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ON FINITE AND ARTINIAN T_o−ALEXANDROFF SPACES

MS Thesis

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Dedication

To

My Parents,

My Brothers,

and My Family.

Contents

Acknowledgments

After God, I am grateful to my parents for their subsidization, love, patience and assistance. Without them, this work would never have come into existence.

I would like to thank Dr. Hisham B. Mahdi, my supervisor, for his many suggestions, guidance, and constant supporting during this thesis.

I would like to express my great thanks to staff members of mathematical department of iugaza and to my teachers who taught me.

Abstract

In this thesis, we study Alexandroff spaces which are topological spaces in which arbitrary intersection of open sets is open.

We study, among other things, the two papers titled "On finite T_o- topological spaces" by A. E. El-Atik et al. [4] and "On T_o- Alexandroff spaces" by H. B. Mahdi et al. [12]. We compare the study of the two papers and study the results of [12] which are generalization of those of [4]. Then we determine the results in [12] which are not included in [4]. These results are still true in [4], since each finite space is Alexandroff.

The heart of this study lies in the generalization of results of finite spaces in [4] to T_o- Alexandroff spaces. Certainly, we study other concepts such as generalized continuity, dimension and supratoplogies, and we get some new results on T_o- Alexandroff spaces.

Introduction

In this thesis, we are looking at the Alexandroff topological spaces which are introduced the first time by by P. A. Alexaandroff in 1937 [5]. These topologies satisfy the property that an arbitrary intersection of open sets is open, or equivalently, each element has a minimal neighborhood base. This property need not be true in general.

It was shown in [5] that T_o- Alexandroff Spaces are in one-to-one correspondence with posets, via the relation called *(Alexandroff) specialization order*

$$x \leq y \text{ if and only if } x \in \overline{\{y\}},$$

and each one is completely determined by the other.

Mahdi and Elatrash introduce a special class of T_o- Alexandroff spaces called *Artinian T_o- Alexandroff spaces*. These are the ones that their corresponding posets satisfy the ascending chain condition (ACC). This class contains the class of T_o- finite spaces, and the class of T_o- locally finite Spaces. It follows that the results of Artinian T_o- Alexandroff is still true in T_o- finite and locally finite spaces. Hence, the results of Artinian will be generalization of the same results of the other two spaces.

A. E. El-Atik [4] focused his study on finite T_o- spaces because of the existence of isolated points (= the maximal points in the corresponding poset). That is, the set of all isolated points is always non-empty set. This important property is still true in Artinian T_o- Alexandroff spaces. More precisely, the existence of isolated points play an important role for the results of the two classes (finite and Artinian T_o- Alexandroff spaces).

Part of this thesis is comparing study between two papers, the paper of [4] which concentrate on finite T_o- spaces and the more general study of [12]. We study the results of [12] which are generalization of the same results of [4]. Then we determine those results of [12] which are not included in [4]. These results are still true in [4], since each finite space is Alexandroff. Furthermore, we study other concepts such as generalized continuity, dimension and supratoplogy on T_o- Alexandroff spaces.

We introduce this thesis in four chapters. Chapter one includes preliminaries and definitions that will be used in the remainder of the thesis. Section one contains basic definitions, concepts and illustrated examples of a partially ordered set (poset). In the second section, we talk about essentialy concepts and theories that are going to be used in this research. More precisely we give the definitions of a topological spaces, open sets, closed sets, closure and interior, dense, nowhere dense, and finally, we study the separation axioms. In the third section, we study a special class of topological spaces called Alexandroff spaces. We include this section definitions, theorems and some of the basic concepts that are well know about Alexandroff spaces. Then we look at special class of Artinian T_o- Alexandroff spaces which is introduced by Mahdi and Elatrash in [12]. We give some illustrated examples.

Chapter two contains three sections. Some of the basic concepts and ideas about Artinian T_o- Alexandroff and T_o- finite spaces that are involved in [4] and [12] is studied. In section one, we will looking exclusively at closure, interior, cluster points, and boundary points of a subset A in T_o-Alexandroff space X, the most of T_o- finite spaces. Section two focuses on the results of [12] of generalized open sets on Artinian T_o-Alexandroff spaces such as preopen semi-open and $\alpha-$open sets. We characterize submaximality, extremally disconnectedness, nodec, and other topological concepts that are related to generalized open sets. Finally, we study semi-closure, semi-interior, pre-closure and pre-interior of a set A. In section three, we study the dimension on T_o-Alexandroff spaces. We get new results such as:

1. A T_o-Alexandroff Space X has zero dimension if and only if X has a base of clopen sets.

2. Let $(X, \tau(\leq))$ be a T_o-Alexandroff space. Then $dimX \leq 1$ if and only if every singleton in X is either maximal or minimal.

In chapter three, we are looking at weaker forms of continuity of a function on Artinian T_o-Alexandroff spaces. In section one, we study semi-continuity and pre-continuity. Upper semicontinuous multifunction and lower semicontinuous multi-functionon T_o-Alexandroff Spaces are studied in section two of this chapter. We give definitions, examples and some of previous results. In this chapter, we get the following new results:

1. A function between T_o-Alexandroff Spaces is continuous if and only if it is order-preserving.

2. Let X and Y be Artinian T_o-Alexandroff, then the function $f : X \to Y$ is continuous at x if and only if $f(\uparrow x) \subseteq \uparrow f(x)$

3. Let X be an Artinian T_o-Alexandroff space and let Y be a T_o-Alexandroff space. The function $f : X \to Y$ is precontinuous if and only if for all $x \in X$, $f(\hat{x}) \subseteq \uparrow f(x)$.

4. Let X be an Artinian T_o-Alexandroff space and let Y be a T_o-Alexandroff space. The function $f : X \to Y$ is semicontinuous if and only if for all $x \in X$, there is $y \in \hat{x}$ such that $f(y) \in \uparrow f(x)$. In equivalent form, for all $x \in X$ $f(\hat{x}) \bigcap \uparrow f(x) \neq \phi$.

   Moreover if X and Y are T_o-Alexandroff spaces and if $F : X \to \mathbf{P}(Y)$ is a multifunction. Then

5. F is upper semicontinuous multifunctionat x if and only if for all $y \geq x$, $F(y) \subseteq \uparrow F(x)$.

6. F is lower semicontinuous multifunctionat x_0 if and only if $F(x) \bigcap \uparrow y \neq \phi$ for all $x \in \uparrow x_0$ and $y \in F(x_0)$.

In the last chapter, we study a supratoplogy on Artinian T_o-Alexandroff spaces. A subclass $\tau^* \subseteq \mathbf{P}(X)$ is called a supratoplogy on X if $X \in \tau^*$ and τ^* is closed under arbitrary union. A supratoplogy τ^* induces a class of m-sets of subset of X. We deals with the questions under when τ_m - the class of all m-sets - forms a topology on X, and when $\tau_m = \tau^*$.

Further, we study the m-interior and m-closure of a set A. In section two of this chapter, we will more on to study m-continuity between an m-topological space X and a topological space Y. We give a characterization of m-continuity and we are looking at the relation of m-continuity and other types of continuity.

Chapter 1

PRELIMINARIES AND DEFINITIONS

In this chapter, we are looking at posets and Alexandroff spaces. Most of the definitions and concepts introduced here can be understanding without any real knowledge of them. Then, we are going to study the relation between Alexandroff topologies which are satisfying the separation axiom T_0 and posets.

1.1 Partially Ordered Sets

Definition 1.1.1. [6] A relation $\leq$ on a set P is called a *partial order* (*simply order*) if it satisfies the following:

(i) $x \leq x$ (*reflexivity*),

(ii) $x \leq y$ and $y \leq x$ imply $x = y$ (*anti-symmetry*),

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity).

The set P together with a partial order $\leq$ is called a *partially order set* (briefly, a poset).

Definitions 1.1.2. [6] Let P be a poset. Then

(i) an element $x \in P$ is called *a maximal element* if whenever $x \leq z$ then $x = z$, and y is *a minimal element* if whenever $z \leq y$ then $y = z$. The set of all maximal (resp. minimal) elements is denoted by M (resp m).

(ii) two elements x, y in P are *comparable* if either $x \leq y$ or $y \leq x$, otherwise they are *incomparable.*

(iii) a subset C of P is *a chain* if any two elements of C are comparable. Alternative names of a chain are *linearly ordered set* and *totaly ordered set.*

(iv) a subset $C \subseteq P$ is called anti-chain if all elements of C are incomparable.

(v) a subset D of a poset P is called *down set* if whenever $x \in D$ and $y \leq x$, then we have $y \in D$.

(vi) a subset U of a poset P is called *up set* if whenever $x \in U$ and $y \geq x$, then we have $y \in U$.

(vii) for $x \in P$, we define the set $\downarrow x = \{y \in P : y \leq x\}$ and the set $\uparrow x = \{y \in P : y \geq x\}$.

($viii$) for a subset A of P, we define the set $\downarrow A = \{y \in P : (\exists x \in A) y \leq x\}$ and the set $\uparrow A = \{y \in P : (\exists x \in A) y \geq x\}$.

Remarks 1.1.3. [6]

(1) Using the transitive of $\leq$, it is easy to prove that the two sets $\uparrow x$ and $\uparrow A$ are up sets, and the two sets $\downarrow x$ and $\downarrow A$ are down sets.

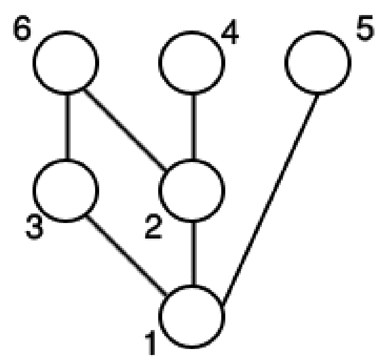
(2) For any $A \subseteq P$, we have $\uparrow A = \bigcup_{x \in A} \uparrow x$ and $\downarrow A = \bigcup_{x \in A} \downarrow x$.

(3) In any poset P, $\uparrow x$ is usually called an *ideal* and $\downarrow x$ is called a *filter.*

Definition 1.1.4. [6] If $C = \{c_o, c_1, c_2, \cdots, c_n\}$ is a finite chain in P with $\mid C \mid= n+1$, then we say that the *length of* C is equal n. The *length* of a poset P (denoted by $\ell(P)$) is the length of the longest chain in P.

Posets can be represented geometrically by the help of a diagram. To draw the diagram of a poset, each element of P represented by a small circle (or a dot). Then two comparable elements are joined by lines in such a way that if $a \leq b$, then a lies below b in the diagram. Finally, non-comparable elements are not joined and there is no horizontal lines in the diagram.

Example 1.1.5. Let $X = \{1, 2, 3, 4, 5, 6\}$ with the division relation then the diagram of the poset X is



Definitions 1.1.6. [6] Let $(P, \leq)$ be a poset. Then

(i) we say that P satisfies the *ascending chain condition* (briefly, ACC), if for each increasing chain $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$ in P, then there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = \cdots$

(ii) we say that P satisfies the *descending chain condition* (briefly, DCC), if for each decreasing chain $x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots$ in P, then there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = \cdots$

(iii) we say that P is of *finite chain condition* (briefly, FCC), if it satisfies both ACC and DCC.

Example 1.1.7. The natural number under the usual order satisfies DCC. It does not satisfy ACC, since $2 \leq 4 \leq 6 \leq \cdots$ is infinite increasing chain in $\mathbb{N}$.

Definitions 1.1.8. [6]

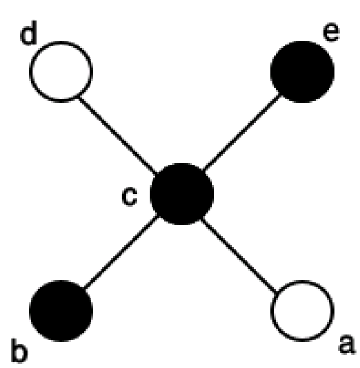
(i) Let $(P, \leq)$ be a poset, and let $\top \in P$. We say that $\top$ is *maximum element* of P if $x \leq \top$ for all $x \in P$. Alternative name of maximum element is *top element* and *largest element*.

(ii) An element $\perp \in P$ is called *minimum element* of P if $\perp \leq x$ for all $x \in P$. The element $\perp$ is also called *bottom element* or *least element*.

Remark 1.1.9. A poset P may has no maximum (resp. minimum) element. It is clearly that P has a maximum (resp. minimum) if and only if $\mid M \mid = 1$ (resp. $\mid m \mid = 1$), where $\mid M \mid$ is the cardinality of the set M.

Definition 1.1.10. [12] Let $(P, \leq)$ be a poset and let A be a subset of P. Then the order of P induces an order in A in the following natural way : if $a, b \in A$ then $a \leq b$ as elements in A if and only if $a \leq b$ as elements in P. In this case, we define $M(A)$ (resp. $m(A)$) to be the set of all *maximal* (resp. *minimal*) elements under the induced order.

Example 1.1.11. If $X = \{a, b, c, d, e\}$ has a relation as it is shown on the following Figure:



Then $M = \{d, e\}$ and $m = \{a, b\}$.

Let $A = \{b, c, e\}$ then $M(A) = \{e\}$ and $m(A) = \{b\}$.

1.2 Topology And Topological Concepts

This thesis is talking about a certain type of topology - named T_o-Alexandroff - and its relation with poset as we referred in the introduction. Things to be more obvious, talking about essential concepts and theories of topology is very necessary. These concepts are going to be used in the remainder of this thesis.

Definition 1.2.1. Let $X \neq \phi$ be a set. Then a *topology* on X is a subset τ of $\mathcal{P}(X)$ obeying the following axioms :

(i) X and ϕ belong to τ.

(ii) If U_1 and U_2 belong to τ, then $U_1 \bigcap U_2$ belongs to τ.

(iii) If $\{U_\alpha : \alpha \in \Delta\}$ is an indexed family of sets, each of which belongs to τ, then $\bigcup_{\alpha\in\Delta} U_\alpha$ belongs to τ.

The pair (X, τ) is called a *topological space*. An element of τ is called an *open* subset of X.

Example 1.2.2. (i) Let X be any set and let τ the collection of all subsets of X. Then τ is a topology for X. This topology is called the *discrete topology.*

(ii) Let X be any set and let $\tau = \{\phi, X\}$. Then τ is a topology on X, called the *indiscrete topology.*

(iii) let $X = \mathbb{R}$ be the set of real numbers. Define

$$L_a = \{x \in \mathbb{R} : x < a\} = (-\infty, a)$$

to be the left ray of real numbers. Consider the collection

$$\tau = \{L_a : a \in \mathbb{R}\} \bigcup \{\mathbb{R}\} \bigcup \{\phi\} .$$

Then τ form a topology on $\mathbb{R}$. This topology is called the *left ray topology* on $\mathbb{R}$.

(iv) let $X = \mathbb{R}$ be the set of real numbers, and let τ be the collection of subset of $\mathbb{R}$ consisting of ϕ and all sets U having the following property: for each $x \in U$ there exists an open interval (a, b) containing x such that $(a, b) \subseteq U$. Then τ forms a topology on $\mathbb{R}$. This topology is called the *standard topology* on $\mathbb{R}$.

(v) Let $X \neq \phi$. Define

$$\tau = \{U \subseteq X : U^c \text{ is finite }\} \bigcup \{\phi\}$$.

Then τ forms a topology on X. This topology is called the *cofinite topology* for X. If X is finite, the cofinite topology coinsides with the discrete topology.

Definition 1.2.3. Let (X, τ) be a topological space. A subset $A \subseteq X$ is *closed* if $X - A$ is open subset of X.

Theorem 1.2.4. *If $\Im$ is the collection of closed sets in a topological space X, then*

(i) *X and ϕ belongs to $\Im$.*

(ii) *Any intersection of members of $\Im$ belongs to $\Im$.*

(iii) *Any finite union of members of $\Im$ belongs to $\Im$.*

Definition 1.2.5. Let (X, τ) be a topological space and $A \subseteq X$. Then the *closure* of A is the intersection of all closed set in X which contain A and is denoted by $\overline{A}$.

Theorem 1.2.6. *Let (X, τ) be a topological space and let A and B be subsets of X. Then*

(i) *$\overline{A}$ is the smallest closed set containing A.*

(ii) *$\overline{\phi} = \phi$.*

(iii) *$\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$, and $\overline{A \bigcap B} \subseteq \overline{A} \bigcap \overline{B}$.*

(iv) *if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.*

(v) *A is closed if and only if* $\overline{A} = A$.

Definitions 1.2.7. Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called

(i) an *interior point* of A if there exists an open set U containing x such that $U \subseteq A$. The set of interior points of A is called the *interior* of A and is denoted by A^o or $Int(A)$.

(ii) an *exterior point* of A if there exists an open set U containing x such that $U \bigcap X = \phi$. The set of exterior points of A is called the *exterior* of A and is denoted by $Ext(A)$.

(iii) a *boundary point* of A if every open set in X containing x contains at least one point of A, and at least one point of A^c. The set of boundary points of A is called the *boundary* of A and is denoted by $bd(A)$.

Theorem 1.2.8. *Let* (X, τ) *be a topological space and* $A \subseteq X$. *Then each point in* X *belongs to one and only one of the sets* A^o, $Ext(A)$, *or* $bd(A)$.

Example 1.2.9. Let $A = [0, 1]$ be a subset of $\mathbb{R}$. For the standard topology on $\mathbb{R}$, we have the following:

(i) $\overline{A} = [0, 1]$

(ii) $A^o = (0, 1)$

(iii) $Ext(A) = (-\infty, 0) \bigcup (1, \infty)$

(iv) $bd(A) = \{0, 1\}$.

