..., q_n)$ is given by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}.$$

Definition 2.24. [18] A Poisson bracket on a smooth manifold M is an operation that assigns a smooth real-valued function $\{F, H\}$ on M to each pair F, H of smooth, real-valued functions, with the basic properties:

(a) Bilinearity: $\{cF + c\,'P, H\} = c\{F, H\} + c\,'\{P, H\}$,
$\{F, c\ H + c\,'P\} = c\{F, H\} + c\,'\{F, P\}$,
for constants $c, c\,' \in \mathbf{R}$,

(b) Skew-Symmetry: {F,H}=-{H,F},

(c) Jacobi Identity: $\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = 0$,

(d) Leibniz' Rule: $\{F, H \cdot P\} = \{F, H\} \cdot P + H \cdot \{F, P\}$.

Chapter 3

First Painlevé hierarchy

In this chapter, we will derive Korteweg-de Vries hierarchy from the Kadomtsev-Petviashvili hierarchy and then we derive a first Painlevé hierarchy from Korteweg-de Vries hierarchy [19].

3.1 Korteweg-de Vries hierarchy from Kadomtsev-Petviashvili hierarchy

Consider the Lax operator of the Kadomtsev-Petviashvili (KP) hierarchy

$$L = \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots . \tag{3.1}$$

The operator L obeys the Lax equation of the KP hierarchy

$$\partial_n L = [B_n, L], \qquad B_n = (L^n)_{\geq 0}, \tag{3.2}$$

where $\partial_n = \partial/\partial t_n$, $n \in \mathbb{N}$, and the commutator B_n, L denote byC $[B_n, L] = B_nL - LB_n$. In equation (3.2), when $n = 1$, we take $t_1 = x$ [26].

Definition 3.1. Let q and p be a pair of coprime positive integers. The string equation of type (q, p) takes the commutator form

$$[Q, P] = 1 \tag{3.3}$$

for a pair of ordinary differential operators

$$Q = \partial_x^q + g_2\partial_x^{q-2} + ... + g_q, \qquad P = \partial_x^p + f_2\partial_x^{p-2} + ... + f_p.$$

We can derive the Korteweg-de Vries (KdV) hierarchy from the KP hierarchy (3.2) under the condition

$$(L^2)_{<0} = 0. \tag{3.4}$$

Let

$$Q = L^2. \tag{3.5}$$

Using equation (3.1), we obtain

$$\begin{aligned}
L^2(f) &= L(Lf) \\
&= (\partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + u_4\partial_x^{-3}....)(f_x + u_2\partial_x^{-1}f + u_3\partial_x^{-2}f + ...) \\
&= f_{xx} + \partial_x u_2\partial_x^{-1}f + u_2 f + \partial_x(u_3\partial_x^{-2}f + u_4\partial_x^{-3}f + ...) \\
&+ u_2 f + u_2\partial_x^{-1}(u_2\partial_x^{-1}f + u_3\partial_x^{-2}f + ...) \\
&+ u_3\partial_x^{-2}(f_x + u_2\partial_x^{-1}f + u_3\partial_x^{-2}f + ...) \\
&+ u_4\partial_x^{-2}(f_x + u_2\partial_x^{-1}f + u_3\partial_x^{-2}f + ...).
\end{aligned}$$

Using the constraint (3.4), we obtain

$$Q(f) = (L^2(f))_{\geq 0} = f_{xx} + 2u_2 f.$$

Thus we have

$$Q = \partial_x^2 + u, \text{ where } u = 2u_2. \tag{3.6}$$

Proposition 3.2. *All even flows are trivial in the sense that*

$$\partial_{2n}L = [L^{2n}, L] = 0.$$

Proof. By equation(3.2), we have

$$\partial_{2n}L = [B_{2n}, L], \quad B_{2n} = (L^{2n})_{\geq 0} = ((L^2)^n)_{\geq 0}.$$

Substituting $L^{2n} = (L^{2n})_{<0} + (L^{2n})_{\geq 0}$ into $[L^{2n}, L]$ and using the linearity of [,], we have

$$[L^{2n}, L] = [(L^{2n})_{<0}, L] + [(L^{2n})_{\geq 0}, L]. \tag{3.7}$$

By equation (3.7), we have

$$\partial_{2n}L = [(L^{2n})_{\geq 0}, L] = [L^{2n}, L] - [(L^{2n})_{<0}, L].$$

Now the constraint (3.4), $(L^2)_{<0} = 0$, implies

$$(L^{2n})_{<0} = ((L^2)^n)_{<0} = ((L^2)_{<0})^n = 0,$$

and hence

$$[(L^{2n})_{<0}, L] = 0.$$

Since L^2 and L are commute, then we obtain

$$\partial_{2n} L = [L^{2n}, L] = (L^{2n})L - L(L^{2n}) = 0.$$

□

We will show that the Lax equation for the KP hierarchy (3.2) under the condition of equation (3.4) reduces to the Lax equations of the KdV hierarchy

$$\partial_{2n+1} Q = [B_{2n+1}, Q], \qquad B_{2n+1} = (L^{2n+1})_{\geq 0} = (Q^{n+\frac{1}{2}})_{\geq 0}. \tag{3.8}$$

Let

$$\phi = 1 + \sum_{i=1}^{\infty} u_i \partial^{-1}, \qquad \partial_n \phi = -(\phi \partial^n \phi^{-1})_{<0} \phi, \quad L = \phi \partial \phi^{-1}. \tag{3.9}$$

Using (3.9), we obtain

$$\begin{aligned}
\partial_{2n+1} L^2 &= \partial_{2n+1}[\phi \partial \phi^{-1} \phi \partial \phi^{-1}] \\
&= \partial_{2n+1}[\phi \partial^2 \phi^{-1}] \\
&= (\partial_{2n+1}\phi)\partial^2 \phi^{-1} + \phi \partial_{2n+1}(\partial^2 \phi^{-1}) \\
&= (\partial_{2n+1}\phi)\partial^2 \phi^{-1} + \phi \partial^2 (\partial_{2n+1} \phi^{-1})
\end{aligned}$$

Since ∂_{2n+1} and ∂^2 commute, we obtain

$$\partial_{2n+1} L^2 = (\partial_{2n+1}\phi)\partial^2 \phi^{-1} + \phi \partial^2 \partial_{2n+1} \phi^{-1}.$$

Equation (3.9) and $\partial_{2n+1}\phi^{-1} = \phi^{-1}(\phi \partial^{2n+1} \phi^{-1})_{<0}$ implies

$$\partial_{2n+1} L^2 = -(\phi \partial^{2n+1} \phi^{-1})_{<0} \phi \partial^2 \phi^{-1} + \phi \partial^2 \phi^{-1} (\phi \partial^{2n+1} \phi^{-1})_{<0}.$$

Since $L^2 = \phi \partial^2 \phi^{-1}$, $L^{2n+1} = \phi \partial^{2n+1} \phi^{-1}$, we have

$$\begin{aligned}
\partial_{2n+1} L^2 &= -(L^{2n+1})_{<0} L^2 + L^2 (L^{2n+1})_{<0} \\
&= -[(L^{2n+1})_{<0}, L^2].
\end{aligned}$$

Since $L^{2n+1} = (L^{2n+1})_{\geq 0} + (L^{2n+1})_{<0}$ and $[\ ,\ ]$ is linear, we have

$$\partial_{2n+1} L^2 = -[L^{2n+1}, L^2] + [(L^{2n+1})_{\geq 0}, L^2].$$

Since L^{2n+1} and L^2 commute, we obtain

$$\begin{aligned}
\partial_{2n+1} L^2 &= [(L^{2n+1})_{\geq 0}, L^2] \\
&= [(Q^{n+\frac{1}{2}})_{\geq 0}, Q] \\
&= [B_{2n+1}, Q], \quad B_{2n+1} = (Q^{n+\frac{1}{2}})_{\geq 0}.
\end{aligned}$$

or

$$\partial_{2n+1} Q = [B_{2n+1}, Q].$$

Define:

$$2R_{n+1,x} = [B_{2n+1}, Q], \tag{3.10}$$

where $R_{n,x}$ is the derivative of R_n with respect to x.

Note that

$$R_0 = 1,\ R_1 = \frac{u}{2},\ R_2 = \frac{1}{8}u_{xx} + \frac{3}{8}u^2,\ R_3 = \frac{1}{32}u_{xxxx} + \frac{3}{16}uu_{xx} + \frac{5}{32}u_x^2 + \frac{5}{16}u^3, \cdots . \tag{3.11}$$

If we choose P to be a linear combination of B_{2n+1} as

$$P = B_{2g+1} + c_1 B_{2g-1} + ... + c_g B_1 \tag{3.12}$$

with constant coefficients $c_1,\ c_2, ..., c_g$ in equation (3.3), we have

$$[Q, P] = -[P, Q] = -[B_{2g+1} + c_1 B_{2g-1} + ... + c_g B_1, Q].$$

By linearity of [,], we have

$$[Q, P] = -[B_{2g+1},\ Q] - c_1[B_{2g-1}, Q] - ... - c_g[B_1, Q].$$

From equation (3.10), we obtain

$$[Q, P] = -2R_{g+1,x} - 2c_1 R_{g,x} - ... - c_g 2R_{1,x}.$$

Using equation (3.3), we have

$$2R_{g+1,x} + 2c_1 R_{g,x} + ... + c_g 2R_{1,x} + 1 = 0. \tag{3.13}$$

Integrating equation (3.13) with respect to x, we obtain

$$2R_{g+1} + 2c_1 R_g + ... + 2c_g R_1 + x = 0. \tag{3.14}$$

Equation (3.14) is the first Painlevé hierarchy.

By equation (3.10), equation (3.8) and equation (3.6), we have

$$\begin{aligned} 2R_{2n+1,x} &= [B_{2n+1}, Q] \\ &= \partial_{2n+1} Q \\ &= \partial_{2n+1}(\partial_x^2 + u) \\ &= \partial_{2n+1}\partial_x^2 + \partial_{2n+1} u. \end{aligned}$$

Since $\partial_{2n+1}\partial_x^2 = 0$,

$$\partial_{2n+1} u = 2R_{2n+1,x}, \tag{3.15}$$

which is the evolution equation for u.

3.2 Orlov-Schulman operator

In this section, we will define the Orlov-Schulman operator and the dressing operator and we will derive the relation between them .

The Orlov-Schulman operator is an infinite-order pseudo-differential operator of the form

$$M = \sum_{n=2}^{\infty} n t_n L^{n-1} + x + \sum_{n=1}^{\infty} v_n L^{-n-1}, \tag{3.16}$$

where L was given in (3.1) and v_n is constant.
The Orlov-Schulman operator M [19] obeys the Lax equations

$$\partial_n M = [B_n, M], \tag{3.17}$$

and the commutation relation [19]

$$[L, M] = 1. \tag{3.18}$$

The existence of this operator can be shown by the language of the auxiliary linear system

$$L\psi = z\psi, \qquad \partial_n \psi = B_n \psi, \tag{3.19}$$

of the KP hierarchy. The solution of the linear system (3.19) has the form [19]

$$\psi = W \exp(xz + \sum_{n=2}^{\infty} t_n z^n), \tag{3.20}$$

where

$$W = 1 + \sum_{j=1}^{\infty} w_j z^{-j}, \tag{3.21}$$

is a pseudo-differential operator and it is called the dressing operator. The operator W satisfies the following two equations

$$\partial_n W = -(W \partial_x^n W^{-1})_{<0} W, \tag{3.22}$$

and

$$L = W \partial_x W^{-1}, \qquad B_n = (W \partial_x^n W^{-1})_{\geq 0}. \tag{3.23}$$

We can define M as

$$M = W (\sum_{n=2}^{\infty} n t_n \partial_x^{n-1} + x) W^{-1}, \tag{3.24}$$

which satisfies the foregoing equations (3.17), (3.18) and the auxiliary linear equation

$$M\psi = \partial_z \psi. \tag{3.25}$$

3.3 First Painlevé hierarchy

In this section, we will derive first Painlevé (PI) hierarchy from the KP hierarchy [19].

The string equation (3.3) is derived from the KP hierarchy under the constraints

$$(Q)_{<0} = 0, \qquad (P)_{<0} = 0, \tag{3.26}$$

on the operators

$$Q = L^2, \qquad P = \frac{1}{2}ML^{-1} = \sum_{n=1}^{\infty} nt_n L^{n-2} + \sum_{n=1}^{\infty} v_n L^{-n-2}. \tag{3.27}$$

The equations (3.26) and (3.27) lead to the following consequences:

1. As it follows from the commutation relation (3.18), the operators P and Q, obey the commutation relation $[Q, P] = 1$.

2. Under the first constraint (3.26), Q becomes the Lax operator $\partial_x^2 + u$ of the KdV hierarchy.

Note that in the following equations, we assume that $t_2 = t_4 = ... = 0$.

3. The constraint (3.26) and (3.27), imply that P is a differential operator of the form

$$P = \frac{1}{2}(ML^{-1})_{\geq 0} = \sum_{n=1}^{\infty} \frac{2n+1}{2} t_{2n+1} B_{2n+1}. \tag{3.28}$$

By equation (3.1), we have

$$L^{-1} = q_1 \partial_x^{-1} + q_2 \partial_x^{-2} + q_3 \partial_x^{-3} + \cdots .$$

It follows that

$$(L^{-n-1})_{\geq 0} = 0, \; n = 0, 1, 2, 3, \cdots .$$

By equation (3.27), we have

$$\begin{aligned} (\tfrac{1}{2}ML^{-1})_{\geq 0} &= (\sum_{n=1}^{\infty} nt_n L^{n-2})_{\geq 0} + (\sum_{n=1}^{\infty} v_n L^{-n-2})_{\geq 0} \\ &= (\sum_{n=1}^{\infty} nt_n L^{n-2})_{\geq 0}. \end{aligned}$$

Setting $t_2 = t_4 = ... = t_{2n} = ... = 0$, and using $B_n = (L^n)_{\geq 0}$ from constraint (3.28), we have

$$(\tfrac{1}{2}ML^{-1})_{\geq 0} \quad = \quad \sum_{n=1}^{\infty} nt_{2n+1}B_{2n-1}.$$

If we set

$$t_{2g+3} = \frac{2}{2g+1},\ t_{2g+5} = t_{2g+7} = ... = 0,$$

then the differential operator of equation (3.28) can be rewritten as

$$P = B_{2g+1} + c_1 B_{2g-1} + c_2 B_{2g-3} + ... + c_g B_1,$$

where $c_n = c_n(t) = \dfrac{2n+1}{2} t_{2n+1}, \quad n = 1, ..., g.$

From the above definition of $c_g = c_g(t)$, we can compute the first Painlevé hierarchy

.

At $g = 1$, we have $c_1 = c_1(t) = \frac{3}{2}t_3$.

Substituting $c_1(t)$ into equation (3.14), we have

$$2R_2 + 2c_1R_1 + x = 0.$$

Substituting $R_2,\ R_1$ from equation (3.10) and setting $t_3 = 2$ into the above equation, we have

$$\frac{1}{4}u_{xx} + \frac{3}{4}u^2 + 3u + x = 0,$$

which is the first Painlevé equation.

Similarly when $g = 2$, we have

$$c_1(t) = \frac{3}{2}t_3, c_2(t) = \frac{5}{2}t_5.$$

Substituting $c_1(t)$ and $c_2(t)$ into equation (3.14), we have

$$2R_3 + 2c_1R_2 + 2c_2R_1 + x = 0.$$

Substituting $R_3,\ R_2,\ R_1$ from equation (3.10) and setting $t_3 = 2$ into the above equation, we get

$$\frac{1}{16}u_{xxxx} + \frac{3}{8}uu_{xx} + \frac{5}{16}u_x^2 + \frac{5}{8}u^3 + \frac{3t_3}{8}u_{xx} + \frac{9t_3}{8}u^2 + u + x = 0,$$

which is the second number of the first Painlevé hierarchy.

4. By equations (3.19) , (3.25) and (3.27), we obtain

$$Q\Psi = z^2\Psi, \qquad P\Psi = \frac{1}{2}z^{-1}\partial_z\Psi. \tag{3.29}$$

Substituting $\lambda = z^2$ into equation (3.29), we have

$$Q\Psi = \lambda\Psi, \quad P\Psi = \partial_\lambda\Psi. \tag{3.30}$$

From the KP hierarchy, the string equation (3.3) of type (2,2g+1) along with g extra commuting flows

$$\partial_{2n+1}Q = [B_{2n+1}, Q], \qquad \partial_{2n+1}P = [B_{2n+1}, P], \qquad n = 1, ..., g. \tag{3.31}$$

The system (3.31) is the PI hierarchy.
By differentiating P with respect to t_{2n+1} in equation (3.27), we have

$$\begin{aligned}\partial_{2n+1}P &= \partial_{2n+1}(\tfrac{1}{2}ML^{-1})\\ &= \tfrac{1}{2}(\partial_{2n+1}M)L^{-1} + \tfrac{1}{2}M(\partial_{2n+1}L^{-1}).\end{aligned}$$

Using equation (3.17) and $\partial_{2n+1}L^{-1} = L^{-1}\partial_{2n+1}L)L^{-1}$ we get

$$\partial_{2n+1}P = \tfrac{1}{2}([B_{2n+1}, M])L^{-1} - \tfrac{1}{2}ML^{-1}(\partial_{2n+1}L)L^{-1}.$$

Now equation (3.2), implies that

$$\begin{aligned}\partial_{2n+1}P &= \tfrac{1}{2}([B_{2n+1}, M])L^{-1} - \tfrac{1}{2}ML^{-1}[B_{2n+1}, L]L^{-1})\\ &= \tfrac{1}{2}B_{2n+1}ML^{-1} - \tfrac{1}{2}MB_{2n+1}L^{-1} - PB_{2n+1} + \tfrac{1}{2}MB_{2n+1}L^{-1}\\ &= B_{2n+1}P - PB_{2n+1} = [B_{2n+1}, P].\end{aligned}$$

3.4 Matrix Lax formalism of commuting flows

We will rewrite the PI hierarchy as a 2×2 matrix Lax equation [19].

The second part of equation (3.19) can be rewritten as

$$\partial_{2n+1}\Psi = U_n(\lambda)\Psi, \tag{3.32}$$

where

$$U_n(\lambda) = \begin{pmatrix} A_n(\lambda) & B_n(\lambda) \\ \Gamma_n(\lambda) & -A_n(\lambda) \end{pmatrix}, \tag{3.33}$$

$$\Psi = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix},$$

$$\begin{aligned} B_n(\lambda) &= R_n(\lambda), \\ A_n(\lambda) &= \tfrac{-1}{2} R_n(\lambda)_x, \\ \Gamma_n(\lambda) &= \tfrac{-1}{2} R_n(\lambda)_{xx} + (\lambda - u) R_n(\lambda), \end{aligned} \tag{3.34}$$

$$R_n(\lambda) = \lambda^n + \lambda^{n-1}\rho_1 + \lambda^{n-2}\rho_2 + ... + \rho_n, \; n = 0, 1, 2, \cdots, \tag{3.35}$$

where $\rho_i = R_i, = R_i$ define in equation (3.11). By differentiate Ψ with respect to t_{2n+1}, we obtain

$$\begin{aligned} \partial_{2n+1}\Psi &= \partial_{2n+1} \begin{pmatrix} \psi \\ \psi_x \end{pmatrix} \\ &= \begin{pmatrix} \partial_{2n+1}\psi \\ \partial_{2n+1}\psi_x \end{pmatrix}. \end{aligned}$$

Since ∂_x and ∂_{2n+1} commute, we obtain

$$\partial_{2n+1}\Psi = \begin{pmatrix} \partial_{2n+1}\psi \\ \partial_x \partial_{2n+1}\psi \end{pmatrix}.$$

By equation (3.19), we obtain

$$\partial_{2n+1}\Psi = \begin{pmatrix} B_{2n+1}\psi \\ \partial_x B_{2n+1}\psi \end{pmatrix}. \tag{3.36}$$

Defined [19]

$$B_{2n+1}\psi = R_n(\lambda)\psi_x - \frac{1}{2} R_n(\lambda)_x \psi. \tag{3.37}$$

Differentiating equation (3.37) with respect to x, we have

$$\begin{aligned}\partial_x(B_{2n+1}\psi) &= \partial_x(R_n(\lambda)\psi_x - \tfrac{1}{2}R_n(\lambda)_x\psi) \\ &= R_n(\lambda)_x\psi_x + R_n(\lambda)\psi_{xx} - \tfrac{1}{2}R_n(\lambda)_{xx}\psi - \tfrac{1}{2}R_n(\lambda)_x\psi_x.\end{aligned} \tag{3.38}$$

Substituting $(\lambda - u)\psi = \psi_{xx}$, [19] into equation (3.38), we get

$$\begin{aligned}\partial_x(B_{2n+1}\psi) &= R_n(\lambda)_x\psi_x + R_n(\lambda)(\lambda - u)\psi - \tfrac{1}{2}R_n(\lambda)_{xx}\psi - \tfrac{1}{2}R_n(\lambda)_x\psi_x \\ &= \tfrac{1}{2}R_n(\lambda)_x\psi_x + [R_n(\lambda)(\lambda - u) - \tfrac{1}{2}R_n(\lambda)_{xx}]\psi.\end{aligned} \tag{3.39}$$

Substituting $\partial_x(B_{2n+1}\psi)$ and $B_{2n+1}\psi$ from equations (3.39) and (3.37) into equation (3.36), we get

$$\partial_{2n+1}\Psi = \begin{pmatrix} R_n(\lambda)\psi_x - \frac{1}{2}R_n(\lambda)\psi \\ \frac{1}{2}R_n(\lambda)_x\psi_x + (R_n(\lambda)(\lambda - u) - \frac{1}{2}R_n(\lambda)_{xx})\psi \end{pmatrix}.$$

From equation (3.34), we obtain

$$\begin{aligned}\partial_{2n+1}\Psi &= \begin{pmatrix} B_n(\lambda)\psi_x + A_n(\lambda)\psi \\ -A_n(\lambda)\psi_x + \Gamma_n(\lambda)\psi, \end{pmatrix} \\ &= \begin{pmatrix} A_n(\lambda) & B_n(\lambda) \\ \Gamma_n(\lambda) & -A_n(\lambda) \end{pmatrix}\begin{pmatrix} \psi \\ \psi_x \end{pmatrix} \\ &= U_n(\lambda)\Psi.\end{aligned}$$

In the special case of the equation (3.32) at $n = 0$ $(t_1 = x)$, we obtain

$$\partial_1\Psi = \partial_x\Psi = \begin{pmatrix} A_0(\lambda) & B_0(\lambda) \\ \Gamma_0(\lambda) & -A_0(\lambda) \end{pmatrix}\begin{pmatrix} \psi \\ \psi_x \end{pmatrix},$$

$$\begin{aligned}B_0(\lambda) &= R_0(\lambda) = 1, \\ A_0(\lambda) &= \tfrac{-1}{2}R_0(\lambda)_x = 0, \\ \Gamma_0(\lambda) &= \tfrac{-1}{2}R_0(\lambda)_{xx} + (\lambda - u)R_0(\lambda) = \lambda - u.\end{aligned}$$

This implies that

$$\partial_x\Psi = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}\Psi. \tag{3.40}$$

