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Abstract

In this thesis, a new class of generalized soft open sets in soft topological spaces, called soft βc-open sets, is introduced and studied. In particular, the class of soft βc-open sets contained properly in the class of soft β-open sets. This class is incomparable with the classes of soft open sets, soft pre-open sets and soft semi-open sets. We prove that in a soft locally indiscrete space every soft semi-open set is a soft βc-open set. We also introduce and study the concepts of soft βc-interior, soft βc-closure. Finally, we introduce the concepts of soft βc-continuous function, soft βc-open function, soft βc-closed function and soft βc-irresolute function. We prove that in a soft locally indiscrete space f_{pu} is a soft continuous function if and only if f_{pu} is a soft βc-continuous function.

Introduction

Molodtsove [25] initiated a novel concept of soft set theory, which is a completely new approach for modeling vagueness and uncertainly. He successfully applied the soft set theory into several directions such as smoothness of functions, game theory, Riemann Integration, theory of measurement, and so on. Soft set theory and its applications have shown great development in recent years. This is because of the general nature of parametrization expressed by a soft set. Shabir and Naz [31] introduced the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. The authors in [11] continued the study of properties of soft topological space. Weaker forms of soft open sets were first studied by Chen in [16]. He investigate soft semi-open sets in soft topological spaces and studied some properties of them. Arockiarani and Arokialancy are defined soft β-open sets and continued to study other weaker forms of soft open sets in soft topological space. Later, Akdag and Ozkan [5] defined soft α-open sets [4].

The aim of this thesis is to introduce a new type of soft β-open sets namely soft βc-open sets and establish some of their soft topological applications.
This thesis comprises four chapters.

In chapter one, we give an introduction to the concepts of soft sets, and give some preliminaries that will be used in the reminder of this thesis. This chapter consists of three sections. In the first section, we give a historical note a bout soft set theory. We define the concept of soft set, and we give some examples to illustrate this concept. Then, we discuss what is the new in the soft set theory. In section two, we prepare the logical operations on soft sets such as soft complement, soft union, soft inclusion,... . Third section is devoted to the concepts of the functions between soft sets.

Chapter two studying the concept of soft topological space. It includes four sections. In the first one, we give a basic definitions and theorems that related to the soft topological space. we give some examples to illustrate this definition. In section two, we talk a bout soft bases and soft subbases. We study the relations between these concepts. In section three, we study soft continuous with illustrated examples. In section four, we study the soft separation axioms and we prove some theorems on them.

In chapter three, we study the generalized soft open sets and it includes four sections. In section one, we study and investigate the concepts of soft semi-open and soft pre-open sets. Then in section two, we study and investigate the concepts of soft α-open and soft β-open sets. We give a comparable between these types of soft open sets. In section three, we study the concepts of soft semi-interior (soft semi-closure), soft pre-interior (soft pre-closure), soft α-interior (soft α-closure) and soft β-interior (soft β-closure). In section four, we study the concepts of soft semi-continuous function, soft pre-continuous function, soft α-continuous function and soft β-continuous function. We study the relations between theses types of soft continuity.

In the last chapter, we introduce and study a new type of soft β-open sets, called soft βc-open sets in three sections. The class of soft βc-open sets is incomparable with the classes of soft semi-open sets and soft pre-open sets. We prove that in a soft locally indiscrete space, every soft semi-open sets is a soft βc-open sets. We introduce and study the concepts of soft βc-interior and soft βc-closure of a given soft set. At the end of this chapter we introduce and investigate the concepts of soft βc-continuous functions.

Chapter 1

Preliminaries

1.1 Introduction to Soft Sets

Most of our traditional tools for formal modeling, reasoning, and computing are crisp, deterministic, and precise in characters. However, there are many complicated problems in economics, engineering, environment, social science, medical science, etc., that involve data which are not always all crisp. We cannot successfully use classical methods because of various types of uncertainties present in these problems. There are theories like as theory of probability, theory of fuzzy sets, theory of intuitionistic fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets which can be considered as mathematical tools for dealing with uncertainties. But all these theories have their inherent difficulties as pointed out in [25]. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theories. Consequently, Molodtsove [25] initiated the concept of soft theory as a mathematical tool for dealing with uncertainties which is free from the above difficulties [24]. There is no limited condition to the description of objects. Many of the established paradigms appear as special cases of Soft Set Theory, so researchers can choose the form of parameters they need, which greatly simplifies the decision making process and make the process more efficient in the absence of partial information. Moreover, soft sets represent a powerful tool for decision making about information systems, data mining and drawing conclusions from data, especially in those cases where some uncertainty exists in the data. Its efficiency in dealing with uncertainty problems is as a result of its parameterized concept [29].

Definition 1.1.1. [25] Let X be an initial universe set and E be a set of parameters. Let $\mathcal{P}(X)$ denoted the power set of X and A be a nonempty subset of E. A pair (F, A) denoted by F_A is called *a soft set* over X, if F is a mapping given by $F : A \to \mathcal{P}(X)$. Shortly, a soft set over X is a parameterized family of subsets of the universe X. The family of all these soft sets over X denoted by $SS(X)_A$. For a particular $e \in A$, the collection $\{F(e) : F_A \textit{ is a soft set}\}$ is considered to be the *set of e-approximate elements* of the soft sets. If $e \notin A$, then $F(e) = \emptyset$.

As an illustration, to these concepts let us consider the following two examples:

Example 1.1.2. [24] Let U be the set of houses under consideration. E is the set of parameters, each parameter is a word or a sentence as follows:
$E = \{expensive,\ beautiful,\ wooden,\ cheap,\ in\ the\ green\ surroundings,\ modern,$ $in\ good\ repair,\ in\ bad\ repair\}$. A set is pointed out expensive houses, beautiful houses, wooden houses, etc. The soft set (F, E) describes the "*attractiveness of the houses*" which Mr. X (say) is going to buy. This example can be considered as follows:
Suppose that there are six houses in the universe U given by $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and let $A = \{e_1, e_2, e_3, e_4, e_5\}$, where

e_1 stands for the parameter '*expensive*',
e_2 stands for the parameter '*beautiful*',
e_3 stands for the parameter '*wooden*',
e_4 stands for the parameter '*cheap*',
e_5 stands for the parameter '*in the green surroundings*'.

Suppose that $F(e_1) = \{h_2, h_4\}$, $F(e_2) = \{h_1, h_3\}$, $F(e_3) = \{h_3, h_4, h_5\}$, $F(e_4) = \{h_1, h_3, h_5\}$ and $F(e_5) = \{h_1\}$. The soft set (F, A) is a parameterized family $\{F(e_i), i = 1, 2, 3, 4, 5\}$ of subsets of the set X which gives us a collection of approximate description of an object.

Consider the mapping F which is "*houses(.)*" where dot (.) is to be filled up by a parameter $e \in A$. Therefore, $F(e_1)$ means "*houses(expensive)*" whose functional value is the set $\{h_2, h_4\}$. Thus we can view the soft set (F, A) as a collection of approximation as following: $(F, A) = \{expensive\ houses = \{h_2, h_4\},\ beautiful\ houses = \{h_1, h_3\},\ wooden\ houses = \{h_3, h_4, h_5\},\ cheap\ houses = \{h_1, h_3, h_5\},\ in\ green\ surroundings = \{h_1\}\}$. For purpose of storing a soft set in a computer, we can represent a soft set (F, A) in the form of the following table:

U	'Expensive'	'Beautiful'	'Wooden'	'Cheap'	'In the green surroundings'
h_1	0	1	0	1	1
h_2	1	0	0	0	0
h_3	0	1	1	1	0
h_4	1	0	1	0	0
h_5	0	0	1	1	0
h_6	0	0	0	0	0

Figure 1.1: Tabular representation of a soft set

Definition 1.1.3. [7] A *topology* on a set X is a collection τ of subsets of X, called the open sets satisfying:
(1) Any union of elements of τ belongs to τ,
(2) Any finite intersection of elements of τ belongs to τ,
(3) $\emptyset$ and X belong to τ.

Example 1.1.4. [25] Let (X, τ) be a topological space. That is, X is a set and τ is a family of subsets of X, called the open sets of X, satisfying the three famous conditions of topology on X. Then, for a fixed $x \in X$, the family of open sets $T(x)$ of point x, where $T(x) = \{V \in \tau : x \in V\}$, can be considered as the soft set (T, τ).

The way of setting (or describing) any object in the soft set theory principally differs from the way in which we use classical mathematics [25]. In classical mathematics, we construct a mathematical model of an object and define the notion of the exact solution of this model. Usually the mathematical model is too complicated and we cannot find the exact solution. So, in the second step, we introduce the notation of approximate solution and calculate that solution. In the soft set theory, we have the opposite approach to this problem. The initial description of the object has an approximate nature, and we do not need to introduce the notion of exact solution. The absence of any restrictions on the approximate description in soft set theory makes this theory very convenient and easily applicable in practice. We can use any parametrization we prefer: with the help of words and sentences, real numbers, functions, mappings, and so on. It means that the problem of setting the membership function or any similar problem does not arise in the soft set theory [25].

1.2 Logical Operations on Soft Sets

Definition 1.2.1. [8] The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$ where, $F^c : A \to P(X)$ is a mapping given by $F^c(e) = X - F(e)$, for all $e \in A$. Let us call F^c to be the *soft complement function* of F. Clearly $(F^c)^c$ is the same as F and $((F, A)^c)^c = (F, A)$.

Definition 1.2.2. [24] For $A \subseteq B$ of parameters, let $(F, A), (G, B)$ be soft sets over the same universe X. Then (F, A) is *a soft subset* of (G, B), denoted by $(F, A)\tilde{\subseteq}(G, B)$, if $A \subseteq B$ and $F(e) \subseteq G(e)$, $\forall e \in A$. In this case, (G, B) is said to be *a soft superset* of (F, A).