Theorem 1.2.10. *Let* (X, τ) *be a topological space and let* A *and* B *be subsets of* X. *Then* (*i*) $A^{^o}$ *is the largest open set contained in* A.

(*ii*) $X^{^o} = X$.

(*iii*) $(A \bigcap B)^{^o} = A^{^o} \bigcap B^{^o}$, *and* $A^{^o} \bigcup B^{^o} \subseteq (A \bigcup B)^{^o}$.

(*iv*) *if* $A \subseteq B$, *then* $A^{^o} \subseteq B^{^o}$.

(*v*) A *is open if and only if* $A^{^o} = A$.

Theorem 1.2.11. *Let* (X, τ) *be a topological space and* $A \subseteq X$. *Then*

(*i*) $\overline{A} = A \bigcup Bd(A)$.

(*ii*) $Bd(A) = \overline{A} \bigcap \overline{X - A}$.

Definitions 1.2.12. Let X be a topological space and $A \subseteq X$. Then

(*i*) x is a *cluster point* of A if $U \bigcap A - \{x\} \neq \phi$ for all open sets U contains x. The set of all cluster points is denoted by A'.

(*ii*) x is *isolated point* if $\{x\}$ is open. If $x \in A$, then x is *isolated point* of A if x is *isolated point* in the subspace A.

(*iii*) a point $x \in X$ is called *pure* if $\{x\}$ is either open or closed. Otherwise it is *mixed*.

(*iv*) A is a *dense set* if $\overline{A} = X$.

(*v*) A is a *nowhere dense* if $\overline{A}^{^o} = \phi$.

(*vi*) A is *dense-in-itself* if A contains no isolated points.

(*vii*) A set A is *perfect* if A is closed and dense-in-itself.

Theorem 1.2.13. *Let* (X, τ) *be a topological space and* $A \subseteq X$. *Then* $\overline{A} = A \bigcup A'$.

Definition 1.2.14. Let X be a topological space and $x \in X$. A *neighborhood* of x is a set U which contains an open set V containing x. Thus, evidently, U is a *neighborhood* of x if and only if $x \in U^o$. The collection $\mathbb{U}_x$ of all *neighborhoods* of x is a *neighborhood system* of x.

Definition 1.2.15. Let X be a topological space and $x \in X$. A *neighborhood base* of x is a subcollection $\mathbb{B}_x$ taken from the *neighborhood system* $\mathbb{U}_x$, having the property that each $U \in \mathbb{U}_x$ contains some $V \in \mathbb{B}_x$. That is, $\mathbb{U}_x$ must be determined by $\mathbb{B}_x$ as follows:

$$\mathbb{U}_x = \{U \subseteq X : V \subseteq U \text{ for some } V \in \mathbb{B}_x\}$$

Definition 1.2.16. Let (X, τ) be a topological space. A *base* for τ is a collection $\mathbb{B}$ of subsets of X such that

(i) each member of $\mathbb{B}$ is also a member of τ.

(ii) if $U \in \tau$ and $U \neq \phi$, then U is a union of members of $\mathbb{B}$.

Example 1.2.17. The collection $\mathbb{B}$ of all open intervals in $\mathbb{R}$ serves as a base for the standard topology on $\mathbb{R}$.

Theorem 1.2.18. *Let (X, τ) be a topological space. Then the collection $\mathbb{B}$ of members of τ is a base for τ if and only if*

(i) $X = \bigcup_{B \in \mathbb{B}} B$.

(ii) *whenever $B_1, B_2 \in \mathbb{B}$ with $p \in B_1 \bigcap B_2$, there is some $B_3 \in \mathbb{B}$ with $p \in B_3 \subseteq B_1 \bigcap B_2$.*

Theorem 1.2.19. *If $\mathbb{B}$ is a collection of open sets in X, then $\mathbb{B}$ is a base for τ if and only if for each $x \in X$, the collection $\mathbb{B}_x = \{B \in \mathbb{B} : x \in B\}$ is a neighborhood base of x.*

Example 1.2.20. For each $x \in \mathbb{R}$. The collection $\mathbb{B}_x = \{(x-\epsilon, x+\epsilon) : \epsilon > 0\}$ in $\mathbb{R}$ serves as a neighborhood base of x for the standard topology on $\mathbb{R}$.

Definition 1.2.21. If (X, τ) is a topological space and $A \subseteq X$, the collection $\tau_A = \{G \bigcap A : G \in \tau\}$ is a topology for A, called the *relative topology* for A.

Theorem 1.2.22. *If (X, τ) is a topological space and $A \subseteq X$, then $F \subseteq A$ is closed in A if and only if $F = K \bigcap A$, where K is closed in X.*

Definitions 1.2.23. Let (X, τ) be a topological space. Then

(*i*) X is T_o-*space* if whenever x and y are distinct points in X, there is an open set containing one and not the other.

(*ii*) X is $T_{1/2}-$*space* if every singleton is either open or closed.

(*iii*) X is T_1-*space* if whenever x and y are distinct points in X, there is a neighborhood of each not containing the other.

(*iv*) X is T_2-*space* (*Hausdorff space*) if whenever x and y are distinct points in X, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

The following implication holds:

$$T_2-\textbf{space} \Rightarrow T_1-\textbf{space} \Rightarrow T_{1/2}-\textbf{space} \Rightarrow T_o-\textbf{space}$$

Theorem 1.2.24. *Let (X, τ) be a topological space. Then*

(*i*) *X is T_o-space if and only if for each pair of distinct points $x, y \in X$, $\overline{\{x\}} \neq \overline{\{y\}}$.*

(*ii*) *X is T_1-space if and only if each singleton set in X is closed.*

Example 1.2.25. Let $\mathbb{R}$ be the set of real numbers with the left ray topology. For any two distinct points x, y where $x < y$, there is an open set of the form $(-\infty, (x+y)/2)$ containing x but not y, which means that the space is T_o-space. However, there is no way of getting an open set containing y to exclude x so that our space is not T_1-space.

Example 1.2.26. Let X be any infinite set with the cofinite topology. Since one-point sets are closed, X is a T_1-space. But no two nonempty open sets are disjoint, so X can not be T_2-space.

1.3 Alexandroff Spaces

Alexandroff spaces was first studied in 1937 by P. Alexandroff [5] under the name Diskrete Raume (discrete space). Now discrete spaces are spaces where singletons are open.

Definition 1.3.1. [5] An *Alexandroff Space* is a topological space in which arbitrary intersection of open sets is open. Equivalently, each singleton has a minimal neighborhood base.

For this definition, any discrete space is Alexandroff, and any finite space is also Alexandroff.

Definition 1.3.2. A topological space (X, τ) is called *locally finite* if each element x of X is contained in a finite open set and a finite closed set.

Each finite space is locally finite and each locally finite is Alexandroff, so the results of Alexandroff spaces include finite and locally finite spaces.

Theorem 1.3.3. *(a) Each finite space is locally finite.*

(b) Each locally finite is Alexandroff.

Proof. (a) If X is finite then any neighborhood and its complement is finite. So X is locally finite.

(b) If X is locally finite, and $x \in X$, so there exists a finite neighborhood U_x of x.
Let $V(x) = \bigcap\{U : \text{is finite neighborhood of } x\}$ and $\mathbb{U} = \{U : \text{is finite neighborhood of } x\}$.
So $x \in V(x)$.
Claim: $V(x)$ is open.
To see this, let $U \in \mathbb{U}$ be a finite neighborhood of x. Clearly $V(x) \subseteq U$. If $V(x) = U$, we done, so suppose $V(x) \neq U$ so there exist $y_{i1}, y_{i2}, y_{i3}, \cdots, y_{im} \in U$

and not in $V(x)$ such that $V(x) = U - \{y_{i1}, y_{i2}, y_{i3}, \cdots, y_{im}\}$. This implies that there exist $U_{i1}, U_{i2}, U_{i3}, \cdots, U_{im} \in \mathbb{U}$ such that $y_{ij} \notin U_{ij}$. Thus $U \bigcap U_{i1} \bigcap U_{i2} \bigcap U_{i3} \bigcap \cdots U_{im} \subseteq V(x)$. Since $V(x) = \bigcap_{U \in \mathbb{U}} \mathbb{U} \subseteq U \bigcap U_{i1} \bigcap U_{i2} \bigcap U_{i3} \bigcap \cdots U_{im}$. Thus $V(x) = U \bigcap U_{i1} \bigcap U_{i2} \bigcap U_{i3} \bigcap \cdots U_{im}$ which is open. Therefore $V(x)$ is the smallest neighborhood base of x, then X is Alexandroff.

□

Theorem 1.3.4. *Let $(P, \leq)$ be a poset, then $\mathbb{B} = \{\uparrow x : x \in P\}$, is a base for a topology on P.*

Proof. It is clearly that $\bigcup_{x \in P} \uparrow x = P$. So suppose that, if $z \in \uparrow x \bigcap \uparrow y$ then $z \geq x$ and $z \geq y$ and so $z \in \uparrow z \subseteq \uparrow x \bigcap \uparrow y$. Then $\mathbb{B}$ is a base. □

Theorem 1.3.5. *This topology in Theorem 1.3.4 is Alexandroff topology satisfies the separation axiom T_o and denoted by $\tau(\leq)$.*

Proof. Let $x \neq y$ in P, then we have three cases

(1) x and y are incomparable then $y \notin \uparrow x$.

(2) $x \leq y$ then $x \notin \uparrow y$.

(3) $y \leq x$ then $y \notin \uparrow x$.

Then this topology satisfies T_o. To prove this topology is Alexandroff. Since $\{\uparrow x_\alpha : \alpha \in \Delta\}$ is a family of open set of $\tau(\leq)$, we want to prove $\bigcap_{\alpha \in \Delta} \uparrow x_\alpha$ is open set. Now for each $z \in \bigcap_{\alpha \in \Delta} \uparrow x_\alpha$ then $z \in \uparrow x_\alpha$ for all $\alpha \in \Delta$, $\uparrow z \subseteq \uparrow x_\alpha$ for all $\alpha \in \Delta$, and so $\uparrow z \subseteq \bigcap_{\alpha \in \Delta} \uparrow x_\alpha$. Since $\uparrow z$ is open, then $\bigcap_{\alpha \in \Delta} \uparrow x_\alpha$ is open. Therefore $\tau(\leq)$ is Alexandroff. □

Corollary 1.3.6. *This base in Theorem 1.3.4 is minimal.*

Proof. Claim: for each $x \in P \uparrow x = G$ where $G = \bigcap\{U : U \in \tau, x \in U\}$.

Proof the claim:

It is clearly that $x \in \uparrow x \subseteq G$ (by Theorem 1.3.4 and 1.3.5). On the other hand, since $\uparrow x$ is open then $\uparrow x \subseteq \{U : U \in \tau, x \in U\}$, this implies that $G \subseteq \uparrow x$.Then we have $\mathbb{B} = \{\uparrow x : x \in P\}$ is minimal base. □

In the other hand, if (X, τ) is Alexandroff space, we define the a pre-order $\leq_\tau$, called *specialization pre-order*, by $a \leq_\tau b$ if and only if $a \in \overline{\{b\}}$.

Lemma 1.3.7. *This pre-order is reflexive and transitive.*

Proof. Since $x \in \overline{\{x\}}$, then $x \leq_\tau x$, and so the pre-order is reflexive. To prove the transitivity, let $x \leq_\tau y$ and $y \leq_\tau z$, then $x \in \overline{\{y\}}$ and $y \in \overline{\{z\}}$. This implies $\overline{\{y\}} \subseteq \overline{\{z\}}$ and $x \in \overline{\{z\}}$. Then $x \leq_\tau z$. □

Lemma 1.3.8. *This pre-order is partial order if and only if X is T_o.*

Proof. ($\Rightarrow$) Suppose that the pre-order is partial order. Let $x \neq y$ in P, then we have three cases

(1) x and y are incomparable then $y \notin \uparrow x$.

(2) $x \leq y$ then $x \notin \uparrow y$.

(3) $y \leq x$ then $y \notin \uparrow x$.

Then X is T_o.

($\Leftarrow$) Suppose X is T_o. Since by above lemma the pre-order is reflexive and transitive, we want prove anti-symmetry. $x \leq_\tau y$ and $y \leq_\tau x$, then $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. This implies $x \in \overline{\{y\}} \subseteq \overline{\{x\}}$ and $y \in \overline{\{x\}} \subseteq \overline{\{y\}}$, then $\overline{\{x\}} = \overline{\{y\}}$ and $a = b$. Since if $a \neq b$ then there exist open set V such that $a \in V$ and $b \notin V$. Then $b \in V^c$. Since V^c is closed, then $\overline{\{b\}} \subseteq V^c$ and $\overline{\{a\}} \subseteq V^c$. Then $a \in V^c$ contradiction. Therefore this pre-order is partial order □

If $(X, \leq)$ is a poset and if $\tau(\leq)$ is its induced T_o-Alexandroff topology, then the specialization order of $\tau(\leq)$ is the order $\leq$ itself; that is, $\leq_{\tau(\leq)}=\leq$. On the other hand, if (X, τ) is a T_o- Alexandroff space and if $\leq_\tau$is it is specialization order, then the induced topology by the specialization order is the topology τ itself; that is, $\tau(\leq_\tau) = \tau$. We will denote $(X, \tau(\leq))$ to be a T_o-Alexandroff space (X, τ) together with its specialization order $\leq$, and where the corresponding poset is $(X, \leq)$[12].

For a T_o-Alexandroff space $(X, \tau(\leq))$, and for $x \in X$, the collection consists of one set $\{\uparrow x\}$ is the minimal neighborhood base for x and sometimes denoted by $\{V(x)\}$.

Lemma 1.3.9. *[5] Let (X, τ) be a T_o- Alexandroff space and $\leq$ is it (Alexandroff) specialization order. Then $x \leq y$ if and only if $y \in V(x)$*

Theorem 1.3.10. *For each T_o- Alexandroff space (X, τ), there is a corresponding poset $(X, \leq_\tau)$, in one to one and onto way, where each one of them is completely determine by the other.*

Proof. (a) Let (X, τ) be T_o- Alexandroff space. Suppose $\leq$ be the specialization order and set $\tau'_{\leq}$ be the topology induced by $\leq$.

Claim: $\tau = \tau'_{\leq}$.

By Lemma 1.3.9 if $y \in V(x)$ if and only if $x \leq y$ if and only if $y \in \uparrow x$. thus $V(x) = \uparrow x$.

(b) If $(X, \leq)$ be a poset and $\tau(\leq)$ be it is T_o- Alexandroff space. Define $\leq'$ to the specialization order.

Claim: $a \leq b$ if and only if $a \leq' b$.

If $a \leq b$ if and only if $b \in \uparrow a = V(a)$ in $\tau(\leq)$ if and only if $a \leq' b$ (By Lemma 1.3.9).

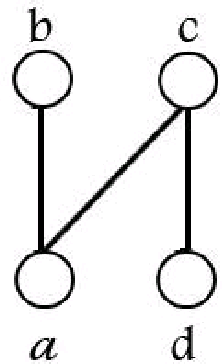
□

Theorem 1.3.11. *[5] Let $(X, \tau(\leq))$ be a T_o- Alexandroff Space, and $x, y \in X$. Then $x \leq y$ if and only if for all $U \in \tau$, $x \in U$ implies $y \in U$.*

The specialization order plays important role in these studies, since the authers characterize topological proparties on T_o- Alexandroff Spaces by looking at their specialization order.

Examples 1.3.12. [12]

(1) Let $X = \{a, b, c, d\}$ with the partial order $a \leq b$, $a \leq c$ and $d \leq c$ as shown in figure below



Then the corresponding T_o-Alexandroff topology is:

$$\tau = \{\phi, X, \{a, b, c\}, \{b\}, \{c\}, \{d, c\}, \{b, c, d\}, \{b, c\}\}$$

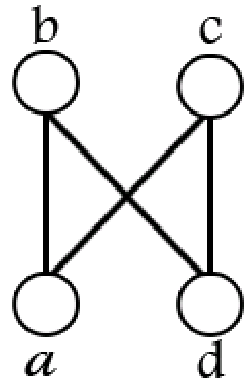
with the minimal base

$$\mathbb{B} = \{\{a, b, c\}, \{b\}, \{c\}, \{d, c\}\}.$$

(2) Let $X = \{a, b, c, d\}$ with the T_o-Alexandroff topology

$$\tau = \{\phi, X, \{a, b, c\}, \{b\}, \{c\}, \{b, d, c\}, \{b, c\}\}.$$

We can find the specialization order as follows : $\overline{\{a\}} = \{a\}$, $\overline{\{d\}} = \{d\}$, $\overline{\{b\}} = \{b, a, d\}$ and $\overline{\{c\}} = \{c, a, d\}$, so $a \leq b$, $d \leq b$, $a \leq c$ and $d \leq c$ and hence the figure of the corresponding poset of X is



Definitions 1.3.13. [12] Let $(X, \tau(\leq))$ be a T_o- Alexandroff Space. If the corresponding poset $(X, \leq)$ satisfies

(i) ACC, then X is called *Artinian space.*

(ii) DCC, then X is called *Noetherian space.*

(iii) FCC, then X is called *generalized locally finite space* (*g-locally finite space*).