Define:

$$V(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{pmatrix}, \tag{3.41}$$

where

$$\begin{aligned}
\gamma(\lambda) &= \alpha(\lambda)_x + (\lambda - u)\beta(\lambda) = \tfrac{-1}{2}\beta(\lambda)_{xx} + (\lambda - u)\beta(\lambda), \\
\beta(\lambda) &= R_g(\lambda) + c_1(t)R_{g-1}(\lambda) + c_2(t)R_{g-2}(\lambda) + ... + c_g(t)R_0(\lambda), \\
\alpha(\lambda) &= -\tfrac{1}{2}\beta(\lambda)_x.
\end{aligned} \tag{3.42}$$

The second equation (3.30) can be thus converted to the matrix form

$$\partial_\lambda \Psi = V(\lambda)\Psi. \tag{3.43}$$

Then we can rewrite $V(\lambda)$ as linear combination of $U_n(\lambda)$ as

$$V(\lambda) = U_g(\lambda) + c_1(t)U_{g-1}(\lambda) + c_2(t)U_{g-2}(\lambda) + ... + c_g(t)U_0(\lambda). \tag{3.44}$$

Substituting $\beta(\lambda), \alpha(\lambda), \gamma(\lambda)$ from equation (3.42) into equation (3.41), we have

$$V(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{pmatrix} = \begin{pmatrix} -\tfrac{1}{2}\beta(\lambda)_x & \beta(\lambda) \\ \tfrac{-1}{2}\beta(\lambda)_{xx} + (\lambda - u)\beta(\lambda) & \tfrac{1}{2}\beta(\lambda)_x) \end{pmatrix}.$$

The definition of $\beta(\lambda)$ in equation (3.42), implies that

$$\begin{aligned}
V(\lambda) &= \begin{pmatrix} -\tfrac{1}{2}R_g(\lambda)_x - \tfrac{1}{2}c_1(t)R_{g-1}(\lambda)_x - ... - \tfrac{1}{2}c_g(t)R_0(\lambda)_x & R_g(\lambda) + ... + c_g(t)R_0(\lambda) \\ \\ -\tfrac{1}{2}R_g(\lambda)_{xx} - \tfrac{1}{2}c_1(t)R_{g-1}(\lambda)_{xx} - ... - \tfrac{1}{2}c_g(t)R_0(\lambda)_{xx} & \\ +(\lambda - u)(R_g(\lambda) + c_1(t)R_{g-1} + ... + c_g(t)R_0(\lambda)) & \tfrac{1}{2}R_g(\lambda)_x + ... + \tfrac{1}{2}c_g(t)R_0(\lambda)_x \end{pmatrix} \\
&= \begin{pmatrix} -\tfrac{1}{2}R_g(\lambda)_x & R_g(\lambda) \\ -\tfrac{1}{2}R_g(\lambda)_{xx} + (\lambda - u)R_g(\lambda) & \tfrac{1}{2}R_g(\lambda)_x \end{pmatrix} \\
&+ \begin{pmatrix} -\tfrac{1}{2}c_1(t)R_{g-1}(\lambda)_x & c_1(t)R_{g-1}(\lambda) \\ -\tfrac{1}{2}c_1(t)R_{g-1}(\lambda)_{xx} + (\lambda - u)c_1(t)R_{g-1}(\lambda) & \tfrac{1}{2}c_1(t)R_{g-1}(\lambda)_x \end{pmatrix} \\
&+ ... + \begin{pmatrix} -\tfrac{1}{2}c_g(t)R_0(\lambda)_x & c_g(t)R_0(\lambda) \\ -\tfrac{1}{2}c_g(t)R_0(\lambda)_{xx} + (\lambda - u)c_g(t)R_0(\lambda) & \tfrac{1}{2}c_g(t)R_0(\lambda)_x \end{pmatrix} \\
&= U_g(\lambda) + c_1(t)U_{g-1}(\lambda) + c_2(t)U_{g-2}(\lambda) + ... + c_g(t)U_0(\lambda).
\end{aligned}$$

Equations (3.32) and (3.43) gives

$$\partial_{2n+1}\partial_\lambda \Psi = \partial_{2n+1}(V(\lambda)\Psi) = (\partial_{2n+1}V(\lambda))\Psi + V(\lambda)U_n(\lambda)\Psi. \tag{3.45}$$

Equation (3.32) and (3.43) gives

$$\partial_\lambda\partial_{2n+1}\Psi = \partial_\lambda(U_n(\lambda))\Psi + U_n(\lambda)\partial_\lambda\Psi = U_n'(\lambda)\Psi + U_n(\lambda)V(\lambda)\Psi. \tag{3.46}$$

Using $\partial_\lambda\partial_{2n+1}\Psi = \partial_{2n+1}\partial_\lambda\Psi$, equations (3.45) and (3.46), we obtain

$$[U_n'(\lambda) + U_n(\lambda)V(\lambda) - V(\lambda)U_n]\Psi = (\partial_{2n+1}V(\lambda))\Psi.$$

It follows that

$$\partial_{2n+1}V(\lambda) = [U_n(\lambda), V(\lambda)] + U_n'(\lambda), \quad n = 0, 1, 2, ..., g, \tag{3.47}$$

where $U_n'(\lambda) = \partial_\lambda U_n(\lambda)$.
Equation (3.47) is the PI hierarchy.
If we take the special case $n = 0, (t_1 = x)$, then we find

$$\partial_x V(\lambda) = [U_0(\lambda), V(\lambda)] + U_0'(\lambda), \quad U_0'(\lambda) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Chapter 4

Second Painlevé Hierarchy

In this chapter, we will derive the Modified Korteweg-de Vries hierarchy from the Korteweg-de Vries hierarchy. Then we will derive the second Painlevé Hierarchy from the Modified Korteweg-de Vries hierarchy [9].

4.1 Korteweg-de Vries hierarchy

In this section, we will use the Lenard recursion relation to defined the Korteweg-de Vries hierarchy. Then we will give the explicit form of the first and the second members of this hierarchy.

The KdV hierarchy is given by [9]

$$U_{t_{2n+1}} + \partial_x \ell_{n+1}\{U\} = 0, \qquad n \geq 0 \tag{4.1}$$

where ℓ_n satisfies the Lenard recursion relation

$$\partial_x \ell_{n+1}\{U\} = (\partial_{xxx} + 4U\partial_x + 2U_x)\ell_n\{U\}. \tag{4.2}$$

Now substituting $\ell_0\{U\} = \frac{1}{2}$ into equation (4.2) with $n = 0$, we find

$$\partial_x \ell_1\{U\} = (\partial_{xxx} + 4U\partial_x + 2U_x)\ell_0\{U\} = U_x.$$

Thus integration with respect to x yields

$$\ell_1\{U\} = U. \tag{4.3}$$

Substituting $\ell_1\{U\}$ from equation (4.3) into equation (4.1), we obtain the equation

$$U_{t_1} + U_x = 0. \tag{4.4}$$

When $n = 1$, the Lenard recursion relation (4.2) gives

$$\begin{aligned}\partial_x \ell_2\{U\} &= (\partial_{xxx} + 4U\partial_x + 2U_x)(\ell_1\{U\}) \\ &= (\partial_{xxx} + 4U\partial_x + 2U_x)(U) \\ &= U_{xxx} + 6UU_x.\end{aligned}$$

Integration of both sides with respect to x gives

$$\ell_2\{U\} = U_{xx} + 3U^2. \tag{4.5}$$

Substituting $\ell_1\{U\}$ from equation (4.5) into equation (4.1), we have the KdV equation

$$U_{t_3} + 6UU_x + U_{xxx} = 0. \tag{4.6}$$

When $n = 2$, the Lenard recursion relation (4.2) reads

$$\begin{aligned}\partial_x \ell_3\{U\} &= (\partial_{xxx} + 4U\partial_x + 2U_x)\ell_2\{U\} \\ &= (\partial_{xxx} + 4U\partial_x + 2U_x)(U_{xx} + 3U^2) \\ &= U_{5x} + 10UU_{xxx} + 20U_xU_{xx} + 30U^2U_x\end{aligned}$$

where U_{5x} is the fifth order derivative of U with respect to x. As a result, we get

$$\ell_3\{U\} = U_{4x} + 10U^3 + 5U_x^2 + 10UU_{xx}. \tag{4.7}$$

Substituting $\ell_3\{U\}$ from equation (4.7) into equation (4.1), then the KdV hierarchy gives

$$U_{t_5} + U_{5x} + 10UU_{xxx} + 20U_xU_{xx} + 30U^2U_x = 0, \tag{4.8}$$

4.2 Modified Korteweg-de Vries hierarchy

In this section, we will derive the Modified Korteweg-de Vries hierarchy from KdV hierarchy.

Under the map $U(x,t) = W_x(x,t) - W^2(x,t)$, the KdV equation (4.6) becomes

$$\partial_t(W_x - W^2) + 6(W_x - W^2)(W_x - W^2)_x + (W_x - W^2)_{xxx} = 0. \tag{4.9}$$

Equation (4.9) can be written explicitly as

$$\begin{aligned} W_{xt} &- 2WW_t + 6W_xW_{xx} - 6W^2W_{xx} - 12WW_x^2 + 12W^3W_x + W_{xxxx} \\ &- 6W_xW_{xx} - 2WW_{xxx} = 0. \end{aligned} \tag{4.10}$$

We note that equation (4.10) is equivalent to the equation

$$(\partial_x - 2W)W_t - 6(\partial_x - 2W)W^2W_x + (\partial_x - 2W)W_{xxx}) = 0,$$

or

$$(\partial_x - 2W)(W_t - 6W^2W_x + W_{xxx}) = 0. \tag{4.11}$$

Therefore, if $W(x,t)$ satisfies the modified Korteweg-de Vries (MKdV) equation

$$W_t - 6W^2W_x + W_{xxx} = 0, \tag{4.12}$$

then $U = W_x - W^2$ satisfies the KdV equation (4.6).
Now we will generalize this result to the KdV hierarchy.
Substituting $U = W_x - W^2$ into equation (4.1), we have

$$\partial_{t_{2n+1}}[W_x - W^2] + \partial_x\ell_{n+1}\{W_x - W^2\} = 0. \tag{4.13}$$

Using equation (4.2), we can compute $\partial_x\ell_{n+1}\{W_x - W^2\}$ as follows:

$$\begin{aligned} \partial_x\ell_{n+1}\{W_x - W^2\} &= (\partial_{xxx} + 4(W_x - W^2)\partial_x + 2(W_x - W^2)_x)(\ell_n\{W_x - W^2\}) \\ &= \partial_x(\partial_{xx}\ell_n\{W_x - W^2\}) + 4W_x\partial_x\ell_n\{W_x - W^2\} - 4W^2\partial_x\ell_n\{W_x - W^2\} \\ &+ 2W_{xx}\ell_n\{W_x - W^2\} - 4WW_x\ell_n\{W_x - W^2\}. \end{aligned}$$

It follows that

$$\partial_x\ell_{n+1}\{W_x - W^2\} = (\partial_x - 2W)(\partial_{xx}\ell_n\{W_x - W^2\} + 2W\partial_x\ell_n\{W_x - W^2\} + 2W_x\ell_n\{W_x - W^2\}). \tag{4.14}$$

Substituting $\partial_x \ell_{n+1}\{W_x - W^2\}$ from equation (4.14) into equation (4.13), we have

$$\partial_{t_{2n+1}}(W_x - W^2) + (\partial_x - 2W)(\partial_{xx}\ell_n\{W_x - W^2\} + 2W\partial_x\ell_n\{W_x - W^2\} + 2W_x\ell_n\{W_x - W^2\}) = 0,$$

or

$$\begin{aligned} \partial_x\partial_{t_{2n+1}}W &- 2W\partial_{t_{2n+1}}W + (\partial_x - 2W)(\partial_{xx}\ell_n\{W_x - W^2\} + 2W\partial_x\ell_n\{W_x - W^2\} \\ &+ 2W_x\ell_n\{W_x - W^2\}) = 0. \end{aligned} \tag{4.15}$$

We note

$$\partial_{t_{2n+1}}(W_x - W^2) = \partial_x\partial_{t_{2n+1}}W - 2W\partial_{t_{2n+1}}W = (\partial_x - 2W)\partial_{t_{2n+1}}W.$$

Thus equation (4.15) can be written as

$$(\partial_x - 2W)(\partial_{t_{2n+1}}W + \partial_{xx}\ell_n + 2W\partial_x\ell_n + 2W_x\ell_n) = 0. \tag{4.16}$$

Equation (4.16) implies that

$$\partial_{t_{2n+1}}W + \partial_x(\partial_x + 2W)\ell_n(W_x - W^2) = 0, \tag{4.17}$$

which is the MKdV hierarchy.
At n=1, we obtain equitation (4.12).

4.3 Second Painlevé hierarchy

In this section, we will derive the second Painlevé hierarchy from the MKdV hierarchy.

Let ℓ_n be given by equation (4.2) and let $\hat{\ell_n}$ be given by

$$\partial_z \hat{\ell}_{n+1}\{V_z - V^2\} = [\partial_{zzz} + (4V_z - V^2)\partial_z + 2\partial_z(V_z - V^2)]\hat{\ell_n}\{V_z - V^2\}. \tag{4.18}$$

Substituting

$$W(x, t_3) = \frac{V(z)}{(3t_3)^{\frac{1}{3}}},\ z = \frac{x}{(3t_3)^{\frac{1}{3}}}$$

into the MKdV equation (4.12) with $t = t_3$, we have

$$\frac{-V'(z)x}{(3t_3)^{\frac{5}{3}}} - \frac{V(z)}{(3t_3)^{\frac{4}{3}}} - \frac{6V^2(z)V'(z)}{(3t_3)^{\frac{4}{3}}} + \frac{V'''(z)}{(3t_3)^{\frac{4}{3}}} = 0. \tag{4.19}$$

Multiplying equation (4.19) by $(3t_3)^{\frac{4}{3}}$ and substituting $z = \dfrac{x}{(3t_3)^{\frac{1}{3}}}$ into equation (4.19), we obtain

$$V'''(z) - 6V^2(z)V'(z) - zV'(z) - V(z) = 0. \tag{4.20}$$

Integrating equation (4.20) with respect to z, we get the second Painlevé equation (P_{II})

$$V''(z) = 2V^3(z) + zV(z) + \alpha_1, \tag{4.21}$$

where α_1 is a constant.
Now we use the substitutions

$$W(x, t_{2n+1}) = \frac{V(z)}{((2n+1)t_{2n+1})^{\frac{1}{2n+1}}}, \qquad z = \frac{x}{((2n+1)t_{2n+1})^{\frac{1}{2n+1}}}, \tag{4.22}$$

to compute the second Painlevé hierarchy. Using equation (4.22), we can compute $W_x - W^2$ and obtain

$$W_x - W^2 = \frac{V' - V^2(z)}{((2n+1)t_{2n+1})^{\frac{2}{2n+1}}}. \tag{4.23}$$

Moreover

$$W_{t_{2n+1}} = \frac{\dfrac{-xV`(z)}{(2n+1)t_{2n+1}} - \dfrac{V}{((2n+1)t_{2n+1})^{\frac{2n}{2n+1}}}}{((2n+1)t_{2n+1})^{\frac{2}{2n+1}}}$$

$$= \frac{-V(z)}{((2n+1)t_{2n+1})^{\frac{2n+2}{2n+1}}} - \frac{V`(z)z}{((2n+1)t_{2n+1})^{\frac{2n+2}{2n+1}}}. \tag{4.24}$$

Lemma 4.1.

$$\ell_k[U] = \frac{1}{((2n+1)t_{2n+1})^{\frac{2k}{2n+1}}}\hat{\ell_k}[V` - V^2]. \tag{4.25}$$

Proof. We will prove this lemma by induction.

If $k = 1$, equation (4.2) and equation (4.23) imply

$$\ell_1\{U\} = U = W_x - W^2 = \frac{V`(z) - V^2(z)}{((2n+1)t_{2n+1})^{\frac{2}{2n+1}}}.$$

By the definition of $\hat{\ell_k}$, we have

$$\ell_1\{U\} = \frac{\hat{\ell_1}(V` - V^2(z))}{((2n+1)t_{2n+1})^{\frac{2}{2n+1}}}.$$

Thus the relation (4.25) is true for $k = 1$.

Assume that equation (4.25) is true for $k = m$; that is

$$\ell_m[U] = \frac{1}{((2n+1)t_{2n+1})^{\frac{2m}{2n+1}}}\hat{\ell_m}[V` - V^2]. \tag{4.26}$$

We want to prove that equation (4.25) is true for $k = m + 1$.

Equation (4.22) implies

$$\partial_x \ell_{m+1}[U] = \partial_z \ell_{m+1}[U]\frac{dz}{dx} = \frac{1}{((2n+1)t_{2n+1})^{\frac{1}{2n+1}}}\partial_z \ell_{m+1}[U]. \tag{4.27}$$

Moreover, we have

$$\partial_x^3 + 4U\partial_x + 2U_x = \partial_z^3\left(\frac{dz}{dx}\right)^3 + 4(W_x - W^2)\partial_z\frac{dz}{dx} + 2(W_x - W^2)_x.$$

Thus, using equation (4.23) and equation (4.22), the above equation becomes

$$\partial_x^3 + 4U\partial_x + 2U_x = \frac{1}{((2n+1)t_{2n+1})^{\frac{3}{2n+1}}}\left(\partial_z^3 + 4[V`(z) - V^2(z)]\partial_z + 2[V`(z) - V^2(z)]_z\right). \tag{4.28}$$

From equations (4.27) , (4.28) and (4.2), we have

$$\begin{aligned}\partial_z \ell_{m+1}\{U\} &= ((2n+1)t_{2n+1})^{\frac{1}{2n+1}}(\partial_{xxx}+4U\partial_x+2U_x)\ell_m\{U\}\\ &= \frac{1}{((2n+1)t_{2n+1})^{\frac{2}{2n+1}}}\{\partial_z^3+4[V\text{`}(z)-V^2(z)]\partial_z+2[V\text{`}(z)-V^2(z)]_z\}\ell_m\{U\}.\end{aligned}$$

Substituting $\ell_m\{U\}$ from equation (4.26) into the above equation, we get

$$\begin{aligned}\partial_z \ell_{m+1}\{U\} &= \frac{1}{((2n+1)t_{2n+1})^{\frac{2}{2n+1}}}\{\partial_z^3+4[V\text{`}(z)-V^2(z)]\partial_z+2[V\text{`}(z)-V^2(z)]_z\}\\ &\times \{\frac{1}{((2n+1)t_{2n+1})^{\frac{2m}{2n+1}}}\hat{\ell_m}[V\text{`}-V^2]\}\\ &= \frac{1}{((2n+1)t_{2n+1})^{\frac{2m+2}{2n+1}}}\partial_z\hat{\ell_m}[V\text{`}-V^2].\end{aligned}$$

□

Remark 4.2. We obtain the P_{II} hierarchy by substituting $W(z), z$ and $\partial_{t_{2n+1}}W$ from equation (4.22) ,(4.23) and (4.24) respectively into equation (4.17). More precisely, equation (4.17) gives

$$\begin{aligned}&\frac{-V(z)}{((2n+1)t_{2n+1})^{\frac{2n+2}{2n+1}}}-\frac{V\text{`}(z)z}{((2n+1)t_{2n+1})^{\frac{2n+2}{2n+1}}}\\ &+\frac{1}{((2n+1)t_{2n+1})^{\frac{2}{2n+1}}}(\partial_z(\partial_z+2V)(\frac{1}{((2n+1)t_{2n+1})^{\frac{2n}{2n+1}}}\hat{\ell_n}[V\text{`}-V^2])=0.\end{aligned} \tag{4.29}$$

Multiplying equation (4.29) by $((2n+1)t_{2n+1})^{\frac{2(n+1)}{2n+1}}$, we get

$$\partial_z(\partial_z+2V)\hat{\ell_n}[V\text{`}-V^2]-V-zV\text{`}=0. \tag{4.30}$$

By integrating equation (4.30) with respect to z, we find the P_{II} hierarchy

$$(\partial_z+2V)\hat{\ell_n}[V\text{`}-V^2]=zV+\alpha_n, \qquad n\geq 1, \tag{4.31}$$

where α_n is a constant.

Example 4.3. When $n=1$, equation (4.31) becomes

$$(\partial_z+2V)\hat{\ell_1}[V\text{`}-V^2]=zV+\alpha_1. \tag{4.32}$$

Substituting $\hat{\ell_1}[V\text{`}-V^2]=V\text{`}-V^2$ into equation (4.31), we have

$$V\text{``}(z)=2V^3(z)+zV(z)+\alpha_1,$$

which is the first member of the second Painlevé hierarchy.

Example 4.4. When $n = 2$, equation (4.31) becomes

$$(\partial_z + 2V)\hat{\ell}_2[V' - V^2] = zV + \alpha_2. \tag{4.33}$$

Substituting $\hat{\ell}_1[V' - V^2] = (V' - V^2)_{zz} + 3(V' - V^2)^2$ into equation (4.33), we have

$$(\partial_z + 2V)[(V' - V^2)_{zz} + 3(V' - V^2)^2] = zV + \alpha_2.$$

This implies

$$(\partial_z + 2V)[V''' - 2(V')^2 - 2VV'' + 3(V')^2 - 6V'V^2 + 3V^4] = zV + \alpha_2.$$

As a result, we find

$$V'''' - 10V^2V'' - 10V(V')^2 + 6V^5 = zV + \alpha_2,$$

which is the second member of the second Painlevé hierarchy.

Chapter 5

The Hamiltonian structure for first Painlevé hierarchy

In this chapter, we will use the the Mumford system and the Spectral Darboux coordinates to derive the Hamiltonian structure of first Painlevé hierarchy [19].

5.1 Equation of spectral curve

In this section, we will study some lemmas concerning the KdV hierarchy.

Let $V(\lambda)$ be a matrix define by

$$V(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{pmatrix}, \tag{5.1}$$

where $\alpha(\lambda),\ \beta(\lambda)$ and $\gamma(\lambda)$ are polynomial of the form

$$\begin{aligned} \alpha(\lambda) &= \alpha_1\lambda^{g-1} + \alpha_2\lambda^{g-2} + ... + \alpha_g, \\ \beta(\lambda) &= \lambda^g + \beta_1\lambda^{g-1} + ... + \beta_g, \\ \gamma(\lambda) &= \lambda^{g+1} + \gamma_1\lambda^g + ... + \gamma_{g+1}. \end{aligned}$$

We define the spectral curve of the matrix $V(\lambda)$ by the characteristic equation

$$det(\mu I - V(\lambda)) = \mu^2 + detV(\lambda) = 0. \tag{5.2}$$

We can write equation (5.2) more explicitly in the from

$$\mu^2 = h(\lambda) = \alpha(\lambda)^2 + \beta(\lambda)\gamma(\lambda). \tag{5.3}$$

This implies that $h(\lambda)$ is polynomial of λ with degree $2g+1$.
We can translate the KdV hierarchy to the isospectral Lax equations

$$\partial_{2n+1}V(\lambda) = [U_n(\lambda), V(\lambda)]. \tag{5.4}$$

Equation (5.4) is called the Mumford system.