Definition 1.2.3. [24] Two soft sets (F, A) and (G, B) over a common universe set X are said to be *soft equal* if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A). That is, $A = B$ and $\forall e \in A$, $F(e) = G(e)$.

Definition 1.2.4. [24] A soft set (F, A) over X is said to be *a null soft set*, denoted by Φ_A , if for all $e \in A$, $F(e) = \emptyset$.

Definition 1.2.5. [24] A soft set (F, A) over X is said to be an *absolute soft set*, denoted by X_A, if for all $e \in A$, $F(e) = X$. Clearly, $X_A^c = \Phi_A$.

Definition 1.2.6. [26] The *difference between two soft sets* (F, A) and (G, A) over a common universe X, denoted by $(F, A) - (G, A)$ is the soft set (H, A) where for all $e \in A$, $H(e) = F(e) - G(e)$.

Definition 1.2.7. [24] The *union of two soft sets* (F, A) and (G, B) over a common universe X is the soft set (H, C), where $C = A \cup B$, and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & , e \in A - B \\ G(e) & , e \in B - A \\ F(e) \cup G(e) & , e \in A \cap B\,. \end{cases}$$

We write $(F, A)\tilde{\cup}(G, B) = (H, C)$.

Remark 1.2.8. Since it has been noted that, $F(e) = \emptyset\ \forall e \in E - A$, the previous definition is equivalent to, $H(e) = F(e) \cup G(e)\ \forall e \in A \cup B$.

Definition 1.2.9. [24] The *intersection of two soft sets* (F, A) and (G, B) over a common universe X is the soft set (H, C), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$. We write $(F, A)\tilde{\cap}(G, B) = (H, C)$.

Definition 1.2.10. [24] Let I be an arbitrary indexed set and $L = \{(F_i, A) : i \in I\}$ be a subfamily of $SS(X)_A$.

(1) *The union* of L is the soft set (H, A), where $H(e) = \bigcup_{i \in I} F_i(e)$ for each $e \in A$. We write $\tilde{\bigcup}_{i \in I}(F_i, A) = (H, A)$.

(2) *The intersection* of L is the soft set (M, A), where $M(e) = \bigcap_{i \in I} F_i(e)$ for each $e \in A$. We write $\tilde{\bigcap}_{i \in I}(F_i, A) = (M, A)$.

Theorem 1.2.11. [8] If (F, A) and (G, A) are two soft sets in $SS(X)_A$, then

(1) $((F, A)\tilde{\cup}(G, A))^c = (F, A)^c \tilde{\cap}(G, A)^c$.

(2) $((F, A)\tilde{\cap}(G, A))^c = (F, A)^c \tilde{\cup}(G, A)^c$.

Theorem 1.2.12. [37] Let I be an arbitrary index set and $\{(F_i, A) : i \in I\}$ be a subfamily of $SS(X)_A$. Then

(1) $[\tilde{\bigcup}_{i \in I}(F_i, A)]^c = \tilde{\bigcap}_{i \in I}(F_i, A)^c$.

(2) $[\tilde{\bigcap}_{i \in I}(F_i, A)]^c = \tilde{\bigcup}_{i \in I}(F_i, A)^c$.

We call these two properties, De Morgan's Law.

Theorem 1.2.13. [8] Let (F, A) and (G, A) be soft sets in $SS(X)_A$. Then the following are true:

(1) $(F, A)\tilde{\cap}\phi_A = \phi_A$.

(2) $(F, A)\tilde{\cap}X_A = (F, A)$.

(3) $(F, A)\tilde{\cup}\phi_A = (F, A)$.

(4) $(F, A)\tilde{\cup}X_A = X_A$.

Theorem 1.2.14. [37] Let (F, A) and (G, A) be soft sets in $SS(X)_A$. Then the following are true:

(1) $(F, A)\tilde{\subseteq}(G, A)$ iff $(F, A)\tilde{\cap}(G, A) = (F, A)$.

(2) $(F, A)\tilde{\subseteq}(G, A)$ iff $(F, A)\tilde{\cup}(G, A) = (G, A)$.

Theorem 1.2.15. Let (F, A), (G, A), (H, A) and $(S, A) \in SS(X)_A$. Then the following are true:

(1) If $(F, A)\tilde{\cap}(G, A) = \phi_A$, then $(F, A)\tilde{\subseteq}(G, A)^c$ [37].

(2) $(F, A)\tilde{\cup}(F, A)^c = X_A$ [9].

(3) If $(F, A)\tilde{\subseteq}(G, A)$ and $(G, A)\tilde{\subseteq}(H, A)$, then $(F, A)\tilde{\subseteq}(H, A)$ [37].

(4) $(F, A)\tilde{\subseteq}(G, A)$ iff $(G, A)^c\tilde{\subseteq}(F, A)^c$.

Definition 1.2.16. [26] Let (F, A) be a soft set over X and $x \in X$. We say that $x \in (F, A)$ read as x *belong to the soft set* (F, A) whenever $x \in F(e)$ for all $e \in A$.

Definition 1.2.17. [26] Let $x \in X$. Then the soft set (F, A) over X, where $F(e) = \{x\}$ $\forall e \in A$, is called the *singleton soft point* and denoted by x_A or (x, A).

Definition 1.2.18. [17] *A soft point* is a soft set x_e where $x \in X$ and $e \in A$ defined by $x_e(e) = \{x\}$ and $x_e(e') = \emptyset \ \forall \ e' \neq e$ in A. The soft point $x_e(e) = \{x\}$ is denoted by x_e.

Definition 1.2.19. [17] The soft point x_e *is said to be in the soft set* (G, A), denoted by $x_e \tilde{\in} (G, A)$, if for the element $e \in A$, $\{x\} \subseteq G(e)$.

Note that any soft point $x_e \tilde{\in} X_A$.

Theorem 1.2.20. Let $x_e \tilde{\in} X_A$ and $(G, A) \tilde{\subseteq} X_A$. Then the following are true:

(1) If $x_e \tilde{\in} (G, A)$, then $x_e \tilde{\notin} (G, A)^c$ [37].

(2) If $x_e \tilde{\notin} (G, A)^c$, then $x_e \tilde{\in} (G, A)$.

Proof. We only prove (2). Assume for some $e \in A$, $x_e \tilde{\notin} (G, A)^c = (G^c, A)$, then $\{x\} \nsubseteq X - G(e)$. This implies that $\{x\} \subseteq G(e)$. Therefore, we have $x_e \tilde{\in} (G, A)$. □

Remark 1.2.21. Let x_e be a soft point and x_A be a singleton soft point over a common universe X. Then $x_e \tilde{\in} x_A$.

Theorem 1.2.22. [28] The union of any collection of soft points can be considered as a soft set and every soft set can be expressed as a union of all soft points belonging to it.

Definition 1.2.23. [14] Two soft point x_e and $y_{e'}$ are not equal if they are *not equal as soft sets*. That is, if $x \neq y$ or $e \neq e'$.

Definition 1.2.24. [20] Two soft sets (G, A), (H, A) in $SS(X)_A$ are said to be *soft disjoint*, if $(G, A) \tilde{\cap} (H, A) = \Phi_A$. That is, if $G(e) \cap H(e) = \phi$, for all $e \in A$.

1.3 Functions Between Soft Sets

Definition 1.3.1. [22] Let $SS(X)_A$ and $SS(Y)_B$ be the families of soft sets on X and Y respectively, $u : X \to Y$ and $p : A \to B$ be mappings. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a mapping. If $(F, A) \in SS(X)_A$, Then the image of (F, A) under f_{pu}, written as $f_{pu}(F, A) = (f_{pu}(F), p(A))$, is soft set in $SS(Y)_B$ such that

$$f_{pu}(F)(b) = \begin{cases} \cup_{a \in p^{-1}(b)} \; u(F(a)) & , \; p^{-1}(b) \neq \emptyset \,, \\ \\ \emptyset & , \text{ otherwise} \,. \end{cases}$$

for all $b \in B$. *The soft function* f_{pu} will be injective if p and u are injectives, and surjective if p and u are surjectives.

Definition 1.3.2. [22] Let $SS(X)_A$ and $SS(Y)_B$ be the families of soft sets on X and Y respectively, $u : X \to Y$ and $p : A \to B$ be mappings. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a mapping. If $(G, B) \in SS(Y)_B$, Then the inverse image of (G, B) under f_{pu}, written as $f_{pu}^{-1}(G, B) = (f_{pu}^{-1}(G), p^{-1}(B))$, is soft set in $SS(X)_A$ such that

$$f_{pu}^{-1}(G)(a) = \begin{cases} u^{-1}(G(p(a))) & , p(a) \in B\,, \\ \\ \emptyset & , \text{otherwise}\,. \end{cases}$$

for all $a \in A$.

Theorem 1.3.3. [22] Let $SS(X)_A$ and $SS(Y)_B$ be families of soft sets. For the soft function $f_{pu} : SS(X)_A \to SS(Y)_B$, the following statements hold,

(a): $f_{pu}(X_A)\tilde{\subseteq}Y_B$. If f_{pu} is surjective, then the equality holds.

(b): If $(F, A)\tilde{\subseteq}(G, A)$ in $SS(X)A$, then $f_{pu}(F, A)\tilde{\subseteq}f_{pu}(G, A)$.

(c): $f_{pu}[(F, A)\tilde{\cup}(G, A)] = f_{pu}(F, A)\tilde{\cup}f_{pu}(G, A)$ and $f_{pu}[(F, A)\tilde{\cap}(G, A)]\tilde{\subseteq}f_{pu}(F, A)\tilde{\cap}f_{pu}(G, A)$ $\forall$ (F, A),$(G, A) \in SS(X)_A$. If f_{pu} is injective, then the equality holds.