Remark 1.3.14. If $(X, \tau(\leq))$ is a locally finite T_o-space, then the corresponding poset $(X, \leq)$ is both ACC and DCC. Hence $(X, \tau(\leq))$ is g-locally finite space. The converse is not true as the following example proves.

Example 1.3.15. Consider the T_o-Alexandroff space $\{\perp\} \bigcup \mathbb{N}$ with anti-chain order on $\mathbb{N}$. The corresponding poset is of FCC, but $\uparrow \perp = X$ is not finite, so X is not locally finite.

Definition 1.3.16. [12]

(i) Let $(X, \tau(\leq))$ be an Artinian space, and let M be the set of all maximal elements of the corresponding poset. If $x \in X$, then we define $\hat{x} = \uparrow x \bigcap M$.

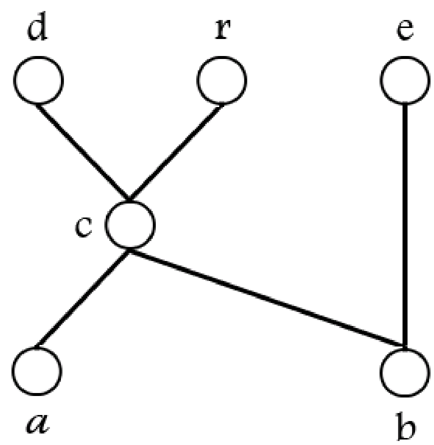
(ii) Let $(X, \tau(\leq))$ Noetherian space, and let m be the set of all minimal elements of the corresponding poset. If $x \in X$, we define $\check{x} = \downarrow x \bigcap m$.

Theorem 1.3.17. *[5] Let $(X, \tau(\leq))$ be a T_o-Alexandroff Space equipped with its specialization order $\leq$, and let $A \subseteq X$. Then*

(*a*) *A is open if and only if A is up set in the corresponding poset. That is, $A = \uparrow A$.*

(*b*) *A is closed if and only if A is a down set in the corresponding poset. That is, $A = \downarrow A$.*

Example 1.3.18. Consider the poset X with the partial order in Figure



Then $M = \{e, r, d\}$ and $m = \{a, b\}$

(i) The set $A = \{e, b\} = \downarrow A$ is down set, then A is closed set.

(ii) The set $B = \{a, c, d, r\} = \uparrow B$ is up set, then B is open set.

(iii) The set $C = \{d, c\}$ is not up and not down set, then C is not open and not closed set.

(iv) Since $\hat{x} = \uparrow x \bigcap M$ then $\hat{a} = \uparrow a \bigcap M = \{d, r\}$, $\hat{b} = \uparrow b \bigcap M = \{d, e, r\}$, and $\hat{c} = \uparrow c \bigcap M = \{d, r\}$.

(v) Since $\check{x} = \downarrow x \bigcap m$ then $\check{d} = \downarrow d \bigcap m = \{a, b\}$, and $\check{e} = \downarrow e \bigcap m = \{b\}$.

Chapter 2

BETWEEN T_o-FINITE AND ARTINIAN SPACES

El-Attik in [4] is talking about finite T_o- topological spaces while Mahdi and El Atrash in [12] are talking about arbitrary T_o-Alexandroff spaces. The technique used in the two studies are similar. This thesis studies the results of the two studies and generalizes some of the theories on the finite case to T_o-Alexandroff.

2.1 Theorems of Some Topological Properties

In this section, we shall study, among other things the two papers titled (On finite T_o- topological spaces) by A.E.El-Atik et al. [4] and (On T_o- Alexandroff spaces) by H.B.Mahdi et al. [12]. Beside comparing the results of the papers [4] and [12]. We study the results of [12] which are generalization of the same results of [4]. Then we determine those results in [12] which are not included in [4] which are still true in [4], since each finite space is Alexandroff. Moreover, we will study the results in [4] which are not included in [12] and generalized them to T_o- Alexandroff spaces.

Proposition 2.1.1. *[12] Let $(X, \tau(\leq))$ be a T_o-Alexandroff space, and let $A \subseteq X$.*

Then

(i) *for* $x \in X$, $\overline{\{x\}} = \downarrow x$.

(ii) $A^{o} = \{x \in A : \uparrow x \subseteq A\} = \bigcup\{\uparrow x : \uparrow x \subseteq A\}$.

(iii) $\overline{A} = \cup\{\downarrow x : x \in A\}$.

(iv) $A^{'} = \overline{A} \backslash \{x : x \textit{ is maximal in } A\}$.

(v) $bd(A) = \cup\{\downarrow x : x \in A\} \backslash \{x : \uparrow x \subseteq A\}$.

Proof. (i) Since $\overline{\{x\}}$ is the smallest closed set containing x and $\downarrow x$ is a closed set containing x, so we have $\overline{\{x\}} \subseteq \downarrow x$. Now let $y \in \downarrow x$, so $y \leq x$. If $y \in \overline{\{x\}}^{c}$, which is an open set then $\uparrow y \bigcap \overline{\{x\}} = \phi$. Since $x \in \uparrow y$, we get that $x \notin \overline{\{x\}}$, which is a contradiction. So $y \in \overline{\{x\}}$, and hence we have $\downarrow x \subseteq \overline{\{x\}}$. Therefore $\downarrow x = \overline{\{x\}}$.

(ii) Let $x \in \bigcup\{\uparrow x : \uparrow x \subseteq A\}$, then $x \in \uparrow x \subseteq A$. Since $\uparrow x$ is open set in A and A^{o} is the largest open set contained in A so we have $\uparrow x \subseteq A^{o}$ for every $\uparrow x \subseteq A$. This implies that $\bigcup\{\uparrow x : \uparrow x \subseteq A\} \subseteq A^{o}$. On the other hand, let $x \in A^{o}$, then $x \in \uparrow x \subseteq \bigcup\{\uparrow x : \uparrow x \subseteq A\}$. Therefore $A^{o} \subseteq \bigcup\{\uparrow x : \uparrow x \subseteq A\}$.

(iii) If $x \in A$, then $\overline{\{x\}} = \downarrow x \subseteq \overline{A}$, so $\cup\{\downarrow x : x \in A\} \subseteq \overline{A}$. On the other hand, if $x \in A$ then $x \in \downarrow x \subseteq \cup\{\downarrow x : x \in A\}$, so $A \subseteq \cup\{\downarrow x : x \in A\}$, which is a closed set. Therefore $\overline{A} \subseteq \cup\{\downarrow x : x \in A\}$.

(iv) If $x \in A^{'}$ then $x \in \overline{A}$ and $\uparrow x \bigcap A \backslash \{x\} \neq \phi$, so x is not maximal in A, and hence $A^{'} \subseteq \overline{A} \backslash \{x : x \text{ is maximal in } A\}$. For the other inclusion, if $y \in \overline{A}$, and y is not maximal in A, we have that $\uparrow y \bigcap A \neq \phi$. If $\uparrow y \bigcap A = \{y\}$, then y is a maximal in A, and this is not true so we have that $y \in A^{'}$.

(v) Since $bd(A) = \overline{A} \backslash A^{^o} = \overline{A} \bigcap \overline{A^c}$, so it is a closed set and $bd(A) = \cup\{\downarrow x : x \in A\} \backslash \{x : \uparrow x \subseteq A\}$. If $A^{^o} = \phi$ then $bd(A) = \overline{A}$.

□

Corollary 2.1.2. *Let $(X, \tau(\leq))$ be a T_o-Alexandroff space, and let $A \in X$. Then $bd(\uparrow x) = \downarrow x - \{x\}$.*

Corollary 2.1.3. *[4] In a T_o-finite space X, let $a \in X$, then*

$$\overline{\{a\}} = \{b : b \leq a\} = \downarrow a.$$

Theorem 2.1.4. *[12] Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space, and A is a subset of X. Then*

(i) $A^{^o} = \phi$ if and only if $A \bigcap M = \phi$.

(ii) $\overline{A} = \cup\{\downarrow x : x \in M(A)\} = \downarrow M(A)$.

(iii) $A^{'} = \cup\{\downarrow x \backslash \{x\} : x \in M(A)\}$.

(iv) A is dense if and only if $M \subseteq A$.

(v) A is nowhere dense if and only if $M \bigcap A = \phi$.

(vi) if $\mid M \mid = 1$, then any subset is either dense or nowhere dense.

Proof. (i) $(\Rightarrow)$ Let $A^{^o} = \phi$, and suppose to contrary that $x \in A \bigcap M$. So x is maximal element of X which is in A. Hence $\uparrow x = \{x\} \subseteq A$ which is a contradiction. So $A \bigcap M = \phi$.

$(\Leftarrow)$ Suppose that $A \bigcap M = \phi$. If $y \in A^{^o}$, then $\uparrow y \subseteq A$. Since X satisfies the ACC, we get a maximal element z in X such that $y \leq z$ and so $z \in \uparrow y \subseteq A$. But this implies that $z \in A \bigcap M$, which is a contradiction.

(ii) If $x \in A$, then there exists a maximal element y in A such that $x \leq y$, so $\downarrow x \subseteq \downarrow y$. This implies that $\cup\{\downarrow x : x \in A\} \subseteq \cup\{\downarrow x : x \in M(A)\}$. On the other hand if $x \in M(A)$, then $\overline{\{x\}} = \downarrow x \subseteq A$ so $\cup\{\downarrow x : x \in M(A)\} \subseteq \overline{A}$.

(iii) Since X satisfies the ACC, then we have that

$A' = \overline{A} \backslash \{x : x \text{ is maximal in } A\}$

$= \cup\{\downarrow x : x \in M(A)\} \backslash M(A)$

$= \cup\{(\downarrow x) \backslash \{x\} : x \in M(A)\}$.

(iv) $(\Rightarrow)$ Suppose that A is dense, and let $x \in M$. Then $\uparrow x \bigcap A \neq \phi$. But $\uparrow x = \{x\}$ so $x \in A$.

$(\Leftarrow)$ Suppose that $M \subseteq A$, so $M(A) = M$. By part(2), $\overline{A} = \cup\{\downarrow x : x \in M(A)\} = \cup\{\downarrow x : x \in M\} = X$.

(v) $(\Rightarrow)$ Suppose that A is nowhere dense. Then $\overline{A}^{o} = \phi$. By part(1), $M \bigcap \overline{A} = \phi$, and hence $M \bigcap A = \phi$.

$(\Leftarrow)$ Suppose that $M \bigcap A = \phi$. So no maximal element of X in A. By Proposition 2.1.1 part(3) no maximal element of X in $\overline{A}$, and hence $\overline{A}^{o} = \phi$.

(vi) Suppose that $M = \{\top\}$, and let $A \subset X$. Then either $\top \in A$ or $\top \notin A$. Hence by parts (1) and (2) either A is dense or nowhere dense.

□

Corollary 2.1.5. *Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space. Then for each dense subset D of X, D^{o} is dense.*

Proof. Let $D \subseteq X$ be a dense set then $M \subseteq D$, and hence $M \subseteq D^{o}$, so D^{o} is dense. □

Corollary 2.1.6. *[4] Let X be a T_o-finite topological space X, and $A \subseteq X$. Then*

(1) $A^{^{o}}$ *is the set of all points* $a \in A$ *such that for every* $b \geq a$ *it is also true that* $b \in A$. *In the other words , the elements of* A^o *are all upper bounds of* X *belonging to* A.

(2) *the closure* $\overline{A}$ *of* A, *is the set of points* $b \in X$ *such that* $b \leq a$ *for some* $a \in A$. *In the other words,* $\overline{A}$ *is the set of all lower bounds of points of* A.

(3) *the boundary* $bd(A)$ *of* A *is the set of all elements* $b \in X$ *such that* $b \leq a$ *for some* $a \in A$ *and* $b \leq c$ *for some* $c \notin A$.

(4) *the exterior* $ext(A)$ *of* A, *is the set of all points* $x \in X$ *such that* $b \geq x$ *implies* $b \notin A$.

(5) *the Closure of an interior* $\overline{A^o}$ *of* A, *is the set of all* x *in* X *such that there exists a maximal element* $a \in A$ *with* $a \geq x$.

(6) *the interior of a closure* $\overline{A}^{o}$ *of* A, *is the points* $x \in X$ *such that all maximal elements* $y \geq x$ *belong to* A.

(7) A *is dense if and only if it contains all open points of* X.

(8) A *is nowhere dense if there is no set has a maximal element in* A.

(9) A *is dense-in-itself if* A *contains all of its lower bounds.*

Theorem 2.1.7. *Let* $(X, \tau(\leq))$ *be an Artinian* T_o-*Alexandroff space and* $x \in X$. *Then* $x \notin M$ *if and only if the set* $\downarrow x$ *is perfect.*

Proof. ($\Rightarrow$) Let $x \notin M$, and let $A = \downarrow x$. Then A is closed. Moreover since for all $y \in A$, $\uparrow y \neq \{y\}$, then $\{y\}$ is not open set and hence y is not isolated point. Therefore A is dense-in-itself, and hence A is perfect.

($\Leftarrow$) If $A = \downarrow x$ is perfect then x is not isolated point. Hence $\uparrow x \neq \{x\}$ then $x \notin M$. □

Definitions 2.1.8. [15] Let X be a topological space. Then X is called

(i) *scattered* if no subset of X is dense-in-itself.

(ii) α-*scattered* if it has a dense set of isolated points.

Theorem 2.1.9. *[12] Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space. Then*

(i) *X is scattered.*

(ii) *X is α-scattered.*

Proof. (i) Let A is a subset of an Artinian T_o-Alexandroff space X, then by Proposition 2.1.1 part(5) we have that $bd(A) \bigcap M = \phi$. If $x \in A$ such that $\hat{x} \nsubseteq A$, then $x \notin A^o$, so we have A is not dense-in-itself. Hence X is scattered.

(ii) Let X be Artinian. So, the set M is dense. Hence X is α-scattered.

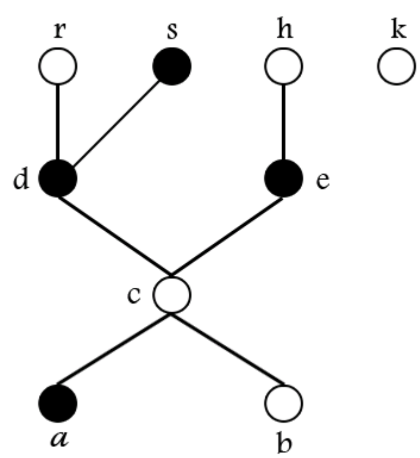
□

Corollary 2.1.10. *[4]*

(i) *A finite T_o space contains an isolated points.*

(ii) *Every open set in a finite T_o space contains an isolated point.*

Example 2.1.11. [12] Consider the poset X with the partial order shown in the figure



So X satisfies the ACC and $M = \{r, s, h, k\}$.

If $A = \{a, d, e, s\}$, then

(i) $A^o = \uparrow s = \{s\}$.

(ii) $\overline{A} = \downarrow s \bigcup \downarrow e = \{a, b, c, d, e, s\}$.

(iii) $bd(A) = \overline{A} \backslash A^o = \{a, b, c, d, e\}$ which is a closed set that contains no maximal element of X.

(iv) $A' = \downarrow s \backslash \{s\} \bigcup \downarrow e \backslash \{e\} = \{a, b, c, d\}$.

If $B = \{a, d, e\}$, then

(i) $B^o = \phi$.

(ii) $\overline{B} = \downarrow d \bigcup \downarrow e = \{a, b, c, d, e\}$.

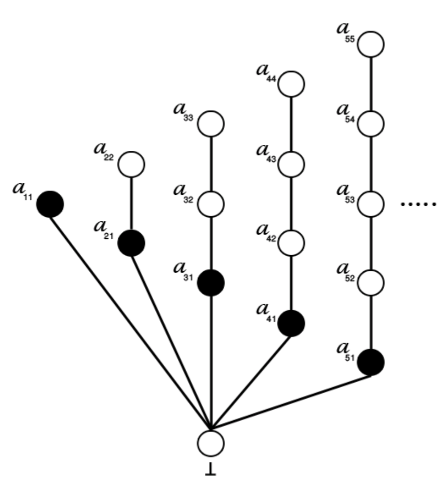
(iii) $bd(B) = \overline{B} = \{a, b, c, d, e\}$.

(iv) $A' = \downarrow d \backslash \{d\} \bigcup \downarrow e \backslash \{e\} = \{a, b, c\}$.

Note that $M \bigcap B = \phi$, so by Theorem 2.1.4 part(5), B is nowhere dense.

If $C = \{r, s, h, k, a, b\}$, then $M \subseteq C$, and hence by Theorem 2.1.4 part(4), C is a dense subset.

Example 2.1.12. [12] Consider the poset X with the partial order shown in figure:



So X satisfies the both ACC and DCC where the set $M = \{a_{11}, a_{22}, a_{33}, a_{44}, ...\}$ and any subset contains M is dense.

Let $A = \{a_{11}, a_{21}, a_{31}, a_{41}, ...\}$. Then

(i) $A^o = \{a_{11}\}$.

(ii) $\overline{A} = A \bigcup \{\perp\}$.

(iii) $bd(A) = \{\perp, a_{21}, a_{31}, a_{41}, ...\}$.

(iv) $A^{'} = \{\perp\}$.

Note that the only set that contains $\perp$ in its interior is X itself.