Lemma 5.1. *There is a* 2×2 *matrix*

$$\Phi(\lambda) = \begin{pmatrix} 1+O(\lambda^{-1}) & O(\lambda^{-1}) \\ w_1 + O(\lambda^{-1}) & 1+O(\lambda^{-1}) \end{pmatrix}$$

of Laurent series of λ *that satisfies the equations*

$$\partial_{2n+1}\Phi(\lambda) = U_n(\lambda)\Phi(\lambda) - \Phi(\lambda)\lambda^n\Lambda, \quad n = 1,2,3,..., \tag{5.5}$$

where $\Lambda = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$, *and* $O(\lambda^{-1}) = c_1\lambda^{-1} + c_2\lambda^{-2} + c_3\lambda^{-3} + \cdots$.

Proof. Let $\Psi(z)$ be the special solution of the auxiliary linear equations (3.19). We can rewrite equations (3.20) as $\psi = W(z)\exp(\xi(z))$, where $W(z)$ is defined by (3.21), and $\xi(z) = \sum_{n=0}^{\infty} t_{2n+1}z^{2n+1}$.
The associated vector-valued function

$$\Psi(z) = \begin{pmatrix} \psi(z) \\ \psi(z)_x \end{pmatrix} = \begin{pmatrix} W(z) \\ zW(z) + W(z)_x \end{pmatrix} \exp(\xi(z)) \tag{5.6}$$

satisfies the auxiliary linear equations (3.32) with $\lambda = z^2$ and U_n defined by equation (3.33) for all $n = 0,1,2,\cdots$. Since λ remains invariant as $z \longrightarrow -z$ and $\Psi(-z)$ is a solution of these linear equations, $\begin{pmatrix} \Psi(z) & \Psi(-z) \end{pmatrix}$ is also a solution of the auxiliary linear equations (3.32).
By equation (3.32) and (3.33), we have

$$
\begin{aligned}
&\partial_{2n+1}\begin{pmatrix} \Psi(z) & \Psi(-z) \end{pmatrix} \\
&= \begin{pmatrix} \partial_{2n+1}\Psi(z) & \partial_{2n+1}\Psi(-z) \end{pmatrix} \\
&= \begin{pmatrix} A_n(\lambda)\psi(z) + B_n(\lambda)\psi(z)_x & A_n(\lambda)\psi(-z) + B_n(\lambda)\psi(-z)_x \\ \Gamma_n(\lambda)\psi(z) - A_n(\lambda)\psi(z)_x & \Gamma_n(\lambda)\psi(-z)_x - A_n(\lambda)\psi(-z)_x \end{pmatrix} \\
&= \begin{pmatrix} A_n(\lambda) & B_n(\lambda) \\ \Gamma_n(\lambda) & -A_n(\lambda) \end{pmatrix}\begin{pmatrix} \psi(z) & \psi(-z) \\ \psi(z)_x & \psi(-z)_x \end{pmatrix} \\
&= U_n(\lambda)\begin{pmatrix} \Psi(z) & \Psi(-z) \end{pmatrix}.
\end{aligned}
$$

If $\Upsilon(\lambda) = \begin{pmatrix} \Psi(z) & \Psi(-z) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\Psi(z)+\Psi(-z)}{2} & \frac{\Psi(z)-\Psi(-z)}{2z} \end{pmatrix}$,
then $\Upsilon(\lambda)$ is a solution of auxiliary linear equation (3.32).
Differentiating $\Upsilon(\lambda)$ with respect to t_{2n+1}, we have

$$
\begin{aligned}
\partial_{2n+1}\Upsilon(\lambda) &= \begin{pmatrix} \dfrac{\partial_{2n+1}\Psi(z) + \partial_{2n+1}\Psi(-z)}{2} & \dfrac{\partial_{2n+1}\Psi(z) - \partial_{2n+1}\Psi(-z)}{2z} \end{pmatrix} \\
&= \begin{pmatrix} \frac{A_n(\lambda)\psi(z)+B_n(\lambda)\psi(z)_x+A_n(\lambda)\psi(-z)+B_n(\lambda)\psi(-z)_x}{2} & \frac{A_n(\lambda)(\Psi(z)-\Psi(-z))+B_n(\lambda)(\Psi(z)-\Psi(-z)_x)}{2z} \\ \frac{\Gamma_n(\lambda)\psi(z)-A_n(\lambda)\psi(z)_x+\Gamma_n(\lambda)\psi(-z)-A_n(\lambda)\psi(-z)_x}{2} & \frac{\Gamma_n(\lambda)(\Psi(z)+\Psi(-z))-A_n(\lambda)(\Psi(z)-\Psi(-z))}{2z} \end{pmatrix} \\
&= \begin{pmatrix} A_n(\lambda) & B_n(\lambda) \\ \Gamma_n(\lambda) & -A_n(\lambda) \end{pmatrix}\begin{pmatrix} \dfrac{\psi(z)+\psi(-z)}{2} & \dfrac{\psi(z)-\psi(-z)}{2z} \\ \dfrac{\psi(z)_x+\psi(-z)_x}{2} & \dfrac{\psi(z)_x-\psi(-z)_x}{2z} \end{pmatrix}.
\end{aligned}
$$

So it follows by equation (3.33), that

$$
\begin{aligned}
\partial_{2n+1}\Upsilon(\lambda) &= U_n(\lambda)\begin{pmatrix} \frac{\psi(z)}{2} & \frac{\psi(z)}{2z} \\ \frac{\psi(z)_x}{2} & \frac{\psi(z)_x}{2z} \end{pmatrix} + U_n(\lambda)\begin{pmatrix} \frac{\psi(-z)}{2} & -\frac{\psi(-z)}{2z} \\ \frac{\psi(-z)_x}{2} & -\frac{\psi(-z)_x}{2z} \end{pmatrix} \\
&= U_n(\lambda)\begin{pmatrix} \frac{\Psi(z)+\Psi(-z)}{2} & \frac{\Psi(z)-\Psi(-z)}{2z} \end{pmatrix} = U_n(\lambda)\Upsilon(\lambda).
\end{aligned} \tag{5.7}
$$

Now we can compute

$$
\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}\begin{pmatrix} \exp(\zeta(z)) & 0 \\ 0 & \exp(-\zeta(z)) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1}
$$

by the definition of the matrix exponential. Thus

$$\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix} \begin{pmatrix} \exp(\zeta(z)) & 0 \\ 0 & \exp(-\zeta(z)) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1}$$

$$= \exp\left(\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix} \begin{pmatrix} \zeta(z) & 0 \\ 0 & -\zeta(z) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1} \right)$$

$$= \exp \begin{pmatrix} 0 & \frac{\zeta(z)}{z} \\ z\zeta(z) & 0 \end{pmatrix}.$$

Using the definition of $\zeta(z)$ and $\lambda = z^2$, we have

$$\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix} \begin{pmatrix} \exp(\zeta(z)) & 0 \\ 0 & \exp(-\zeta(z)) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1}$$

$$= \exp \begin{pmatrix} 0 & \sum_{n=0}^{\infty} t_{2n+1} z^{2n} \\ \sum_{n=0}^{\infty} t_{2n+1} z^{2(n+1)} & 0 \end{pmatrix}$$

$$= \exp\left((\textstyle\sum_{n=0}^{\infty} t_{2n+1}) \begin{pmatrix} 0 & \lambda^n \\ \lambda^{n+1} & 0 \end{pmatrix} \right).$$

Since

$$\Lambda^{2n+1} = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}^{2n+1} = \begin{pmatrix} 0 & \lambda^n \\ \lambda^{n+1} & 0 \end{pmatrix},$$

we have

$$\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix} \begin{pmatrix} \exp(\zeta(z)) & 0 \\ 0 & \exp(-\zeta(z)) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1} = \exp \left(\sum_{n=0}^{\infty} t_{2n+1} \Lambda^{2n+1} \right). \tag{5.8}$$

Define:

$$\Phi(\lambda) = \begin{pmatrix} W(z) & W(-z) \\ zW(z) & -zW(-z) + W(-z)_x \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1}. \tag{5.9}$$

Then using equation (5.6), we obtain

$$\begin{pmatrix} \Psi(z) & \Psi(-z) \end{pmatrix} = \begin{pmatrix} W(z) & W(-z) \\ zW(z) + W(z)_x & -zW(-z) + W(-z)_x \end{pmatrix} \begin{pmatrix} \exp(\zeta(z)) & 0 \\ 0 & \exp(-\zeta(z)) \end{pmatrix}. \tag{5.10}$$

Assume that

$$\Upsilon(\lambda) \;=\; \Phi(z)\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}\begin{pmatrix} \exp(\zeta(z)) & 0 \\ 0 & \exp(-\zeta(z)) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1}.$$

Then equation (5.9) and (5.10) give

$$\begin{aligned}\Upsilon(\lambda) &= \begin{pmatrix} W(z) & W(-z) \\ zW(z) & -zW(-z)+W(-z)_x \end{pmatrix}\begin{pmatrix} \exp(\zeta(z)) & 0 \\ 0 & \exp(-\zeta(z)) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \Psi(z) & \Psi(-z) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix}^{-1}.\end{aligned}$$

Using equation (5.8), we can rewrite $\Upsilon(\lambda)$ as

$$\Upsilon(\lambda) = \Phi(\lambda)\exp\Big(\sum_{n=0}^{\infty} t_{2n+1}\Lambda^{2n+1}\Big). \tag{5.11}$$

Thus equation (5.11), implies that

$$\partial_{2n+1}\Phi(\lambda) \;=\; (\partial_{2n+1}\Upsilon(\lambda))\exp\Big(-\sum_{n=0}^{\infty} t_{2n+1}\Lambda^{2n+1}\Big) + \Upsilon(\lambda)\partial_{2n+1}\exp\Big(-\sum_{n=0}^{\infty} t_{2n+1}\Lambda^{2n+1}\Big).$$

By equation (5.7) and equation (5.11), we have

$$\begin{aligned}\partial_{2n+1}\Phi(\lambda) &= U_n(\lambda)\Phi(\lambda)\exp\Big(\sum_{n=0}^{\infty} t_{2n+1}\Lambda^{2n+1}\Big)\exp\Big(-\sum_{n=0}^{\infty} t_{2n+1}\Lambda^{2n+1}\Big) \\ &+ \Phi(\lambda)\exp\Big(\sum_{n=0}^{\infty} t_{2n+1}\Lambda^{2n+1}\Big)\exp\Big(-\sum_{n=0}^{\infty} t_{2n+1}\Lambda^{2n+1}\Big)(-\Lambda^{2n+1}).\end{aligned}$$

Since $\Lambda^{2n+1} = \lambda^n\Lambda$, we have

$$\partial_{2n+1}\Phi(\lambda) = U_n(\lambda)\Phi(\lambda) - \Phi(\lambda)\lambda^n\Lambda.$$

□

Now will prove that

$$U(\lambda) = \Phi(\lambda)\Lambda\Phi(\lambda)^{-1} \tag{5.12}$$

satisfies the lax equation

$$\partial_{2n+1}U = [U_n(\lambda), U(\lambda)]. \tag{5.13}$$

Differentiating equation (5.12) with respect to t_{2n+1}, we have

$$\begin{aligned}\partial_{2n+1}U(\lambda) &= \partial_{2n+1}(\Phi(\lambda)\Lambda\Phi(\lambda)^{-1}) \\ &= (\partial_{2n+1}\Phi(\lambda))\Lambda\Phi(\lambda)^{-1} + \Phi(\lambda)(\partial_{2n+1}\Lambda)\Phi(\lambda)^{-1} \\ &+ \Phi(\lambda)\Lambda(\partial_{2n+1}\Phi(\lambda)^{-1}).\end{aligned}$$

Using equation (5.5), we have

$$\begin{aligned}\partial_{2n+1}U(\lambda) &= U_n(\lambda)\Phi(\lambda)\Lambda\Phi(\lambda)^{-1} - \Phi(\lambda)\lambda^n\Lambda^2\Phi(\lambda)^{-1} \\ &- \Phi(\lambda)\Lambda\Phi(\lambda)^{-1}[U_n(\lambda)\Phi(\lambda) - \Phi(\lambda)\lambda^n\Lambda]\Phi(\lambda)^{-1}.\end{aligned}$$

Using equation (5.12), we obtain

$$\begin{aligned}\partial_{2n+1}U(\lambda) &= U_n(\lambda)U(\lambda) - \Phi(\lambda)\lambda^n\Lambda^2\Phi(\lambda)^{-1} \\ &- \Phi(\lambda)\Lambda\Phi(\lambda)^{-1}U_n + \Phi(\lambda)\Lambda\lambda^n\Lambda\Phi(\lambda)^{-1} \\ &= U_n(\lambda)U(\lambda) - U(\lambda)U_n(\lambda) \\ &= [U_n(\lambda), U(\lambda)],\end{aligned}$$

and hence we complete the proof.

Lemma 5.2. *The matrix elements of* $\begin{pmatrix} A(\lambda) & B(\lambda) \\ \Gamma(\lambda) & -A(\lambda) \end{pmatrix}$ *are Laurent series of* λ *of the form*

$$A(\lambda) = O(\lambda^{-1}), \qquad B(\lambda) = 1 + O(\lambda^{-1}), \qquad \Gamma(\lambda) = \lambda + O(\lambda^0), \tag{5.14}$$

that satisfy the following conditions:

$$\begin{aligned}A_n(\lambda) &= (\lambda^n A(\lambda))_{\geq 0}, \\ B_n(\lambda) &= (\lambda^n B(\lambda))_{\geq 0}, \\ \Gamma_n(\lambda) &= (\lambda^n \Gamma(\lambda))_{\geq 0} - R_{n+1}, \ \ n = 1, 2, 3, \cdots\end{aligned} \tag{5.15}$$

$$A(\lambda)^2 + B(\lambda)\Gamma(\lambda) = \lambda. \tag{5.16}$$

Proof. From equation (5.12), we have $U(\lambda) = \Phi(\lambda)\Lambda\Phi(\lambda)^{-1}$. Let

$$\Phi(\lambda) = \begin{pmatrix} F(\lambda) & G(\lambda) \\ H(\lambda) & F(\lambda) \end{pmatrix},$$

where

$$\begin{aligned}F(\lambda) &= 1 + f_1\lambda^{-1} + f_2\lambda^{-2} + \cdots, \\ H(\lambda) &= w_1 + h_1\lambda^{-1} + h_2\lambda^{-2} + \cdots, \\ G(\lambda) &= g_1\lambda^{-1} + g_2\lambda^{-2} + \cdots.\end{aligned} \tag{5.17}$$

Then

$$U(\lambda) = \begin{pmatrix} F(\lambda) & G(\lambda) \\ H(\lambda) & F(\lambda) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} F(\lambda) & G(\lambda) \\ H(\lambda) & F(\lambda) \end{pmatrix}^{-1}$$

$$= \frac{1}{F^2 - HG} \begin{pmatrix} \lambda FG - FH & -\lambda G^2 + F^2 \\ \lambda F^2 - H^2 & -\lambda FG + FH \end{pmatrix}.$$

Define: $A(\lambda)$, $B(\lambda)$ and $\Gamma(\lambda)$ as

$$\begin{aligned} A(\lambda) &= \frac{\lambda FG - FH}{F^2 - HG}, \\ B(\lambda) &= \frac{-\lambda G^2 + F^2}{F^2 - HG}, \\ \Gamma(\lambda) &= \frac{\lambda F^2 - H^2}{F^2 - HG}. \end{aligned}$$

Then using equation (5.17), we obtain

$$\begin{aligned} B(\lambda) &= 1 + (\frac{g_1 w - g_1^2}{2f_2 - g_1 w_1})\lambda^{-1} + ... = 1 + O(\lambda^{-1}), \\ A(\lambda) &= O(\lambda^{-1}) \quad \text{if } \ w_1 = g_1, \\ \Gamma(\lambda) &= \lambda + (w_1^2 - w_1 g_1) + (g_1 h_1 + w_1 g_2 - 2w_1 h_2)\lambda^{-1} + ... = \lambda + O(\lambda^0). \end{aligned}$$

Thus we can write $U(\lambda)$ as

$$U(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ \Gamma(\lambda) & -A(\lambda) \end{pmatrix}. \tag{5.18}$$

We can rewrite (5.5) in the form

$$\begin{aligned} U_n(\lambda) &= \partial_{2n+1}\Phi(\lambda).\Phi(\lambda)^{-1} + \Phi(\lambda)\lambda^n \Lambda \Phi(\lambda)^{-1} \\ &= \partial_{2n+1}\Phi(\lambda).\Phi(\lambda)^{-1} + \lambda^n U(\lambda), \end{aligned} \tag{5.19}$$

and hence we get

$$(\partial_{2n+1}\Phi(\lambda))\Phi(\lambda)^{-1} = U_n(\lambda) - \lambda^n U(\lambda).$$

Using equations (3.33) and (5.18), we have

$$(\partial_{2n+1}\Phi(\lambda))\Phi(\lambda)^{-1} = \begin{pmatrix} A_n(\lambda) - \lambda^n A(\lambda) & B_n(\lambda) - \lambda^n B(\lambda) \\ \Gamma_n(\lambda) - \lambda^n \Gamma(\lambda) & -A_n(\lambda) + \lambda^n A(\lambda) \end{pmatrix}. \tag{5.20}$$

By equation (5.17), we obtain

$$(\partial_{2n+1}\Phi(\lambda))\Phi(\lambda)^{-1} = \begin{pmatrix} O(\lambda^{-1}) & O(\lambda^{-1}) \\ \partial_{2n+1}w_1 + O(\lambda^{-1}) & O(\lambda^{-1}) \end{pmatrix}. \tag{5.21}$$

By comparing (5.21) and (5.20) and using $\partial_{2n+1}w_1 = -R_{n+1}$, we obtain

$$\begin{aligned} A_n(\lambda) - \lambda^n A(\lambda) &= O(\lambda^{-1}), \\ B_n(\lambda) - \lambda^n B(\lambda) &= O(\lambda^{-1}), \\ \Gamma_n(\lambda) - \lambda^n \Gamma(\lambda) &= O(\lambda^{-1}) - R_{n+1}. \end{aligned} \tag{5.22}$$

Therefore

$$\begin{aligned} A_n(\lambda) &= O(\lambda^{-1}) + \lambda^n A(\lambda), \\ B_n(\lambda) &= O(\lambda^{-1}) + \lambda^n B(\lambda), \\ \Gamma_n(\lambda) &= \partial_{2n+1}w_1 + O(\lambda^{-1}) + \lambda^n \Gamma(\lambda). \end{aligned} \tag{5.23}$$

Since the smallest n in equation (3.33) and equation (3.34) is $n = 0$, we have

$$\begin{aligned} A_n(\lambda) &= [O(\lambda^{-1}) + \lambda^n A(\lambda)]_{\geq 0} = [\lambda^n A(\lambda)]_{\geq 0} \\ B_n(\lambda) &= [O(\lambda^{-1}) + \lambda^n B(\lambda)]_{\geq 0} = [\lambda^n B(\lambda)]_{\geq 0} \\ \Gamma_n(\lambda) &= [O(\lambda^{-1}) + \lambda^n \Gamma(\lambda) - R_{n+1}]_{\geq 0} = [\lambda^n \Gamma(\lambda)]_{\geq 0} - R_{n+1}, \quad n = 1, 2, 3, \cdots . \end{aligned}$$

Since $\det(U(\lambda)) = \det(\Phi(\lambda)\Lambda\Phi(\lambda)^{-1})$, we obtain

$$A(\lambda)^2 + B(\lambda)\Gamma(\lambda) = \det(\Phi(\lambda))\det(\Lambda)\det(\Phi(\lambda)^{-1}) = \det(\Phi(\lambda))\det(\Phi(\lambda)^{-1})\det(\Lambda) = \lambda.$$

□

Remark 5.3. Since $B_n(\lambda)$ is equal to the auxiliary polynomial of $R_n(\lambda)$ defined in equation (3.35), the second equation of (5.15) implies that $B(\lambda)$ is a generating function of all ρ_n's:

$$B(\lambda) = 1 + \rho_1\lambda^{-1} + \rho_2\lambda^{-2} + \cdots .$$

Using equation (5.15), we get

$$B_n(\lambda) = \lambda^n + \lambda^{n-1}w_1 + \lambda^{n-2}w_2 + ... + w_n. \tag{5.24}$$

Substuating $B_n(\lambda) = R_n(\lambda)$ in equation (5.24), we have

$$B_n(\lambda) = \lambda^n + \lambda^{n-1}\rho_1 + \lambda^{n-2}\rho_2 + ... + \rho_n, \qquad \rho_j = w_j.$$

Thus we obtain

$$B(\lambda) = 1 + \sum_{j=0}^{\infty} \rho_j\lambda^{-j} = 1 + \rho_n\lambda^{-1} + \rho_n\lambda^{-2} + ... \ .$$

Remark 5.4. In equation (5.13) at $n = 0$, we have

$$\begin{aligned} \partial_x A(\lambda) &= \Gamma(\lambda) - (\lambda - u)B(\lambda), \\ \partial_x B(\lambda) &= -2A(\lambda), \\ \partial_x \Gamma(\lambda) &= 2(\lambda - u)A(\lambda). \end{aligned} \tag{5.25}$$

Since $t_1 = x$ and $U_0(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}$ and by equation (5.13), we have

$$\partial_x U(\lambda) = [U_0(\lambda), U(\lambda)].$$

From equation (5.18) and definition of commuttator, we have

$$\begin{aligned} \begin{pmatrix} \partial_x A(\lambda) & \partial_x B(\lambda) \\ \partial_x \Gamma(\lambda) & -\partial_x A(\lambda) \end{pmatrix} &= U_0(\lambda)U(\lambda) - U(\lambda)U_0(\lambda) \\ &= \begin{pmatrix} \Gamma(\lambda) & -A(\lambda) \\ (\lambda - u)A(\lambda) & (\lambda - u)B(\lambda) \end{pmatrix} - \begin{pmatrix} (\lambda - u)B(\lambda) & A(\lambda) \\ -(\lambda - u)A(\lambda) & \Gamma(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} \Gamma(\lambda) - (\lambda - u)B(\lambda) & -2A(\lambda) \\ 2(\lambda - u)A(\lambda) & (\lambda - u)B(\lambda) - \Gamma(\lambda) \end{pmatrix}. \end{aligned}$$

Therefore, we have equation (5.25).
The first two equations in (5.25) can be solved in $A(\lambda)$, $\Gamma(\lambda)$ as

$$A(\lambda) = \frac{-1}{2}B(\lambda)_x, \quad \Gamma(\lambda) = \frac{-1}{2}B(\lambda)_{xx} + (\lambda - u)B(\lambda). \tag{5.26}$$

As a result, the third equation of (5.25) can be rewritten as

$$\frac{1}{2}B(\lambda)_{xx} - 2(\lambda - u)B(\lambda)_x + u_x B(\lambda) = 0. \tag{5.27}$$

Remark 5.5. Assume that $A(\lambda)$ and $\Gamma(\lambda)$ are defined in equations (5.14), (5.16) and substituting $\Gamma(\lambda)$ from equation (5.26), into equation (5.16), we obtain

$$(\frac{-1}{2}B(\lambda)_x)^2 + B(\lambda)(\frac{-1}{2}(B(\lambda)_{xx}) + (\lambda - u)B(\lambda)) = \lambda.$$

5.2 The structure of $h(\lambda)$

In this section, we will study the structure of $h(\lambda)$ which produced by the spectral curve of matrix $V(\lambda)$.