Theorem 1.3.4. [22] Let $SS(X)_A$ and $SS(Y)_B$ be families of soft sets. For the soft function $f_{pu} : SS(X)_A \to SS(Y)_B$, the following statements hold:

(a): $f_{pu}^{-1}((G, B)^c) = (f_{pu}^{-1}(G, B))^c$ $\forall$ $(G, B) \in SS(Y)_B$.

(b): $f_{pu}(f_{pu}^{-1}((G, B)))\tilde{\subseteq}(G, B)$ $\forall$ $(G, B) \in SS(Y)_B$. If f_{pu} is surjective, then the equality holds.

(c): $(F, A)\tilde{\subseteq}f_{pu}^{-1}(f_{pu}((F, A)))$ $\forall$ $(F, A) \in SS(X)_A$. If f_{pu} is injective, then the equality holds.

(d): $f_{pu}^{-1}(Y_B) = X_A$ and $f_{pu}(\phi_A) = \phi_B$.

(e): If $(F,B)\tilde{\subseteq}(G,B)$ in $SS(Y)_B$, then $f_{pu}^{-1}(F,B)\tilde{\subseteq}f_{pu}^{-1}(G,B)$.

(f): $f_{pu}^{-1}[(F,B)\tilde{\cup}(G,B)] = f_{pu}^{-1}(F,B)\tilde{\cup}f_{pu}^{-1}(G,B)$ and $f_{pu}^{-1}[(F,B)\tilde{\cap}(G,B)] = f_{pu}^{-1}(F,B)\tilde{\cap}$ $f_{pu}^{-1}(G,B)$ $\forall$ (F,B),$(G,B) \in SS(Y)_B$.

Definition 1.3.5. [13] Let (F,A) and (G,B) be two soft sets over X_1 and X_2, respectively. *The cartesian product* $(F,A) \times (G,B)$ is defined by $(F \times G)_{(A\times B)}$ where $(F\times G)_{(A\times B)}(e,k) = (F,A)(e)\times(G,B)(k)$, $\forall$ $(e,k) \in A\times B$. According to this definition, the soft set $(F,A)\times(G,B)$ is soft set over $X_1 \times X_2$ and its parameter universe is $A\times B$. The cartesian product of three or more non-null soft sets can be defined by generalizing the definition of the cartesian product of two soft sets.

Example 1.3.6. [13] Consider the soft set (F,A) which describes the "*cost of the houses*" and the soft set (G,B) which describes the "*attractiveness of the houses*". Suppose that $X = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}\}$, $A = \{very\ costly,\ costly,\ cheap\}$ and $B = \{beautiful,\ in\ the\ green\ surroundings,\ cheap\}$. Let $F(very\ costly) = \{h_2, h_4, h_7, h_8\}$, $F(costly) = \{h_1, h_3, h_5\}$, $F(cheap) = \{h_6, h_9, h_{10}\}$, and $G(beautiful) = \{h_2, h_3, h_7\}$, $G(in\ the\ green\ surroundings) = \{h_6, h_5, h_8\}$, $G(cheap) = \{h_6, h_9, h_{10}\}$. Now $(F,A) \times (G,B) = (H, A\times B)$ where a typical element will look like $H(very\ costly,\ beautiful) = \{h_2,h_4,h_7,h_8\}\times\{h_2,h_3,h_7\} = \{(h_2,h_2),(h_2,h_3),(h_2,h_7),(h_4,h_2),(h_4,h_3),(h_4,h_7),(h_7,h_2)$ $,(h_7,h_3),(h_7,h_7),(h_8,h_2),(h_8,h_3),(h_8,h_7)\}$.

Definition 1.3.7. [13] Let (F_1,E_1) and (F_2,E_2) be two soft sets over X_1 and X_2, respectively and $p_i : X_1 \times X_2 \to X_i$, $q_i : E_1 \times E_2 \to E_i$ be projection mappings in classical meaning. Then the soft mappings (p_i, q_i), $i \in \{1,2\}$, is called *soft projection mapping* from $X_1\times X_2$ to X_i and defined by $(p_i,q_i)((F_1,E_1)\times(F_2,E_2)) = (p_i,q_i)((F_1\times F_2),(E_1\times E_2)) = (p_i(F_1\times F_2), q_i(E_1\times E_2)) = (F,E)_i$.

Theorem 1.3.8. [27] Let (F_1,E_1), $(G_1,E_1) \in SS(X)_{E_1}$ and (F_2,E_2), $(G_2,E_2) \in SS(Y)_{E_2}$. Then:

(1): $\phi_{E_1} \times (F_2,E_2) = (F_1,E_1)\times\phi_{E_2} = \phi_{E_1\times E_2}$,

(2): $((F_1,E_1)\times(F_2,E_2))\tilde{\cap}((G_1,E_1)\times(G_2,E_2)) = ((F_1,E_1)\tilde{\cap}(G_1,E_1))\times((F_2,E_2)\tilde{\cap}(G_2,E_2))$.

Chapter 2

Soft Topological Spaces

2.1 Definitions and Basic Information

Definition 2.1.1. [31] Let τ be a collection of soft sets over a universe X with the fixed set of parameters A. Then τ is said to be *a soft topology* on X, if

(1) Φ_A, X_A belong to τ.

(2) the union of any number of soft sets in τ belongs to τ.

(3) the intersection of any two soft sets in τ belongs to τ.

The triple (X, τ, A) is called *a soft topological space over* X. Simply, if no confusion hold, we can use X rather than (X, τ, A). The members of τ are called *soft open sets*. A soft complement of a soft open set (F, A) is called *a soft closed set* in (X, τ, A). If (F, A) belongs to τ, we write $(F, A)\tilde{\in}\tau$. A soft set (F, A) which is both soft open set and soft closed set is called *soft clopen set*.

Example 2.1.2. Suppose that there are four alternatives in the universe of houses $X = \{h_1, h_2, h_3, h_4\}$ and consider $A = \{a_1(cotton), a_2(woollen)\}$ be the set of parameters showing the material of the dresses. Let (F_1, A),(F_2, A),(F_3, A),(F_4, A) be four soft sets over the common universe X which describe the goodness of the dresses, where

$$\begin{aligned}(F_1, A) &= \{(a_1, \{h_1\}), (a_2, \{h_2\})\},\\ (F_2, A) &= \{(a_1, \{h_2\}), (a_2, \{h_1\})\},\\ (F_3, A) &= \{(a_1, \{h_1, h_2\}), (a_2, \{h_1, h_2\})\},\\ (F_4, A) &= \{(a_1, \{h_1, h_2, h_4\}), (a_2, \{h_1, h_2, h_4\})\}.\end{aligned}$$

Then $\tau = \{X_A, \phi_A, (F_1, A), (F_2, A), (F_3, A), (F_4, A)\}$ defines a soft topology on X.

Definition 2.1.3. [31] Let X be an initial universe set, A a set of parameters and $\tau = \{\phi_A, X_A\}$. Then τ is called the *soft indiscrete topology* on X and (X, τ, A) is said to be the soft indiscrete space.

Definition 2.1.4. [31] Let X be an initial universe set, A a set of parameters and let $\tau = SS(X)_A$. Then τ is called the *soft discrete topology* on X and (X, τ, A) is said to be the soft discrete space.

Theorem 2.1.5. [31] Let (X, τ, A) be a soft topological space over X. Then the collection $\tau_e = \{F(e) : (F, A)\tilde{\in}\tau\}$, for each $e \in A$ defines a topology on X.

This theorem shows that corresponding to each parameter $e \in A$, we have a topology τ_e on X. Thus a soft topology on X gives a parameterized family of topologies on X. For more illustration see the following example:

Example 2.1.6. In Example 2.1.2, it can be easily seen that $\tau_{a_1} = \{\emptyset, X, \{h_1\}, \{h_2\}, \{h_1, h_2\}, \{h_1, h_2, h_4\}\} = \tau_{a_2}$ is a topology on X.

Remark 2.1.7. [31] It is known that the intersection of two soft topologies over the same universe X is a soft topology, whereas the union may or may not be a soft topology .

Definition 2.1.8. [15] Let (X, τ_1, A) and (X, τ_2, A) be soft topological spaces. Then

(1) if $\tau_1 \tilde{\subseteq} \tau_2$, then τ_2 is *soft finer* than τ_1.

(2) if $\tau_1 \tilde{\subset} \tau_2$, then τ_1 is *soft strictly finer* than τ_1.

(3) if either $\tau_1 \tilde{\subseteq} \tau_2$ or $\tau_2 \tilde{\subseteq} \tau_1$, then τ_1 is *comparable* with τ_2.

Definition 2.1.9. [31] Let (X, τ, A) be a soft topological space over X and (F, A) a soft set in $SS(X)_A$. The soft point $x_e \tilde{\in} X_A$ is called a soft interior point of a soft set (F, A), if there exists a soft open set (H, A) such that $x_e \tilde{\in} (H, A) \tilde{\subseteq} (F, A)$. The *soft interior of a soft set* (F, A) is denoted by $(F, A)^\circ$ and is defined as the union of all soft open sets contained in (F, A). Clearly $(F, A)^\circ$ is the largest soft open set contained in (F, A).

Theorem 2.1.10. [37] Let $x_e \tilde{\in} X_A$ for all $e \in A$ and (G, A) be a soft open set in a soft topological space (X, τ, A). Then every soft point $x_e \tilde{\in} (G, A)$ is a soft interior point.