2.2 Generalized Open Sets on T_o-Alexandroff Spaces

Generalized open sets such as preopen, semi-open, α-open, γ-open, and some other concepts have been considered. The name of preopen was used the first time by Mashhour, Abd El-Monsef, and El-Deeb [2]. The definition of a preopen set was introduced by Corson and Michael [11] under the name of locally dense. The other types of generalization of open sets such as α-open, semi-open, β-open, b-open were studied by Njastad, Levine, and others. They introduced the concepts of preclosed, semi-closed, α-closed, then they derived the semi-interior, semi-closure, pre-interior, pre-closure, α-interior, and α-closure [12].

Definitions 2.2.1. A subset A of a space (X, τ) is called

(*i*) *semi-open set* [20] if $A \subseteq \overline{A^o}$, and *semi-closed set*[22] if A^c is semi-open. Thus A is semi-closed if and only if $\overline{A}^o \subseteq A$. If A is both semi-open and semi-closed then A is called *semi-regular*[7].

(*ii*) *preopen set* [2] if $A \subseteq \overline{A}^o$, and *preclosed set* [19] if A^c is preopen. Thus A is preclosed if and only if $\overline{A^o} \subseteq A$.

(*iii*) α*-open set* [21] if $A \subseteq \overline{A^o}^o$, and α*-closed set* [14] if A^c is α-open. Thus A is α-closed if and only if $\overline{\overline{A}^o} \subseteq A$.

(*iv*) γ*-open set* [4] if A is a union of semi-open and preopen sets.

Notations 2.2.2. For a topological space X and for a subset A of X, the following notations have the following meaning:

(1) $SO(X)$ is the class of all semi-open sets.

(2) $PO(X)$ is the class of all preopen sets.

(3) τ_α is the class of all α-open sets.

(4) $\gamma(X)$ is the class of all γ-open sets.

(5) $pInt(A)$ is the largest preopen set contained in A.

(6) $sInt(A)$ is the largest semi-open set contained in A.

(7) $pCl(A)$ is the smallest preclosed set contains A.

(8) $sCl(A)$ is the smallest semi-closed set contains A.

Njastad [21] proved that τ_α is a topology on X. In general, $SO(X)$ and $PO(X)$ need not be topologies on X.

Proposition 2.2.3. *A subset A of a space (X, τ) is*

(*i*) *preopen[18] if and only if $A = U \bigcap D$ where U is open set and D is dense set.*

(*ii*) *preclosed [18] if and only if $A = F \bigcup D^c$ where F is closed and D is dense .*

(*iii*) *α-open [14] if and only if it is semi-open and preopen.*

Definitions 2.2.4. (1) A space (X, τ) is *resolvable*[8] if and only if $X = D \bigcup D^c$ where both D and D^c are dense.

(2) A subset A is *resolvable* [8] if the subspace $(A, \tau|_A)$ is resolvable.

(3) A space (X, τ) is *irresolvable* [8] if it is not resolvable.

(4) A space (X, τ) is *strongly irresolvable* [16] if no nonempty open set is resolvable.

(5) A space (X, τ) is *hereditarily irresolvable* [8] if no nonempty subset is resolvable.

(6) A space (X, τ) is *nodec* [17] if all nowhere dense sets are closed.

(7) A space (X, τ) is *hyperconnected* [17] if every open subset of X is dense .

(8) A space (X, τ) is *hyperdisconnected* if it is not *hyperconnected*.

(9) A space (X, τ) is *submaximal* [13] if each dense subset is open.

(10) A space (X, τ) is *disconnected* if and only if there are disjoint nonempty open sets H and K such that $X = H \bigcup K$. If X is not disconnected, we say that X is *connected*.

(11) A subset A is *connected* if the subspace $(A, \tau|_A)$ is connected.

(12) Let (X, τ) be a topological space. If $x \in X$, the largest connected subset C_x of X containing x is called the *component* of x.

Remark 2.2.5. The following implications hold in any topological space:

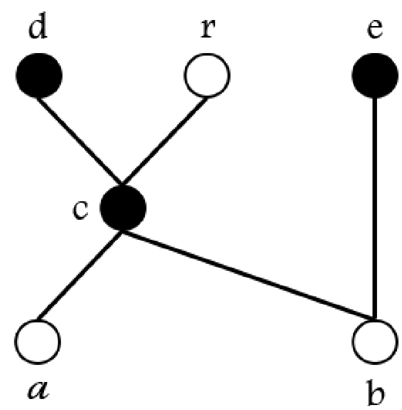
submaximal⇒ hereditarily irresolvable⇒ strongly irresolvable⇒ irresolvable

Proposition 2.2.6. *[12] Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space. If A is a preopen set, then each maximal element in A belongs to M; that is, $M(A) \subseteq M$.*

Proof. If A is preopen, by definition we have $A \subseteq \overline{A}^o$. Let $x \in M(A)$. Since X satisfies the ACC, a maximal element $y \in X$ exists such that $x \leq y$. If $x \neq y$, then $y \notin A$. Therefore by Theorem 2.1.4 part(2), we have $y \notin \overline{A}$, and hence $y \notin \overline{A}^o$. But $x \in A \subseteq \overline{A}^o$ so $y \in \uparrow x \subseteq \overline{A}^o$ which is a contradiction. Thus $x = y$. □

The following example shows that the converse of the above proposition is not true; that is, if each maximal element of A is maximal in X, then A need not be preopen.

Example 2.2.7. [12] Let $X = \{a, b, c, d, r, e\}$ be a poset with the partial order as shown in the figure



Let $A = \{c, d, e\}$, then $M(A) = \{d, e\}$ and $M = \{d, r, e\}$. So the maximal elements of A are in M. Note that $\overline{A} = \downarrow d \bigcup \downarrow e = \{a, b, c, d, e\}$, $\overline{A}^{o} = \{d, e\}$. Hence $A \nsubseteq \overline{A}^{o}$. Therefore A is not preopen.

Theorem 2.2.8. *[12] Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space. Then the set A is preclosed if and only if $\downarrow x \subseteq A$ for all $x \in A \bigcap M$.*

Proof. $(\Rightarrow)$ Suppose that A is preclosed, then by Proposion 2.2.3 we have $A = F \bigcup D^c$ where F is closed and D is dense. Since D is dense then $M \subseteq D$. Let $x \in A \bigcap M$ so $x \in F \bigcup D^c$ and $x \in M$ and this implies that $x \in F$. So $\downarrow x \subseteq F \subseteq A$ for all $x \in A \bigcap M$.

$(\Leftarrow)$ Suppose that for all $x \in A \bigcap M$, $\downarrow x \subseteq A$. Set $F = \bigcup\{\downarrow x : x \in A \bigcap M\}$ and $D^c = A \backslash F$ so $F \subseteq A$ which is closed and $M \bigcap D^c = \phi$ and hence D is dense. Moreover $A = F \bigcup D^c$ and therefore A is preclosed. □

Corollary 2.2.9. *[12] Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space. Then*

(i) the set A is preopen if and only if $\downarrow x \bigcap A = \phi$ for all $x \in A^c \bigcap M$. Equivalently, A is preopen if and only if $\hat{x} \subseteq A$ for all $x \in A$.

(ii) if X contains a top element $\top$, then a nonempty subset A is preopen if and only if $\top \in A$ if and only if A is dense.

Proof. (i) $(\Rightarrow)$ Let $x \in A$, and suppose to contrary that $\hat{x} \bigcap A^c \neq \phi$. Let $y \in \hat{x} \bigcap A^c$ then $y \in M \bigcap A^c$ and $x \in \downarrow y \bigcap A$. Hence A^c is not preclosed and A is not preopen, this is a contradiction. Therefore $\hat{x} \bigcap A^c = \phi$.

($\Leftarrow$) Suppose that $\hat{x} \subseteq A$ for all $x \in A$ and let $y \in A^c \bigcap M$. If $\downarrow y \bigcap A \neq \phi$ then there exists $x \in A$ such that $y \in \hat{x}$ and hence $\hat{x} \nsubseteq A$ this is a contradiction.

(ii) ($\Rightarrow$) Suppose that A is preopen and let $x \in A$. Then by part(1)we have $\hat{x} = \{\top\} \subseteq A$ then A is dense.

($\Leftarrow$) Suppose that A is dense then $\overline{A} = X$, so we have that $A \subseteq \overline{A}^o$. Hence A is preopen.

□

As a consequence of this theorem, we get the following result of [4] as a corollary.

Corollary 2.2.10. *[4] In a finite T_0-space X. Every preopen set in X is semi-open. The converse may be not true.*

Proposition 2.2.11. *[4] In a finite T_0-space X, semi-open sets coincide with γ-open sets.*

The following implications show the relation between these notions in finite T_0-spaces.

$$\begin{array}{ccc} \text{openness} & \Longrightarrow & \text{preopenness} \\ & & \Downarrow \\ \gamma\text{-openness} & \Longleftrightarrow & \text{semi-openness} \end{array}$$

The following examples shows that semi-open set need not be preopen.

Example 2.2.12. Let $X = \{x, y, z\}$ be a set equipped with a topology τ of a minimal neighborhoods $U_x = \{x\}$,$U_y = \{y\}$ and $U_z = X$. If $A = \{y, z\}$. Then $\overline{A}^o = \{y\}$ and $\overline{A^o} = \{y, z\}$. So $A \subseteq \overline{A^o}$ but $A \nsubseteq \overline{A}^o$, hence A is semi-open but not preopen.

The following Theorem is one of the main results of [12] because of its corollaries.

Theorem 2.2.13. *[12] Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space. Then a set A is a semi-open if and only if $M(A) \subseteq M$.*

Proof. $(\Rightarrow)$ Suppose that A is semi-open and let $x \in M(A)$. Since A is semi-open then $A \subseteq \overline{A^o}$ then $x \in \overline{A^o}$. So by Theorem 2.1.4, there exists z which is a maximal in A^o such that $x \in\downarrow z$. Since x is maximal in A, we get $x = z$ and hence $x \in A^o$ so $\hat{x} \subseteq A$. Finally, if $y \in \hat{x}$, then $y \in M$ and $x \leq y$ in A. Since x is a maximal in A, we have $x = y$ in M.

$(\Leftarrow)$ Let $x \in A$, and choose y as maximal in A such that $x \leq y$, so $y \in M$ and $\uparrow y = \{y\} \in A$ which implies that $y \in A^o$ and $x \in\downarrow y \subseteq \overline{A^o}$. Therefore $A \subseteq \overline{A^o}$. Thus, A is semi-open. □

Corollary 2.2.14. *[12] Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space. Then $PO(X) \subseteq SO(X)$; that is if A is preopen then A is semi-open.*

Proof. Suppose that A is preopen, then by Proposition 2.2.6 we have that $M(A) \subseteq M$, and by Theorem 2.2.13, A is semi-open. □

As a consequence of this corollary, we have that if A is preclosed then A is semi-closed. To see this, suppose that A is preclosed, then A^c is preopen and by Corollary 2.2.14, A^c is semi-open. Therefore A is semi-closed.[12]

Corollary 2.2.15. *[12] Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space. Then $PO(X) = \tau_\alpha$. that is, a set A is preopen if and only if A is α-open.*

Proof. If A is preopen, then A is semi-open and hence it is α-open. For the converse, if A is α-open, then it is both semi-open and preopen. Hence we done. □

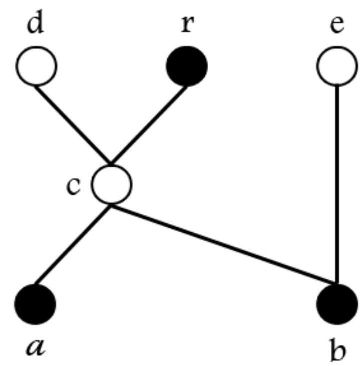
As a consequence of this corollary, a set is α-closed if and only if it is preclosed.

Corollary 2.2.16. *[12] Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space. Then a subset A is a semi-closed if and only if $\forall x \notin A$, there exists $y \in M \backslash A$ such that $x \leq y$. Equivalently, A is semi-closed if and only if $\forall x \notin A, \hat{x} \nsubseteq A$.*

Proof. (⇒) Let A be a semi-closed and let $x \in A^c$. If x is maximal in A^c then $x \in M$ so take $y = x$. In the other case If x is not maximal in A^c, we get a maximal element y in A^c such that $x \leq y$ and by theorem $y \in M \bigcap A^c$.

(⇐) Suppose that $\forall x \in A^c$ there exists $y \in M \bigcap A^c$ such that $x \leq y$. Let z be a maximal element in A^c. Then there exists $y \in M \bigcap A^c$ such that $z \leq y$. But $y \in A^c$ and z is maximal in A^c. Hence we get that $z = y$ and hence $z \in M$ so A^c is semi-open, then A is semi-closed. □

Example 2.2.17. Let $X = \{a, b, c, d, r, e\}$ be a poset with the partial order as shown the figure below:



Let $A = \{a, b, r\}$. If we use the definition of semi-open, $A^c = \{c, d, e\}$ and $\overline{(A^c)^o} = \{a, b, c, d, e\}$. So $A^c \subseteq \overline{(A^c)^o}$. Then A^c is semi-open, and hence A is semi closed. Note that if we use Theorem 2.2.13, we have that A^c is semi-open since $M(A^c) = \{d, e\} \subseteq M$. Therefore A is semi closed.

Proposition 2.2.18. *[12]Let $(X, \tau(\leq))$ be an Artinian T_o−Alexandroff space and let A be a subset of X . Then:*

(*i*) $sCl(A) \subseteq pCl(A)$.

(*ii*) $pInt(A) \subseteq sInt(A)$.

Proof. (*i*) Since $pCl(A)$is preclosed set contains A, so it is semi-closed contains A, and hence $sCl(A) \subseteq pCl(A)$.

(ii) Since $pInt(A)$ is preopen set inside A so it is semi-open in A and hence $pInt(A) \subseteq sInt(A)$.

□

Theorem 2.2.19. *[12] Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space and A is a subset of X. Then*

(i) $pInt(A) = \{x \in A : \hat{x} \subseteq A\}$.

(ii) $sInt(A) = \{x \in A : \hat{x} \bigcap A \neq \phi\}$.

(iii) $pCl(A) = A \bigcup \{\downarrow x : x \in A \bigcap M\}$.

(iv) $sCl(A) = A \bigcup \{x \in A : \hat{x} \subseteq A\}$.

Proof. (i) If $y \in A \bigcap M$ then $\hat{y} = \{y\} \subseteq A$ and so $y \in \{x \in A : \hat{x} \subseteq A\}$. Let $y \in M \backslash \{x \in A : \hat{x} \subseteq A\}$, so $y \notin A$. If $z \in \downarrow y$ then $y \in \hat{z}$ which implies that $z \notin \{x \in A : \hat{x} \subseteq A\}$, so $\downarrow y \cap \{x \in A : \hat{x} \subseteq A\} = \emptyset$, and hence by Corollary 2.2.9 part(1), $\{x \in A : \hat{x} \subseteq A\}$ is preopen which is a subset of A. Now if U is preopen in A, and if $z \in U$ such that $z \notin \{x \in A : \hat{x} \subseteq A\}$, then $\hat{z} \cap A^c \neq \emptyset$, say $r \in \hat{z} \cap A^c$, so we get $r \in M \backslash U$, moreover $z \in \downarrow r \cap U$ which contradicts that U is preopen. Therefore $z \in \{x \in A : \hat{x} \subseteq A\}$ and hence $pInt(A) = \{x \in A : \hat{x} \subseteq A\}$.

(ii) Let $z \in M(\{x \in A : \hat{x} \cap A \neq \emptyset\})$ so $\hat{z} \cap A \neq \emptyset$. Let $w \in \hat{z} \cap A$, so $w \in M$ and $\hat{w} = \{w\} \subseteq A$, therefore $w \in \{x \in A : \hat{x} \cap A \neq \emptyset\}$. Since $w \geq z$ and z is maximal in $\{x \in A : \hat{x} \cap A \neq \emptyset\}$, we get that $z = w$, therefore $z \in M$ and $\{x \in A : \hat{x} \cap A \neq \emptyset\}$ is semi-open set contained in A. Now suppose that S is semi-open set contained in A. If $x \in S$ then by ACC there is $y \in M(S)$ such that $x \leq y$, so $y \in M$, hence $\hat{x} \cap A \neq \emptyset$, therefore $S \subseteq \{x \in A : \hat{x} \cap A \neq \emptyset\}$ and hence $sInt(A) = \{x \in A : \hat{x} \cap A \neq \emptyset\}$.

(iii) Let $z \in A \cup \{\downarrow x : x \in A \cap M\}$ where $z \in M$. If $z \notin A$ then $z \in \downarrow x$ for some $x \in A \cap M$ which contradicts that $z \in M$, so $z \in A \cap M$ and hence $\downarrow z \subseteq \cup\{\downarrow x : x \in A \cap M\}$ which implies that $A \cup \{\downarrow x : x \in A \cap M\}$ is preclosed contains A. Let B be another preclosed set contains A, and let $x \in A \cap M$, so $x \in B$ and hence $\downarrow x \subseteq B$, therefore $A \cup \{\downarrow x : x \in A \cap M\} \subseteq B$ and hence $pCl(A) = A \cup \{\downarrow x : x \in A \cap M\}$.