Let us recall equation (3.44), which reads

$$\begin{aligned} \alpha(\lambda) &= A_g(\lambda) + c_1(t)A_{g-1}(\lambda) + ...c_g(t)A_0(\lambda), \\ \beta(\lambda) &= B_g(\lambda) + c_1(t)B_{g-1}(\lambda) + ...c_g(t)B_0(\lambda), \\ \gamma(\lambda) &= \Gamma_g(\lambda) + c_1(t)\Gamma_{g-1}(\lambda) + ...c_g(t)\Gamma_0(\lambda). \end{aligned} \tag{5.28}$$

Equation (5.28) and equation (5.3) yield

$$\begin{aligned} h(\lambda) &= (\sum_{i=0}^{g} c_i(t)A_{g-i}(\lambda))^2 + (\sum_{i=0}^{g} c_i(t)B_{g-i}(\lambda))(\sum_{i=0}^{g} c_i(t)\Gamma_{g-i}(\lambda))^2 \\ &= \sum_{i=0}^{g}(\sum_{j=0}^{g} c_i(t)c_j(t)A_{g-j}(\lambda))A_{g-i}(\lambda) + \sum_{i=0}^{g}(\sum_{j=0}^{g} c_i(t)c_j(t)B_{g-j}(\lambda))\Gamma_{g-i}(\lambda) \\ &= \sum_{i=0}^{g}\sum_{j=0}^{g}(c_i(t)c_j(t)A_{g-j}(\lambda)A_{g-i}(\lambda) + B_{g-j}(\lambda)\Gamma_{g-i}(\lambda)). \end{aligned}$$

Hence, $h(\lambda)$ can be expressed as

$$h(\lambda) = \sum_{m,n=0}^{g} c_m(t)c_n(t)(A_{g-n}(\lambda)A_{g-m}(\lambda) + B_{g-m}(\lambda)\Gamma_{g-n}(\lambda)), \text{ where } c_0 = 1. \tag{5.29}$$

$h(\lambda)$ can be written as [19]

$$h(\lambda) = \sum_{m,n=0}^{g} c_m(t)c_n(t)\lambda^{2g-m-n}(A(\lambda)^2 + B(\lambda)\Gamma(\lambda)) - 2\sum_{m=0}^{g} c_m(t)R_{g+1-m}\lambda^g + O(\lambda^{-1}). \tag{5.30}$$

Theorem 5.6. $h(\lambda)$ *can be expressed as*

$$\begin{aligned} h(\lambda) = \lambda^{2g+1} &+ 2c_1(t)\lambda^{2g} + (2c_2(t) + c_1(t)^2)\lambda^{2g-2} + ... \\ &+ \sum_{m=0}^{g} c_m(t)c_{g-m}(t)\lambda^{g+1} + (\sum_{m=1}^{g-1} c_m(t)c_{g+1-m}(t) + x)\lambda^g + O(\lambda^{g-1}). \end{aligned} \tag{5.31}$$

Proof. Substituting $A(\lambda)^2 + B(\lambda)\Gamma(\lambda)$ from equation (5.16), and $2R_{g+1} + 2c_1 R_g + ... + 2c_g R_1 + x = 0$ into equation (5.30), we have

$$\begin{aligned}
h(\lambda) &= \sum_{m,n=0}^{g} c_m(t)c_n(t)\lambda^{2g-m-n+1} - 2(R_{g+1} + 2c_1 R_g + ... + 2c_g R_1)\lambda^g + O(\lambda^{-1}) \\
&= \sum_{m=0}^{g} c_m[\lambda^{2g-m+1} + c_1\lambda^{2g-m} + c_2\lambda^{2g-m-1} + ... + c_g\lambda^{g-m+1}] + x\lambda^g + O(\lambda^{-1}) \\
&= [\lambda^{2g+1} + c_1\lambda^{2g} + c_2\lambda^{2g-1} + ... + c_g\lambda^{g+1}] \\
&+ c_1[\lambda^{2g} + c_1\lambda^{2g-1} + c_2\lambda^{2g-2} + ... + c_g\lambda^g] \\
&+ c_2[\lambda^{2g-1} + c_1\lambda^{2g-2} + c_2\lambda^{2g-3} + ... + c_g\lambda^{g-1}] + ... \\
&+ x\lambda^g + O(\lambda^{-1}) \\
&= \lambda^{2g+1} + 2c_1\lambda^{2g} + (2c_2(t) + c_1(t)^2)\lambda^{2g-2} + ... \\
&+ (c_g + c_1c_{g-1} + c_2c_{g-1} + c_3c_{g-3} + ...)\lambda^{g+1} + (c_1c_g + c_2c_{g-1} + ...)\lambda^g + x\lambda^g + O(\lambda^{-1}) \\
&= \lambda^{2g+1} + 2c_1(t)\lambda^{2g} + (2c_2(t) + c_1(t)^2)\lambda^{2g-2} + ... \\
&+ \sum_{m=0}^{g} c_m(t)c_{g-m}(t)\lambda^{g+1} + (\sum_{m=1}^{g-1} c_m(t)c_{g+1-m}(t) + x)\lambda^g + O(\lambda^{g-1}).
\end{aligned}$$

□

Let $I_0(\lambda)$ denote the part of $h(\lambda)$ consisting of $\lambda^{2g+1}, \lambda^{2g}, ..., \lambda^g$ and $I_1,\ I_2, ...,\ I_g$ denote the coefficients of $\lambda^{g-1}, \lambda^{g-2}, ..., \lambda^0$ respectively, then

$$h(\lambda) = I_0 + I_1\lambda^{g-1} + I_2\lambda^{g-2} + ... + I_g, \tag{5.32}$$

where $I_0(\lambda)$ is a kinematical quantity that is independent of the solution of the PI hierarchy in equation (3.13).

Remark 5.7. In the case of the Mumford system (5.4), these coefficients $I_1,\ I_2, ...,\ I_g$ are Hamiltonians of commuting flows. More precisely, it is not these coefficients but their suitable linear combinations $H_1, ..., H_g$.

5.3 Hamiltonian structure of PI hierarchy

In this section, we will define new Hamiltonian $H_1, ..., H_g$ that used to derive the Hamiltonian form of PI hierarchy.

By the definition of the Poisson bracket $\{f, g\}(V(\lambda)) =< V(\lambda), [df, dg] >$, we can define the Poisson structure between the element of the matrix $V(\lambda)$ by

$$\{V_{ij}(\lambda), V_{kl}(\mu)\} = \frac{(V_{il}(\lambda) - V_{il}(\mu))\delta_{jk} - (V_{kj}(\lambda) - V_{kj}(\mu))\delta_{il}}{\lambda - \mu}, \tag{5.33}$$

where δ_{jk} is the Kronecker delta [28].
The Poisson structures between the elements of the matrix $V(\lambda)$ are given by

$$\begin{array}{cc} \{\alpha(\lambda), \alpha(\mu)\} = 0, & \{\beta(\lambda), \beta(\mu)\} = 0, \\ \{\alpha(\lambda), \beta(\mu)\} = \frac{\beta(\lambda)-\beta(\mu)}{\lambda-\mu}, & \{\alpha(\lambda), \gamma(\mu)\} = -\frac{\gamma(\lambda)-\gamma(\mu)}{\lambda-\mu}, \\ \{\beta(\lambda), \gamma(\mu)\} = 2\frac{\alpha(\lambda)-\alpha(\mu)}{\lambda-\mu}, & \{\gamma(\lambda), \gamma(\mu)\} = 0. \end{array} \tag{5.34}$$

Using equation (5.33), the definition of $V(\lambda)$ in equation (3.41) and the definition of δ_{ij}, we have

$$\begin{aligned} \{V_{11}(\lambda), V_{11}(\mu)\} &= \{\alpha(\lambda), \alpha(\mu)\} \\ &= \frac{(V_{11}(\lambda) - V_{11}(\mu))\delta_{11} - (V_{11}(\lambda) - V_{11}(\mu))\delta_{11}}{\lambda - \mu} \\ &= \frac{(\alpha(\lambda) - \alpha(\mu))\delta_{11} - (\alpha(\lambda) - \alpha(\mu))\delta_{11}}{\lambda - \mu} = 0. \end{aligned}$$

Similarly, we can compute the following:

$$\{\beta(\lambda), \beta(\mu)\} = \frac{(\beta(\lambda) - \beta(\mu))\delta_{21} - (\beta(\lambda) - \beta(\mu))\delta_{12}}{\lambda - \mu} = 0.$$

$$\begin{aligned} \{\alpha(\lambda), \beta(\mu)\} &= \frac{(\beta(\lambda) - \beta(\mu))\delta_{11} - (\alpha(\lambda) - \alpha(\mu))\delta_{12}}{\lambda - \mu} \\ &= \frac{\beta(\lambda) - \beta(\mu)}{\lambda - \mu}. \end{aligned}$$

$$\{\alpha(\lambda),\gamma(\mu)\} = \frac{(\alpha(\lambda)-\alpha(\mu))\delta_{12}-(\gamma(\lambda)-\gamma(\mu))\delta_{11}}{\lambda-\mu}$$

$$= -\frac{\gamma(\lambda)-\gamma(\mu)}{\lambda-\mu}.$$

$$\{\beta(\lambda),\gamma(\mu)\} = \frac{(\alpha(\lambda)-\alpha(\mu))\delta_{22}-(-\alpha(\lambda)+\alpha(\mu))\delta_{11}}{\lambda-\mu}$$

$$= 2\frac{\alpha(\lambda)-\alpha(\mu)}{\lambda-\mu}.$$

$$\{\gamma(\lambda),\gamma(\mu)\} = \frac{(\gamma(\lambda)-\gamma(\mu))\delta_{12}-(\gamma(\lambda)-\gamma(\mu))\delta_{21}}{\lambda-\mu} = 0.$$

Now we define the Poisson structure between $V(\lambda), h(\mu)$ by

$$\{V(\lambda),h(\mu)\} = [V(\lambda),\frac{V(\mu)}{\lambda-\mu}+\beta(\mu)E_{21}]. \tag{5.35}$$

Lemma 5.8.

$$\{V(\lambda),I_{n+1}\} = [V_n(\lambda),V(\lambda)], \tag{5.36}$$

where $V_n(\lambda)$ is a matrix of the form

$$V_n(\lambda) = \begin{pmatrix} \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & -\alpha_n(\lambda) \end{pmatrix}, \tag{5.37}$$

with the matrix elements

$$\begin{aligned} \alpha_n(\lambda) &= (\lambda^{n-g}\alpha(\lambda))_{\geq 0}, \\ \beta_n(\lambda) &= (\lambda^{n-g}\beta(\lambda))_{\geq 0}, \\ \gamma_n(\lambda) &= (\lambda^{n-g}\gamma(\lambda))_{\geq 0}-\beta_{n+1}, \qquad n=0,1,2,...,g-1. \end{aligned}$$

Proof. I_{n+1} can be extracted from $h(\lambda)$ by a contour integral of the form

$$I_{n+1} = \oint \frac{d\mu}{2\pi i}\mu^{n-g}h(\mu),$$

where the contour is understood to be a circle around $\mu=\infty$. The same contour integral applied to equation (5.35) yields the Poisson bracket in question

$$\{V(\lambda),I_{n+1}\} = \oint \frac{d\mu}{2\pi i}\mu^{n-g}\{V(\lambda),h(\mu)\} = [V(\lambda),\oint \frac{d\mu}{2\pi i}\frac{\mu^{n-g}V(\mu)}{\lambda-\mu}-\beta(\mu)E_{21}].$$

By equation (5.35), we obtain

$$\begin{aligned}
\{V(\lambda), I_{n+1}\} &= \oint \frac{d\mu}{2\pi i}\mu^{n-g}[V(\lambda), \frac{V(\mu)}{\lambda-\mu} + \beta(\mu)E_{21}] \\
&= V(\lambda)\oint \frac{d\mu}{2\pi i}\mu^{n-g}\frac{V(\mu)}{\lambda-\mu} + V(\lambda)\oint \frac{d\mu}{2\pi i}\mu^{n-g}\beta(\mu)E_{21} \\
&- \oint \frac{d\mu}{2\pi i}\mu^{n-g}\frac{V(\mu)}{\lambda-\mu}V(\lambda) - \oint \frac{d\mu}{2\pi i}\mu^{n-g}\beta(\mu)E_{21}V(\lambda).
\end{aligned}$$

Since $\oint \frac{d\mu}{2\pi i}\frac{f(\mu)}{\lambda-\mu} = -(f(\lambda))_{\geq 0}$, we have

$$\begin{aligned}
\{V(\lambda), I_{n+1}\} &= V(\lambda)(-(\lambda^{n-g}V(\lambda))_{\geq 0}) + V(\lambda)\oint \frac{d\mu}{2\pi i}\mu^{n-g}\beta(\mu)E_{21} \\
&- (-(\lambda^{n-g}V(\lambda))_{\geq 0})V(\lambda) - \oint \frac{d\mu}{2\pi i}\mu^{n-g}\beta(\mu)E_{21}V(\lambda) \\
&= V(\lambda)(-(\lambda^{n-g}V(\lambda))_{\geq 0} + \oint \frac{d\mu}{2\pi i}\mu^{n-g}\beta(\mu)E_{21}) \\
&- \left(-(\lambda^{n-g}V(\lambda))_{\geq 0} + \oint \frac{d\mu}{2\pi i}\mu^{n-g}\beta(\mu)E_{21}\right)V(\lambda) \\
&= [(\lambda^{n-g}V(\lambda))_{\geq 0} - \oint \frac{d\mu}{2\pi i}\mu^{n-g}\beta(\mu)E_{21}), V(\lambda)].
\end{aligned}$$

Now

$$\begin{aligned}
\oint \frac{d\mu}{2\pi i}\mu^{n-g}\beta(\mu) &= \oint \frac{d\mu}{2\pi i}\mu^{n-g}\beta(\mu)\frac{\lambda-\mu}{\lambda-\mu} \\
&= \lambda\oint \frac{d\mu}{2\pi i}\mu^{n-g}\frac{\beta(\mu)}{\lambda-\mu} - \oint \frac{d\mu}{2\pi i}\mu^{n-g+1}\frac{\beta(\mu)}{\lambda-\mu} \\
&= -\lambda(\lambda^{n-g}\beta(\lambda))_{\geq 0} + (\lambda^{n-g+1}\beta(\lambda))_{\geq 0} \\
&= \beta_{n+1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\{V(\lambda), I_{n+1}\} &= [(\lambda^{n-g}V(\lambda))_{\geq 0} - \beta_{n+1}E_{21}, V(\lambda)] \\
&= [V(\lambda), -(\lambda^{n-g}V(\lambda))_{\geq 0} + \beta_{n+1}E_{21}].
\end{aligned}$$

Let

$$V_n(\lambda) = (\lambda^{n-g}V(\lambda))_{\geq 0} - \beta_{n+1}E_{21}.$$

Then we have

$$\{V(\lambda), I_{n+1}\} = [V_n(\lambda), V(\lambda)].$$

□

We can replace $[V_n(\lambda), V(\lambda)]$ in lemma 5.8 by $[U_n(\lambda), V(\lambda)]$ as it is explained by the following lemma.

Lemma 5.9.

$$\begin{aligned} V_0(\lambda) &= U_0(\lambda) \\ V_n(\lambda) &= U_n(\lambda) + c_1(t)U_{n-1}(\lambda) + ... + c_n(t)U_0(\lambda), \quad n = 1, 2, ..., g. \end{aligned} \tag{5.38}$$

Proof. Since

$$\begin{aligned} \beta_n(\lambda) &= B_n(\lambda) + c_1(t)B_{n-1}(\lambda) + ... + c_n(t)B_0(\lambda) \\ &= R_n(\lambda) + c_1(t)R_{n-1}(\lambda) + ... + c_n(t)R_0(\lambda), \end{aligned}$$

$\alpha(\lambda),\ \gamma(\lambda)$ are connected with $\beta(\lambda)$ as

$$\alpha(\lambda) = -\frac{1}{2}\beta(\lambda)_x, \qquad \gamma(\lambda) = -\frac{1}{2}\beta(\lambda)_{xx} + (\lambda - \mu)\beta(\lambda),$$

and

$$\alpha_n(\lambda) = -\frac{1}{2}\beta_n(\lambda)_x, \qquad \gamma_n(\lambda) = -\frac{1}{2}\beta_n(\lambda)_{xx} + (\lambda - \mu)\beta_n(\lambda),$$

we obtain

$$\alpha_n(\lambda) = -\tfrac{1}{2}\beta_n(\lambda)_x = -\tfrac{1}{2}B_n(\lambda)_x - \tfrac{1}{2}c_1(t)B_{n-1}(\lambda)_x - ... - \tfrac{1}{2}c_n(t)B_0(\lambda)_x.$$

Using equation (5.26), we get

$$\alpha_n(\lambda) = A_n(\lambda) + c_1(t)A_{n-1}(\lambda) + ... + c_n(t)A_0(\lambda). \tag{5.39}$$

By the same way, we find

$$\begin{aligned} \gamma_n(\lambda) &= -\tfrac{1}{2}\beta_n(\lambda)_{xx} + (\lambda - \mu)\beta_n(\lambda) \\ &= -\tfrac{1}{2}B_n(\lambda)_{xx} - \tfrac{1}{2}c_1(t)B_{n-1}(\lambda)_{xx} - \cdots - \tfrac{1}{2}c_n(t)B_0(\lambda)_{xx} \\ &+ (\lambda - \mu)B_n(\lambda) + (\lambda - \mu)c_1(t)B_{n-1}(\lambda) + \cdots + (\lambda - \mu)c_n(t)B_0(\lambda) \\ &= (-\tfrac{1}{2}B_n(\lambda)_{xx} + (\lambda - \mu)B_n(\lambda)) + (-\tfrac{1}{2}c_1(t)B_{n-1}(\lambda)_{xx} + (\lambda - \mu)c_1(t)B_{n-1}(\lambda)) + \cdots \\ &+ (-\tfrac{1}{2}c_n(t)B_0(\lambda)_{xx} + (\lambda - \mu)c_n(t)B_0(\lambda)). \end{aligned}$$

Thus

$$\gamma_n(\lambda) = \Gamma_n(\lambda) + c_1(t)\Gamma_{n-1}(\lambda) + ... + c_n(t)\Gamma_0(\lambda). \tag{5.40}$$

Substituting $\alpha_n(\lambda)$ and $\gamma_n(\lambda)$ from equations (5.39) and (5.40) into equation (5.37), we find

$$V_n(\lambda) = \begin{pmatrix} A_n(\lambda)+c_1(t)A_{n-1}(\lambda)+...+c_n(t)A_0(\lambda) & B_n(\lambda)+c_1(t)B_{n-1}(\lambda)+...+c_n(t)B_0(\lambda) \\ \Gamma_n(\lambda)+c_1(t)\Gamma_{n-1}(\lambda)+...+c_n(t)\Gamma_0(\lambda) & -A_n(\lambda)-c_1(t)A_{n-1}(\lambda)-...-c_n(t)A_0(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} A_n(\lambda) & B_n(\lambda) \\ \Gamma_n(\lambda) & -A_n(\lambda) \end{pmatrix} + c_1(t) \begin{pmatrix} A_{n-1}(\lambda) & B_{n-1}(\lambda) \\ \Gamma_{n-1}(\lambda) & -A_{n-1}(\lambda) \end{pmatrix}$$

$$+ \quad ... + c_n \begin{pmatrix} A_0(\lambda) & B_0(\lambda) \\ \Gamma_0(\lambda) & A_0(\lambda) \end{pmatrix}.$$

Using the definition of $U_n(\lambda)$ in equation (3.33), we have

$$V_n(\lambda) \quad = \quad U_n(\lambda) + c_1(t)U_{n-1}(\lambda) + ... + c_n(t)U_0.$$

□

In lemma (5.9), we define new Hamiltonian $H_1, ..., H_g$ by the linear equations

$$I_1 = H_1, \quad I_{n+1} = H_{n+1} + c_1(t)H_n + ... + c_n(t)H_1, \quad n = 1, 2, .., g-1. \tag{5.41}$$

Note that H_{g+1} is not defined (because I_{g+1} does not exist). The formula (5.36) of the Poisson brackets of $V(\lambda)$ and I_n can be converted to the form

$$\{V(\lambda), H_{n+1}\} = [U_n(\lambda), V(\lambda)]. \tag{5.42}$$

We can see this by substituting $c_1(t) = c_2(t) = ... = c_n(t) = 0$ into equation (5.41) and substituting I_{n+1} from equation (5.41) into equation (5.36). As a result we find

$$\{V(\lambda), H_{n+1}\} = \{V(\lambda), I_{n+1}\} = [V_n(\lambda), V(\lambda)] = [U_n(\lambda), V(\lambda)].$$

Theorem 5.10. *Except for the t_{2g+1} flow, the matrix Lax equations (3.47) of the PI hierarchy can be written in the Hamiltonian form*

$$\partial_{2n+1}V(\lambda) = \{V(\lambda), H_{n+1}\} + U_n'(\lambda), \quad n = 0, 1, 2, ..., g-1, \tag{5.43}$$

with the Hamiltonian defined by (5.41).

Proof. By equation (3.47) and (5.42), we have

$$\partial_{2n+1}V(\lambda) = [U_n(\lambda), V(\lambda)] + U_n'(\lambda) = \{V(\lambda), H_{n+1}\} + U_n'(\lambda).$$

□

Remark 5.11. $I_0(\lambda)$ is a central element "a Casimir function" of the Poisson algebra.

5.4 Spectral Darboux coordinates

In this section, we will construct Spectral Darboux coordinates. We can reconstruct the L-matrix $V(\lambda)$ from this coordinates and we will use it to define the Hamiltonian H_{n+1} .