Definition 2.1.11. [37] Let (X, τ, A) be a soft topological space. Then a soft set (G, A) in $SS(X)_A$ is called *a soft neighborhood* (briefly, *a soft nhood*) of the soft point $x_e \tilde{\in} X_A$, if there exists a soft open set (H, A) such that $x_e \tilde{\in} (H, A) \tilde{\subseteq} (G, A)$. The soft neighborhood system of a soft point x_e, denoted by $N_\tau(x_e)$, is the family of all its soft neighborhoods.

Definition 2.1.12. [37] Let (X, τ, A) be a soft topological space over X. Then a soft set (G, A) in $SS(X)_A$ is called *a soft neighborhood* (briefly, *a soft nhood*) of the soft set (F, A), if there exists a soft open set (H, A) such that $(F, A) \tilde{\subseteq} (H, A) \tilde{\subseteq} (G, A)$.

Theorem 2.1.13. [37] The neighborhood system $N_\tau(x_e)$ at x_e in a soft topological space (X, τ, A) has the following properties:

(1) If $(G, A) \in N_\tau(x_e)$, then $x_e \tilde{\in} (G, A)$.

(2) If $(G, A) \in N_\tau(x_e)$ and $(G, A) \tilde{\subseteq} (H, A)$, then $(H, A) \in N_\tau(x_e)$.

(3) If $(G, A), (H, A) \in N_\tau(x_e)$, then $(G, A) \tilde{\cap} (H, A) \in N_\tau(x_e)$.

(4) If $(G, A) \in N_\tau(x_e)$, then there is a soft set $(M, A) \in N_\tau(x_e)$ such that $(G, A) \in N_\tau(y_{e'})$ for each $y_{e'} \tilde{\in} (M, A)$.

Definition 2.1.14. [31] Let (X, τ, A) be a soft topological space over X and (F, A) a soft set over X. Then the *soft closure* of (F, A), denoted by $\overline{(F, A)}$, is the intersection of all soft closed supersets of (F, A). Clearly $\overline{(F, A)}$ is the smallest soft closed set in (X, τ, A) which contains (F, A).

Corollary 2.1.15. Let (X, τ, A) be a soft topological space and let (F, A) and (G, A) be soft sets over X. Then

(1) (F, A) is soft closed iff $(F, A) = \overline{(F, A)}$ [31].

(2) (G, A) is soft open iff $(G, A) = (G, A)^\circ$ [37].

Theorem 2.1.16. [37] A soft set (G, A) is soft open if and only if for each soft set (F, A) contained in (G, A), (G, A) is a soft neighborhood of (F, A).

Theorem 2.1.17. [37] Let (X, τ, A) be a soft topological space and let (F, A) and (G, A) be soft sets over X. Then

(1) $\left(\overline{(G, A)}\right)^c = ((G, A)^c)^\circ$.

(2) $((G, A)^\circ)^c = \overline{(G, A)^c}$.

Theorem 2.1.18. [34] Let (X, τ, A) be a soft topological space and (F, A), (G, A) soft sets over X. Then

(1) $(X_A)^\circ = X_A$ and $(\phi_A)^\circ = \phi_A$.

(2) $(G, A)^\circ \tilde{\subseteq} (G, A)$.

(3) $[(G, A)^\circ]^\circ = (G, A)^\circ$.

(4) $(G, A) \tilde{\subseteq} (F, A)$ implies $(G, A)^\circ \tilde{\subseteq} (F, A)^\circ$.

(5) $[(G, A) \tilde{\cap} (F, A)]^\circ = (G, A)^\circ \tilde{\cap} (F, A)^\circ$.

(6) $(G, A)^\circ \tilde{\cup} (F, A)^\circ \tilde{\subseteq} [(G, A) \tilde{\cup} (F, A)]^\circ$.

Theorem 2.1.19. [31] Let (X, τ, A) be a soft topological space over X and (F, A), (G, A) soft sets over X. Then

(1) $\overline{\phi_A} = \phi_A$ and $\overline{X_A} = X_A$.

(2) $(F, A) \tilde{\subseteq} \overline{(F, A)}$.

(3) $\overline{\overline{(F, A)}} = \overline{(F, A)}$.

(4) $(F, A) \tilde{\subseteq} (G, A)$ implies $\overline{(F, A)} \tilde{\subseteq} \overline{(G, A)}$.

(5) $\overline{(F, A) \tilde{\cup} (G, A)} = \overline{(F, A)} \tilde{\cup} \overline{(G, A)}$.

(6) $\overline{(F,A)\tilde{\cap}(G,A)}\tilde{\subseteq}\overline{(F,A)}\tilde{\cap}\overline{(G,A)}$.

Theorem 2.1.20. [15] Let (X,τ,A) be a soft topological space and (F,A) a soft sets in $SS(X)_A$. Then $x_e\tilde{\in}\overline{(F,A)}$ if and only if every soft open set (G,A) containing x_e intersect (F,A).

Definition 2.1.21. [31] Let (X,τ,A) be a soft topological space over X and (F,A) a soft set over X. *Associate with* (F,A) a soft set over X, denoted by $(\overline{F},A)$ and defined as $\overline{F}(e)=\overline{F(e)}$, where $\overline{F(e)}$ is the closure of $F(e)$ in τ_e for each $e\in A$.

Theorem 2.1.22. [31] Let (X,τ,A) be a soft topological space over X and (F,A) be a soft set over X. Then $(\overline{F},A)\tilde{\subseteq}\overline{(F,A)}$.

Corollary 2.1.23. [31] Let (X,τ,A) be a soft topological space over X and (F,A) a soft set over X. Then $(\overline{F},A)=\overline{(F,A)}$ if and only if $(\overline{F},A)^c\tilde{\in}\tau$.

Theorem 2.1.24. [31] let (X,τ,A) be a soft topological space over X, (F,A) a soft set over X and $x_e\tilde{\in}X_A$. If x_e is a soft interior point of (F,A) then x is an interior point of $F(e)$ in (X,τ_e), for each $e\in A$.

Definition 2.1.25. [15] Let (X,τ,A) be a soft topological space, $(F,A)\in SS(X)_A$, and $x_e\tilde{\in}(F,A)$. If every soft nhood of x_e intersect (F,A) in some point other than x_e itself, then x_e is called *a soft limit point* of (F,A). The set of all soft limit points of (F,A) is denoted by $(F,A)'$. In other words, if (X,τ,A) is a soft topological space, $(F,A)\in SS(X)_A$ and $x_e\tilde{\in}(F,A)$, then $x_e\tilde{\in}(F,A)'\Leftrightarrow(C,A)\tilde{\cap}((F,A)-\{x_e\})\neq\phi_A$ for all $(C,A)\in N_\tau(x_e)$.

Theorem 2.1.26. Let (X,τ,A) be a soft topological space, then for any soft set over X

(1) If (G,A) is a soft open set, then $\overline{(F,A)}\tilde{\cap}(G,A)\tilde{\subseteq}\overline{((F,A)\tilde{\cap}(G,A))}$.

(2) If (M,A) is a soft closed set, then $((F,A)\tilde{\cup}(M,A))^\circ\tilde{\subseteq}(F,A)^\circ\tilde{\cup}(M,A)$.

Proof. (1) Let $x_e\tilde{\in}\overline{(F,A)}\tilde{\cap}(G,A)$, then $x_e\tilde{\in}\overline{(F,A)}$ and $x_e\tilde{\in}(G,A)$. Let (O,A) be any soft open set containing x_e, then $(O,A)\tilde{\cap}(G,A)$ is a soft open set containing x_e. Since $x_e\tilde{\in}\overline{(F,A)}$, $((O,A)\tilde{\cap}(G,A))\tilde{\cap}(F,A)\neq\phi_A$ which implies $(O,A)\tilde{\cap}((G,A)\tilde{\cap}(F,A))\neq\phi_A$. Hence, $x_e\tilde{\in}\overline{((F,A)\tilde{\cap}(G,A))}$.

(2) Using Theorem 2.1.17 and part (1). □

Theorem 2.1.27. If $\{(G_\alpha, A) : \alpha \in \Delta\}$ is a collection of soft sets, then the following hold:

(1) $\tilde{\bigcup}_{\alpha\in\Delta}\overline{(G_\alpha, A)}\tilde{\subseteq}\overline{\tilde{\bigcup}_{\alpha\in\Delta}(G_\alpha, A)}$.[19]

(2) $\tilde{\bigcup}_{\alpha\in\Delta}(G_\alpha, A)^\circ\tilde{\subseteq}\left(\tilde{\bigcup}_{\alpha\in\Delta}(G_\alpha, A)\right)^\circ$.[19]

(3) $\overline{\tilde{\bigcap}_{\alpha\in\Delta}(G_\alpha, A)}\tilde{\subseteq}\tilde{\bigcap}_{\alpha\in\Delta}\overline{(G_\alpha, A)}$.

(4) $\left(\tilde{\bigcap}_{\alpha\in\Delta}(G_\alpha, A)\right)^\circ\tilde{\subseteq}\tilde{\bigcap}_{\alpha\in\Delta}(G_\alpha, A)^\circ$.

Theorem 2.1.28. [15] Let (X, τ, A) be a soft topological space and $(F, A), (G, A) \in SS(X)_A$. Then:

(1) $(F, A)'\tilde{\subseteq}\overline{(F, A)}$.

(2) $(F, A)\tilde{\subseteq}(G, A)$ implies $(F, A)'\tilde{\subseteq}(G, A)'$.

(3) $((F, A)\tilde{\cap}(G, A))'\tilde{\subseteq}(F, A)'\tilde{\cap}(G, A)'$.

(4) $((F, A)\tilde{\cup}(G, A))' = (F, A)'\tilde{\cup}(G, A)'$.