(iv) If $z \notin A \cup \{x : \hat{x} \subseteq A\}$ then $z \notin A$ and $z \notin \{x : \hat{x} \subseteq A\}$, so $\hat{z} \cap A^c \neq \emptyset$. If $r \in \hat{z} \cap A^c$ then $r \notin A$ and $r \notin \{x : \hat{x} \subseteq A\}$ ($r \in M$ so $\hat{r} = \{r\}$). Therefore $\hat{z} \nsubseteq A \cup \{x : \hat{x} \subseteq A\}$ and by Corollary 2.2.16, $A \cup \{x : \hat{x} \subseteq A\}$ is semi-closed contains A. If C is another semi-closed set contains A, and if $y \in \{x : \hat{x} \subseteq A\}$ not in C, then $\hat{y} \subseteq A \subseteq C$. By ACC there exists $r \in M(C^c)$ such that $y \leq r$. Since C^c is semi-open, we get $r \in M$ and so $r \in \hat{y} \cap C^c$ which contradicts that $\hat{y} \subseteq A \subseteq C$. Therefore $A \cup \{x : \hat{x} \subseteq A\} \subseteq C$ and $sCl(A) = A \cup \{x : \hat{x} \subseteq A\}$.

□

Corollary 2.2.20. *Let (X, τ) be a topological space and let $A \subset X$. Then*

(i) *$A \subseteq sCl(A)$.*

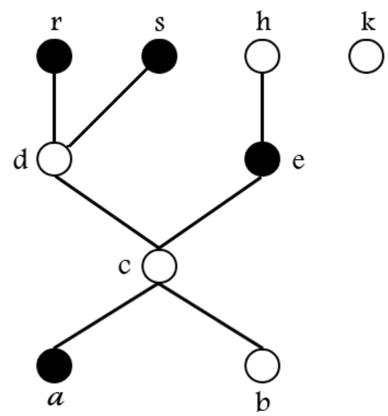
(ii) *A is semi-closed if and only if $A = sCl(A)$.*

Corollary 2.2.21. *Let (X, τ) be a topological space and let $A \subset X$. Then*

(i) *$A \subseteq PCl(A)$.*

(ii) *A is preclosed if and only if $A = pCl(A)$.*

Example 2.2.22. Consider the poset $X = \{a, b, c, d, e, h, k, r, s\}$ with the partial order shown the figure below:



X satisfies the ACC and $M = \{r, s, h, k\}$. Let $A = \{a, e, r, s\}$. Then $\hat{a} = \uparrow a \bigcap M = \{r, s, h\}$, $\hat{e} = \uparrow e \bigcap M = \{h\}$, $\hat{r} = \uparrow r \bigcap M = \{r\}$, $\hat{s} = \uparrow a \bigcap M = \{s\}$. Now $\hat{e} = \{h\} \nsubseteq A$, we get that A is not preopen, and $e \in M(A)$, so $M(A) \nsubseteq M$ and A is not semi-open. Moreover, we have the following :

(1) $\hat{x} \subseteq A$ for all $x = r, s$, then $pInt(A) = \{x \in A : \hat{x} \subseteq A\} = \{r, s\}$.

(2) $\hat{x} \bigcap A \neq \phi$ for all $x = r, s, a$, then $sInt(A) = \{x \in A : \hat{x} \bigcap A \neq \phi\} = \{a, r, s\}$.

(3) $A \bigcap M = \{r, s\}$, then $pCl(A) = A \bigcup \downarrow r \bigcup \downarrow s = A \bigcup \{\downarrow x : x \in A \bigcap M\} = \{a, b, c, e, d, r, s\}$.

(4) $\hat{r} \subseteq A$ for all $x = r, s, d$,then $sCl(A) = A \bigcup \{x \in A : \hat{x} \subseteq A\} = A \bigcup \{r, s\} = \{a, e, d, r, s\}$.

Theorem 2.2.23. *[12] Let $(X, \tau(\leq))$ be a T_o-Alexandroff space. Then all the following are equivalent:*

(1) *X is submaximal.*

(2) *Each element of X is either maximal or minimal; that is, each element of X is pure.*

(3) *X is nodec.*

Proof. $(1 \Rightarrow 2)$ Suppose that X is submaximal, and there exists $x_1 < x_2 < x_3$ in X, then we get a set $X \backslash \{x_2\}$ which is dense subset but not open which is a contradiction.

($2 \Rightarrow 1$) Suppose that each element of X is either maximal or minimal, so X satisfies both ACC and DCC. If U is a dense subset, then by Theorem2.1.4 part(4) we have $M \subseteq U$ so U is up set and so open set (to see this, if $x \in U$, and $y \in X$ such that $x \leq y$, then either $x = y$ or $y \in M \subseteq U$). Therefore X is submaximal.

($2 \Rightarrow 3$) If each element of X is either maximal or minimal, we get X is Artinian, and by Theorem 2.1.4 part (5), a subset A is nowhere dense if and only if $A \bigcap M = \phi$, so $A \subseteq m$ and hence A is closed.

($3 \Rightarrow 2$) If there exist $a, b, c \in X$ such that $a < b < c$ then $\{b\}$ is nowhere dense subset which is not closed, so X is not nodec which is a contradiction.

□

Corollary 2.2.24. *If a T_o-Alexandroff space is submaximal, then it is both Artinian and Noetherian.*

Definition 2.2.25. [21] Let (X, τ) be a topological space. Then the space is *extremally disconnected* if the closure of every open set is open.

By Corollary 2.2.14, a preopen set is semi-open. A semi-open set need not be preopen.But is there a condition on T_o-Alexandroff space that makes $PO(X) = SO(X)$?. The answer of this question is not direct. Njastad [21] showed that $SO(X)$ is a topology if and only if (X, τ) is extremally disconnected,in which case, $SO(X) = \tau_\alpha$. In an Artinian T_o-Alexandroff space, Corollary 2.2.15 shows that $PO(X) = \tau_\alpha$, so we getthe following theorem.[12]

Theorem 2.2.26. *[12] In an Artinian T_o-Alexandroff spaces, the following are equivalent:*

(1) (X, τ) *is extremelly disconnected.*

(2) $PO(X) = SO(X)$.

(3) *For all* $x \in X$, $\mid \hat{x} \mid= 1$, *that is* $\forall x \in X$ *there exists exactly one element* $y \in M$ *such that* $x \leq y$.

Proof. $(1 \Leftrightarrow 2)$ Obvious.

$(2 \Rightarrow 3)$ If there exists $x_0 \in X$ such that $\mid \hat{x_0} \mid\geq 2$, then there are two different elements $y, z \in\uparrow x_0 \bigcap M$ and hence $S = \{x_0, y\}$ is semi-open set that is not preopen.

$(3 \Rightarrow 2)$ Let S be semi-open and x be maximal in S^c. If $\downarrow x \bigcap S \neq \phi$, then there exists $y \in S$ such that $y \leq x$. By ACC of X there exists a maximal element $z \in S$ such that $z \geq y$. Since S is semi-open, $z \in M$ and so $x, z \in \hat{y}$. Therefore $\mid \hat{y} \mid\geq 2$.

□

Theorem 2.2.27. *[12] Let* $(X, \tau(\leq))$ *be a* T_o*−Alexandroff space. Then*

(i) *if* X *is a chain, then* X *is hyperconnected.*

(ii) *if* X *contains a maximum element* $\top$ *, then* X *is hyperconnected.*

(iii) *if* X*is Artinian, then* X *is hyperconnected if and only if* X *contains a top element* $\top$.

Proof. (i) Let $U \in \tau(\leq))$ be a nonempty open set, so U is an up set. Let $x \in U \subseteq \overline{U}$, and let $y \in X$. Since X is a chain, either $x \leq y$ or $x \geq y$. If $x \leq y$ then $y \in U \subseteq \overline{U}$, since U is an up set, and if $x \geq y$, then $y \in \overline{U}$, since $\overline{U}$ is a down set. Therefore $\overline{U} = X$. Hence U is dense.

(ii) If U is open set contains $\top$, then $\downarrow \top \subseteq \overline{U}$. Therefore $\overline{U} = X$.

(*iii*) ($\Rightarrow$) Suppose by contrapositive that $\mid M \mid\geq 2$, so there exist two elements $x_1 \neq x_2$ in M, and hence $\{x_1\}$ is open subset which is not dense.

($\Leftarrow$) If $M = \{\top\}$, then M is a subset of every open set, which implies by Theorem 2.1.4 part (4) every open set is dense, and hence X is hyperconnected.

□

2.3 The Dimension In Artinian T_o-Alexandroff Spaces

Let $(P, \leq)$ be a poset. If C is an infinite chain in P, we set the length of C - denoted by $\ell(C)$- to be ∞. If $C = \{c_0, c_1, c_2, c_3, \cdot\cdot\cdot, c_n\}$ is a finite chain, with $|C| = n + 1$, then we define the length of C to be n. If a poset P contains an infinite chain, we define the length of the poset P - denoted by $\ell(P)$- to be ∞. If all chain in P are finite, we define $\ell(P) = sup\{\ell(C) : C \text{ is a chain in } P\}$.

Example 2.3.1. (1) Let $\mathbb{R}$ be the set of real numbers with usual order $(\leq)$, since $[0, 1)$ is infinite chain in $\mathbb{R}$, then $\ell([0, 1)) = \infty$, while $\ell\{0, 1\} = 1$. Now, $\ell(\mathbb{R}) = \infty$.

(2) In Example 2.1.12, we have

$\ell(P) = \sup\{n : n \in \mathbb{N}\} = \infty$.

Definition 2.3.2. [10] Let X be a topological space. We say $dimX = -1$ if and only if $X = \phi$. Let n be a positive integer such that the dimension is defined for each $k \leq n - 1$. Then $dimX = n$ if X has a base β such that $dim(bd(B)) \leq n - 1$ for all $B \in \beta$, and there exists B_0 with $dim(bd(B_0)) = n - 1$.

Remark 2.3.3. (1) If $dim(bd(B)) \leq n - 1$ for all $B \in \beta$, then we must have $dimX \leq n$.

(2) For an Alexandroff Space X, each base β contains the minimal base $\{\uparrow x : x \in X\}$ of X, so we can use the minimal base of X to find $dimX$.

Example 2.3.4. Let X be a discrete space. Consider the base β consists of singleton sets. Each set $B \in \beta$ is a singleton set with $bd(B) = \emptyset$ so $dim(bd(B)) = -1$ for all $B \in \beta$. Hence $dimX = 0$.

Proposition 2.3.5. *Let X be a T_o-Alexandroff Space. If the minimal base β has the property that $dim(bd(\uparrow x)) \leq n-1$ for each $x \in B$ and if there exists $x_0 \in X$ with $dim(bd(\uparrow x_0)) = n-1$ then $dimX = n$.*

Corollary 2.3.6. *[4] In finite T_o-space X, if the minimal base β fulfils $dim(bd(U_x)) \leq n-1$ for each $U_x \in \beta$, $dim(X) \leq n$.*

Proposition 2.3.7. *A T_o-Alexandroff Space X has dimension zero if and only if X has abase of clopen sets.*

Proof. ($\Rightarrow$) Suppose X has dimension zero then by definition of dimension, there is a base β where, $bd(B) = \phi$ for all $B \in \beta$. Since $bd(B) = bd(B^c) = \phi$, so we have B and B^c are closed and so B is clopen set .
($\Leftarrow$) Suppose X has abase of clopen set. It clear $bd(B) = \phi$ if and only if B is clopen set. Since dimension zero means existence of a base with $bd(B) = \phi$ we are done. □

Proposition 2.3.8. *Every Artinian T_o-Alexandroff Space X with base β of clopen sets is discrete.*

Proof. Suppose $(X, \leq)$ is the corresponding poset. Let $x, y \in X$. If $x \neq y$ and $x \leq y$ then $V(y) = \uparrow y \in \beta$ is not clopen, which is a contradiction. So, we have $(X, \leq)$ is anti-chain and hence X is discrete. □

By above proposition we have the following corollary in T_o-finite space X:

Corollary 2.3.9. *[4] Every finite T_o-space with base β of clopen sets is discrete.*

Finiteness and T_o-axiom are necessary, as example illustrates:

Example 2.3.10. [4] Let $X = \{x, y, z\}$ with a topology τ of a minimal neighborhoods $U_x = \{x\}$ and $U_y = U_z = \{y, z\}$. The topology has a base $\{\{x\}, \{y, z\}\}$ with clopen sets, although X is not discrete because X is not T_o .

Theorem 2.3.11. *Let X be a T_o-Alexandroff Space. Then X has zero-dimensional space if and only if X is discrete.*

Proof. Let X be a T_o-Alexandroff Space.

($\Rightarrow$) Suppose X has zero-dimensional. Then by Proposition 2.3.7 X has a base of clopen sets, and by Proposition 2.3.8, X is discrete .

($\Leftarrow$) Follows from Example 2.3.4. □

Lemma 2.3.12. *Let $(X, \tau(\leq))$ be a T_o-Alexandroff space and let $X = C \bigcup V$ such that $c \in C$ is minimal and $v \in V$ is maximal. Then each of C and V is a discrete subspace of X.*

Proof. Each $c \in C$ is minimal point in C, so it is closed. Since arbitrary union of closed sets is closed, $C\backslash\{c\}$ is also closed. So $\{c\}$ is an open set in C. Thus C is a discrete subspace of X. Similarly each $v \in V$ is maximal in V, so it is open, and $V\backslash\{v\}$ is also open, since it is union of open sets, and hence $\{v\}$ is closed. Thus, V is a discrete. □

As a consequence of the preceding lemma, we have the following result of [4] as a corollary.

Corollary 2.3.13. *[4] Let X be a finite T_o-space and let $X = C \bigcup V$ such that $c \in C$ is closed and $v \in V$ is open. Then each of C and V is a discrete subspace of X.*

Theorem 2.3.14. *Let $(X, \tau(\leq))$ be a T_o-Alexandroff space. Then $dimX \leq 1$ if and only if every singleton in X is either maximal or minimal. Moreover, if there exist two distinct $x, y \in X$ such that $x \leq y$, then $dimX = 1$.*

Proof. ($\Rightarrow$) Suppose to contrary that, X has three distinct points $x < y < z$ ($x < y$ means $x \leq y$ and $x \neq y$). By Corollary 2.1.2, $bd(\uparrow z) = \downarrow z - \{z\}$ so $x, y \in bd(\uparrow z)$.

Since $dim X \leq 1$, we get either $dim(bd(\uparrow z)) = 0$ or $bd(\uparrow z) = \phi$. But $x, y \in bd(\uparrow z)$, so $bd(\uparrow z) \neq \phi$ and hence $dim(bd(\uparrow z)) = 0$, then $bd(\uparrow z)$ is discrete subspace. Therefore the induced order on $bd(\uparrow z)$ is the anti chain. But $x, y \in bd(\uparrow z)$ and $x < y$ which is a contradiction.

($\Leftarrow$) If each singleton is either maximal or minimal, then the elements of $bd(\uparrow x)$ is either minimal in X or $bd(\uparrow x)$ is empty set, and hence $dim(bd(\uparrow x)) = 0$ or $dim(bd(\uparrow x)) = -1$. Thus, $dim X \leq 1$.

Now, if there exist two distinct points $x \leq y$ in X, then $bd\{\uparrow y\} = \downarrow y - \{y\}$ is minimal points with discrete topology so $dim(bd\{\uparrow y\}) = 0$. Thus by definition of dimension, $dim X = 1$. □

Remark 2.3.15. In [12], Mahdi, proves that X is submaximal if and only if each singleton is either maximal or minimal if and only if X is $T_{1/2}-$space. So, in T_o-Alexandroff space, $dim X \leq 1$ if and only if X is submaximal space.

Corollary 2.3.16. *[4] Let X be a a finite T_o-space. Then $dim X \leq 1$ if and only if every singleton in X is either open or closed.*

Chapter 3

CONTINUITY IN ARTINIAN T_o-ALEXANDROFF SPACES

In this chapter , we are looking at several types of continuity related to generalized open sets. Moreprecisely, we study continuity, precontinuity, semi-continuity, α-open and γ-open. We characterize then on Artinian T_o-Alexandroff Spaces. After that, we study two types of functions called upper semicontinuous multifunctionand upper semicontinuous multifunction.

3.1 Some Weaker Forms of Continuity in Artinian T_o-Alexandroff spaces

In [12], the authors studied some of generalized open sets such as preopen, semi-open and α-open. They characterize each type of them in T_o-Alexandroff Spaces. In this section, we will apply these characterize on continuities such as semi-continuity, precontinuity and α-continuity.

Definitions 3.1.1. Let $f : X \to Y$ be a function from a topological space X to a topological space Y, and let $\mathcal{A}$ be a collection of subsets of X we say that the function f is $\mathcal{A}$-*continuous* if $f^{-1}(V) \in \mathcal{A}$, for all open set V in Y.

Remark 3.1.2. As special cases of the above definition, we have that

(i) f is *continuous* if $\mathcal{A} = \tau$.

(ii) f is *precontinuous* [18] if $\mathcal{A} = PO(X)$.

(iii) f is *semicontinuous* [20] if $\mathcal{A} = SO(X)$.

(iv) f is γ-*continuous*[3] if $\mathcal{A} = \gamma(X)$.

(v) f is α-*continuous*[3] if $\mathcal{A} = \tau_\alpha$.

In general, there is no relation between different types of continuity. In the case, where X and Y are T_o-finite spaces, El-Attik, prove that $\tau \subseteq PO(X) \subseteq SO(X) = \gamma(X)$, so we have the following implications:

$$\begin{array}{ccc} \text{continuous} & \Longrightarrow & \text{precontinuous} \\ & & \Downarrow \\ \gamma\text{-continuous} & \Longleftrightarrow & \text{semicontinuous} \end{array}$$

The following example shows that γ-continuous need not precontinuous in general.