The construction of "Spectral Darboux coordinates" is also parallel to the case of the Mumford system. These coordinates consist of the roots $\lambda_1, ..., \lambda_g$ of $\beta(\lambda)$ and the values $\mu_1, ..., \mu_g$ of $\alpha(\lambda)$ at these roots of $\beta(\lambda)$:

$$\beta(\lambda) = \prod_{j=1}^{g}(\lambda - \lambda_j), \quad \mu_j = \alpha(\lambda_j), \quad j = 1, 2, ..., g. \tag{5.44}$$

To avoid delicate problems, the following consideration is limited to a domain of the phase space where λ_j's are distinct. λ_j and μ_j satisfy the equation $\mu_j^2 = h(\lambda_j)$.
As it follows from (5.34), these new variables satisfy the canonical Poisson relations [29]

$$\{\lambda_j, \lambda_k\} = 0, \quad \{\mu_j, \mu_k\} = 0, \quad \{\lambda_j, \mu_k\} = \delta_{jk}, \; j, k = 1, 2, ..., g. \tag{5.45}$$

Thus λ_j's and μ_j's may be literally called Darboux coordinates.
We can reconstruct the L-matrix $V(\lambda)$ from these new coordinates. We use the familiar Lagrange interpolation formula

$$f(\lambda) = \sum_{j=1}^{g} \frac{f(\lambda_j)}{\beta'(\lambda_j)} \frac{\beta(\lambda)}{(\lambda - \lambda_j)}, \tag{5.46}$$

which holds for any polynomial $f(\lambda) = f_1\lambda^{g-1} + ... + f_g$ of degree less than g.
Since

$$\frac{\beta(\lambda)}{\lambda - \lambda_j} = -\frac{\partial\beta(\lambda)}{\partial\lambda_j} = -\sum_{n=1}^{g} \frac{\partial\beta_n}{\partial\lambda_j}\lambda^{g-n}, \tag{5.47}$$

we have

$$f_n = -\sum_{j=1}^{g} \frac{f(\lambda_j)}{\beta'(\lambda_j)} \frac{\partial\beta_n}{\partial\lambda_j}, \tag{5.48}$$

for the coefficients of $f(\lambda)$ as well. Note that β_n's are being functions of λ_j's.
We apply formula (5.48) to $\alpha(\lambda)$ and obtain the explicit formula

$$\alpha_n = -\sum_{j=1}^{g} \frac{\mu_j}{\beta'(\lambda_j)} \frac{\partial\beta_n}{\partial\lambda_j}. \tag{5.49}$$

Define:

$$f(\lambda) = -\sum_{n=1}^{g} I_n \lambda^{g-n} = h(\lambda) - I_0(\lambda), \tag{5.50}$$

where

$$I_n = -\sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} \frac{\partial \beta_n}{\partial \lambda_j}. \tag{5.51}$$

For instance at $n = 1$, we have $\frac{\partial \beta_1}{\partial \lambda_j} = 1$.

Thus we obtain $I_1 = -\sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)}$.

By the definition of $I_0(\lambda)$ and equation (5.31), we find

$$\begin{aligned} I_0(\lambda) &= \lambda^{2g+1} + 2c_1(t)\lambda^{2g} + (2c_2(t) + c_1(t)^2)\lambda^{2g-2} + ... \\ &+ \sum_{m=0}^{g} c_m(t)c_{g-m}(t)\lambda^{g+1} + (\sum_{m=1}^{g-1} c_m(t)c_{g+1-m}(t) + x)\lambda^g. \end{aligned}$$

Since $\alpha(\lambda)$ and $h(\lambda)$ are reconstructed, we can recover $\gamma(\lambda)$ by equation (5.3) as

$$\gamma(\lambda) = \frac{h(\lambda) - \alpha(\lambda)^2}{\beta(\lambda)}.$$

Lemma 5.12. *If*

$$\beta_n(\lambda) = \lambda^n + \beta_1\lambda^{n-1} + ... + \beta_n, \tag{5.52}$$

then

$$\frac{\partial \beta_n}{\partial \lambda_j} = -\beta_{n-1}(\lambda_j), \qquad n = 1, ..., g. \tag{5.53}$$

Proof. Start by equation (5.44)

$$\begin{aligned} \frac{\partial \beta(\lambda)}{\partial \lambda_j} &= \frac{\partial \prod_{i=1}^{g}(\lambda - \lambda_i)}{\partial \lambda_j} \\ &= -\prod_{i=1, i\neq j}^{g} (\lambda - \lambda_i) = -\frac{\prod_{i=1}^{g}(\lambda - \lambda_i)}{\lambda - \lambda_j} \end{aligned}$$

$$\frac{\partial \beta(\lambda)}{\partial \lambda_j} = -\frac{\beta(\lambda)}{\lambda - \lambda_j}. \tag{5.54}$$

By equation (5.54) and $\beta(\lambda_j) = 0$, we have

$$\frac{\partial \beta(\lambda)}{\partial \lambda_j} = -\frac{\beta(\lambda) - \beta(\lambda_j)}{\lambda - \lambda_j}.$$

By dividing $\lambda^n - \lambda_j^n$ by $\lambda - \lambda_j$, we get

$$-\beta_n \frac{\lambda^n - \lambda_j^n}{\lambda - \lambda_j} = -\beta_n(\lambda^{n-1} + \lambda_j \lambda^{n-2} + ... + \lambda_j^{n-2}\lambda + \lambda_j^{n-1}).$$

This leads to the identity

$$\begin{aligned} \frac{\partial \beta(\lambda)}{\partial \lambda_j} &= -\lambda^{g-1} - (\lambda_j + \beta_1)\lambda^{g-2} - .... - (\lambda_j^{g-1} + \beta_1 \lambda_j^{g-2} + ... + \beta_{g-1}) \\ &= -\lambda^{g-1} - \beta_1(\lambda_j)\lambda^{g-2} - ..... - \beta_{g-1}(\lambda_j). \end{aligned} \tag{5.55}$$

By equation (5.47), we obtain

$$\frac{\beta(\lambda)}{\lambda - \lambda_j} = -\frac{\partial \beta_1}{\partial \lambda_j}\lambda^{g-1} - \frac{\partial \beta_2}{\partial \lambda_j}\lambda^{g-2} - ... - \frac{\partial \beta_g}{\partial \lambda_j}. \tag{5.56}$$

By equations (5.55) and (5.56), we obtain

$$\frac{\partial \beta_n}{\partial \lambda_j} = -\beta_{n-1}(\lambda_j), \qquad n = 1, ..., g.$$

□

By these identities, we can rewrite (5.51) as

$$I_{n+1} = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} \beta_n(\lambda_j), \qquad n = 0, 1, ..., g-1, \tag{5.57}$$

$$\beta_n(\lambda) = R_n(\lambda) + c_1(t) R_{n-1}(\lambda) + ... + c_n(t) R_0(\lambda). \tag{5.58}$$

Choosing $c_1(t) = ... = c_n(t) = 0$, we have $\beta_n(\lambda) = R_n(\lambda)$.
Comparing this linear relation with the linear relation (5.41) among $I_n's$ and $H_n's$, we find that the Hamiltonian H_{n+1} can be expressed as

$$H_{n+1} = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} R_n(\lambda_j), \qquad n = 0, 1, ..., g-1.$$

Now we compute the Hamiltonian for $n = 0, 1, 2$, and write the $I_n's$ as linear combination of Hamiltonian $H_n's$.

Example 5.13. When $n = 0$, we have

$$H_1 = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} R_0(\lambda_j).$$

By equation (5.58), we have

$$H_1 = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} \beta_0(\lambda_j).$$

Using equation (5.57), we obtain

$$H_1 = I_1. \tag{5.59}$$

When $n = 1$, we have

$$H_2 = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} R_1(\lambda_j).$$

By equation (5.58), we have

$$H_2 = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} (\beta_1(\lambda_j) - c_1(t)).$$

Using (5.57), we obtain

$$H_2 = I_2 - c_1(t)I_1 = I_2 - c_1(t)H_1. \tag{5.60}$$

As a result, we obtain

$$I_2 = H_2 + c_1(t)H_1.$$

When $n = 2$, we have

$$H_3 = \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} R_2(\lambda_j).$$

By equation (5.58), we have

$$\begin{aligned} H_3 &= \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} (\beta_2(\lambda_j) - c_1(t)R_1(\lambda_j) - c_2(t)) \\ &= \sum_{j=1}^{g} \frac{\mu_j^2 - I_0(\lambda_j)}{\beta'(\lambda_j)} (\beta_2(\lambda_j) - c_1(t)(\beta_1(\lambda_j) - c_1(t)) - c_2(t)). \end{aligned}$$

In this case, equation (5.57) gives

$$\begin{aligned} H_3 &= I_3 - c_1(t)I_2 - c_2(t)I_1 + c_1^2(t)I_1 \\ &= I_3 - c_1(t)(H_2 + c_1(t)H_1) - c_2(t)H_1 + c_1^2(t)H_1 \\ &= I_3 - c_1(t)H_2 - c_2(t)H_1 \end{aligned}$$

Which implies

$$I_3 = H_3 + c_1(t)H_2 + c_2(t)H_1.$$

Now we will drive the the Hamiltonian structure of the first and second members of PI hierarchy.

Example 5.14. When $g = 1$, by equation (3.42), we have

$$\beta(\lambda) = R_1(\lambda) + c_1(t)R_0(\lambda).$$

Substituting $R_1(\lambda)$ and $R_0(\lambda)$ from equation (3.35) and (3.10) into above equation, we obtain

$$\beta(\lambda) = \lambda + \frac{u}{2} + c_1(t). \tag{5.61}$$

By equation (3.42), we obtain

$$\begin{aligned} \alpha(\lambda) &= -\frac{1}{2}\beta(\lambda)_x = \frac{-1}{4}u_x. \\ \gamma(\lambda) &= \frac{-1}{2}\beta(\lambda)_{xx} + (\lambda - u)\beta(\lambda) \\ &= \frac{-1}{4}u_{xx} + \lambda^2 + \frac{u}{2}\lambda + c_1(t)\lambda - \lambda u - \frac{u^2}{2} - c_1(t)u. \end{aligned} \tag{5.62}$$

Substituting $\beta(\lambda), \alpha(\lambda)$ and $\gamma(\lambda)$ from equations (5.61) and (5.62) into equation (5.3), we obtain

$$\begin{aligned} h(\lambda) &= \lambda^3 + 2\lambda^2 c_1(t) + \lambda(\frac{-1}{4}\lambda u_{xx} - uc_1(t) - \frac{3}{4}u^2 + c_1(t)^2) \\ &+ \frac{1}{16}u_x^2 - \frac{1}{8}uu_{xx} - u^2 c_1(t) - \frac{1}{4}u^3 - \frac{1}{4}c_1(t)u_{xx} + uc_1(t)^2. \end{aligned}$$

Since from equation (5.32), we have

$$I_1 = \frac{1}{16}u_x^2 - \frac{1}{8}uu_{xx} - u^2 c_1(t) - \frac{1}{4}u^3 - \frac{1}{4}c_1(t)u_{xx} + uc_1(t)^2.$$

From equation (5.59), we have

$$H_1 = \frac{1}{16}u_x^2 - \frac{1}{8}uu_{xx} - u^2 c_1(t) - \frac{1}{4}u^3 - \frac{1}{4}c_1(t)u_{xx} + uc_1(t)^2,$$

which is the Hamiltonian structure of the first member of PI hierarchy.

Example 5.15. When $g = 2$, equation (3.42) gives

$$\beta(\lambda) = R_2(\lambda) + c_1(t)R_1(\lambda) + c_2(t)R_0(\lambda).$$

Substituting $R_2(\lambda), R_1(\lambda)$ and $R_0(\lambda)$ from equation (3.35) and (3.10) into above equation, we obtian

$$\beta(\lambda) = \lambda^2 + \lambda(\frac{u}{2} + c_1(t)) + \frac{1}{8}u_{xx} + \frac{3}{8}u^2 + c_1(t)\frac{u}{2} + c_2(t). \tag{5.63}$$

By equation (3.42), we obtain

$$\begin{aligned}
\alpha(\lambda) &= -\frac{\lambda u_x}{4} - \frac{1}{16}u_{xxx} - \frac{3}{8}uu_x - c_1(t)\frac{u_x}{4}. \\
\gamma(\lambda) &= \lambda^3 - \frac{1}{2}\lambda^2 u + \lambda^2 c_1(t) - \frac{1}{8}u_{xx}\lambda - \frac{1}{8}u^2\lambda - \frac{1}{2}c_1(t)\lambda u - \frac{1}{2}uu_{xx} \\
&+ c_2(t)\lambda - \frac{3}{8}u^3 - \frac{1}{2}c_1(t)u^2 - uc_2(t) - \frac{1}{16}u_{xxx} - \frac{3}{8}u_x^2 - \frac{1}{4}u_{xx}c_1(t).
\end{aligned} \tag{5.64}$$

Substituting $\beta(\lambda), \alpha(\lambda)$ and $\gamma(\lambda)$ from equations (5.63) and (5.64) into equation (5.3), we obtain

$$\begin{aligned}
h(\lambda) &= \lambda^5 + 2\lambda^4 c_1(t) + \lambda^3(c_1(t)^2 + 2c_2(t)) \\
&+ \lambda^2[\frac{-5}{16}u_x^2 - \frac{5}{8}u^3 - \frac{1}{16}u_{xx} - uc_2(t) - \frac{5}{8}uu_{xx} - \frac{3}{4}u^2c_1(t) - \frac{1}{4}u_{xx}c_1(t) + 2c_1(t)c_2(t)] \\
&+ \lambda[c_2(t)^2 + \frac{1}{32}u_xu_{xxx} - \frac{15}{64}u^4 - \frac{1}{64}u_{xx}^2 - \frac{3}{4}uu_{xx}c_1(t) - c_1(t)c_2(t)u - \frac{3}{4}u^2c_1(t)^2] \\
&+ \lambda[-\frac{1}{32}uu_{xxx} - \frac{1}{4}u^2c_2(t) - \frac{7}{8}u^3c_1(t) - \frac{1}{4}c_1(t)^2u_{xx} - \frac{5}{16}u^2u_{xx} - \frac{1}{4}u_x^2c_1(t) - \frac{1}{16}c_1(t)u_{xxx}] \\
&+ \frac{1}{32}c_1(t)u_xu_{xxx} - \frac{1}{32}c_1(t)uu_{xxx} - c_2(t)^2u - \frac{1}{4}c_1(t)u_{xx}c_2(t) - \frac{13}{32}c_1(t)u_{xx}u^2 - \frac{3}{64}u_x^2u_{xx} \\
&- \frac{15}{64}u^3u_{xx} - \frac{1}{32}c_1(t)u_{xx}^2 + \frac{1}{16}c_1(t)^2u_x^2 - \frac{1}{128}u_{xxx}u_{xx} - \frac{1}{16}c_2(t)u_{xxx} - \frac{3}{4}c_2(t)u^3 - \frac{1}{4}c_1(t)^2u^3 \\
&- \frac{3}{8}c_2(t)u_x^2 - \frac{1}{16}uu_{xx}^2 - \frac{3}{128}u^2u_{xxx} - \frac{3}{8}u^4c_1(t) - \frac{1}{8}c_1(t)^2uu_{xx} - \frac{5}{8}c_2(t)uu_{xx} \\
&+ \frac{1}{250}u_{xxx}^2 + \frac{3}{64}uu_xu_{xxx} - \frac{9}{64}u^5 + c_1(t)c_2(t)u^2.
\end{aligned}$$

Since from equation (5.32), we have

$$\begin{aligned}
I_1 &= c_2(t)^2 + \frac{1}{32}u_xu_{xxx} - \frac{15}{64}u^4 - \frac{1}{64}u_{xx}^2 - \frac{3}{4}uu_{xx}c_1(t) - c_1(t)c_2(t)u - \frac{3}{4}u^2c_1(t)^2 \\
&- \frac{1}{32}uu_{xxx} - \frac{1}{4}u^2c_2(t) - \frac{7}{8}u^3c_1(t) - \frac{1}{4}c_1(t)^2u_{xx} - \frac{5}{16}u^2u_{xx} - \frac{1}{4}u_x^2c_1(t) - \frac{1}{16}c_1(t)u_{xxx}.
\end{aligned}$$

and

$$\begin{aligned}
I_2 = \quad & \frac{1}{32}c_1(t)u_x u_{xxx} - \frac{1}{32}c_1(t)uu_{xxx} - c_2(t)^2 u - \frac{1}{4}c_1(t)u_{xx}c_2(t) - \frac{13}{32}c_1(t)u_{xx}u^2 - \frac{3}{64}u_x^2 u_{xx} \\
- \quad & \frac{15}{64}u^3 u_{xx} - \frac{1}{32}c_1(t)u_{xx}^2 + \frac{1}{16}c_1(t)^2 u_x^2 - \frac{1}{128}u_{xxx}u_{xx} - \frac{1}{16}c_2(t)u_{xxx} - \frac{3}{4}c_2(t)u^3 - \frac{1}{4}c_1(t)^2 u^3 \\
- \quad & \frac{3}{8}c_2(t)u_x^2 - \frac{1}{16}uu_{xx}^2 - \frac{3}{128}u^2 u_{xxx} - \frac{3}{8}u^4 c_1(t) - \frac{1}{8}c_1(t)^2 uu_{xx} - \frac{5}{8}c_2(t)uu_{xx} \\
+ \quad & \frac{1}{250}u_{xxx}^2 + \frac{3}{64}uu_x u_{xxx} - \frac{9}{64}u^5 + c_1(t)c_2(t)u^2.
\end{aligned}$$

From equation (5.60), we have

$$H_2 = I_2 - c_1(t)I_1$$

which is the Hamiltonian structure of the second member of the PI hierarchy.

Chapter 6

The Hamiltonian structure of second Painlevé hierarchy

In this chapter, we introduce canonical coordinates $P_1, ..., P_n, Q_1, ..., Q_n$ and a Hamiltonian function $\mathcal{H}^{(n)}(P_1, ..., P_n, Q_1, ..., Q_n, z)$ such that the n-th member of the second Painlevé hierarchy is given by the equation [27]

$$\frac{dP_k}{dz} = -\frac{\partial \mathcal{H}^{(n)}}{\partial Q_k}, \qquad \frac{dQ_k}{dz} = \frac{\partial \mathcal{H}^{(n)}}{\partial P_k}, \ k = 1, 2, ..., n.$$

6.1 Isomonodromic problem for the P_{II} Hierarchy

The isomonodromic deformation problem for the P_{II} Hierarchy (4.31) is given by:

$$\begin{aligned} \frac{\partial \Psi}{\partial z} &= B\Psi = \begin{pmatrix} -\lambda & w \\ w & \lambda \end{pmatrix} \Psi, \\ \frac{\partial \Psi}{\partial \lambda} &= A^{(n)}\Psi = \tfrac{1}{\lambda} \left[\begin{pmatrix} -\lambda z & -\alpha_n \\ -\alpha_n & \lambda z \end{pmatrix} + M^{(n)} + \sum_{l=1}^{n-1} t_l M^{(l)} \right] \Psi, \end{aligned} \tag{6.1}$$

where

$$M^{(l)} = \begin{pmatrix} \sum_{j=1}^{2l+1} A_j^{(l)} \lambda^j & \sum_{j=1}^{2l} B_j^{(l)} \lambda^j \\ \sum_{j=1}^{2l} C_j^{(l)} \lambda^j & -\sum_{j=1}^{2l+1} A_j^{(l)} \lambda^j \end{pmatrix}$$

with

$$\begin{array}{rcl} A_{2l+1}^{(l)} &=& 4^l, \qquad A_{2k}^{(l)} = 0, \qquad k = 0, ..., l, \\ A_{2k+1}^{(l)} &=& \frac{4^{k+1}}{2}\{\ell_{l-k}[w_z - w^2] - \frac{d}{dz}(\frac{d}{dz} + 2w)\ell_{l-k-1}[w_z - w^2]\}, \quad k = 0, ..., l-1, \\ B_{2k+1}^{(l)} &=& \frac{4^{k+1}}{2}\frac{d}{dz}(\frac{d}{dz} + 2w)\ell_{l-k-1}[w_z - w^2], \quad k = 0, ..., l-1, \\ B_{2k}^{(l)} &=& -4^k(\frac{d}{dz} + 2w)\ell_{l-k}[w_z - w^2], \quad k = 1, ..., l, \\ C_{2k+1}^{(l)} &=& -B_{2k+1}^l, \quad k = 0, ..., l-1, \\ C_{2k}^{(l)} &=& B_{2k}^l, \quad k = 0, ..., l. \end{array} \tag{6.2}$$

By equation (6.1), we have

$$\begin{array}{rcl} \frac{\partial^2\Psi}{\partial z\partial\lambda} &=& \frac{\partial(\frac{\partial\Psi}{\partial\lambda})}{\partial z} \\ &=& \frac{\partial(A^{(n)}\Psi)}{\partial z} \\ &=& \frac{\partial A^{(n)}}{\partial z}\Psi + A^{(n)}\frac{\partial\Psi}{\partial z}. \end{array} \tag{6.3}$$

Substituting $\frac{\partial\Psi}{\partial z}$ from equation (6.1) into equation (6.3), we have

$$\frac{\partial^2\Psi}{\partial z\partial\lambda} = \frac{\partial A^{(n)}}{\partial z}\Psi + A^{(n)}B\Psi. \tag{6.4}$$

Similarity

$$\frac{\partial^2\Psi}{\partial\lambda\partial z} = \frac{\partial B}{\partial\lambda}\Psi + BA^{(n)}\Psi. \tag{6.5}$$

Equation (6.5), yield

$$\frac{\partial A^{(n)}}{\partial z} - \frac{\partial B}{\partial\lambda} = [B, A^{(n)}]. \tag{6.6}$$

$A^{(n)}$ can be simplified by introducing some new notations.
Define

$$\begin{array}{rcl} a_{2k+1}^{(n)} &=& \sum_{l=1}^{n} t_l A_{2k+1}^{(l)}, \quad k = 1, 2, ..., n, \quad a_1^{(n)} = \sum_{l=1}^{n} t_l A_{2k+1}^{(l)} - z, \\ b_{2k+1}^{(n)} &=& \sum_{l=1}^{n} t_l B_{2k+1}^{(l)}, \quad k = 0, 1, ..., n-1, \\ b_{2k}^{(n)} &=& \sum_{l=1}^{n} t_l B_{2k}^{(l)}, \qquad k = 1, 2, ..., n, \qquad b_0^{(n)} = -\alpha_n, \end{array} \tag{6.7}$$

where $t_n = 1$. Then $A^{(n)}$ can be written as [27]

$$A^{(n)}(\lambda) := \begin{pmatrix} \sum_{k=0}^{n} a_{2k+1}^{(n)} \lambda^{2k} & \sum_{k=0}^{2n} b_k^{(n)} \lambda^{k-1} \\ \sum_{k=0}^{2n} (-1)^k b_k^{(n)} \lambda^{k-1} & -\sum_{k=0}^{n} a_{2k+1}^{(n)} \lambda^{2k} \end{pmatrix}. \tag{6.8}$$

6.2 Coadjoint orbit interpretation

In this section, we will define the total derivative of any function and study another lemma of the coadjoint action. In the end of this section, the Poisson brackets between the coefficients $a_{2k+1}^{(n)}, b_{2k}^{(n)}$ and $b_{2k+1}^{(n)}$ are computed.

Given any function f of $\lambda, z, w_z, w_{zz}, ...$, we will denote the total derivative of f with respect to z by $\partial_z f$. That is

$$\partial_z f := \frac{\partial f}{\partial z} + \frac{\partial f}{\partial w} w_z + \frac{\partial f}{\partial w_z} w_{zz} + ...$$

The explicit derivative of f with respect to z is denoted by $\partial_z^w f$, where $w, w_z, w_{zz}, ...$ are treated as independent variables. Analogously, $\partial_\lambda f$ denotes the total derivative of f with respect to λ.