(5) (F, A) is a soft closed set iff $(F, A)'\tilde{\subseteq}(F, A)$.

(6) $(F, A)\tilde{\cup}(F, A)' = \overline{(F, A)}$.

Definition 2.1.29. [15] Let (X, τ, A) be a soft topological space, and $(F, A) \in SS(X)_A$. Then, the *soft boundary* of (F, A), denoted $(F, A)^b$, is defined by $(F, A)^b = \overline{(F, A)}\tilde{\cap}\overline{(F, A)^c}$.

Theorem 2.1.30. [15] Let (X, τ, A) be a soft topological space, and $(F, A) \in SS(X)_A$. Then:

(1) $(F, A)^b\tilde{\subseteq}\overline{(F, A)}$.

(2) $(F, A)^b = ((F, A)^c)^b$.

(3) $(F, A)^b = \overline{(F, A)} - (F, A)^\circ$.

Definition 2.1.31. [1] Let (X, τ, A) be a soft topological space and $(F, A) \in SS(X)_A$. Then, the *soft exterior* of (F, A), denoted by $(F, A)^e$, is defined as $(F, A)^e = ((F, A)^c)^{\circ}$. Thus, x_e is called a *soft exterior point* of (F, A) if there exists a soft open set (C, A) such that $x_e \tilde{\in} (C, A) \tilde{\subseteq} (F, A)^c$. We observe that $(F, A)^e$ is the largest soft open set contained in $(F, A)^c$.

Theorem 2.1.32. [1] Let (F, A) and (G, A) be soft sets of a soft topological space (X, τ, A). Then:

(1) $(F, A)^{\circ} = ((F, A)^c)^e$.

(2) $[(F, A) \tilde{\cup} (G, A)]^e = (F, A)^e \tilde{\cup} (G, A)^e$.

(3) $(F, A)^e \tilde{\cap} (G, A)^e \tilde{\subseteq} [(F, A) \tilde{\cap} (G, A)]^e$.

(4) $\left((F, A)^b\right)^c = (F, A)^{\circ} \tilde{\cup} \left((F, A)^c\right)^{\circ} = (F, A)^{\circ} \tilde{\cup} (F, A)^e$.

(5) $(F, A)^{\circ} = (F, A) - (F, A)^b$.

(6) $(F, A)^b \tilde{\cap} (F, A)^{\circ} = \phi_A$.

(7) $(F, A)^b \tilde{\cap} (F, A)^e = \phi_A$.

(8) (F, A) is a soft open if and only if $(F, A)^b \tilde{\cap} (F, A) = \phi_A$.

(9) (F, A) is a soft closed if and only if $(F, A)^b \tilde{\subseteq} (F, A)$.

(10) $[(F, A) \tilde{\cup} (G, A)]^b \tilde{\subseteq} [(F, A) \tilde{\cap} (G, A)]^b \tilde{\cup} [(F, A)^b \tilde{\cap} \overline{(G, A)^c}\ ]$.

(11) $[(F, A) \tilde{\cap} (G, A)]^b \tilde{\subseteq} [(F, A)^b \tilde{\cap} \overline{(G, A)}\ ] \tilde{\cup} [(G, A)^b \tilde{\cap} \overline{(F, A)}\ ]$.

(12) $\left(\left((F, A)^b\right)^b\right)^b = \left((F, A)^b\right)^b$.

(13) $(F, A)^b = \phi_A$ if and only if (F, A) is a soft clopen set.

2.2 Soft Bases and Soft Subbases

Definition 2.2.1. [15] Let (X, τ, A) be a soft topological space and $\mathcal{B}\tilde{\subseteq}\tau$. If every element of τ can be written as the union of elements of $\mathcal{B}$, then $\mathcal{B}$ is called *a soft basis* for the topology τ. Each element of $\mathcal{B}$ is called a *soft basic element*.

Theorem 2.2.2. [15] Let (X, τ, A) be a soft topological space and $\mathcal{B}$ be a soft basis for τ. Then, τ equals the collection of all soft unions of elements of $\mathcal{B}$.

Theorem 2.2.3. [12] Let (X, τ, A) be a soft topological space and $\mathcal{B}$ a collection of soft open sets of $SS(X)_A$. Then $\mathcal{B}$ is a basis if for every $x_e\tilde{\in}X_A$ and for every soft open set (F, A) containing x_e there exists (G, A) in $\mathcal{B}$ such that that $x_e\tilde{\in}(G, A)\tilde{\subseteq}(F, A)$, then $\mathcal{B}$ is a soft basis for τ.

Theorem 2.2.4. [12] Let (X, τ, A) be a soft topological space and $\mathcal{B}$ a collection of soft open sets of $SS(X)_A$. Then $\mathcal{B}$ is a soft basis for the soft topology τ if and only if

(1) X_A is a soft set union of members of $\mathcal{B}$ and,

(2) if (F, A), (G, A) in $\mathcal{B}$ and $x_e\tilde{\in}(F, A)\tilde{\cap}(G, A)$, then there exists $(H, A) \in \mathcal{B}$ such that $x_e\tilde{\in}(H, A)\tilde{\subseteq}(F, A)\tilde{\cap}(G, A)$.

Theorem 2.2.5. [15] Let (X, τ, A) and (X, τ', A) be soft topological spaces, and $\mathcal{B}$, $\mathcal{B}'$ soft bases for τ and τ' respectively. If $\mathcal{B}'\tilde{\subseteq}\mathcal{B}$ then τ is a soft finer than τ'.

Definition 2.2.6. [28] A collection $\mathcal{S}$ of members of a soft topology τ is said to be *soft subbase* for τ if the collection of all finite intersection of members of $\mathcal{S}$ forms a basis for τ.

Theorem 2.2.7. [18] Let (X, τ, A) be a soft topological space over X and $Y \subseteq X$. Then $\tau_Y = \{(F_Y, A) = Y_A\tilde{\cap}(F, A)|(F, A)\tilde{\in}\tau\}$ is a soft topological space over Y.

Definition 2.2.8. [18] Let (X, τ, A) be a soft topological space over X and $Y \subseteq X$. Then $\tau_Y = \{(F_Y, A) = Y_A\tilde{\cap}(F, A)|(F, A)\tilde{\in}\tau\}$ is said to be the *soft relative topology* on Y, where $F_Y(e) = Y \cap F(e)$, for all $e \in A$. (Y, τ_Y, A) is called a *soft subspace* of (X, τ, A).

Theorem 2.2.9. [18] Let (X, τ, A) be a soft topological space over X and $Y \subseteq X$. Then $(Y, (\tau_Y)_e)$ is a subspace of (X, τ_e), for each $e \in A$.

Theorem 2.2.10. [18] Let (Y, τ_Y, A) be a soft subspace of a soft topological space (X, τ, A) and (F, A) a soft open set in (Y, τ_Y, A). If $Y_A \tilde{\in} \tau$, then $(F, A) \tilde{\in} \tau$.

Theorem 2.2.11. [18] Let (Y, τ_Y, A) be a soft subspace of a soft topological space (X, τ, A) and (F, A) a soft set over X. Then

(1) (F, A) is soft open in (Y, τ_Y, A) if and only if $(F, A) = Y_A \tilde{\cap} (G, A)$, for some soft open set (G, A) in (X, τ, A).

(2) (F, A) is soft closed in (Y, τ_Y, A) if and only if $(F, A) = Y_A \tilde{\cap} (K, A)$, for some soft closed set (K, A) in (X, τ, A).

Theorem 2.2.12. [18] Let (Y, τ_Y, A) be a soft subspace of a soft topological space (X, τ, A) and $\mathcal{B}$ a soft basis for τ. Then the collection $\mathcal{B}_Y = \{Y \tilde{\cap} (G, A) : (G, A) \in \mathcal{B}\}$ is a soft basis for the soft subspace topology on Y.

2.3 Soft Continuity

Definition 2.3.1. [37] Let (X, τ, A) and (Y, τ', B) be two soft topological spaces and $f_{pu} : SS(X)_A \to SS(Y)_B$ a soft function. For each soft nhood (H, B) of $f(x)_e$, if there exists a soft nhood (F, A) of x_e such that $f_{pu}(F, A) \tilde{\subseteq} (H, B)$, then f_{pu} is called *soft continuous* function at x_e. If f_{pu} is a soft continuous function for all soft points x_e, then f_{pu} is called *soft continuous function*.

Theorem 2.3.2. [37] Let (X, τ, A) and (Y, τ', B) be two soft topological spaces and $f_{pu} : SS(X)_A \to SS(Y)_B$ a soft function. Then the following conditions are equivalent:

(1) $f_{pu} : SS(X)_A \to SS(Y)_B$ is a soft continuous function.

(2) For each soft open set (M, B) over Y, $f_{pu}^{-1}(M, B)$ is a soft open set over X.

(3) For each soft closed set (H, B) over Y, $f_{pu}^{-1}(H, B)$ is a soft closed set over X.

(4) For each soft set (F, A) over X, $f_{pu}(\ \overline{(F, A)}\ ) \tilde{\subseteq} \overline{f_{pu}(F, A)}$.

(5) For each soft set (G, B) over Y, $\overline{f_{pu}^{-1}(G, B)} \tilde{\subseteq} f_{pu}^{-1}(\overline{(G, B)})$.

(6) For each soft set (K, B) over Y, $f_{pu}^{-1}((K, B)^{\circ}) \tilde{\subseteq} \left(f_{pu}^{-1}(K, B)\right)^{\circ}$.