Example 3.1.3. Let $X = Y = \{x, y, z\}$ be two topological spaces where τ_x has the minimal neighborhoods $U_x = \{x\}$, $U_y = \{y\}$ and $U_z = \{x, y, z\}$, and τ_y has the minimal neighborhoods $V_x = \{x\}$, $V_y = \{x, y\}$ and $V_z = \{x, y, z\}$. Suppose that $f : X \longrightarrow Y$ be a function defined by $f(x) = x$, $f(y) = z$ and $f(z) = y$. Then we have the following

(i) $f^{-1}(V_x) = \{x\} \in \gamma(X)$.

(ii) $f^{-1}(V_y) = \{x, z\} \in \gamma(X)$.

(iii) $f^{-1}(V_z) = X \in \gamma(X)$.

Then we have f is γ-continuous. Moreover $\overline{f^{-1}(V_y)} = \overline{\{x,z\}} = \{x,z\}$ and $(\overline{f^{-1}(V_y)})^o =$ $\{x\}$. Then $f^{-1}(V_y)$ is not preopen, because $f^{-1}(V_y) \nsubseteq (\overline{f^{-1}(V_y)})^o$. Therefore f is not precontinuous.

Mahdi and Elatrash in [12] prove that in Artinian T_o-Alexandroff spaces, $PO(X) = \tau_\alpha \subseteq SO(X)$. Hence we can get easily the following implication on Artinian T_o-Alexandroff spaces.

$$\begin{array}{ccc} \text{continuous} & \Longrightarrow & \text{precontinuous} \\ & & \Updownarrow \\ \text{semicontinuous} & \Longleftarrow & \alpha\text{-continuous} \end{array}$$

Theorem 3.1.4. *Let X and Y be Artinian T_o-Alexandroff spaces, then the function $f : X \to Y$ is continuous at x if and only if $f(\uparrow x) \subseteq \uparrow f(x)$; that is, for all $x, a \in X$, if $a \geq x$ then $f(a) \geq f(x)$.*

Proof. $(\Rightarrow)$ Let $f : X \longrightarrow Y$ be a continuous function, and let $x \in X$. Since $\uparrow f(x)$ is a neighborhood of $f(x)$, then by continuity of f, we have $f^{-1}(\uparrow f(x))$ is open neighborhood of x. Since $\uparrow x$ is the smallest neighborhood of x, so we have $\uparrow x \subseteq f^{-1}(\uparrow f(x))$ and hence $f(\uparrow x) \subseteq \uparrow f(x)$.

$(\Leftarrow)$ Let $x \in X$ be arbitrary, and let W be an open neighborhood of $f(x)$. Then $\uparrow f(x) \subseteq W$, so by assumption $f(\uparrow x) \subseteq \uparrow f(x) \subseteq W$. Hence $\uparrow x \subseteq f^{-1}(W)$, which implies that f is continuous. □

Corollary 3.1.5. *[4] Let X and Y be T_o-finite topological spaces, then the function $f : X \to Y$ is continuous at x if and only if $f(U_x) \subseteq U_{f(x)}$.*

Theorem 3.1.6. *(i)Let $\{O_\alpha : \alpha \in I\}$ be a collection of semi-open sets in X. Then $\bigcup\{O_\alpha : \alpha \in I\}$ is semi-open set in X.*

(*ii*) *Let* $\{B_\alpha : \alpha \in I\}$ *be a collection of preopen sets in* X*. Then* $\bigcup\{B_\alpha : \alpha \in I\}$ *is preopen set in* X*.*

Theorem 3.1.7. *Let* $(X, \tau(\leq))$ *and* $(Y, \tau(\leq))$ *be a* T_o-*Alexandroff spaces, and let* $f : X \to Y$ *be a function. Then the following statements are equivalent.*

(*i*) f *is semi-continuous on* X*.*

(*ii*) $f^{-1}(\downarrow y)$ *is semi-closed for every* $y \in Y$*.*

(*iii*) $f^{-1}(\uparrow y)$ *is semi-open in* X *for every element* $\{\uparrow y : y \in Y\}$ *of a base for* Y*.*

(*iv*) *For every* $p \in X$ *and* $\uparrow f(p)$ *is an open set in* Y *containing* $f(p)$*, then there exists a semi-open set* O *in* X *such that* $p \in O$ *and* $f(O) \subseteq\uparrow f(p)$*.*

(*v*) $f(sCl(A)) \subseteq \overline{f(A)}$ *for every subset* A *of* X*.*

(*vi*) $sCl(f^{-1}(B)) \subseteq f^{-1}(\overline{B})$ *for every subset* B *of* Y*.*

Proof. ((*i*) $\Rightarrow$ (*ii*)) Suppose f is semi-continuous on X and since $\downarrow y$ be a closed set in Y, for all $y \in Y$. Then $(\downarrow y)^c$ is an open set in Y. By semi-continuity property of f now implies that $f^{-1}((\downarrow y)^c) = (f^{-1}(\downarrow y))^c$ is semi-open in X. Therefore, by definition of semi-open, $f^{-1}(\downarrow y)$ is semi-closed in X.

((*ii*) $\Rightarrow$ (*i*)) Assume the statement (*ii*) holds. Let $y \in Y$, then $\uparrow y$ is an open set in Y. Then $(\uparrow y)^c$ is closed. Hence by assumption, $f^{-1}((\uparrow y)^c) = (f^{-1}(\uparrow y))^c$ is semi-closed. By definition of semi-open, $f^{-1}(\uparrow y)$ is semi-open in X. Therefore f is semi-continuous on X.

((*i*) $\Rightarrow$ (*iii*)) Since $\{\uparrow y : y \in Y\}$ is a base for Y, and $\uparrow y$ is open set in Y. By semi-continuity property of f, it follows that $f^{-1}(\uparrow y)$ is semi-open in X.

((*iii*) $\Rightarrow$ (*i*)) Suppose $f^{-1}(\uparrow y_\alpha)$ is semi-open in X for every member $\uparrow y_\alpha$ of a base $\{\uparrow y_\alpha : \alpha \in I\}$ for Y. Since $\uparrow y$ is an open set in Y. Then there exists $J \subset I$

such that $\uparrow y = \bigcup\{\uparrow y_\alpha : \alpha \in J\}$. Hence $f^{-1}(\uparrow y) = \bigcup\{f^{-1}(\uparrow y_\alpha) : \alpha \in J\}$. By Theorem 3.1.6 part (i) $f^{-1}(\uparrow y)$ is semi-open set in X. Therefore f is semi-continuous on X.

$((i) \Rightarrow (iv))$ Assume that f is semi-continuous on X. Let $p \in X$ and $\uparrow f(p)$ is an open set in Y containing $f(p)$. Put $O = f^{-1}(\uparrow f(p))$. Since f is semi-continuous, O is semi-open set in X. Moreover, $p \in O$ and $f(O) = f(f^{-1}(\uparrow f(p))) \subseteq \uparrow f(p)$. Therefore, (iv) holds.

$((iv) \Rightarrow (v))$ Suppose statement (iv) holds. Let A be a subset of X, $p \in sCl(A)$, and $\uparrow f(p)$ is an open set in Y containing $f(p)$. Then by assumption, there exists a semi-open set O containing p such that $f(O) \subseteq \uparrow f(p)$. Now, since $p \in sCl(A)$, $O \bigcap A \neq \phi$. Consequently $\phi = f(O \bigcap A) \subseteq f(O) \bigcap f(A) \subseteq \uparrow f(p) \bigcap f(A)$. This shows that $f(p) \in \overline{f(A)}$. Therefore $f(sCl(A)) \subseteq \overline{f(A)}$, and so (v) holds.

$((v) \Rightarrow (vi))$ Suppose statement (v) holds. Let $B \subseteq Y$ and set $A = f^{-1}(B)$. Then by assumption $f(sCl(A)) \subseteq \overline{f(A)}$. Therefore

$$sCl(f^{-1}(B)) = sCl(A) \subseteq f^{-1}(f(sCl(A))) \subseteq f^{-1}(\overline{f(A)}).$$

But $f^{-1}(\overline{f(A)}) = f^{-1}(\overline{f(f^{-1}(B))})$, then $sCl(f^{-1}(B)) \subseteq f^{-1}(\overline{B})$. This shows that (vi) holds.

$((vi) \Rightarrow (ii))$ Suppose statement (vi) holds. Since $\downarrow y$ be a closed set in Y for all $y \in Y$. By assumption $sCl(f^{-1}(\downarrow y)) \subseteq f^{-1}(\overline{\downarrow y}) = f^{-1}(\downarrow y)$. By Corollary2.3.12 part (i) we get $sCl(f^{-1}(\downarrow y)) = f^{-1}(\downarrow y)$. Therefore, by Corollary2.3.12 part (ii), $f^{-1}(\downarrow y)$ is semi-closed in X. This shows that (ii) holds.

□

Theorem 3.1.8. *Let $(X, \tau(\leq))$ and $(Y, \tau(\leq))$ be a T_o-Alexandroff spaces, and let $f : X \to Y$ be a function. Then the following statements are equivalent.*

(i) *f is precontinuous on X.*

(ii) $f^{-1}(\downarrow y)$ *is preclosed for every* $y \in Y$*.*

(iii) $f^{-1}(\uparrow y)$ *is preopen in* X *for every element* $\{\uparrow y : y \in Y\}$ *of a base for* Y*.*

(iv) For every $p \in X$ *and* $\uparrow f(p)$ *is an open set in* Y *containing* $f(p)$*, then there exists a preopen set* O *in* X *such that* $p \in B$ *and* $f(B) \subseteq \uparrow f(p)$*.*

(v) $f(pCl(A)) \subseteq \overline{f(A)}$ *for every subset* A *of* X*.*

(vi) $pCl(f^{-1}(B)) \subseteq f^{-1}(\overline{B})$ *for every subset* B *of* Y*.*

Proof. Mimic to Proof Theorem 3.1.7 □

Theorem 3.1.9. *Let* X *be an Artinian* T_o-*Alexandroff space and let* Y *be a* T_o-*Alexandroff space. The function* $f : X \to Y$ *is precontinuous if and only if for all* $x \in X$*,* $f(\hat{x}) \subseteq \uparrow f(x)$*.*

Proof. ($\Rightarrow$) Suppose f is precontinuous. Since for all $x \in X$, $\uparrow f(x)$ is open set in Y. Then by precontinuity of f we have that $f^{-1}(\uparrow f(x))$ is preopen set in X for all $x \in X$, and by Corollary 2.2.9 part(i) we have $\hat{x} \subseteq f^{-1}(\uparrow f(x))$ for all $x \in X$. Therefore $f(\hat{x}) \subseteq \uparrow f(x)$ for all $x \in X$.

($\Leftarrow$) Suppose that $f(\hat{x}) \subseteq \uparrow f(x)$ for all $x \in X$ and suppose that W is an open set in Y. Let $x \in f^{-1}(W)$, so $f(x) \in W$ and hence $\uparrow f(x) \subseteq W$. Therefore $f(\hat{x}) \subseteq \uparrow f(x) \subseteq W$ and hence $\hat{x} \subseteq f^{-1}(W)$. Since $x \in f^{-1}(W)$ is arbitrary, by Corollary 2.2.9, $f^{-1}(W)$ is preopen set in X.

□

Theorem 3.1.10. *Let* X *be an Artinian* T_o-*Alexandroff space and let* Y *be a* T_o-*Alexandroff space. The function* $f : X \to Y$ *is semicontinuous if and only if for all* $x \in X$*, there is* $y \in \hat{x}$ *such that* $f(y) \in \uparrow f(x)$*. In equivalent form, for all* $x \in X$*,* $f(\hat{x}) \bigcap \uparrow f(x) \neq \phi$*.*

Proof. (⇒) Let $x \in X$ be arbitrary. Since $\uparrow f(x)$ is open in Y, $f^{-1}(\uparrow f(x))$ is semi-open set in X containing x. Since X is Artinian, there exists $y \in M(f^{-1}(\uparrow f(x)))$ such that $x \leq y$. By Theorem 2.2.13, $M(f^{-1}(\uparrow f(x))) \subseteq M$. Thus $y \in M$ and $x \leq y$. Therefore $y \in \hat{x}$, and $f(y) \in \uparrow f(x)$.

(⇐) Let W be an open set in Y, and let $x \in M(f^{-1}(W))$. By assumption,there exists $y \in \hat{x}$ sauch that , $f(y) \in \uparrow f(x)$. Since $x \in f^{-1}(W)$, we get $f(x) \in W$, and hence $\uparrow f(x) \subseteq W$. So, we get that $f(y) \in W$, and so $y \in f^{-1}(W)$. Now, $x \leq y$ in $f^{-1}(W)$ and x is maximal, so $x = y$. Moreover, $y \in \hat{x} \subseteq M$, we get that $x \in M$. Hence $M(f^{-1}(\uparrow f(x))) \subseteq M$. By Theorem 2.2.13, $f^{-1}(W)$ is semi-open set in X. □

Definition 3.1.11. A function $f : X \to Y$ from a topological space X into a topological space Y is called :

(i) preopen [18] if the image of each open set in X is preopen in Y

(ii) semiopen [20] if the image of each open set in X is semiopen in Y

(iii) γ-open [3] if the image of each open set in X is γ-open in Y

(iv) α-open [3] if the image of each open set in X is α-open in Y

In general, there is no relation between different types of open. In finite T_o-space we have the following implications

$$\begin{array}{ccc} \text{open function} & \Longrightarrow & \text{preopen function} \\ & & \Downarrow \\ \gamma\text{-open function} & \Longleftrightarrow & \text{semiopen function} \end{array}$$

and in Artinian T_o-Alexandroff spaces, we have that

$$\begin{array}{ccc} \text{open function} & \Longrightarrow & \text{preopen function} \\ & & \Updownarrow \\ \text{semiopen function} & \Longleftarrow & \alpha\text{-open function} \end{array}$$

Definitions 3.1.12. If $f : (X, \leq_x) \longrightarrow (Y, \leq_y)$ be a function from a poset X into a poset Y, then f is called

(i) *order-preserving (monotone)* if $x_1 \leq x_2$ in X then $f(x_1) \leq f(x_2)$.

(ii) *order-embedding* if $x_1 \leq x_2$ in X if and only if $f(x_1) \leq f(x_2)$.

(iii) *order-isomorphism* if it is order-embedding from X onto Y.

Theorem 3.1.13. *If $f : (X, \leq_x) \longrightarrow (Y, \leq_y)$ be a function from a poset X into a poset Y, and $\tau(\leq_x)$ and $\tau(\leq_y)$ are the corresponding T_o-Alexandroff spaces, then f continuous in the sense of topology if and only if f is order-preserving.*

Proof. $(\Rightarrow)$ Suppose that $a \leq b$ in X. By Theorem 3.1.5 we have $f(\uparrow a) \subseteq \uparrow f(a)$. Since $b \in \uparrow a$, we get $f(b) \in f(\uparrow a) \subseteq \uparrow f(a)$, then $f(a) \leq f(b)$.
$(\Leftarrow)$ Let $x \in X$ and suppose that $b \in \uparrow x$, then $x \leq b$ and hence $f(x) \leq f(b)$. Therefore $f(b) \in \uparrow f(x)$ which implies that $f(\uparrow x) \subseteq \uparrow f(x)$ and by Theorem 3.1.5, f is continuous. $\square$

3.2 Continuity of Multifunction in Artinian T_o- Alexandroff Spaces

Definition 3.2.1. [10] Let X and Y be two nonempty sets and $\mathbf{P}(Y)$ be the power set of Y. A *multifunction* is a function $F : X \to \mathbf{P}(Y)$.

Definition 3.2.2. [4] A multifunction $F : X \to \mathbf{P}(Y)$ from a topological space X into a topological space Y is called:

(1) *upper semicontinuous multifunction* at point $x_o \in X$ if for every open set V in Y such that $F(x_0) \subset V$, there exists an open set U containing x_o such that $F(U) = \bigcup_{x_i \in U} F(x_i) \subset V$.

(2) *lower semicontinuous multifunction* at point $x_o \in X$ if for every open set V in Y such that $F(x_0) \bigcap V \neq \phi$, there exists an open set U containing x_0 such that $F(x) \bigcap V \neq \phi$ for each $x \in U$.

(3) *continuous multifunction* at a point $x_o \in X$ if it is both upper semicontinuous and lower semicontinuous multifunction at point $x_o \in X$.

Definition 3.2.3. Let $F : X \to \mathbf{P}(Y)$ be a multifunction and we always assume that $F(x) \neq \phi$ for all $x \in X$. For each $G \subseteq Y$, we define upper and lower inverse sets respectively by

$$F(G^+) = \{x \in X : F(x) \subseteq G\} \text{ and}$$
$$F(G^-) = \{x \in X : F(x) \bigcap G \neq \phi\}.$$

Clearly $F(G^+) \subseteq F(G^-)$ for each subset G of Y. Moreover $F(Y^+) = F(Y^-) = X$

Theorem 3.2.4. *Let $F : X \to \boldsymbol{P}(Y)$ be a multifunction. Then*

(i) F is upper semicontinuous multifunction if and only if $F(G^+)$ is open, for each open set G in Y.

(ii) *F is lower semicontinuous multifunction if and only if $F(G^-)$ is open, for each open set G in Y.*

Proof. (i) ($\Rightarrow$) Let $G \in \tau_Y$ and let $x \in F(G^+)$, so $F(x) \subseteq G$. Since F is upper semicontinuous multifunction, there exists an open set U_x in X containing x such that $F(U_x) \subseteq G$. Hence we get $U_x \subseteq F(G^+)$. Therefore $F(G^+) = \bigcup_{x \in F(G^+)} U_x$, arbitrary union of open sets in X, is open.