Now, we will show that

$$\partial_z^w A^{(n)} = \partial_\lambda B. \tag{6.9}$$

Since $a_1^{(n)}$ is the unique element of $A^{(n)}$ that depends explicitly on z, by equation (6.8), we obtain

$$\partial_z^w A^{(n)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, from equation (6.1), we obtain

$$\partial_\lambda B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So that equation (6.6) can be written as

$$(\partial_z - \partial_z^w) A^{(n)} = [B, A^{(n)}]. \tag{6.10}$$

We are now going to interpret the evolution along $(\partial_z - \partial_z^w)$ as a vector field on a coadjoint orbit of an element of an appropriate twisted loop algebra. Let LG be the group of smooth maps f from S^1 to SL_2, S^1 is circle with radius one, such that

$$f(\lambda)\sigma_1(f(-\lambda))^{-1} = I, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and λ is considered as a parameter on S^1. The subgroup maps of the form $f = I + \lambda^{-2n-2} f_\infty$, where f_∞ is holomorphic outside S^1 is denoted by $L_{2n+2}G$. Let $\mathfrak{g}_{2n+2}$ be the Lie algebra:

$$\mathfrak{g}_{2n+2} = \left(X(\lambda) = \sum_{-\infty}^{-2n-2} X_i \lambda^i | X_i \in gl(2,\mathbb{C}), X(\lambda)\sigma_1 = \sigma_1 X(-\lambda) \right).$$

Define G by the quotient of these two groups:

$$G = LG/L_{2n+2}G,$$

whose Lie algebra is given by

$$\mathfrak{g} = \left(X(\lambda) = \sum_{-\infty}^{\infty} X_i \lambda^i | X_i \in gl(2,\mathbb{C}), X(\lambda)\sigma_1 = \sigma_1 X(-\lambda) \right) / \mathfrak{g}_{2n+2},$$

with Lie bracket defined by

$$[X(\lambda), \tilde{X}(\lambda)] = \sum_{i=-2n-1}^{\infty} \left(\sum_{k=-2n-1}^{i+2n+1} [X_k(\lambda), \tilde{X}_{i-k}(\lambda)] \right) \lambda^i \text{ mod } \mathfrak{g}_{2n+2}.$$

This implies that $[X(\lambda), \tilde{X}(\lambda)] \in \mathfrak{g}$ and satisfies the Jacobi identity. The dual space $\mathfrak{g}^*$ can be identified with

$$\mathfrak{g}^* = \left(\Xi(\lambda) = \sum_{-\infty}^{\infty} \Xi_i \lambda^i | i \in \mathbb{N}, \Xi_i \in gl(2,\mathbb{C}), \quad \Xi(\lambda)\sigma_1 = -\sigma_1 \Xi(-\lambda) \right) / \mathfrak{g}^*_{2n+2},$$

$$\mathfrak{g}^*_{2n+2} = \left(X(\lambda) = \sum_{2n+1}^{\infty} X_i \lambda^i | X_i \in gl(2,\mathbb{C}), \quad X(\lambda)\sigma_1 = \sigma_1 X(-\lambda) \right).$$

Define $< \Xi(\lambda), X(\lambda) >$ by

$$< \Xi(\lambda), X(\lambda) > := Tr(\text{Res} X(\lambda)\Xi(\lambda)), \qquad \forall X(\lambda) \in \mathfrak{g}, \ \Xi(\lambda) \in \mathfrak{g}^*, \tag{6.11}$$

where Res indicates the formal residue, i.e. the coefficient of the λ^{-1} term.
Consider the subalgebra

$$\mathfrak{g}_- = \left(X(\lambda) = \sum_{-\infty}^{-1} X_i \lambda^i | X(\lambda)\sigma_1 = \sigma_1 X(-\lambda) \right) / \mathfrak{g}_{2n+2}. \tag{6.12}$$

The dual space of (6.12) can be identified with

$$\mathfrak{g}^*_- = \left(\Xi(\lambda) = \sum_{0}^{\infty} \Xi_i \lambda^i | \Xi_i \in gl(2,\mathbb{C}), \quad \Xi(\lambda)\sigma_1 = -\sigma_1 \Xi(-\lambda) \right) / \mathfrak{g}^*_{2n+2}. \tag{6.13}$$

An element X in the Lie algebra $\mathfrak{g}$ acts on an element $\Xi \in \mathfrak{g}^*$, by the coadjoint action

$$< \mathrm{ad}^*_X \Xi \ , Y >:= - < \Xi, [X,Y] >=< [X,\Xi], Y > \tag{6.14}$$

for any $Y \in \mathfrak{g}$.

By equation (6.11), we have

$$\begin{aligned} < \mathrm{ad}^*_X \Xi \ , Y > &= - < \Xi, [X,Y] > \\ &= -\mathrm{Tr}\ \mathrm{Res}([X,Y]\Xi) \\ &= -\mathrm{Tr}\ \mathrm{Res}(XY\Xi - YX\Xi). \end{aligned}$$

Since Res Tr(A)=Tr Res(A) for any matrix A, we have

$$< \mathrm{ad}^*_X \Xi \ , Y > = -\mathrm{Res}\ \mathrm{Tr}(XY\Xi - YX\Xi).$$

Using the linearity of the Trace operator, we obtain

$$< \mathrm{ad}^*_X \Xi \ , Y > = -\mathrm{Res}(\mathrm{Tr}(XY\Xi) - \mathrm{Tr}(YX\Xi)).$$

Using $\mathrm{Tr}(YX\Xi) = \mathrm{Tr}(\Xi YX) = \mathrm{Tr}(X\Xi Y)$ and $\mathrm{Tr}(XY\Xi) = \mathrm{Tr}(Y\Xi X) = \mathrm{Tr}(\Xi XY)$, we find

$$\begin{aligned} < \mathrm{ad}^*_X \Xi \ , Y > &= -\mathrm{Res}\ \mathrm{Tr}(\Xi XY - X\Xi Y) \\ &= -\mathrm{Res}\ \mathrm{Tr}([\Xi, X]Y) \\ &= - < Y, [\Xi, X] > \\ &= < [X,\Xi], Y > . \end{aligned}$$

This shows that for every $X \in \mathfrak{g}$, $\Xi \in \mathfrak{g}^*$

$$[X,\Xi] = \mathrm{ad}^*_X \Xi \in \mathfrak{g}^*.$$

When we restrict the coadjoint action to the subalgebra $\mathfrak{g}_-$ and to its dual space $\mathfrak{g}^*_-$, we obtain

$$[X_-, \Xi]_+ = \mathrm{ad}^*_{X_-} \Xi, \qquad \Xi \in \mathfrak{g}^*_-, \qquad X_- \in \mathfrak{g}_-, \tag{6.15}$$

where $(.)_+$ is the projection from $\mathfrak{g}^*$ onto $\mathfrak{g}^*_-$ and $(.)_-$ denotes the projection onto $\mathfrak{g}_-$.

Lemma 6.1. *Given the matrices B and $A^{(n)}$ as in equation (6.1) and (6.2), one has*

$$[B, A^{(n)}] = ad^*_B A, \tag{6.16}$$

where $B = \left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_- \in \mathfrak{g}_-$ *and* $A = (A^{(n)})_+ \in \mathfrak{g}_-^*$*, which is the dynamical part of* $A^{(n)}$*.*

Proof. Using equation (6.8), we have

$$\begin{aligned} -\left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_+ &= \frac{-1}{4^n}\begin{pmatrix} a^{(n)}_{2n+1}\lambda & b^{(n)}_{2n} \\ b^{(n)}_{2n} & -a^{(n)}_{2n+1}\lambda \end{pmatrix} \\ &= \frac{-1}{4^n}\begin{pmatrix} \sum_{l=1}^{n} t_l A^{(l)}_{2n+1}\lambda & \sum_{l=1}^{n} t_l B^{(l)}_{2n} \\ \sum_{l=1}^{n} t_l B^{(l)}_{2n} & -\sum_{l=1}^{n} t_l A^{(l)}_{2n+1}\lambda \end{pmatrix}. \end{aligned}$$

Since $t_1 = t_2 = ... = t_{n-1} = 0,\ t_n = 1$ and $B^{(n)}_{2n} = -4^n w$, we obtain

$$\begin{aligned} -\left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_+ &= \frac{-1}{4^n}\begin{pmatrix} 4^n\lambda & B^{(n)}_{2n} \\ B^{(n)}_{2n} & -4^n\lambda \end{pmatrix} \\ &= \begin{pmatrix} -\lambda & w \\ w & \lambda \end{pmatrix} \\ &= B. \end{aligned}$$

Using $B = \left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_-$ to calculate $[B, A^{(n)}]$, we obtain

$$[B, A^{(n)}] = [\left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_-, A^{(n)}].$$

Using $A^{(n)} = (A^{(n)})_+ + (A^{(n)})_-$ and bilinearity of [,], we have

$$[B, A^{(n)}] = [\left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_-, (A^{(n)})_+] + [\left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_-, (A^{(n)})_-].$$

Since $(A^{(n)})_-$ and $\left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_-$ commute, we have

$$[\left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_-, (A^{(n)})_-] = 0.$$

Hence it follows that

$$\begin{aligned} [B, A^{(n)}] &= [\left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_-, (A^{(n)})_+] \\ &= [\left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_-, A]. \end{aligned}$$

□

Remark 6.2. We want to compute $\mathrm{ad}^*_{X_k} A^{(n)}$ for $X_k = \left(\dfrac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_-$.
For every $Y \in \mathfrak{g}_-$, we have

$$\begin{aligned} < \mathrm{ad}^*_{X_k} A^{(n)}, Y > &= < [\left(\frac{A^{(n)}\lambda^{-2k+1}}{4^k}\right)_-, A^{(n)}], Y > \\ &= < [\left(\frac{A^{(n)}\lambda^{-2k+1}}{4^k}\right)_-, (A^{(n)})_-], Y > + < [\left(\frac{A^{(n)}\lambda^{-2k+1}}{4^k}\right)_-, (A^{(n)})_+], Y > \\ &= < [\left(\frac{A^{(n)}\lambda^{-2k+1}}{4^k}\right)_-, (A^{(n)})_+], Y >. \end{aligned}$$

Similarly we can prove that

$$(\partial_z - \partial_z^w)A^{(n)} = (\partial_z - \partial_z^w)A.$$

Since $(\partial_z - \partial_z^w)A^{(n)} = (\partial_z - \partial_z^w)(A^{(n)})_+ + (\partial_z - \partial_z^w)(A^{(n)})_-$, using equation (6.1), we obtain

$$(\partial_z - \partial_z^w)(A^{(n)})_- = 0.$$

Thus $(\partial_z - \partial_z^w)A^{(n)} = (\partial_z - \partial_z^w)A$.
So, from above, we can write the following lemma:

Lemma 6.3. *The monodromy preserving deformation equation (6.10) is the same as*

$$(\partial_z - \partial_z^w)A^{(n)} = ad_B^* A \tag{6.17}$$

where $A = (A^{(n)})_+ \in \mathfrak{g}_-^*$ *is the dynamical part of* $A^{(n)}$, $B = \left(\frac{A^{(n)}\lambda^{-2n+1}}{4^n}\right)_- \in \mathfrak{g}_-$.

The Poisson structure on $\mathfrak{g}^*$ is given by observing that every $X \in \mathfrak{g}_-$ defines a linear function X_* on $\Xi \in \mathfrak{g}_-^*$:

$$X_* : \begin{array}{l} g_-^* \longrightarrow \mathbb{C} \\ \Xi \longrightarrow < \Xi, X > . \end{array}$$

This fact allows one to identify $\mathfrak{g}^{**}$ with $\mathfrak{g}_-$ and to define the Poisson bracket between two linear functions on $\mathfrak{g}_-^*$ by

$$\{f, h\}(\Xi) = < \Xi, [df, dh] >, \tag{6.18}$$

where the differential df of a function f on $\mathfrak{g}_-^*$ is a linear function $df \in \mathfrak{g}_-^{**} \sim \mathfrak{g}_-$ defined by

$$< df, \delta\Xi >:= f(\Xi + \delta_x(\Xi)) - f(\Xi) + Q(\delta_x(\Xi))^2, \tag{6.19}$$

and $\delta_x(\Xi)) := \mathrm{ad}^*_X \Xi \in \mathfrak{g}^*_-$.

Define the so-called Kostant-Kirillov symplectic structure w by

$$\omega(f, h) = \{f, h\}, \tag{6.20}$$

for every pair of functions on the coadjoint orbit [27].

To compute the Poisson brackets between the coefficients $a^{(n)}_{2k+1}, b^{(n)}_{2k}$ and $b^{(n)}_{2k+1}$, we observe that their differentials are

$$\begin{array}{lcll} da^{(n)}_{2k+1} & = & \frac{1}{2}(\Xi_{11} - \Xi_{22})\lambda^{-(2k+1)} & \text{for } \ 0 \le k \le n, \\ db^{(n)}_{2k+1} & = & \frac{1}{2}(-\Xi_{12} + \Xi_{21})\lambda^{-(2k+1)} & \text{for } \ 0 \le k \le n-1, \\ db^{(n)}_{2k} & = & \frac{1}{2}(\Xi_{12} + \Xi_{21})\lambda^{-2k} & \text{for } \ 0 \le k \le n, \end{array} \tag{6.21}$$

where with a slight abuse of notation, we are calling $a^{(n)}_{2k+1}, b^{(n)}_{2k+1}$, and $b^{(n)}_{2k}$ the elements of $\mathfrak{g}^{**}_- \sim \mathfrak{g}_-$ which applied to Ξ producing the coefficients $a^{(n)}_{2k+1}$, $b^{(n)}_{2k+1}$, and $b^{(n)}_{2k}$ respectively. By using these gradients, we can compute the Poisson brackets between the matrix entries

$$\begin{array}{lcll} \{a^{(n)}_{2m+1}, b^{(n)}_{2l+1}\} & = & -b^{(n)}_{2(m+l+1)} & \text{for } 0 \le m \le n, \ 0 \le l \le n-1, \ m+l \le n-1, \\ \{a^{(n)}_{2m+1}, b^{(n)}_{2l}\} & = & -b^{(n)}_{2(m+l)+1} & \text{for } 0 \le m \le n, \ 1 \le l \le n, \ m+l \le n-1, \\ \{b^{(n)}_{2m}, b^{(n)}_{2l+1}\} & = & a^{(n)}_{2(m+l)+1} & \text{for } 1 \le m \le n, \ 1 \le l \le n-1, \ m+l \le n, \end{array} \tag{6.22}$$

while all other brackets vanish.

By equation (6.8), we can put $\Xi(\lambda) = A^{(n)}(\lambda) \in \mathfrak{g}^*$. Using equation (6.21), we have

$$\begin{array}{lcll} da^{(n)}_{2m+1} & = & \frac{1}{2}\lambda^{-(2m+1)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{for } \ 0 \le m \le n, \\ db^{(n)}_{2l+1} & = & \frac{1}{2}\lambda^{-(2l+1)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{for } \ 0 \le l \le n-1, \\ db^{(n)}_{2k} & = & \frac{1}{2}\lambda^{-2k} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{for } \ 0 \le k \le n. \end{array} \tag{6.23}$$

Using equation (6.23), we can compute the commutation between $da^{(n)}_{2m+1}$ and $db^{(n)}_{2l+1}$ by

$$\begin{array}{rcl} [da^{(n)}_{2m+1}, db^{(n)}_{2l+1}] & = & da^{(n)}_{2m+1} db^{(n)}_{2l+1} - db^{(n)}_{2l+1} da^{(n)}_{2m+1} \\ & = & \frac{-1}{2}\lambda^{-2(m+l+1)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{array}$$

Thus, by equation (6.18), we find that the Poisson brackets between $a^{(n)}_{2m+1}$ and $b^{(n)}_{2l+1}$ are given by

$$\begin{aligned}\{a^{(n)}_{2m+1}, b^{(n)}_{2l+1}\} &= <\Xi, [da^{(n)}_{2m+1}, db^{(n)}_{2l+1}]> \\ &= <\Xi, -\frac{1}{2}\lambda^{-2(m+l+1)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}>.\end{aligned}$$

By equation (6.11), we obtain

$$\begin{aligned}\{a^{(n)}_{2m+1}, b^{(n)}_{2l+1}\} &= \text{Tr Res}\left(\tfrac{-1}{2}\lambda^{-2(m+l+1)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Xi\right) \\ &= \text{Res Tr}\left(\tfrac{-1}{2}\lambda^{-2(m+l+1)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Xi\right).\end{aligned}$$

By definition of Ξ, we have

$$\begin{aligned}\{a^{(n)}_{2m+1}, b^{(n)}_{2l+1}\} &= \tfrac{-1}{2}\text{ Res}\left(2\sum_{k=0}^{n} b^{(n)}_{2k}\lambda^{-2(m+l+1)+2k-1}\right) \\ &= -b^{(n)}_{2(m+l+1)}, \qquad \text{for } \ 0 \le m \le n, \ 0 \le l \le n-1, \ m+l \le n-1.\end{aligned}$$

Similarly, we can calculate the Poisson brackets between $a^{(n)}_{2m+1}$ and $b^{(n)}_{2l}$ by

$$\begin{aligned}[da^{(n)}_{2m+1}, db^{(n)}_{2l}] &= da^{(n)}_{2m+1}db^{(n)}_{2l} - db^{(n)}_{2l}da^{(n)}_{2m+1} \\ &= \tfrac{1}{2}\lambda^{-2(m+l)-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\end{aligned}$$

By equation (6.11), we obtain

$$\begin{aligned}\{a^{(n)}_{2m+1}, b^{(n)}_{2l}\} &= \text{Tr Res}\left(\tfrac{1}{2}\lambda^{-2(m+l)-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Xi\right) \\ &= \text{Res Tr}\left(\tfrac{1}{2}\lambda^{-2(m+l)-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Xi\right).\end{aligned}$$

By definition of Ξ, we have

$$\begin{aligned}\{a^{(n)}_{2m+1}, b^{(n)}_{2l+1}\} &= \tfrac{1}{2}\text{ Res}\left(-2\sum_{k=0}^{n-1} b^{(n)}_{2k+1}\lambda^{-2(m+l)-1+2k}\right) \\ &= -b^{(n)}_{2(m+l)+1} \quad \text{for } 0 \le m \le n, \ 1 \le l \le n, \ m+l \le n-1.\end{aligned}$$

In the same way, we can calculate the Poisson brackets between $b_{2m}^{(n)}$ and $b_{2l+1}^{(n)}$ by

$$\begin{aligned}[db_{2m}^{(n)}, db_{2l+1}^{(n)}] &= db_{2m}^{(n)} db_{2l+1}^{(n)} - db_{2l+1}^{(n)} db_{2m}^{(n)} \\ &= \tfrac{1}{2}\lambda^{-2(m+l)-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned}$$

By equation (6.11), we obtain

$$\begin{aligned}\{b_{2m}^{(n)}, b_{2l+1}^{(n)}\} &= \text{Tr Res}\left(\tfrac{1}{2}\lambda^{-2(m+l)-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Xi\right) \\ &= \text{Res Tr}\left(\tfrac{1}{2}\lambda^{-2(m+l)-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Xi\right).\end{aligned}$$

By definition of Ξ, we have

$$\begin{aligned}\{b_{2m}^{(n)}, b_{2l+1}^{(n)}\} &= \tfrac{1}{2}\text{ Res}\left(2\sum_{k=0}^{n} a_{2k+1}^{(n)} \lambda^{-2(m+l)-1+2k}\right) \\ &= a_{2(m+l)+1}^{(n)}, \quad \text{for } 1 \le m \le n,\ 1 \le l \le n-1,\ m+l \le n.\end{aligned}$$

6.3 Canonical coordinates for the isomonodromic deformations

In this section, we will build the canonical coordinates for the second Painlevé hierarchy by using the general framework of the algebro-geometric darboux coordinates.

In this setting one considers the spectral curve

$$\Gamma(\mu,\lambda) = Q(\mu,\lambda) = [\mu^2 = -\det(A^{(n)}(\lambda))], \tag{6.24}$$

where $Q(\mu,\lambda) = [\det(\mu I - A^{(n)}(\lambda) = 0]$.
Denote by q the λ-projection of the generic point in the divisor of ψ [27]. We fix the following normalization

$$(c_1, c_2).\psi(q) = 1,$$

for some choice of c_1, c_2. The q_j variables are the roots of

$$c_1^2 A_{12}(q_j) - c_1 c_2 (A_{11}(q_j) - A_{22}(q_j)) - c_2^2 A_{21}(q_j) = 0, \tag{6.25}$$

while the p_j variables are the eigenvalues

$$p_j = \left(A_{11}(q_j) - \frac{c_1}{c_2} A_{12}(q_j) \right), \tag{6.26}$$

where $A = [A_{ij}]$.
Choosing the normalization $c_1 = -c_2 = 1$, we get roots $q_1, ..., q_{2n}$ such that $q_{n+j} = -q_j, \quad j = 1, ..., n$. They are the roots of the following equation:

$$\sum_{k=0}^{n-1} (b_{2k+1}^{(n)} + a_{2k+1}^{(n)}) q_j^{2k} + a_{2n+1}^{(n)} q_j^{2n} = 0. \tag{6.27}$$

Substituting $A_{12}, A_{11}, A_{22}, A_{21}$ from equation (6.8) and $c_1 = -c_2 = 1$ into equation (6.25), we have

$$\sum_{k=0}^{n} b_{2k}^{(n)} q_j^{2k-1} + \sum_{k=0}^{n-1} b_{2k+1}^{(n)} q_j^{2k} + \sum_{k=0}^{n} a_{2k+1}^{(n)} q_j^{2k} + \sum_{k=0}^{n} a_{2k+1}^{(n)} q_j^{2k} + \sum_{k=0}^{n-1} b_{2k+1}^{(n)} q_j^{2k} - \sum_{k=0}^{n} b_{2k}^{(n)} q_j^{2k-1} = 0.$$

Which implies that

$$\sum_{k=0}^{n-1} (b_{2k+1}^{(n)} + a_{2k+1}^{(n)}) q_j^{2k} + a_{2n+1}^{(n)} q_j^{2n} = 0.$$

The corresponding p_j are given by

$$p_j = \sum_{k=0}^{n} b_{2k}^{(n)} q_j^{2k-1}.$$

Substituting A_{11}, A_{12} from equation (6.8) and $c_1 = -c_2 = 1$ into equation (6.26), we have

$$p_j = \sum_{k=0}^{n} a_{2k+1}^{(n)} q_j^{2k} + \sum_{k=0}^{n} b_{2k}^{(n)} q_j^{2k-1} + \sum_{k=0}^{n-1} b_{2k+1}^{(n)} q_j^{2k}. \tag{6.28}$$

Substituting $\sum_{k=0}^{n} a_{2k+1}^{(n)} q_j^{2k} = -\sum_{k=0}^{n-1} b_{2k+1}^{(n)} q_j^{2k}$ from equation (6.27) into equation (6.28), we obtain

$$p_j = \sum_{k=0}^{n} b_{2k}^{(n)} q_j^{2k-1}.$$

In the generic case, the coordinates $q_1, ..., q_{2n}, p_1, ..., p_{2n}$ are canonical with respect to the Konstant Kirillov Poisson structure as well

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \qquad \{p_i, q_j\} = \delta_{ij}. \tag{6.29}$$