Example 2.3.3. [10] Let $X = Y = \{h_1, h_2, h_3\}$, $A = B = \{e_1, e_2\}$ and $\tau = \{X_A, \phi_A, (F_1, A), (F_2, A)\}$, $\tau' = \{\phi_B, Y_B, (G_1, B), (G_2, B)\}$ be two soft topologies defined on X and Y respectively, where $F_1(e_1) = \{h_1, h_2\}$, $F_1(e_2) = \{h_3\}$, $F_2(e_1) = X$, $F_2(e_2) = \{h_3\}$ and $G_1(e_1) = \{h_1\}$, $G_1(e_2) = \{h_3\}$, $G_2(e_1) = \{h_1, h_3\}$, $G_2(e_1) = \{h_2, h_3\}$. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft function such that $u : X \to Y$ defines as $u(h_1) = u(h_2) = h_1$, $u(h_3) = h_3$ and $P : A \to B$ is the identity function, then since $f_{pu}^{-1}(G_1, B) = (F_1, A)$ and $f_{pu}^{-1}(G_2, B) = (F_2, A)$, f_{pu} is a soft continuous function.

Definition 2.3.4. [10] Let (X, τ, A) and (Y, τ', B) be two soft topological spaces and $f_{pu} : SS(X)_A \to SS(Y)_B$ a soft function.

(a) If the image $f_{pu}(F, A)$ of each soft open set (F, A) over X is a soft open set over Y, then f_{pu} is called a *soft open function*.

(b) If the image $f_{pu}(H, A)$ of each soft closed set (H, A) over X is a soft closed set over Y, then f_{pu} is called a *soft closed function*.

Remark 2.3.5. [10] The concepts of soft function, soft open and soft closed functions are all independent of each other as illustrated in the following examples:

Example 2.3.6. Let (X, τ, A) be a soft indiscrete topological space and (X, τ', A) a soft discrete topological space. If $f_{pu} : SS(X)_A \to SS(X)_A$ is a soft function where $u : X \to X$ and $p : A \to A$ are identity functions, then f_{pu} is a soft open and a soft closed function. But it is not a soft continuous function.

Example 2.3.7. [10] Let $X = Y = \{h_1, h_2, h_3\}$, $A = B = \{e_1, e_2\}$ and $\tau = \{\phi_X, X_A, (F_1, A), (F_2, A), ..., (F_7, A)\}$, $\tau' = \{\phi_B, Y_B, (G_1, B), (G_2, B), (G_3, B), (G_4, B)\}$ be two soft topologies defines on X and Y respectively, where

$$\begin{aligned}(F_1, A) &= \{(e_1, \{h_2\}), (e_2, \{h_1\})\},\\ (F_2, A) &= \{(e_1, \{h_1, h_3\}), (e_2, \{h_2, h_3\})\},\\ (F_3, A) &= \{(e_1, \{h_2\}), (e_2, X)\},\\ (F_4, A) &= \{(e_2, \{h_1\})\},\\ (F_5, A) &= \{(e_1, \{h_1, h_3\}), (e_2, X)\},\\ (F_6, A) &= \{(e_2, \{h_2, h_3\})\},\\ (F_7, A) &= \{(e_2, X)\}.\end{aligned}$$

and

$$\begin{aligned}(G_1, B) &= \{(e_1, \{h_2\}), (e_2, \{h_1\})\},\\ (G_2, B) &= \{(e_1, \{h_2, h_3\}), (e_2, \{h_1, h_2\})\},\\ (G_3, B) &= \{(e_1, \{h_1, h_2\}), (e_2, Y)\},\\ (G_4, B) &= \{(e_1, \{h_2\}), (e_2, \{h_1, h_2\})\}.\end{aligned}$$

If $f_{pu} : SS(X)_A \to SS(X)_A$ is a function such that $u : X \to Y$ defines as $u(h_1) = u(h_2) = u(h_3) = h_1$ and $p : A \to B$ is the identity map, then $f_{pu}^{-1}(G_1, B) = (F_7, A)$, $f_{pu}^{-1}(G_2, B) = (F_7, A)$, $f_{pu}^{-1}(G_3, B) = X_A$ and $f_{pu}^{-1}(G_4, B) = (F_7, A)$. Hence, f_{pu} is a soft continuous function. Since $f_{pu}(F_1, A) = \{(e_1, \{h_1\}), (e_2, \{h_1\})\} \tilde{\notin} \tau'$, f_{pu} is not soft open function. Also since $f_{pu}((F_1, A)^c) = f_{pu}(\{(e_1, \{h_1, h_3\}), (e_2, \{h_2, h_3\})\}) = \{(e_1, \{h_1\}), (e_2, \{h_1\})\}$ is not soft closed set over Y, f_{pu} is not soft closed function.

Example 2.3.8. [10] Let $X = \{h_1, h_2, h_3\}$, $Y = \{a, b\}$ and $A = B = \{e_1, e_2\}$ and $\tau = \{\phi_A, X_A, (F_1, A), (F_2, A)\}$, $\tau' = \{\phi_B, Y_B, (G_1, B), (G_2, B)\}$ be two soft topologies defined on X and Y respectively, where $(F_1, A) = \{(e_1, \{h_1, h_2\}), (e_2, \{h_3\})\}$, $(F_2, A) = \{(e_1, X), (e_2, \{h_3\})\}$ and $(G_1, B) = \{(e_1, Y), (e_2, \{b\})\}$, $(G_1, B) = \{(e_1, \{a\}), (e_2, \{b\})\}$. If $f_{pu} : SS(X)_A \to SS(X)_A$ is a soft function such that $u : X \to Y$ defines as $u(h_1) = \{a\}$, $u(h_2) = u(h_3) = \{b\}$ and $p : A \to B$ is the identity map, then $f_{pu}(F_1, A) = (G_1, B)$ and $f_{pu}(F_2, A) = (G_2, B)$. Hence, f_{pu} is a soft open function. Since $f_{pu}((F_1, A)^c) = f_{pu}(\{(e_1, \{h_3\}), (e_2, \{h_1, h_2\})\} = \{(e_1, \{b\}), (e_2, Y)\}$ is not a soft closed set over Y, f_{pu} is not a soft closed function. Also since $f_{pu}^{-1}(G_1, B) = \{(e_1, X), (e_2, \{h_2, h_3\})\}\tilde{\notin}\tau$, f_{pu} is not a soft continuous function.

Example 2.3.9. [10] Let $X = \{h_1, h_2, h_3\}$, $Y = \{a, b\}$ and $A = B = \{e_1, e_2\}$ and $\tau = \{\phi_A, X_A, (F_1, A), (F_2, A), (F_3, A)\}$, $\tau' = \{\phi_B, Y_B, (G_1, A), (G_2, A)\}$ be two soft topologies defined on X and Y respectively, where $(F_1, A) = \{(e_1, \{h_1, h_3\}), (e_2, \{h_2\})\}$, $(F_2, A) = \{(e_1, X), (e_2, \{h_2, h_3\})\}$, $(F_3, A) = \{(e_1, \{h_3\}), (e_2, \{h_2\})\}$ and $(G_1, A) = \{(e_2, \{a\})\}$, $(G_2, B) = \{(e_1, \{a\}), (e_2, Y)\}$. If $f_{pu} : SS(X)_A \to SS(X)_A$ is a soft function such that $u : X \to Y$ defines as $u(h_3) = \{b\}$, $u(h_1) = u(h_2) = \{a\}$ and $p : A \to B$ the identity map, then $f_{pu}((F_1, A)^c) = f_{pu}(\{(e_1, \{h_2\}), (e_2, \{h_1, h_3\})\}) = \{(e_1, \{a\}), (e_2, Y)\}$, $f_{pu}((F_2, A)^c) = f_{pu}(\{(e_2, \{h_1\})\}) = \{(e_2, \{a\})\}$ and $f_{pu}((F_3, A)^c) = f_{pu}(\{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_3\})\}) = \{(e_1, \{a\}), (e_2, Y)\}$. Hence, f_{pu} is a soft closed function. Since $f_{pu}(F_1, A) = \{(e_1, Y), (e_2, \{a\})\}\tilde{\notin}\tau'$, f_{pu} is not soft open function. Also since $f_{pu}^{-1}(G_1, A) = \{(e_2, \{h_1, h_2\})\}\tilde{\notin}\tau$, f_{pu} is not soft continuous function.

Theorem 2.3.10. *[10] Let (X, τ, A) and (Y, τ', B) be two soft topological spaces and $f : SS(X)_X \to SS(Y)_B$ a soft function.*

(a) f_{pu} is a soft open function if and only if for each soft set (F, A) over X, $f_{pu}((F, A)^\circ)$ $\tilde{\subseteq} (f_{pu}(F, A))^\circ$.

(b) f_{pu} is a soft closed function if and only if for each soft set (G, A) over X, $\overline{f_{pu}(G, A)}$ $\tilde{\subseteq} f_{pu}(\ \overline{(G, A)}\ )$.

Definition 2.3.11. [10] Let (X, τ, A) and (Y, τ', B) be two soft topological spaces and $f_{pu} : SS(X)_A \to SS(Y)_B$ a soft function. If f_{pu} is a bijection, soft continuous and f_{pu}^{-1} is a soft continuous function, then f_{pu} is called *soft homeomorphism* from $SS(X)_A$ to $SS(Y)_B$. When a soft homeomorphism f_{pu} exists between $SS(X)_A$ onto $SS(Y)_B$, we say that $SS(X)_A$ is soft homeomorphic to $SS(Y)_B$.

Theorem 2.3.12. *[10] Let (X, τ, A) and (Y, τ', B) be two soft topological spaces and $f_{pu} : SS(X)_A \to SS(Y)_B$ a bijective soft function. Then the following conditions are equivalent:*

(a) f_{pu} is a soft homeomorphism.

(b) f_{pu} is a soft continuous and soft closed function.

(c) f_{pu} is a soft continuous and soft open function.