($\Leftarrow$)Let $x \in X$ be arbitrary, and let V be any open subset in Y such that $F(x) \subseteq V$. So $x \in F(V^+)$. By assumption $F(V^+)$ is open in X, we get $U = F(V^+) \in \tau_X$ and for all $x_i \in U$, $F(x_i) \subseteq V$. Therefore F is upper semicontinuous multifunction.

(ii) ($\Rightarrow$) Let $G \in \tau_Y$ and let $x \in F(G^-)$, so $F(x) \bigcap G \neq \phi$. Since F is lower semicontinuous multifunction, there exists an open set U_x in X containing x such that $F(x_i) \bigcap G \neq \phi$ for each $x_i \in U_x$. Thus $U_x \subseteq F(G^-)$. Therefore $F(G^-) = \bigcup_{x \in F(G^-)} U_x$, arbitrary union of open sets in X is open.

($\Leftarrow$) Let $x \in X$ be arbitrary, and let V be any open subset in Y such that $F(x) \bigcap V \neq \phi$. So $x \in F(V^-)$. By assumption $U = F(V^-)$ is open in X containing x and for all $x_i \in U$, $F(x_i) \bigcap V \neq \phi$, we get that F is lower semicontinuous multifunction.

□

Theorem 3.2.5. *Let $F : X \to \boldsymbol{P}(Y)$ be a multifunction. Then F is continuous if and only if $F(G_1) \bigcap F(G_2) \in \tau_X$ for each open sets G_1, G_2 in Y.*

Proof. ($\Rightarrow$) Suppose that F is both upper semicontinuous and lower semicontinuous multifunction, and let G_1, G_2 be open sets in Y. By Theorem 3.2.4, $F(G^+)$ and $F(G^-)$ are open in X. Hence $F(G^+) \bigcap F(G^-)$ is open set in X.

($\Leftarrow$) Let G be open in Y. Since $F(Y^+) = F(Y^-) = X$, we get by assumption

$F(G^+) = F(G^+) \bigcap F(Y^-)$ is open in X, and $F(G^-) = F(G^-) \bigcap F(Y^+)$ is open in X. Hence F is both upper semicontinuous and lower semicontinuous multifunction. $\square$

Theorem 3.2.6. *Let X and Y be T_o-Alexandroff spaces, and let $F : X \to \boldsymbol{P}(Y)$ be a multifunction. Then*

(i) F is upper semicontinuous multifunction at x if and only if for all $y \geq x$, $F(y) \subseteq \uparrow F(x)$.

(ii) F is lower semicontinuous multifunction at x if and only if $F(y) \bigcap \uparrow z \neq \phi$ for all $y \in \uparrow x$ and $z \in F(x)$.

Proof. $(i)(\Rightarrow)$ Let F be upper semicontinuous multifunctionat $x \in X$. Let $V = \bigcup_{y \in F(x)} \uparrow y = \uparrow F(x)$, which is open set in Y. Since $F(x) \subseteq V$, by definition of upper semicontinuous multifunction, there exists an open set U containing x such that $\bigcup_{x_i \in U} F(x_i) \subseteq V$. Now $x \in U$, $\uparrow x \subset U$ and hence if $y \geq x$ then $y \in U$. Therefore $F(y) \subseteq V = \uparrow F(x)$.

$(\Leftarrow)$ Suppose for all $y \geq x$, $F(y) \subseteq \uparrow F(x)$. Now, let V be any open set in Y such that $F(x) \subseteq V$, so $\uparrow F(x) \subseteq V$. Take $U = \uparrow x$, we get $x \in U$ and for all $y \in U$, $y \geq x$ and hence $F(y) \subseteq \uparrow F(x) \subseteq V$. Hence F is upper semicontinuous multifunction.

$(ii)(\Rightarrow)$ Let F be lower semicontinuous multifunctionat $x \in X$. Then by definition of lower semicontinuous multifunction, we have that for every open set V in Y such that $F(x) \bigcap V \neq \phi$, there exists an open set U in X with $x \in U$ and $F(y) \bigcap V \neq \phi$ for all $y \in U$. Now for each $z \in F(x)$, take for a special case $V = \uparrow z$. Since $F(x) \bigcap V \neq \phi$, there is an open set U in X with $x \in U$ and $F(y) \bigcap \uparrow z \neq \phi$ for all $y \in \uparrow x$.

$(\Leftarrow)$ Let V be an arbitrary open set in Y with $F(x) \bigcap V \neq \phi$. Take $z \in$

$F(x)\bigcap V$. Then $\uparrow z \subset V$. Choose U to be $\uparrow x$. So from the given, for all $y \in\uparrow x$, $\phi \neq F(y)\bigcap \uparrow z \subset F(y)\bigcap V$. Then $F(y)\bigcap V \neq \phi$ for all $y \in\uparrow x$. Hence F is lower semicontinuous multifunction.

□

As a consequence of this theorem, we get the following result of [4] as a corollary.

Corollary 3.2.7. *[4] For a finite topological spaces X and Y and multifunction $F : X \to \boldsymbol{P}(Y)$, we have:*

(*i*) *F is upper semicontinuous multifunction at x if and only if* $\bigcup\limits_{z\in\ U_x} F(z) \subset \bigcup\limits_{y\in F(x)} U_y$.

(*ii*) *F is lower semicontinuous multifunction at x if and only if $F(x_0)\bigcap U_y \neq \phi$ for all $x_0 \in U_x$ and $y \in F(x)$.*

Theorem 3.2.8. *Let X and Y be T_o-Alexandroff spaces, and let $F : X \to \boldsymbol{P}(Y)$ be a multifunction. Then F is continuous multifunction at $x \in X$ if and only if for all $y \geq x$ we get that*

(*i*) *for all $a \in F(y)$, there exists $z \in F(x)$ such that $a \in\uparrow z$.*

(*ii*) *for all $z \in F(x)$, there exists $a \in F(y)$ such that $a \in\uparrow z$.*

Proof. ($\Rightarrow$) Suppose F is continuous multifunction at $x \in X$, then by Theorem 3.2.6 for all $y \geq x$, $F(y) \subseteq\uparrow F(X)$. If $a \in F(y)$ then $a \in\uparrow F(X)$. Then there exists $z \in F(x)$ such that $a \in\uparrow z$ and $F(y)\bigcap \uparrow z \neq \phi$ for all $z \in F(x)$ so there exist $a_z \in F(y)\bigcap \uparrow z$. $a_z \in F(y)$ and $a_z \in\uparrow z$.
($\Leftarrow$) Suppose that for all $y \geq x$, (i) and (ii) are satisfying. Let $a \in F(y)$ then there exists $z \in F(x)$ such that $a \in\uparrow z \subseteq\uparrow F(x)$ and $F(y) \subseteq\uparrow F(x)$. Hence F is upper semicontinuous multifunction. From (ii) we have for all $z \in F(x)$, there exists $a_z \in F(y)$ such that $a_z \in\uparrow z$. Then $a_z \in F(y)\bigcap \uparrow z$ for all $z \in F(x)$ and

$F(y) \bigcap \uparrow z \neq \phi$. Hence F is lower semicontinuous multifunction. Therefore F is continuous multifunction at $x \in X$. $\square$

Definition 3.2.9. [9] A topological space X is said to be *a connected ordered topological space* if for every three points subset Y in X, there exists $y \in Y$ such that Y meets two connected components of $\{y\}^c$. In other words, for any three points, one of them separates the other two.

Proposition 3.2.10. *[4] Let $X = [0,1]$ with usual topology and let Y be a finite connected ordered topological space. If f is a continuous function from X into itself and if $\pi : X \rightarrow Y$ is a quotient function from X onto Y, then the function g from Y into itself which is defined by $g(y) = \pi f \pi^{-1}(y)$ is continuous.*

Remark 3.2.11. Upper semicontinuous multifunction may be not lower semicontinuous multifunction, and lower semicontinuous multifunction may be not upper semicontinuous multifunction. The following examples illustrate this remark.

Example 3.2.12. [4] Let $X = \{a, b\}$ be the Sierpenski space with minimal neighborhoods $\uparrow a = \{a\}$ and $\uparrow b = \{a, b\}$. Consider the multifunction $F : X \rightarrow \mathbf{P}(X)$ defined by $F(a) = \{a, b\}$ and $F(b) = \{a\}$. Note that

1. F is lower semicontinuous multifunctionat a, since $F(a) \bigcap \uparrow y \neq \phi$, for all $y \in F(a)$.

2. Since $\uparrow b = \{a, b\}$, we have that

    (a) $F(a) \bigcap \uparrow a = \{a, b\} \bigcap \{a\} = \{a\} \neq \phi$.

    (b) $F(b) \bigcap \uparrow a = \{a\} \bigcap \{a\} = \{a\} \neq \phi$.

So by Theorem 3.2.6, we have F is lower semicontinuous multifunction. It is not upper semicontinuous multifunctionsince $\bigcup_{x \in \uparrow b} F(x) = \{a, b\} \nsubseteq \bigcup_{y \in F(b)} \uparrow y = \{a\}$. Therefore F is not upper semicontinuous multifunction.

Now, consider the multifunction $G : X \to p(X)$ defined by $G(a) = \{a\}$, $G(b) = \{a, b\}$. Since $\bigcup_{x\in \uparrow a} G(x) \subseteq\uparrow G(a)$ and $\bigcup_{y\in \uparrow b} G(y) \subseteq\uparrow G(b)$, then G is upper semicontinuous multifunction. It is not lower semicontinuous multifunctionat the point b. Since $\uparrow b = \{a, b\}$, $G(b) = \{a, b\}$ and $G(a) \bigcap \uparrow a = \{a\} \bigcap \{b\} = \phi$.

Example 3.2.13. [4] Let $X = [0, 1]$ with the usual topology, and let $Y = \{a, b, c, d, e\}$ be a set. Let f be a function from X into itself which is defined by

$$f(x) = \begin{cases} 3/4 - 2x, & 0 \leq x \leq 1/4 \\ (1/3)x + 1/6, & 1/4 \leq x \leq 1, \end{cases}$$

and let $\pi : X \to Y$ be the quotient function defined by

$$\pi(X) = \begin{cases} a, & x = 0 \\ b, & 0 < x < 0.5 \\ c, & x = 0.5 \\ d, & 0.5 < x < 1 \\ e, & x = 1. \end{cases}$$

Then Y is a finite connected ordered topological space. The multifunction $g : Y \to P(Y)$ is defined by $g(y) = \pi f \pi^{-1}(y)$. Then $g(a) = \{d\}$, $g(b) = \{b, c, d\}$, $g(c) = \{b\}$, $g(d) = \{b\}$, $g(e) = \{c\}$.

Claim: g is lower semicontinuous multifunction.

(i) g is lower semicontinuous multifunctionat a, since $g(a) = \{d\}$, and $U_a = \{a, b\}$. Then

(a) $g(a) \bigcap U_d = \{d\} \bigcap \{d\} = \{d\} \neq \phi$.

(b) $g(b) \bigcap U_d = \{b, c, d\} \bigcap \{d\} = \{d\} \neq \phi$.

(ii) g is lower semicontinuous multifunctionat b, since $g(b) = \{b, c, d\}$, and $U_b = \{b\}$. Then

(a) $g(b)\bigcap U_b = \{b,c,d\}\bigcap\{b\} = \{b\} \neq \phi$.

(b) $g(b)\bigcap U_c = \{b,c,d\}\bigcap\{c,d\} = \{c,d\} \neq \phi$.

(c) $g(b)\bigcap U_d = \{b,c,d\}\bigcap\{d\} = \{d\} \neq \phi$.

(iii) g is lower semicontinuous multifunctionat c, since $g(c) = \{b\}$, and $U_c = \{c,b\}$. Then

(a) $g(c)\bigcap U_b = \{b\}\bigcap\{b\} = \{b\} \neq \phi$.

(b) $g(b)\bigcap U_b = \{a,b,d\}\bigcap\{a,b,d\} = \{a,b,d\} \neq \phi$.

(iv) g is lower semicontinuous multifunctionat d, since $g(d) = \{b\}$, and $U_d = \{d\}$. Then $g(d)\bigcap U_b = \{b\}\bigcap\{b\} = \{b\} \neq \phi$. Then g is lower semicontinuous multifunctionat d.

(v) g is lower semicontinuous multifunctionat e, since $g(e) = \{c\}$, and $U_e = \{d,e\}$. Then

(a) $g(b)\bigcap U_c = \{b\}\bigcap\{b,c\} = \{b\} \neq \phi$.

(b) $g(e)\bigcap U_c = \{c\}\bigcap\{b,c\} = \{b\} \neq \phi$.

Therefore, from (i) to (v), we have g is lower semicontinuous multifunction. Note that g is not upper semicontinuous multifunction, since the point $y = a$, $U_a = \{a,b\}$, $g(a) = \{d\}$ and $g(U_a) = \{b,c,d\} \nsubseteq \bigcup\limits_{y\in g(a)} U_y = \{d\}$.

Theorem 3.2.14. *Let $\pi : X \to Y$ be a continuous function from a topological space X into a one-dimensional T_o-Alexandroff space Y. If any minimal point y in Y, $\pi^{-1}(y)$ is only one point and for any maximal point $z \in \hat{y}$, $\pi^{-1}(y) \in cl(\pi^{-1}(z))$, then π is an open function.*

Proof. Let W be an open set in X. We want to prove that $\pi(W) = \bigcup\limits_{y\in\pi(W)} \uparrow y$. It is clear that $\pi(W) \subseteq \bigcup\limits_{y\in\pi(W)} \uparrow y$. We want to prove that $\uparrow y \subseteq \pi(W)$ for each

$y \in \pi(W)$. By Theorem 2.3.14 we have two cases:

case 1: If y is a maximal point, then $\uparrow y = \{y\} \subseteq \pi(W)$ and we are done.

case 2: If y is a minimal point, then by the assumption, $\pi^{-1}(y) = x$ is one point. Let z be maximal point such that $z \in \hat{y}$. By hypothesis, $x \in cl(\pi^{-1}(z))$. Since W is an open neighborhood of x, then $\pi^{-1}(z) \bigcap W \neq \phi$. For a point $v \in \pi^{-1}(z) \bigcap W$, we have $z = \pi(v) \in \pi(W)$. Since $\uparrow y = \{y\} \bigcup \hat{y}$, and $z \in \hat{y}$ is arbitrary, Then $\uparrow y \subset \pi(W)$. This completes the proof. □

As a consequence of this theorem, we get the following result of [4] as a corollary.

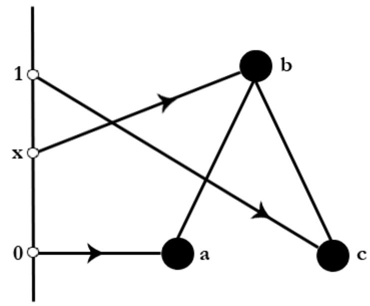
Corollary 3.2.15. *[4] Let $\pi : X \to Y$ be a continuous function from a topological space X into a one-dimensional T_0 finite space Y. If for any closed point in Y, $\pi^{-1}(y)$ is only one point and for any open point $z \in U_y$, $\pi^{-1}(y) \in cl(\pi^{-1}(z))$, then π is an open function.*

The following gives a good example of a function f satisfies the hypothesis of Theorem 3.2.14 which is open and not closed.

Example 3.2.16. [4] Let $X = [0, 1]$ with usual topology and let $Y = \{a, b, c\}$ be a T_o-Alexandroff space with specialization order described as $\uparrow a = \{a, b\}$, $\uparrow b = \{b\}$, and $\uparrow c = \{c, b\}$. Define the function $\pi : X \to Y$ by

$$\pi(x) = \begin{cases} a & x = 0 \\ b & 0 < x < 1 \\ c & x = 1. \end{cases}$$

(see figure below)



Since $\pi^{-1}(\uparrow a) = [0,1)$, $\pi^{-1}(\uparrow c) = (0,1]$ and $\pi^{-1}(\uparrow b) = (0,1)$ are open in X, so π is continuous. Note that $\pi^{-1}(a) = \{0\}$, $\pi^{-1}(c) = \{1\}$. Hence $\pi^{-1}(x)$ is a singleton set for all minimal points of Y. Now the only maximal point of Y is b and $\pi^{-1}(b) = (0,1)$. Therefore $cl(\pi^{-1}(b)) = [0,1]$. Now $b \in \hat{a}$ and $b \in \hat{c}$, so $\{0\} = \pi^{-1}(a) \subseteq [0,1]$ and $\{1\} = \pi^{-1}(c) \subseteq [0,1]$. This implies that π must be open function. We can see directly that f is open if we note that

$$\pi([0,y)) = \{a,b\} = \uparrow a, \quad 0 < y < 1.$$

$$\pi(x,y) = \{b\} = \uparrow b, \quad 0 < x,y < 1.$$

$$\pi((x,1]) = \{c,b\} = \uparrow c, \quad 0 < x < 1.$$

π is not closed because $[1/3, 1/2]$ is closed in X, and $\pi[1/3,1/2] = \{b\}$ is not closed in Y.

Chapter 4

SUPRATOPOLOGY AND ARTINIAN T_o-ALEXANDROFF

4.1 m-set induced by a supratopology

Definition 4.1.1. [1] Let X be a nonempty set. A subclass $\tau^* \subseteq P(X)$ is called a *supratopology* on X if

(i) $X \in \tau^*$.