Theorem 6.4. *Consider the following:*

$$P_k = \Pi_{2k} = \frac{a_{2(n-k)+1}^{(n)} + b_{2(n-k)+1}^{(n)}}{a_{2n+1}^{(n)}}, \qquad Q_k = \sum_{j=1}^{n} \frac{1}{2j} b_{2j}^{(n)} \frac{\partial S_{2j}}{\partial \Pi_{2k}}, \quad k = 1, ..., n,$$

where $S_k = \sum_{j=1}^{2n} q_j^k$ *for* $k = 1, ..., 2n$ *and* $\Pi_1, ..., \Pi_{2n}$ *are the symmetric functions of* $q_1, ..., q_{2n}$:

$$\Pi_1 = q_1 + q_2 + ... + q_{2n}, \qquad \Pi_2 = \sum_{1 \leq j \leq 2n} q_j q_k, ..., \qquad \Pi_{2n} = q_1 q_2 ... q_{2n}.$$ *Then*

(1) $P_1, ..., P_n, Q_1, ..., Q_n$ *are coordinates in the symplectic leaves.*

(2) $P_1, ..., P_n, Q_1, ..., Q_n$ *are canonical, namely*

$$\{P_i, P_j\} = \{Q_i, Q_j\} = 0, \qquad \{P_i, Q_j\} = \delta_{ij}.$$

Proof. We want to prove the second part of our theorem, that is, the coordinates $P_1, ..., P_n, Q_1, ..., Q_n$ are canonical. First let us compute the bracket $\{P_k, P_l\}$

Since the pracket is bilinear, we obtain

$$\begin{aligned}
\{P_k, P_l\} &= \{\frac{a^{(n)}_{2(n-k)+1} + b^{(n)}_{2(n-k)+1}}{a^{(n)}_{2n+1}}, \frac{a^{(n)}_{2(n-l)+1} + b^{(n)}_{2(n-l)+1}}{a^{(n)}_{2n+1}}\} \\
&= \frac{1}{(a^{(n)}_{2n+1})^2}(\{a^{(n)}_{2(n-k)+1}, b^{(n)}_{2(n-l)+1}\} - \{a^{(n)}_{2(n-l)+1}, b^{(n)}_{2(n-k)+1}\}) \\
&+ \frac{1}{(a^{(n)}_{2n+1})^2}(\{a^{(n)}_{2(n-k)+1}, a^{(n)}_{2(n-l)+1}\} + \{b^{(n)}_{2(n-k)+1}, b^{(n)}_{2(n-l)+1}\}).
\end{aligned} \tag{6.30}$$

By equation (6.21), we have

$$[da^{(n)}_{2(n-k)+1}, da^{(n)}_{2(n-l)+1}] = 0 = [b^{(n)}_{2(n-k)+1}, b^{(n)}_{2(n-l)+1}]. \tag{6.31}$$

By equation (6.31) and equation (6.18), we obtain

$$\{a^{(n)}_{2(n-k)+1}, a^{(n)}_{2(n-l)+1}\} = 0 = \{b^{(n)}_{2(n-k)+1}, b^{(n)}_{2(n-l)+1}\}$$

Equation (6.30), gives

$$\{P_k, P_l\} = \frac{1}{(a^{(n)}_{2n+1})^2}(-\{b^{(n)}_{2(n-l)+1}, a^{(n)}_{2(n-k)+1}\} + \{a^{(n)}_{2(n-l)+1} + b^{(n)}_{2(n-k)+1}\}).$$

Substituting $\{b^{(n)}_{2(n-l)+1}, a^{(n)}_{2(n-k)+1}\}$ and $\{a^{(n)}_{2(n-l)+1}, b^{(n)}_{2(n-k)+1}\}$ from equation (6.22) into the above equation, we have

$$\{P_k, P_l\} = \frac{1}{(a^{(n)}_{2n+1})^2}(-b^{(n)}_{2(2n-k-l+1)} + b^{(n)}_{2(2n-k-l+1)}) = 0.$$

Therefore

$$\{P_k, P_l\} = \{\Pi_{2k}, \Pi_{2l}\} = 0, \qquad k, l = 1, .., n. \tag{6.32}$$

To compute the brackets that involve Q_k, we make use of the following formula:

$$\ln(\sum_{j=0}^{\infty} \Pi_j \gamma^j) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{S_k}{k} \gamma^k, \tag{6.33}$$

where γ is an auxiliary variable.

Since in equation (6.27) the roots of the polynomial

$$\sum_{k=0}^{n-1} (b^{(n)}_{2k+1} + a^{(n)}_{2k+1})\lambda^{2k} + a^{(n)}_{2n+1}\lambda^{2n} = 0,$$

are given by $q_1, ..., q_n, -q_1, ..., -q_n$, we obtain

$$\Pi_{2k+1} = S_{2k+1} = \sum_{j=1}^{2n} q_j^{2k+1} = q_1^{2k+1} + q_2^{2k+1} + ... + q_n^{2k+1} - q_1^{2k+1} - q_2^{2k+1} - ... - q_n^{2k+1} = 0.$$

Therefore, by differentiating equation (6.33) with respect to Π_{2k}, we can express $\dfrac{\partial S_{2j}}{\partial \Pi_{2k}}$ as:

$$\frac{\partial S_{2j}}{\partial \Pi_{2k}} = -2j \left[\left(\sum_{i=0}^{n} \Pi_{2i} \gamma^{2i} \right)^{-1} \right]_{2j-2k}, \tag{6.34}$$

where $[X(\gamma)]_{2j-2k}$ is the coefficient of γ^{2j-2k} in $X(\gamma)$ which is a power series in γ near 0 .

By differentiating equation (6.33) with respect to Π_{2k}, we find

$$\frac{1}{\sum_{j=0}^{\infty} \Pi_j \gamma^j} \gamma^{2k} = \frac{\partial S_1}{\partial \Pi_{2k}} \gamma^1 - \frac{1}{2} \frac{\partial S_2}{\partial \Pi_{2k}} \gamma^2 + \frac{1}{3} \frac{\partial S_3}{\partial \Pi_{2k}} \gamma^3 + \cdots.$$

Since $S_{2j+1} = 0$, we obtain

$$\frac{1}{\sum_{j=0}^{\infty} \Pi_j \gamma^j} \gamma^{2k} = -\frac{1}{2} \frac{\partial S_2}{\partial \Pi_{2k}} \gamma^2 - \frac{1}{4} \frac{\partial S_4}{\partial \Pi_{2k}} \gamma^4 - \frac{1}{6} \frac{\partial S_6}{\partial \Pi_{2k}} \gamma^6 - \cdots. \tag{6.35}$$

Dividing the equation (6.35) by γ^{2k}, we obtain

$$\frac{1}{\sum_{j=0}^{\infty} \Pi_j \gamma^j} = -\frac{1}{2} \frac{\partial S_2}{\partial \Pi_{2k}} \gamma^{2-2k} - \frac{1}{4} \frac{\partial S_4}{\partial \Pi_{2k}} \gamma^{4-2k} - \frac{1}{6} \frac{\partial S_6}{\partial \Pi_{2k}} \gamma^{6-2k} - \cdots$$

By comparing the coefficients of γ^{2j-2k}, we have

$$\left[\frac{1}{\sum_{j=0}^{\infty} \Pi_j \gamma^j} \right]_{2j-2k} = -\frac{1}{2j} \frac{\partial S_{2j}}{\partial \Pi_{2k}}.$$

Since $j \leq n$, $j - k \leq n$ and the coefficient of γ^{2j-2k} enter in $(\sum_{i=0}^{n} \Pi_{2i} \gamma^{2i})^{-1}$, we have

$$\frac{\partial S_{2j}}{\partial \Pi_{2k}} = -2j[(\sum_{i=0}^{n} \Pi_{2i} \gamma^{2i})^{-1}]_{2j-2k}.$$

Compute the bracket $\{P_k, Q_l\}$.
By definition of Q_l, $P_k = \Pi_{2k}$ and by Leibniz' Rule of $\{,\}$, we have

$$\{P_k, Q_l\} = \{\Pi_{2k}, \sum_{j=1}^{n} \frac{1}{2j} b_{2j}^{(n)} \frac{\partial S_{2j}}{\partial \Pi_{2l}}\} = \sum_{j=1}^{n} \{\Pi_{2k}, b_{2j}^{(n)}\} \frac{1}{2j} \frac{\partial S_{2j}}{\partial \Pi_{2l}} + \sum_{j=1}^{n} \frac{1}{2j} b_{2j}^{(n)} \{\Pi_{2k}, \frac{\partial S_{2j}}{\partial \Pi_{2l}}\}. \tag{6.36}$$

Thus, by equation (6.34), we have $\frac{\partial S_{2j}}{\partial \Pi_{2l}}$ is a polynomial in Π_{2m} with $m \leq j$, and from equation (6.32), we have the second term in (6.36) is zero .
Then equation (6.36) becomes

$$\begin{aligned} \{P_k, Q_l\} &= \sum_{j=1}^{n} \{\Pi_{2k}, b_{2j}^{(n)}\} \frac{1}{2j} \frac{\partial S_{2j}}{\partial \Pi_{2l}} \\ &= \sum_{j=1}^{n} \{\Pi_{2k}, b_{2j}^{(n)}\} \left[\left(-\sum_{j=0}^{n} \Pi_{2j} \gamma^{2j} \right)^{-1} \right]_{2j-2l} . \end{aligned} \tag{6.37}$$

By definition of Π_{2k} and $b_{2j}^{(n)}$, we have

$$\{\Pi_{2k}, b_{2j}^{(n)}\} = \{\frac{a_{2(n-k)+1}^{(n)} + b_{2(n-k)+1}^{(n)}}{a_{2n+1}^{(n)}}, b_{2j}^{(n)}\}.$$

By linearity of bracket $\{\ ,\ \}$, we obtain

$$\{\Pi_{2k}, b_{2j}^{(n)}\} = \frac{1}{a_{2n+1}^{(n)}} (\{a_{2(n-k)+1}^{(n)}, b_{2j}^{(n)}\} + \{b_{2(n-k)+1}^{(n)}, b_{2j}^{(n)}\}).$$

Substituting $\{a_{2(n-k)+1}^{(n)}, b_{2j}^{(n)}\}$ and $\{b_{2(n-k)+1}^{(n)}, b_{2j}^{(n)}\}$ from equation (6.22) into the above equation, we have

$$\begin{aligned} \{\Pi_{2k}, b_{2j}^{(n)}\} &= \frac{1}{a_{2n+1}^{(n)}} (-b_{2(n-k+j)+1}^{(n)} - a_{2(n-k+j)+1}^{(n)+1}) \\ &= -\Pi_{2(k-j)}. \end{aligned}$$

Therefore

$$\{\Pi_{2k}, b_{2j}^{(n)}\} = -\Pi_{2(k-j)}. \tag{6.38}$$

Substituting $\{\Pi_{2k}, b_{2j}^{(n)}\}$ from equation (6.38) into equation (6.37), we have

$$\begin{aligned} \{P_k, Q_l\} &= -\sum_{j=1}^{n} \Pi_{2(k-j)} \left[\left(-\sum_{j=0}^{n} \Pi_{2j} \gamma^{2j} \right)^{-1} \right]_{2j-2l} \\ &= \sum_{j=1}^{n} \Pi_{2(k-j)} \left[\left(\sum_{j=0}^{n} \Pi_{2j} \gamma^{2j} \right)^{-1} \right]_{2j-2l} . \end{aligned}$$

Since $\{P_k, Q_l\} = \sum_{j=1}^{n} \Pi_{2(k-j)} \left[\left(\sum_{j=0}^{n} \Pi_{2j}\gamma^{2j} \right)^{-1} \right]_{2j-2l} = \sum_{j=l}^{n} \Pi_{2(k-j)} \left[\left(\sum_{j=0}^{n} \Pi_{2j}\gamma^{2j} \right)^{-1} \right]_{2j-2l}$,
we can replac the sum from 1 to n by a sum from l to n in the last equation because the expression

$$\left(\sum_{j=0}^{n} \Pi_{2j}\gamma^{2j} \right)^{-1}$$

does not contain any negative power . Since the expression

$$\sum_{j=l}^{n} \Pi_{2(k-j)} \left[\left(\sum_{j=0}^{n} \Pi_{2j}\gamma^{2j} \right)^{-1} \right]_{2j-2l}$$

is just the coefficient of $\gamma^{2k-2l}, k \le n$ in

$$\left(\sum_{i=0}^{n} \Pi_{2i}\gamma^{2i} \right) \left(\sum_{j=0}^{n} \Pi_{2j}\gamma^{2j} \right)^{-1} = 1,$$

we see that

$$\{P_k, Q_l\} = \delta_{kl}.$$

To compute the brackets between Q_k and Q_l , we note that

$$\{b_{2j}^{(n)}, b_{2i}^{(n)}\} = \{\frac{\partial S_{2j}}{\partial \Pi_{2k}}, \frac{\partial S_{2i}}{\partial \Pi_{2l}}\} = 0. \tag{6.39}$$

The only contributions to the bracket $\{Q_k, Q_l\}$ come from the cross terms

$$\{Q_k, Q_l\} = \sum_{i,j=1}^{n} \{\frac{1}{2j} b_{2j}^{(n)} \frac{\partial S_{2j}}{\partial \Pi_{2k}}, \frac{1}{2i} b_{2i}^{(n)} \frac{\partial S_{2i}}{\partial \Pi_{2l}}\}.$$

By Leibniz' Rule of the bracket, we obtain

$$\{Q_k, Q_l\} = \sum_{i}^{n} \frac{1}{2i} \left(\{\frac{1}{2j} b_{2j}^{(n)} \frac{\partial S_{2j}}{\partial \Pi_{2k}}, b_{2i}^{(n)}\} \frac{\partial S_{2i}}{\partial \Pi_{2l}} + b_{2i}^{(n)} \{\frac{1}{2j} b_{2j}^{(n)} \frac{\partial S_{2j}}{\partial \Pi_{2k}}, \frac{\partial S_{2i}}{\partial \Pi_{2l}}\} \right).$$

By Skew-Symmetry of { , }, we obtain

$$\{Q_k, Q_l\} = \sum_{i}^{n} \frac{1}{2i} \left(-\{b_{2i}^{(n)}, \frac{1}{2j} b_{2j}^{(n)} \frac{\partial S_{2j}}{\partial \Pi_{2k}}\} \frac{\partial S_{2i}}{\partial \Pi_{2l}} - b_{2i}^{(n)} \{\frac{\partial S_{2i}}{\partial \Pi_{2l}}, \frac{1}{2j} b_{2j}^{(n)} \frac{\partial S_{2j}}{\partial \Pi_{2k}}\} \right).$$

By Leibniz' Rule of the bracket, we have

$$\begin{aligned} \{Q_k, Q_l\} &= \sum_{i,j=1}^{n} \frac{1}{4ij} \left(-\{b_{2i}^{(n)}, b_{2j}^{(n)}\} \frac{\partial S_{2j}}{\partial \Pi_{2k}} \frac{\partial S_{2i}}{\partial \Pi_{2l}} - b_{2j}^{(n)} \{b_{2i}^{(n)}, \frac{\partial S_{2j}}{\partial \Pi_{2k}}\} \frac{\partial S_{2i}}{\partial \Pi_{2l}} \right) \\ &+ \sum_{i,j=1}^{n} \frac{1}{4ij} \left(-b_{2i}^{(n)} \{\frac{\partial S_{2i}}{\partial \Pi_{2l}}, b_{2j}^{(n)}\} \frac{\partial S_{2j}}{\partial \Pi_{2k}} - b_{2i}^{(n)} b_{2j}^{(n)} \{\frac{\partial S_{2i}}{\partial \Pi_{2l}}, \frac{\partial S_{2j}}{\partial \Pi_{2k}}\} \right). \end{aligned}$$

From equation (6.39), we obtain

$$\{Q_k, Q_l\} = \sum_{i,j=1}^{n} \frac{1}{4ij}\left(-b_{2j}^{(n)}\{b_{2i}^{(n)}, \frac{\partial S_{2j}}{\partial \Pi_{2k}}\}\frac{\partial S_{2i}}{\partial \Pi_{2l}} - b_{2i}^{(n)}\{\frac{\partial S_{2i}}{\partial \Pi_{2l}}, b_{2j}^{(n)}\}\frac{\partial S_{2j}}{\partial \Pi_{2k}}\right).$$

By Skew-Symmetry of { , }, we have

$$\{Q_k, Q_l\} = \sum_{i,j=1}^{n} \frac{1}{4ij}\left(b_{2j}^{(n)}\{\frac{\partial S_{2j}}{\partial \Pi_{2k}}, b_{2i}^{(n)}\}\frac{\partial S_{2i}}{\partial \Pi_{2l}} - b_{2i}^{(n)}\{\frac{\partial S_{2i}}{\partial \Pi_{2l}}, b_{2j}^{(n)}\}\frac{\partial S_{2j}}{\partial \Pi_{2k}}\right).$$

replacing i by j in the last part, we obtain

$$\{Q_k, Q_l\} = \sum_{i,j=1}^{n} \frac{1}{4ij}\left(b_{2j}^{(n)}\{\frac{\partial S_{2j}}{\partial \Pi_{2k}}, b_{2i}^{(n)}\}\frac{\partial S_{2i}}{\partial \Pi_{2l}} - b_{2j}^{(n)}\{\frac{\partial S_{2j}}{\partial \Pi_{2l}}, b_{2i}^{(n)}\}\frac{\partial S_{2i}}{\partial \Pi_{2k}}\right). \qquad (6.40)$$

The bracket between $\frac{\partial S_{2j}}{\partial \Pi_{2k}}$ and $b_{2i}^{(n)}$ can be computed by

$$\{\frac{\partial S_{2j}}{\partial \Pi_{2k}}, b_{2i}^{(n)}\} = \{-2j\left[\left(\sum_{i=0}^{n} \Pi_{2i}\gamma^{2i}\right)^{-1}\right]_{2j-2k}, b_{2i}^{(n)}\}$$

Since $\left(\sum_{m=0}^{n} \Pi_{2m}\gamma^{2m}\right)^{-1}\left(\sum_{s=0}^{n} \Pi_{2s}\gamma^{2s}\right) = 1$, we obtain

$$\begin{aligned}
\{\frac{\partial S_{2j}}{\partial \Pi_{2k}}, b_{2i}^{(n)}\} &= \{-2j\left[\left(\sum_{i=0}^{n} \Pi_{2i}\gamma^{2i}\right)^{-1}\right]_{2j-2k}\left(\sum_{m=0}^{n} \Pi_{2m}\gamma^{2m}\right)^{-1}\left(\sum_{s=0}^{n} \Pi_{2s}\gamma^{2s}\right), b_{2i}^{(n)}\} \\
&= -2j\{\left[\left(\sum_{m=0}^{n} \Pi_{2m}\gamma^{2m}\right)^{-2}\left(\sum_{s=0}^{n} \Pi_{2s}\gamma^{2s}\right)\right]_{2j-2k}, b_{2i}^{(n)}\} \\
&= -2j\left[\left(\sum_{m=0}^{n} \Pi_{2m}\gamma^{2m}\right)^{-2}\sum_{s=0}^{n}\{\Pi_{2s}, b_{2i}^{(n)}\}\gamma^{2s}\right]_{2j-2k}.
\end{aligned}$$

By equation (6.38), we have

$$
\begin{aligned}
\{\frac{\partial S_{2j}}{\partial \Pi_{2k}}, b_{2i}^{(n)}\} &= 2j\left[\left(\sum_{m=0}^{n}\pi_{2m}\gamma^{2m}\right)^{-2}\sum_{s=0}^{n}\Pi_{2(s-i)}\gamma^{2s}\right]_{2j-2k} \\
&= 2j\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-2}\sum_{s=i}^{n-i}\Pi_{2s}\gamma^{2(s+i)}\right]_{2j-2k} \\
&= 2j\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-2}\sum_{s=i}^{n-i}\Pi_{2s}\gamma^{2s}\right]_{2j-2k-2i} \\
&= 2j\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-2}\sum_{s=0}^{n}\Pi_{2s}\gamma^{2s}\right]_{2j-2k-2i},
\end{aligned}
$$

where in the last line, we replaced the upper and lower limit of the second sum by n , 0 respectively . Because $2j-2k-2i \leq 2(n-i) \leq 2n$, so if $p > n-i$, the coefficient of γ^{2p} in $\sum_{s=0}^{n}\Pi_{2s}\gamma^{2s}$ will not enter in the final expression.

Therefore

$$
\begin{aligned}
&\frac{1}{4ij}\{\frac{\partial S_{2j}}{\partial \Pi_{2k}}, b_{2i}^{(n)}\}\frac{\partial S_{2i}}{\partial \Pi_{2l}} \\
&= \tfrac{1}{4ij}2j\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-2}\sum_{s=0}^{n}\Pi_{2s}\gamma^{2s}\right]_{2j-2k-2i}\left(-2i\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-1}\right]_{2i-2l}\right) \\
&= -\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-3}\sum_{s=0}^{n}\Pi_{2s}\gamma^{2s}\right]_{2j-2k-2i+2i-2l} \\
&= -\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-3}\sum_{s=0}^{n}\Pi_{2s}\gamma^{2s}\right]_{2j-2k-2l}.
\end{aligned}
$$

Similarly, the second term in equation (6.40), tokes the form

$$\begin{aligned}\frac{1}{4ij}\{\frac{\partial S_{2j}}{\partial \Pi_{2l}}, b_{2i}^{(n)}\}\frac{\partial S_{2i}}{\partial \Pi_{2k}} &= -\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-3}\sum_{s=0}^{n}\Pi_{2s}\gamma^{2s}\right]_{2j-2k-2l} \\ &= -\left[\left(\sum_{m=0}^{n}\Pi_{2m}\gamma^{2m}\right)^{-3}\sum_{s=0}^{n}\Pi_{2s}\gamma^{2s}\right]_{2j-2k-2l}.\end{aligned}$$

Since the first and second terms in (6.40) are equal to each other, we obtain

$$\{Q_k, Q_l\} = 0.$$

In summary

$$\{P_k, Q_l\} = \delta_{lk}, \qquad \{P_k, P_l\} = 0, \qquad \{Q_k, Q_l\} = 0.$$

□

Example 6.5. Case $n = 1$. In this case we will show that

$$Q = 4w, \qquad P = \frac{1}{2}(w_z - w^2 - z).$$

First of all

$$Q = Q_1 = \frac{1}{2}\, b_2^{(1)}\frac{\partial S_2}{\partial \Pi_2} = -b_2^{(1)}.$$

By equation (6.2), we have

$$Q = 4w.$$

Next

$$P = P_1 = \frac{a_1^{(1)} + b_1^{(1)}}{a_3^{(1)}}.$$

By equations (6.7),(6.2), and using $t_1 = 1$, we have

$$\begin{aligned} P &= \frac{2(w_z - w^2) - z}{4} \\ &= \tfrac{1}{2}(w_z - w^2 - z).\end{aligned}$$

These coincide with Okamoto s canonical coordinates [10].