2.4 Soft Separation Axioms

Definition 2.4.1. [20] Let (X, τ, A) be a soft topological space. If for each $x_e, y_{e'} \tilde{\in} X_A$ such that $x_e \neq y_{e'}$, there exists either a soft open set (F_1, A) such that $x_e \tilde{\in} (F_1, A)$, $y_{e'} \tilde{\notin} (F_1, A)$ or a soft open set (F_2, A) such that $y_{e'} \tilde{\in} (F_2, A)$, $x_e \tilde{\notin} (F_2, A)$, then (X, τ, A) is called a *soft* T_0*-space.*

Definition 2.4.2. [20] Let (X, τ, A) be a soft topological space. If for each $x_e, y_{e'} \tilde{\in} X_A$ such that $x_e \neq y_{e'}$, there exist two soft open sets (F_1, A) and (F_2, A) such that $x_e \tilde{\in} (F_1, A)$, $y_{e'} \tilde{\notin} (F_1, A)$ and $y_{e'} \tilde{\in} (F_2, A)$, $x_e \tilde{\notin} (F_2, A)$, then (X, τ, A) is called a *soft* T_1*-space.*

Definition 2.4.3. [20] Let (X, τ, A) be a soft topological space. If for each $x_e, y_{e'} \tilde{\in} X_A$ such that $x_e \neq y_{e'}$, there exist two soft open sets (F_1, A) and (F_2, A) such that $x_e \tilde{\in} (F_1, A)$, $y_{e'} \tilde{\in} (F_2, A)$ and $(F_1, A) \tilde{\cap} (F_2, A) = \Phi_A$, then (X, τ, A) is called a *soft* T_2*-space.*

Theorem 2.4.4. [20] Let (X, τ, A) be a soft topological space over X. Then each soft point is a soft closed if and only if (X, τ, A) is a soft T_1-space.

Theorem 2.4.5. [20] Every soft T_2-space is a soft T_1-space and every soft T_1-space is a soft T_0-space.

Remark 2.4.6. [20] The converses of the above theorem need not be true in general as illustrated in the following examples:

Example 2.4.7. Let $X = \{x_1, x_2\}$, $A = \{e_1, e_2\}$ and $\tau = \{\Phi_A, X_A, (F_1, A), (F_2, A), (F_3, A), (F_4, A), (F_5, A)\}$, where

$$\begin{aligned}
(F_1, A) &= \{(e_1, \{x_1\}), (e_2, \{x_2\})\}, \\
(F_2, A) &= \{(e_1, \{x_1\})\}, \\
(F_3, A) &= \{(e_1, \{x_2\})\}, \\
(F_4, A) &= \{(e_1, \{x_1, x_2\})\}, \\
(F_5, A) &= \{(e_1, \{x_1, x_2\}), (e_2, \{x_2\})\}.
\end{aligned}$$

Then (X, τ, A) is a soft topological space over X. There are four of distinct soft points, namely $x_{1_{e_1}} = \{(e_1, \{x_1\})\}$, $x_{2_{e_1}} = \{(e_1, \{x_2\})\}$ and $x_{1_{e_2}} = \{(e_2, \{x_1\})\}$, $x_{2_{e_2}} = \{(e_2, \{x_2\})\}$. Now,

$$x_{1_{e_1}} \neq x_{2_{e_2}},\ x_{1_{e_1}} \tilde{\in} (F_2, A) \text{ and } x_{2_{e_2}} \tilde{\notin} (F_2, A).$$
$$x_{1_{e_1}} \neq x_{2_{e_1}},\ x_{1_{e_1}} \tilde{\in} (F_2, A) \text{ and } x_{2_{e_1}} \tilde{\notin} (F_2, A).$$
$$x_{1_{e_1}} \neq x_{1_{e_2}},\ x_{1_{e_1}} \tilde{\in} (F_2, A) \text{ and } x_{1_{e_2}} \tilde{\notin} (F_2, A).$$
$$x_{2_{e_2}} \neq x_{2_{e_1}},\ x_{2_{e_2}} \tilde{\in} (F_1, A) \text{ and } x_{2_{e_1}} \tilde{\notin} (F_1, A).$$
$$x_{2_{e_2}} \neq x_{1_{e_2}},\ x_{2_{e_2}} \tilde{\in} (F_1, A) \text{ and } x_{2_{e_2}} \tilde{\notin} (F_1, A).$$
$$x_{2_{e_1}} \neq x_{1_{e_2}},\ x_{2_{e_1}} \tilde{\in} (F_3, A) \text{ and } x_{1_{e_2}} \tilde{\notin} (F_3, A).$$

This shows that (X, τ, A) is a soft T_0-space. Clearly (X, τ, A) is not a soft T_1-space.

Example 2.4.8. Let $X = A = \mathbb{R}$ and $\tau = \{\phi_A\} \tilde{\cup} \{(F, A) \tilde{\in} SS(X)_A : (F(e))^c$ *is a countable subset of* X *for every* $e \in A\}$. We will show that τ is a soft topology on X.

(1) Obviously ϕ_A, X_A $\tilde{\in} \tau$.

(2) Let $(F, A), (G, A) \tilde{\in} \tau$. Then for $e \in A$, $(F(e))^c, (G(e))^c$ are two countable subset of X. Since $(F(e))^c \cup (G(e))^c = (F(e) \cap G(e))^c = (((F, A) \cap (G, A))(e))^c$ is a countable subset of X, $(F, A) \tilde{\cap} (G, A) \tilde{\in} \tau$.

(3) Let $(F_i, A) \tilde{\in} \tau$ $\forall i \in I$. Then $(F_i(e))^c$ is a countable subset of X $\forall i \in I$. Since $\bigcap_{i \in I} (F_i(e))^c = \left(\bigcup_{i \in I} (F_i(e)\right)^c = \left(\bigcup_{i \in I} (F_i, A)(e)\right)^c$ is a countable subset of X, $\bigcup_{i \in I} (F_i, A) \tilde{\in} \tau$.

Thus, (X, τ, A) is a soft topological space over X. We will call it *soft co-countable topology*. Since every soft point over X is a soft closed set, by Theorem 2.4.4, (X, τ, A) is a soft T_1-space. On the other hand, if (X, τ, A) is a soft T_2-space, then for every two distinct soft points $x_e, y_{e'}$ there exist $(F, A), (G, A) \tilde{\in} \tau$ such that $x_e \tilde{\in} (F, A), y_{e'} \tilde{\in} (G, A)$ and $(F, A) \tilde{\cap} (G, A) = \phi_A$. Hence, $(F(e))^c \cup (G(e'))^c = X$. A contradiction.

Theorem 2.4.9. [20] Let (X, τ, A) be a soft topological space over X and $Y \subseteq X$. If (X, τ, A) is a soft T_2-space (resp. a soft T_1-space) (resp. soft T_0-space), then (Y, τ_Y, A) is a soft T_2-space (resp. a soft T_1-space) (resp. a soft T_0-space).

Definition 2.4.10. [20] Let (X, τ, A) be a soft topological space over X. If for each (K, A) a soft closed set in (X, τ, A) and for each $x_e \tilde{\in} X_A$ such that $x_e \tilde{\notin} (K, A)$, there exist soft open sets (F_1, A) and (F_2, A) such that $x_e \tilde{\in} (F_1, A)$, $(K, A) \tilde{\subseteq} (F_2, A)$ and $(F_1, A) \tilde{\cap} (F_2, A) = \Phi_A$, then (X, τ, A) is called a *soft regular space.* A soft T_1 regular space is called a *soft T_3-space.*

Theorem 2.4.11. [20] Let (X, τ, A) be a soft topological space over X. Then the following statements are equivalent:

(a) (X, τ, A) is soft regular.

(b) For any soft open set (F, A) in (X, τ, A) and $x_e \tilde{\in} (F, A)$, there is a soft open set (G, A) containing x_e such that $x_e \tilde{\in} \overline{(G, A)} \tilde{\subseteq} (F, A)$.

(c) Each soft point in (X, τ, A) has a soft nhood base consisting of soft closed sets.

Theorem 2.4.12. [20] Let (X, τ, A) be a soft regular space over X. Then every soft subspace of (X, τ, A) is soft regular space.

Theorem 2.4.13. [20] Let (X, τ, A) be a soft topological space over X. A space (X, τ, A) is soft regular if and only if for each $x_e \tilde{\in} X_A$ and a soft closed set (F, A) in (X, τ, A) such that $x_e \tilde{\notin} (F, A)$, there exist soft open sets (F_1, A), (F_2, A) in (X, τ, A) such that $x_e \tilde{\in} (F_1, A)$, $(F, A) \tilde{\subseteq} (F_2, A)$ and $\overline{(F_1, A)} \tilde{\cap} \overline{(F_2, A)} = \Phi_A$.

Theorem 2.4.14. [20] Let (X, τ, A) be a soft topological space over X and $Y \subseteq X$. If (X, τ, A) is a soft T_3-space, then (Y, τ_Y, A) is a soft T_3-space.

Definition 2.4.15. [26] Let (X, τ, A) be a soft topological space over X, (F, A) and (G, A) soft closed sets over X such that $(F, A) \tilde{\cap} (G, A) = \Phi_A$. If there exist soft open sets (F_1, A) and (F_2, A) such that $(F, A) \tilde{\subseteq} (F_1, A)$, $(G, A) \tilde{\subseteq} (F_2, A)$ and $(F_1, A) \tilde{\cap} (F_2, A) = \Phi_A$, then (X, τ, A) is called a *soft normal space.* A soft T_1 normal space is called a *soft T_4-space.*

Theorem 2.4.16. [20] A soft topological space (X, τ, A) is soft normal if and only if for any soft closed set (F, A) and soft open set (G, A) such that $(F, A) \tilde{\subset} (G, A)$, there exists at least one soft open set (H, A) such that $(F, A) \tilde{\subseteq} (H, A) \tilde{\subseteq} \overline{(H, A)} \tilde{\subseteq} (G, A)$.