(ii) τ^* is closed under arbitrary union.

(X, τ^*) is called a *supratopological space*, and the members of τ^* are called *supraopen* sets .

Definition 4.1.2. [1] Let (X, τ) be a topological space and τ^* be a supratopology on X. We call τ^* is a *supratopology associated* with τ if $\tau \subseteq \tau^*$.

Definition 4.1.3. [1] Let (X, τ^*) be a supratopological space. A subset A of X is called an *m-set* with τ^* if $A \bigcap T \in \tau^*$ for all $T \in \tau^*$.

The class of all m-sets with τ^* will be denoted by τ_m.

Remark 4.1.4. [1] Let (X, τ^*) be a supratopological space. If T is any supraopen set of τ^* in X and A is an m-set with τ^*, then $A \bigcap T$ is also a supraopen set.

Lemma 4.1.5. *[23] Let (X, τ^*) be a supratopological space. Then the class τ_m of all m-sets with τ^* is contained in τ^*; that is, $\tau_m \subseteq \tau^*$. Moreover, $X \in \tau_m$.*

Proof. Let A be an m-subset with τ^*. Since X is an element of τ^*, $X \bigcap A = A \in \tau^*$. Since $X \bigcap T = T \in \tau^*$ for any $T \in \tau^*$, $X \in \tau_m$. □

Theorem 4.1.6. *Let (X, τ^*) be a supratopological space. Then the class τ_m of all m-sets with τ^* is a discrete topology if and only if τ^* is a discrete topology on X.*

Proof. ($\Rightarrow$) Since τ_m is the discrete topology and by Lemma 4.1.5, $\tau_m \subseteq \tau^*$, then τ^* is the discrete topology on X.
($\Leftarrow$) Suppose τ^* is the discrete topology on X and let A be any subset of X. Then for all $T \in \tau^*$, $A \bigcap T \in \tau^*$ (since τ^* is the power set). Therefore A is m-set and hence $\tau_m = \mathbb{P}(X)$. □

Theorem 4.1.7. *Let (X, τ^*) be a supratopological space with $\phi \in \tau^*$ and τ_m be the class of all m-sets with τ^*. Then $\tau_m = \tau^*$ if and only if τ^* is a topology on X.*

Proof. ($\Rightarrow$) Suppose $\tau_m = \tau^*$. Since τ^* contains ϕ, X and τ^* is closed under arbitrary union, we need to prove that τ^* is closed under finite intersection, so let $A, B \in \tau^*$. By looking at A as element of τ_m and B as element of τ^*, we have $A \bigcap B \in \tau^*$. Thus τ^* is a topology on X.
($\Leftarrow$) Suppose that τ^* is a topology on X. By Lemma 4.1.5, $\tau_m \subseteq \tau^*$. So let $A \in \tau^*$ be arbitrary and for all $T \in \tau^*$, $A \bigcap T \in \tau^*$ (τ^* is a topology on X). Hence $A \in \tau_m$ which implies that $\tau^* \subseteq \tau_m$. Thus $\tau_m = \tau^*$. □

Example 4.1.8. Let $X = \{a, b, c, d\}$ and $\tau^* = \{\phi, X, \{a\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}\}$. Using Lemma 4.1.5 and the definition of m-set, we find $\tau_m = \{\phi, X, \{a\}, \{b, c, d\}\}$.

Theorem 4.1.9. *[23] Let (X, τ^*) be a supratopological space. Then the class τ_m of all m-sets with τ^* is a supratopology.*

Proof. From Lemma 4.1.5 $X \in \tau_m$. So, let $\{A_\alpha\}$ be a class members of τ_m. By definition of m-set and the supratopology, we have $A_\alpha \bigcap T \in \tau^*$ for all $T \in \tau^*$, then $(\bigcup A_\alpha) \bigcap T = \bigcup(A_\alpha \bigcap T) \in \tau^*$ for all $T \in \tau^*$. Thus $(\bigcup A_\alpha)$ belongs to τ_m and we have τ_m is a supratopology. □

Theorem 4.1.10. *[23] Let (X, τ^*) be a supratopological space with $\phi \in \tau^*$. If a subset A of X is a singleton set and $A \in \tau^*$, then A is an m-set.*

Proof. Since $A \in \tau^*$ is a singleton set then for all $B \in \tau^*$ we have $A \bigcap B$ is either ϕ or $A \in \tau^*$. Thus A is an m-set. □

Theorem 4.1.11. *[23] Let (X, τ^*) be a supratopological space with $\phi \in \tau^*$. Then the class τ_m of all m-subsets of X is a topology on X.*

Proof. Since $\phi \bigcap T = \phi \in \tau^*$ and $X \bigcap T = T \in \tau^*$ for all $T \in \tau^*$, then we have $\phi, X \in \tau_m$. Suppose $A, B \in \tau_m$. By definition of m-set we have $B \bigcap T \in \tau^*$ and $A \bigcap(B \bigcap T) = (A \bigcap B) \bigcap T \in \tau^*$. Thus $A \bigcap B \in \tau_m$. Hence by Theorem 4.1.9, The proof is completed. □

If $\phi \in \tau^*$, then the class τ_m is *m-topology* with τ^* and the members of τ_m are called m-open sets. Moreover (X, τ_m, τ^*) is called *m-topological space*.

Definition 4.1.12. [23] Let (X, τ_m, τ^*) be called m-topological space. A subset B of X is called an *m-closed* set if the complement of B is an m-open set.

In any m-topological space (X, τ_m, τ^*), we have the following

(i) The intersection of any family of m-closed sets is m-closed set.

(ii) The union of finitely many m-closed sets is an m-closed set.

Remark 4.1.13. In a space (X, τ), if τ^* is a supratopology associated with τ , an m-set need not be an open set and vice versa, as the following example shows.

Example 4.1.14. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, b\}\}$ and $\tau^* = \{\phi, X, \{a\}, \{a, b\}, \{b, d\}, \{a, b, d\}\}$. Then τ^* is a supratopology associated with τ. The set $\{a, b, d\}$ is an m-set but it is not an open set, and the set $\{a, b\}$ is an open set but it is not m-set since $\{a, b\} \bigcap \{b, d\} = \{b\} \notin \tau^*$.

Proposition 4.1.15. *[23] Let (X, τ) be a topological space. Then*

(i) *$SO(X)$ and $PO(X)$ are closed with respect to arbitrary union.*

(ii) *For any α-set A in X, $A \bigcap B \in SO(X)$ for all $B \in SO(X)$.*

(iii) *For any α-set A in X, $A \bigcap B \in PO(X)$ for all $B \in PO(X)$.*

By the above proposition, we have the following theorem

Theorem 4.1.16. *[23] Let (X, τ) be a topological space then*

(i) *$SO(X)$ and $PO(X)$ are supratopologies on X associated with τ.*

(ii) *τ_α is τ_m with $SO(X)$.*

(iii) *τ_α is τ_m with $PO(X)$.*

Remark 4.1.17. Since $\phi \in SO(X)$ (resp $PO(X)$), then by Theorem 4.1.11, τ_α is a topology on X and this is another proof.

Definition 4.1.18. [23] Let (X, τ_m, τ^*) be an m-topological space.

(i) The *m-interior* of A is defined as the union of all m-open sets contained in A. The m-interior of A is denoted by $mIntA$ (m-interior of A is the largest m-open sets contained in A).

(ii) The m-closure of A is defined as the intersection of all m-closed sets containing A. The m-closure of A is denoted by $mClA$(The m-closure of A is the smallest m-closed sets containing A).

Theorem 4.1.19. *[23] Let (X, τ_m, τ^*) be an m-topological space and A be a subset of X. Then*

(i) *A is m-open if and only if $A = mIntA$.*

(ii) *A is m-closed if and only if $A = mClA$.*

(iii) *$mCl(mClA) = mClA$ and $mInt(mIntA) = mInt(A)$.*

(iv) *$A \subset B$ implies $mClA \subset mClB$.*

(v) *$mClA \bigcup mClB = mCl(A \bigcup B)$.*

Theorem 4.1.20. *Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space, and let $\tau^* = SO(X)$ be a supratopology associated with $\tau(\leq)$. Then a non-empty set U is m-set if and only if $\hat{x} \subseteq U$, for all $x \in U$.*

Proof. By Theorem 4.1.16, $\tau_m = \tau_\alpha$ and by Corollary 2.2.15, $\tau_\alpha = PO(X)$. Finally, from Corollary 2.2.9 part (i), U is preopen if and only if $\hat{x} \subseteq U$, for all $x \in U$. $\square$

4.2 m-continuity

Definition 4.2.1. [23] Let (X, τ_m, τ^*) be an m-topological space and (Y, μ) be a topological space. A function $f : X \to Y$ is called an *m-continuous* if the inverse image of each open set of Y is an m-open set in X; that is, *m-continuity* is the usual continuous with respect to m-topological space τ_m in X.

Remark 4.2.2. In general, if τ^* is a supratopology with τ, then there is no relation between the continuity and the m-continuity

Example 4.2.3. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, b\}\}$ and $\mu = \{\phi, X, \{a, b, d\}\}$. Now we take a supratopology $\tau^* = \{\phi, X, \{a, b\}, \{b, d\}, \{a, b, d\}\}$ for τ. Then $\tau_m = \{\phi, X, \{a, b, d\}\}$. Let $f : (X, \tau, \tau^*) \to (X, \tau)$ be the identity function . Then f is continuous but it is not m-continuous. But if $f : (X, \tau, \tau^*) \to (X, \mu)$ is the identity function, then f is m-continuous and it is not continuous.

Definition 4.2.4. [23] Let (X, τ^*) be a supratopological space and (Y, μ) be a topological space. A function $f : X \to Y$ is an *S-continuous function* if the inverse image of each open set in Y is a supraopen set in X.

Definition 4.2.5. [23] Let (X, τ^*) and (Y, μ^*) be a superatopological spaces. A function $f : X \to Y$ is an *S^*-continuous function* if the inverse image of each supraopen set in Y is a supraopen set in X.

Definition 4.2.6. [23] Let (X, τ_m, τ^*), (Y, μ_m, μ^*) be an m-topological spaces. A function $f : (X, \tau_m, \tau^*) \to (Y, \mu_m, \mu^*)$ is an *mS-continuous function* if the inverse image of each m-set in Y is a supraopen set in X.

Theorem 4.2.7. *Let (X, τ_m, τ^*) be an m-topological space and $(Y, \mu(\leq))$ be an Artinian T_o–Alexandroff space . If $f : (X, \tau_m, \tau^*) \to (Y, \mu(\leq))$ is a function, then the following statements are equivalent:*

(i) *f is an m-continuous.*

(ii) *For each $y \in Y$, then $f^{-1}(\downarrow y)$ is m-closed in X.*

(iii) *For each $x \in X$, there exists $W \in \tau_m$ such that $x \in W$, $f(W) \subseteq \uparrow f(x)$.*

(iv) *$f(mClA) \subseteq \overline{f(A)}$ for every $A \subseteq X$.*

(v) *$mCl(f^{-1}(B)) \subseteq f^{-1}(\overline{B})$ for every $B \subset Y$.*

Proof. $((i) \Rightarrow (ii))$ For each $y \in Y$, $\downarrow y$ is closed set in Y. Since $(\downarrow y)^c$ is an open set in Y then $f^{-1}((\downarrow y)^c) = (f^{-1}(\downarrow y))^c$ is m-open in X. So $f^{-1}(\downarrow y)$ is m-closed in X.

$((ii) \Rightarrow (i))$ For each $y \in Y$, $\uparrow y$ is open set in Y, then $(\uparrow y)^c$ is closed set in Y. By assumption, $f^{-1}((\uparrow y)^c) = (f^{-1}(\uparrow y))^c$ is m-closed in X, so $f^{-1}(\uparrow y)$ is m-open in X. Thus f is an m-continuous.

$((i) \Rightarrow (iii))$ Let $x \in X$. Then $\uparrow f(x)$ is open set in Y containing $f(x)$. Set $W = f^{-1}(\uparrow f(x))$. Then W is m-open, $x \in W$, and $f(W) \subseteq \uparrow f(x)$.

$((iii) \Rightarrow (iv))$ we will show that for each $b \in mClA$, $f(b) \in \overline{f(A)}$. Since $\uparrow f(b)$ be an open neighborhood of $f(b)$, then there exists $W \in \tau_m$ such that $b \in W$, $f(W) \subseteq \uparrow f(b)$. Since $b \in mClA$, so $W \bigcap A \neq \emptyset$. Then $f(W \bigcap A) \neq \emptyset$, and so $f(W) \bigcap f(A) \neq \emptyset$. Thus $\uparrow f(b) \bigcap f(A) \neq \emptyset$ and $f(b) \in \overline{f(A)}$.

$((iv) \Rightarrow (v))$ Let $A = f^{-1}(B)$ for $B \subseteq Y$. Then $f(mClA) \subseteq \overline{f(A)} \subseteq \overline{B}$. hence $mCl(f^{-1}(B)) \subseteq f^{-1}(\overline{B})$.

$((v) \Rightarrow (ii))$ For each $y \in Y$, $\downarrow y$ is closed set in Y. Then $mCl(f^{-1}(\downarrow y)) \subseteq f^{-1}(\overline{\downarrow y}) = f^{-1}(\downarrow y)$. hence $f^{-1}(\downarrow y)$ is an m-closed in X.

□

Proposition 4.2.8. *If $f : (X, \tau_m, \tau^*) \to (Y, \mu)$ is an m-continuous function and $g : (Y, \mu) \to (Z, \nu)$ is a continuous function, then $g\, of$ is m-continuous.*

Remark 4.2.9. [23] The following implication holds:

(i) m-continuity $\Rightarrow$ S-continuity.

(i) S^*-continuity $\Rightarrow$ mS-continuity.

(iii) If $\tau \subseteq \tau_m$

continuity $\Rightarrow$ m-continuity $\Rightarrow$ S-continuity.

(iv) In $\tau \subseteq \tau_m$ and In $\mu \subseteq \mu_m$

m-continuity $\Rightarrow$ S-continuity $\Leftarrow$ mS-continuity $\Leftarrow$ S^*-continuity.

(v) In $\tau^* = SO(X)$

continuity $\Rightarrow$ m-continuity($= \alpha$-continuity) $\Rightarrow$ semi-continuity.

(vi) In $\tau^* = PO(X)$

continuity $\Rightarrow$ α-continuity $\Rightarrow$ m-continuity $\Rightarrow$ precontinuity.

Bibliography

[1] A. S. Mashhour, A.A.Allam, F.S.MAhmoud and F.H.KHedr, *On supratopological spaces*,Indian J.Pure Appl. Math **14**(1983),502-510.

[2] A. S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb,*On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47-53.

[3] A.E El-Attik,*A study on some types of mappings on topological spaces*, Masters thesis, Tanta University, 1997

[4] A.E El-Attik, M.E. Abd El-Monsef and E.I. Lashin,*On finite T_o- topological spaces*, Proceedings of the ninth Pragu Topological Symposium, (Pragu 2001), 75-90, Topological Atlas, Toronto, 2002.

[5] Alexandroff P., *Diskrete Raume*, Mat. Sb. (N.S.) **2** (1937), 501-518.

[6] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Great Britain, Reprinted 1991.

[7] Di Maio and T. Noiri, *On s-closed spaces*, Indian J. Pure Appl. Math., **18** (3) (1987), 226-233.

[8] E. Hewitt, *A problem of set theoretic topology*, Duke Math. J. **10** (1943), 309-333.

[9] E.Khalimsky, Ralph Kopperman, and Paul R.Meyer, *Computer graphics and connected topogies on finite order sets*, Topology Appl. **36** (1990).

[10] George L.Cain, *Introduction to general topology* , Addison -Wesely Publishing company Advanced Book Program , Reading , MA, 1994.

[11] H. H. Corson and E. Michael, *Metrizability of certain countable unions*, Illinois J. Math. **8** (1964), 351-360.

[12] H. Mahdy and M. El atrash, *On T_0-Alexandroff spaces*, the islamic University Journal, Volume 13, Number 2, pages 1946, June 2005.

[13] I. L. Reilly and M. K. Vamanamurthy, *On some questions concerning preopen sets*, preprint.

[14] J. Dontchev, *Survey on preopen sets*, The Proceedings of the 1998 Yatsushiro Topological conference, 1-18

[15] J. Dontchev, M. Ganster and D. Rose, *α-scattered spaces II*, Houston J. Math., **23** (2)(1997), 231-246.

[16] J. Foran and P. Liebnitz, *A characterization of almost resolvable spaces*, Rend. Circ. Mat. Palermo (2) **40** (1991), 136-141.

[17] J.van Mill and C.F.Mills, *A boojum and other snarks*, Nederl. Acad . Wetensch. Proc, Ser. A, **38** (1980),419-424 .

[18] M. Ganster, *Preopen sets and resolvable spaces*, Kyungpook Math. J. **27**(2) (1987), 135-143.

[19] N. El-Deep, I. A. Hasanein, A. S. Mashhour and T. Noiri, *On p-regular spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie **27**(75) (1983), 311-315.

[20] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36-41.

[21] O. Njastad, *On some classes of nearly open sets*, Paci c J. Math. **15** (1965), 961-970.

[22] S. G. Crossley and S. K. Hildebrand, *Semi-closure*, Texas J. Sci. **22** (1971), 99-112.

[23] W.K.Min and H.S.Chang,*On M-Continity*,Kangweon-Kyungki Math. Jour. **6**(1998),323-329.