Example 6.6. Case $n = 2$. In this case we will show that

$$\begin{aligned} P_1 &= -\tfrac{1}{2}(w^2 - w_z - \tfrac{t_1}{2}), \\ P_2 &= \tfrac{1}{16}[-z + 6w^4 - 12w^2w_z + 2w_z^2 - 4ww_{zz} + 2w_{zzz} + 2t_1(w_z - w^2)], \\ Q_1 &= -8ww_z + 4w_{zz}, \\ Q_2 &= 16w. \end{aligned} \tag{6.41}$$

First By equation (6.2) and using $t_2 = 1$, we have

$$\begin{aligned} P_1 &= \frac{a_3^{(2)} + b_3^{(2)}}{a_5^{(2)}} \\ &= \frac{4t_1 + 8[w_z - w^2]}{16} \\ &= \tfrac{1}{4}(t_1 + 2[w_z - w^2]) \\ &= -\tfrac{1}{2}(w^2 - \tfrac{t_1}{2} - w_z). \end{aligned}$$

Next by definition of P_2, equation (6.7) and equation (6.2), we have

$$\begin{aligned} P_2 &= \Pi_4 = \frac{a_1^{(2)} + b_1^{(2)}}{a_5^{(2)}} \\ &= \frac{t_1(-2w^2) + (2w_z^2 - 4ww_{zz} + 6w^4) - z + t_1(2w_z) + 2(w_{zzz} - 6w^2 w_z)}{16} \\ &= \tfrac{1}{16}[-z + 6w^4 - 12w^2 w_z + 2w_z^2 - 4ww_{zz} + 2w_{zzz} + 2t_1(w_z - w^2)]. \end{aligned}$$

Next by definition of Q_1, and equation (6.7), we have

$$\begin{aligned} Q_1 &= \frac{1}{2} b_2^{(2)} \frac{\partial S_2}{\partial \Pi_2} + \frac{1}{4} b_4^{(2)} \frac{\partial S_4}{\partial \Pi_2} \\ &= -b_2^{(2)} + b_4^{(2)} \Pi_2 \\ &= -t_1(-4w) - (-4(w_{zz} - 2ww_z + 2ww_z - 2w^3)) + (-16w)(-\tfrac{1}{2}(w^2 - w_z - \tfrac{t_1}{2})) \\ &= -8ww_z + 4w_{zz}. \end{aligned}$$

Finally

$$Q_2 = \frac{1}{4} b_4^{(2)} \frac{\partial S_4}{\partial \Pi_4} = -b_4^{(2)} = 16w.$$

6.4 Hamiltonian structure

In this section, we will construct the Hamiltonian function, Hamiltonian vector field of this Hamiltonian function and we will derive the Hamiltonian structure of the second Painlevé hierarchy.

Proposition 6.7. *The vector field*

$$\chi_k(A) := -[(A^n\lambda^{1-2k})_-, A], \tag{6.42}$$

is Hamiltonian with the Hamiltonian function

$$h_k^{(n)} := \frac{1}{2}TrRes(\lambda^{1-2k}(A^n)^2). \tag{6.43}$$

Proof. Let us denote

$$\hat{L}_k = -(A^{(n)}\lambda^{1-2k})_- \in \mathfrak{g}_-. \tag{6.44}$$

Substituting $\hat{L}_k$ from equation (6.44) into equation(6.42), we have

$$\chi_k(A) = [\hat{L}_k, A].$$

To show that it is Hamiltonian and to compute the Hamiltonian function f, we use the following definition:

$$\omega(\chi_k\ , Y)(E) :=< [Y, E], df >, \qquad Y \in \mathfrak{g}_-, E \in \mathfrak{g}_-^*. \tag{6.45}$$

Since $\hat{L}_k \in \mathfrak{g}_-, A \in \mathfrak{g}_-^*$, equation (6.45) gives

$$\omega(\chi_k\ , Y)(A) =< [Y, A], df > . \tag{6.46}$$

By equations (6.20) , (6.18), and by the same method of the proof of equation (6.14), we have

$$\begin{aligned}
\omega(\chi_k\ ,Y)(A) &= \{\chi_k\ ,Y\}(A) \\
&= < A, [d\chi_k, dY] > \\
&= - < [\hat{L}_k, Y], A > \\
&= -\mathrm{TrRes}(A[\hat{L}_k, Y]) \\
&= -\mathrm{TrRes}(A\hat{L}_k Y - AY\hat{L}_k) \\
&= -\mathrm{ResTr}(A\hat{L}_k Y - AY\hat{L}_k) \\
&= -(\mathrm{ResTr}A\hat{L}_k Y - \mathrm{ResTr}AY\hat{L}_k) \\
&= -(\mathrm{ResTr}\hat{L}_k YA - \mathrm{ResTr}\hat{L}_k AY) \\
&= -\mathrm{ResTr}(\hat{L}_k YA - \hat{L}_k AY) \\
&= -\mathrm{ResTr}(\hat{L}_k[Y, A]) \\
&= \mathrm{TrRes}(-\hat{L}_k[Y, A]).
\end{aligned}$$

Thus it follows that

$$\omega(\chi_k\ ,Y)(A) \ = \ < [Y, A]\ , -\hat{L}_k > . \tag{6.47}$$

By equation (6.47) and equation (6.46), we find

$$\omega(\chi_k\ ,Y)(A) =< [Y, A], df >=< [Y, A]\ , -\hat{L}_k > .$$

This shows that if we can prove that there exist f such that $df = \hat{L}_k$ then $[\hat{L}_k, A] = [A, df]$ defines a Hamiltonian vector field of Hamiltonian f . We are now going to show that the Hamiltonian (6.43) is such that $dh_k^{(n)} = -\hat{L}_k$. For every $X \in \mathfrak{g}_-$, $E \in \mathfrak{g}_-^*$, and by equation (6.19), we can identify

$$[X, E] = \mathrm{ad}_X^* = \delta_X E.$$

Using the definition (6.19), we get

$$h_k^{(n)}(A + \delta_x(A)) - h_k^{(n)}(A) + Q(\delta_x(A))^2 =< dh_k^{(n)}, \delta_X A >, \tag{6.48}$$

$\delta_X A = [Y, A]$ which is the contraction. □

We are now going to show that the isomonodromic deformation Hamiltonian for the n-th equation in the P_{II} hierarchy is given by

$$H^{(n)} := -\frac{1}{2\ 4^n}\mathrm{TrRes}\left(\lambda^{1-2n}(A^{(n)})^2\right). \tag{6.49}$$

Theorem 6.8. *Define:*

$$\mathcal{H}^{(n)}(P_1,...,P_n,Q_1,...,Q_n,z) := -\frac{1}{4^n}\left(\sum_{l=0}^{n-1} a_{2l+1}^{(n)} a_{2(n-l)-1}^{(n)} - \sum_{l=0}^{n-1} b_{2l+1}^{(n)} b_{2(n-l)-1}^{(n)} + \sum_{l=0}^{n} b_{2l}^{(n)} b_{2(n-l)}^{(n)}\right) + \frac{Q_n}{4^n}, \tag{6.50}$$

in which we are thinking of $a_{2l+1}^{(n)}, b_{2l+1}^{(n)}, b_{2l}^{(n)}$ *as functions of* $P_1,...,P_n,\ Q_1,...,Q_n,\ t_1,...,t_{n-1},z$. *Then the n-th member of the second Painlevé hierarchy is given by the equation*

$$\frac{dP_k}{dz} = -\frac{\partial \mathcal{H}^{(n)}}{\partial Q_k}, \qquad \frac{dQ_k}{dz} = \frac{\partial \mathcal{H}^{(n)}}{\partial P_k}. \tag{6.51}$$

Proof. By equation (6.49), we have

$$\begin{aligned} H^{(n)} &= -\frac{1}{2\ 4^n}\mathrm{TrRes}\left(\lambda^{1-2n}(A^{(n)})^2\right) \\ &= -\frac{1}{2\ 4^n}\mathrm{ResTr}\left(\lambda^{1-2n}(A^{(n)})^2\right). \end{aligned}$$

By equation (6.8), and definition of the Trace on the matrix, we obtain

$$\begin{aligned} H^{(n)} &= -\frac{1}{2\ 4^n}\mathrm{Res}\lambda^{1-2n}\{2\left(\sum_{k=0}^{n} a_{2k+1}^{(n)}\lambda^{2k}\right)\left(\sum_{k=0}^{n} a_{2k+1}^{(n)}\lambda^{2k}\right) + \left(\sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1}\right)\left(\sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1}\right) \\ &- \left(\sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1}\right)\left(\sum_{k=0}^{n-1} b_{2k+1}^{(n)}\lambda^{2k}\right) + \left(\sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1}\right)\left(\sum_{k=0}^{n-1} b_{2k}^{(n)}\lambda^{2k-1}\right) \\ &+ \left(\sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1}\right)\left(\sum_{k=0}^{n-1} b_{2k+1}^{(n)}\lambda^{2k}\right) - \left(\sum_{k=0}^{n-1} b_{2k+1}^{(n)}\lambda^{2k}\right)\left(\sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1}\right) \\ &- \left(\sum_{k=0}^{n-1} b_{2k+1}^{(n)}\lambda^{2k}\right)\left(\sum_{k=0}^{n-1} b_{2k+1}^{(n)}\lambda^{2k}\right)\}. \end{aligned}$$

Collection the similar elements of the above equation, we have

$$\begin{aligned} H^{(n)} &= -\frac{1}{2\ 4^n}\mathrm{Res}2\lambda^{1-2n}\{\left(\sum_{k=0}^{n} a_{2k+1}^{(n)}\lambda^{2k}\right)\left(\sum_{k=0}^{n} a_{2k+1}^{(n)}\lambda^{2k}\right) + \left(\sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1}\right)\left(\sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1}\right) \\ &- \left(\sum_{k=0}^{n-1} b_{2k+1}^{(n)}\lambda^{2k}\right)\left(\sum_{k=0}^{n-1} b_{2k+1}^{(n)}\lambda^{2k}\right)\}. \end{aligned}$$

Using the definition of Residual, we obtain

$$\begin{aligned} H^{(n)} &= -\frac{1}{4^n}[a_1^{(n)}a_{2n-1}^{(n)} + a_{2(1)+1}^{(n)}a_{2(n-1)-1}^{(n)} + a_{2(2)+1}^{(n)}a_{2(n-2)-1}^{(n)} + ... + a_{2(n-1)+1}^{(n)}a_1^{(n)}] \\ &+ [b_{2(0)}^{(n)}b_{2(n)}^{(n)} + b_{2(1)}^{(n)}b_{2(n-1)}^{(n)} + ... + b_{2(n)}^{(n)}b_0^{(n)}] \\ &- [b_1^{(n)}b_{2(n)-1}^{(n)} + b_3^{(n)}b_{2(n-1)-1}^{(n)} + ... + b_{2(n-1)+1}^{(n)}b_1^{(n)}]. \end{aligned}$$

Therefore

$$H^{(n)} = -\frac{1}{4^n}\left[\sum_{l=0}^{n-1} a_{2l+1}^{(n)}a_{2(n-l)-1}^{(n)} - \sum_{l=0}^{n-1} b_{2l+1}^{(n)}b_{2(n-l)-1}^{(n)} + \sum_{l=0}^{n} b_{2l}^{(n)}b_{2(n-l)}^{(n)}\right],$$

where we are treating $a_j^{(n)}, b_j^{(n)}$ as coordinates on the coadjoint orbit. When expressing this Hamiltonian in our canonical coordinates, we need to take into account a shift h due to the explicit z dependence in the variable P_n. All other canonical coordinates depend on $a_j^{(n)}, b_j^{(n)}$ only. To compute this shift we use the following well-known result [27]:

Lemma 6.9. *Let*

$$\frac{dy_i}{dz} = \{y_i, H(y, z)\} \tag{6.52}$$

be a Hamiltonian system on a Poisson manifold with Poisson brackets $\{,\}$ *and* $y = \phi(x, z)$*, be a local diffeomorphism depending explicitly on* z*. Let the vector field* $\partial_z\phi$ *be a Hamiltonian vector field with Hamiltonian* δH *. Then (6.52) is a Hamiltonian system also in the x-coordinates*

$$\frac{dx_i}{dz} = \{x_i, \hat{H}(x, z)\},$$

where

$$\hat{H}(x, z) = H(\phi(x, z), z) - \delta H(\phi(x, y), z).$$

Let us compute this shift in our case. The only coordinate depending explicitly on z is

$$p_n = \frac{-z}{4^n} + f(a_1^{(n)}, ..., a_{2n+1}^{(n)}, b_0^{(n)}, ..., b_0^{(n)}, ..., b_{2n}^{(n)}).$$

So for $y = (p_1, ..., p_n, Q_1, ..., Q_n)$, we have

$$\delta H^{(n)} = \frac{Q_n}{4^n}$$

which gives (6.50). □

Now we will give the Hamiltonian structure of the first and second members of the second Pailevé hierarchy.

Example 6.10. In the case $n = 1$, we will show that the Hamiltonian function for the first member of the second Pailevé hierarchy is given by

$$\mathcal{H}^{(1)} = 4P^2 + \frac{1}{4}Q + \frac{1}{4}PQ^2 + 2Pz - \frac{1}{2}Q\alpha_1. \tag{6.53}$$

Since $n = 1$ then by example (6.5), we have $Q = 4w, \quad P = \frac{1}{2}(w_z - w^2 - \frac{z}{2})$, and by equation(6.50) , we have

$$\begin{aligned}
\mathcal{H}^{(1)} &= -\frac{1}{4}\left[(a_1^{(1)})^2 - (b_1^{(1)})^2 + 2b_2^{(1)}b_0^{(1)}\right] + \frac{Q}{4} \\
&= -\frac{1}{4}\left[(-2w^2 - z)^2 - (2w_z)^2 + 2\alpha_1(-8w)\right] + \frac{Q}{4} \\
&= -\frac{1}{4}\left[4(-w^2 - \frac{z}{2})^2 - 4w_z^2 + 16\alpha_1 w\right] + \frac{Q}{4} \\
&= -\frac{1}{4}\left[4[(-w^2 - \frac{z}{2}) + w_z][(-w^2 - \frac{z}{2}) - w_z] + 16\alpha_1 w\right] + \frac{Q}{4} \\
&= -\frac{1}{4}\left[8P[(-w^2 - \frac{z}{2}) - w_z] + 16\alpha_1 w\right] + \frac{Q}{4}
\end{aligned}$$

Thus

$$\mathcal{H}^{(1)} = 2Pw^2 + Pz + 2Pw_z - 2\alpha_1 w + \tfrac{Q}{4}. \tag{6.54}$$

Substituting $w = \frac{Q}{4}, \quad w_z = 2P + w^2 + \frac{z}{2}$ into equation(6.54), we have

$$\begin{aligned}
\mathcal{H}^{(1)} &= \tfrac{PQ^2}{8} + Pz + 2P(2P + w^2 + \tfrac{z}{2}) - \tfrac{1}{2}Q\alpha_1 + \tfrac{Q}{4} \\
&= 4P^2 + \tfrac{1}{4}Q + \tfrac{1}{4}PQ^2 + 2Pz - \tfrac{1}{2}Q\alpha_1.
\end{aligned}$$

Now equation (6.51) reads

$$\frac{dP}{dz} = -\frac{\partial\mathcal{H}^{(1)}}{\partial Q}, \qquad \frac{dQ}{dz} = \frac{\partial\mathcal{H}^{(1)}}{\partial P}.$$

As a result, we obtain

$$\frac{dQ}{dz} = \frac{\partial\mathcal{H}^{(1)}}{\partial P} = 8P + \frac{1}{4}Q^2 + 2z, \tag{6.55}$$

$$\frac{dP}{dz} = -\frac{\partial\mathcal{H}^{(1)}}{\partial Q} = \frac{-1}{4} - \frac{1}{2}PQ + \frac{1}{2}\alpha_1. \tag{6.56}$$

Differentiating equation (6.55) with respect to z, we find

$$\begin{aligned}
\frac{d^2Q}{dz^2} &= 8\frac{dP}{dz} + \frac{1}{2}Q\frac{dQ}{dz} + 2 \\
&= 8(\tfrac{-1}{4} - \tfrac{1}{2}PQ + \tfrac{1}{2}\alpha_1) + \tfrac{1}{2}Q(8P + \tfrac{1}{4}Q^2 + 2z) + 2 \\
&= \tfrac{1}{8}Q^3 + Qz + 4\alpha_1.
\end{aligned}$$

Thus we obtain the equation

$$\frac{d^2Q}{dz^2} = \frac{1}{8}Q^3 + Qz + 4\alpha_1. \tag{6.57}$$

Equation (6.57) gives the second Painlevé equation for $w = \frac{1}{4}Q$,

$$\frac{d^2w}{dz^2} = 2w^3 + wz + \alpha_1.$$

Therefore we have derived the Hamiltonian structure for the second Painlevé equation; that is the first member of the P_{II} hierarchy.

Example 6.11. In the case $n = 2$, we will derive the Hamiltonian function for the second member of the second Painlevé hierarchy;

$$\mathcal{H}^{(2)} = -16P_1^3 + 8P_1^2 t_1 - \frac{1}{8}\alpha_2 Q_2 + 2P_1 z + \frac{1}{16}P_2 Q_2^2 + 32P_1 P_2 - P_1 t_1^2 - 8P_2 t_1 - \frac{1}{16}Q_1^2. \tag{6.58}$$

Since $n = 2$, equation (6.50) gives

$$\mathcal{H}^{(2)} = -\frac{1}{16}\left[2a_1^{(2)}a_3^{(2)} - 2b_1^{(2)}b_3^{(2)} + 2b_0^{(2)}b_4^{(2)} + (b_2^{(2)})^2\right] + \frac{Q_2}{16}. \tag{6.59}$$

By equation (6.7), we have

$$\begin{aligned}
a_1^{(2)} &= -2t_1 w^2 + 2w_z^2 - 4ww_{zz} + 6w^4 - z,\\
a_3^{(2)} &= 4t_1 - 8w^2,\\
b_1^{(2)} &= 2w_{zzz} - 12w^2 w_z + 2w_z t_1,\\
b_3^{(2)} &= 8w_z,\\
b_0^{(2)} &= -\alpha_2,\\
b_4^{(2)} &= -16w\\
b_2^{(2)} &= 8w^3 - 4t_1 w - 4w_{zz}.
\end{aligned} \tag{6.60}$$

Substituting $a_1^{(2)}$, $a_3^{(2)}$, $b_1^{(2)}$, $b_3^{(2)}$, $b_0^{(2)}$, $b_4^{(2)}$ and $b_2^{(2)}$ from equation (6.60) into equation (6.59), we obtain

$$\begin{aligned}
\mathcal{H}^{(2)} &= -\frac{1}{16}\left[2(-2t_1 w^2 + 2w_z^2 - 4ww_{zz} + 6w^4 - z)(4t_1 - 8w^2)\right]\\
&+ \frac{1}{16}\left[2(2w_{zzz} - 12w^2 w_z + 2w_z t_1)(8w_z)\right]\\
&- \frac{1}{16}\left[32\alpha_2 w + (8w^3 - 4t_1 w - 4w_{zz})^2\right] + \frac{Q_2}{16}.
\end{aligned} \tag{6.61}$$

By equation (6.41), we can rewrite $w,\ w_z,\ w_{zz}$, and $2w_{zzz}-12w^2w_z+2w_zt_1$ in terms of $P_1,\ P_2,\ Q_1,\ Q_2$ as

$$\begin{aligned}
w_z &= 2P_1+\frac{Q_2^2}{(16)^2}-\frac{t_1}{2},\\
w_{zz} &= \frac{1}{4}\left[Q_1+Q_2P_1+\frac{8Q_2^3}{(16)^3}-\frac{1}{4}Q_2t_1\right],\\
2w_{zzz}-12w^2w_z+2w_zt_1 &= 16P_2+z-3\frac{Q_2^4}{(16)^4}-8P_1^2-\frac{1}{32}P_1Q_2^2+4P_1t_1-\frac{1}{2}t_1+\frac{1}{64}t_1Q_2^2.
\end{aligned} \tag{6.62}$$

Substituting $w,\ w_z,\ w_{zz}$, and $2w_{zzz}-12w^2w_z+2w_zt_1$ from equation (6.62) into equation (6.61), we obtain

$$\mathcal{H}^{(2)}=-16P_1^3+8P_1^2t_1-\frac{1}{8}\alpha_2Q_2+2P_1z+\frac{1}{16}P_2Q_2^2+32P_1P_2-P_1t_1^2-8P_2t_1-\frac{1}{16}Q_1^2.$$

Using equation (6.51), we get

$$\begin{aligned}
\frac{dP_1}{dz} &= -\frac{\partial\mathcal{H}^{(2)}}{\partial Q_1}=\frac{1}{8}Q_1,\\
\frac{dP_2}{dz} &= -\frac{\partial\mathcal{H}^{(2)}}{\partial Q_2}=\frac{1}{8}\alpha_2-\frac{1}{8}P_2Q_2,\\
\frac{dQ_1}{dz} &= \frac{\partial\mathcal{H}^{(2)}}{\partial P_1}=-48P_1^2+16P_1t_1+2z+32P_2-t_1^2,\\
\frac{dQ_2}{dz} &= \frac{\partial\mathcal{H}^{(2)}}{\partial P_2}=\frac{1}{16}Q_2^2+32P_1-8t_1.
\end{aligned} \tag{6.63}$$

As a result, we obtain

$$\frac{d^2Q_2}{dz^2}=\frac{2}{(16)^2}Q_2^3+4P_1Q_2-t_1Q_2+4Q_1. \tag{6.64}$$

Differentiating equation (6.63) with respect to z, we find

$$\frac{d^3Q_2}{dz^3} = \frac{6}{(16)^2}Q_2^2\frac{dQ_2}{dz}+4P_1\frac{dQ_2}{dz}+4Q_2\frac{dP_1}{dz}-t_1\frac{dQ_2}{dz}+4\frac{dQ_1}{dz} \tag{6.65}$$

Substituting $\frac{dQ_2}{dz}$, $\frac{dQ_1}{dz}$ and $\frac{dP_1}{dz}$ from equation (6.63) into (6.65), we have

$$\frac{d^3Q_2}{dz^3}=Q_2^2P_1-\frac{1}{4}Q_2^2t_1+\frac{3}{2048}Q_2^4-64P_1^2+\frac{1}{2}Q_1Q_2+128P_2+4t_1^2+8z. \tag{6.66}$$

By the same way, we can differentiating equation (6.66) with respect to z, we obtain

$$\begin{aligned}\frac{d^4Q_2}{dz^4} &= \frac{5}{32}Q_1Q_2^2+40Q_2P_1^2-24Q_2P_1t_1+\frac{5}{16}Q_2^3P_1+\frac{7}{2}Q_2t_1^2 \\ &- \frac{5}{64}Q_2^3t_1+\frac{3}{8192}Q_2^5+Q_2z-4Q_1t_1+16\alpha_2+8.\end{aligned} \tag{6.67}$$

Substituting P_1, Q_1, O_2, and P_2 from equation (6.41) into equation (6.67), we obtain

$$w_{zzzz} = 10w^2w_{zz}-6w^5-18w^3t_1+10ww_z^2-20ww_zt_1+12wt_1^2+wz-t_1w_{zz}+\alpha_2+\frac{1}{2}. \tag{6.68}$$

Setting $t_1 = 0$,$w = V$ and $\alpha_2 + \frac{1}{2} = \alpha$ into equation(6.68), we gives

$$V'''' - 10V^2V'' - 10V(V')^2 + 6V^5 = zV + \alpha_2,$$

the second member of P_{II} hierarchy. Therefore we have derived the Hamiltonian structure for the second member of the P_{II} hierarchy.

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