Theorem 2.4.17. [20] A soft closed subspace of a soft normal space is soft normal.

Corollary 2.4.18. [20] Every soft closed subspace of a soft T_4-space is a soft T_4-space.

Chapter 3

Generalized Soft Open Sets

3.1 Soft Semi-open and Soft Pre-open Sets

Definition 3.1.1. [16] A soft set (F, A) in a soft topological space (X, τ, A) is called *soft semi-open* if and only if there exists a soft open set (O, A) such that $(O, A)\tilde{\subseteq}(F, A)\tilde{\subseteq}\overline{(O, A)}$.

Theorem 3.1.2. [16] A soft set (F, A) in a soft topological space (X, τ, A) is soft semi-open if and only if $(F, A)\tilde{\subseteq}\overline{(F, A)^{\circ}}$.

Proof. ($\Rightarrow$) Let (F, A) be a soft semi-open set. Then $(O, A)\tilde{\subseteq}(F, A)\tilde{\subseteq}\overline{(O, A)}$, for some soft open set (O, A). But $(O, A)\tilde{\subseteq}(F, A)^{\circ}$, so $\overline{(O, A)}\tilde{\subseteq}\overline{(F, A)^{\circ}}$. Hence $(F, A)\tilde{\subseteq}\overline{(O, A)}\tilde{\subseteq}\overline{(F, A)^{\circ}}$.
($\Leftarrow$) Let $(F, A)\tilde{\subseteq}\overline{(F, A)^{\circ}}$. Then for $(O, A) = (F, A)^{\circ}$, we have $(O, A)\tilde{\subseteq}(F, A)\tilde{\subseteq}\overline{(O, A)}$. □

Remark 3.1.3. [16] Every soft open set in a soft topological space (X, τ, A) is a soft semi-open set. But the converse need not be true in general as illustrated in the following example:

Example 3.1.4. [16] Let $X = \{h_1, h_2, h_3\}$, $A = \{e_1, e_2\}$ and $\tau = \{\phi_A, X_A, (F_1, A), (F_2, A),$

$(F_3, A), (F_4, A), (F_5, A), (F_6, A), (F_7, A)\}$, where

$$\begin{aligned}
(F_1, A) &= \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_2\})\}, \\
(F_2, A) &= \{(e_1, \{h_2\}), (e_2, \{h_1, h_3\})\}, \\
(F_3, A) &= \{(e_1, \{h_2, h_3\}), (e_2, \{h_1\})\}, \\
(F_4, A) &= \{(e_1, \{h_2\}), (e_2, \{h_1\})\}, \\
(F_5, A) &= \{(e_1, \{h_1, h_2\}), (e_2, X)\}, \\
(F_6, A) &= \{(e_1, X), (e_2, \{h_1, h_2\})\}, \\
(F_7, A) &= \{(e_1, \{h_2, h_3\}), (e_2, \{h_1, h_3\})\}.
\end{aligned}$$

Then τ defines a soft topology over X. Consider a soft set (G, A) in (X, τ, A) which is defined as follows:
$G(e_1) = \{h_2, h_3\}$ and $G(e_2) = \{h_1, h_2\}$. Then, for the soft open set (F_3, A), we have $(F_3, A)\tilde{\subseteq}(G, A)$. Since $\overline{(F_3, A)} = X_A$, we have $(F_3, A)\tilde{\subseteq}(G, A)\tilde{\subseteq}\overline{(F_3, A)}$. Then the soft set (G, A) is a soft semi-open set in the soft topological space (X, τ, A), but is not a soft open set because $(G, A)\tilde{\notin}\tau$.

Theorem 3.1.5. [16] Let $\{(F_\alpha, A) : \alpha \in \Delta\}$ be a collection of soft semi-open sets in (X, τ, A). Then $\tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A)$ is soft semi-open set.

Proof. For each $\alpha \in \Delta$, we have a soft open set (O_α, A) such that $(O_\alpha, A)\tilde{\subseteq}(F_\alpha, A)\tilde{\subseteq}\overline{(O_\alpha, A)}$. Then $\tilde{\bigcup}_{\alpha\in\Delta}(O_\alpha, A)\tilde{\subseteq}\tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A)\tilde{\subseteq}\tilde{\bigcup}_{\alpha\in\Delta}\overline{(O_\alpha, A)}\tilde{\subseteq}\overline{\tilde{\bigcup}_{\alpha\in\Delta}(O_\alpha, A)}$, and $\tilde{\bigcup}_{\alpha\in\Delta}(O_\alpha, A)$ is soft open set. □

Remark 3.1.6. [21] The intersection of even two soft semi-open sets need not be soft semi-open set as illustrated in the following example:

Example 3.1.7. [21] Let $X = \{x_1, x_2, x_3\}$, $A = \{e_1, e_2\}$ and $\tau = \{X_A, \phi_A, (F_1, A), (F_2, A), (F_3, A)\}$, where

$$\begin{aligned}
(F_1, A) &= \{(e_1, \{x_1\}), (e_2, \{x_1\})\}, \\
(F_2, A) &= \{(e_1, \{x_2\}), (e_2, \{x_2\})\}, \\
(F_3, A) &= \{(e_1, \{x_1, x_2\}), (e_2, \{x_1, x_2\})\}.
\end{aligned}$$

Then τ defines a soft topology over X. So, the soft sets (G, A) and (H, A) which defines as follows: $G(e_1) = \{x_2, x_3\}$, $G(e_2) = \{x_2, x_3\}$ and $H(e_1) = \{x_1\}$, $H(e_2) = \{x_1, x_3\}$ are soft semi-open sets over X, but their intersection $(G, A)\tilde{\cap}(H, A) = (M, A)$ where $M(e_1) = \{\emptyset\}$ and $M(e_2) = \{x_3\}$ is not soft semi-open set.

Theorem 3.1.8. [16] Let (F, A) be a soft semi-open set in a soft topological space (X, τ, A) and suppose $(F, A)\tilde{\subseteq}(B, A)\tilde{\subseteq}\overline{(F, A)}$. Then (B, A) is a soft semi-open set.

Proof. There exists a soft open set (O, A) such that $(O, A)\tilde{\subseteq}(F, A)\tilde{\subseteq}(B, A)\tilde{\subseteq}\overline{(F, A)}\tilde{\subseteq}\overline{(O, A)}$. Hence $(O, A)\tilde{\subseteq}(B, A)\tilde{\subseteq}\overline{(O, A)}$ and (B, A) is a soft semi-open set. □

Definition 3.1.9. [16] A soft set (B, A) in a soft topological space (X, τ, A) is called *soft semi-closed set* if its complement is soft a semi-open set. Equivalently, there exists a soft closed set (C, A) such that $(C, A)^\circ\tilde{\subseteq}(B, A)\tilde{\subseteq}(C, A)$.

Remark 3.1.10. [16] Every soft closed set in a soft topological space (X, τ, A) is a soft semi-closed set. But the converse need not be true in general as illustrated in the following example:

Example 3.1.11. [16] Consider the soft topological space (X, τ, A) in Example 3.1.4. Then $(B, A) = (G, A)^c$ where $B(e_1) = \{h_1\}$ and $B(e_2) = \{h_3\}$ is a soft semi-closed because (G, A) is a soft semi-open set in Example 3.1.4. So (B, A) is a soft semi-closed set, which is not soft closed set.

Theorem 3.1.12. [16] A soft set (B, A) in a soft topological space (X, τ, A) is a soft semi-closed set if and only if $\left(\overline{(B, A)}\right)^\circ \tilde{\subseteq}(B, A)$.

Proof. ($\Rightarrow$) Let (B, A) be a soft semi-closed set. Then $(F, A)^\circ\tilde{\subseteq}(B, A)\tilde{\subseteq}(F, A)$, for some soft closed set (F, A). But $\overline{(B, A)}\tilde{\subseteq}\overline{(F, A)} = (F, A)$ and $\left(\overline{(B, A)}\right)^\circ \tilde{\subseteq}(F, A)^\circ$. Hence $\left(\overline{(B, A)}\right)^\circ \tilde{\subseteq}(F, A)^\circ\tilde{\subseteq}(B, A)$.
($\Leftarrow$) Let $\left(\overline{(B, A)}\right)^\circ \tilde{\subseteq}(B, A)$. Then for $(F, A) = \overline{(B, A)}$, we have $(F, A)^\circ\tilde{\subseteq}(B, A)\tilde{\subseteq}(F, A)$.
□

Theorem 3.1.13. [16] Let $\{(B_\alpha, A) : \alpha \in \Delta\}$ be a collection of soft semi-closed sets in (X, τ, A). Then $\tilde{\bigcap}_{\alpha\in\Delta}(B_\alpha, A)$ is a soft semi-closed set.

Proof. For each $\alpha \in \Delta$, we have a soft closed set (F_α, A) such that $(F_\alpha, A)^\circ \tilde{\subseteq} (B_\alpha, A) \tilde{\subseteq} (F_\alpha, A)$. Then $\left(\tilde{\bigcap}_{\alpha\in\Delta}(F_\alpha, A)\right)^\circ \tilde{\subseteq} \tilde{\bigcap}_{\alpha\in\Delta}(F_\alpha, A)^\circ \tilde{\subseteq} \tilde{\bigcap}_{\alpha\in\Delta}(B_\alpha, A) \tilde{\subseteq} \tilde{\bigcap}_{\alpha\in\Delta}(F_\alpha, A)$. Since $\tilde{\bigcap}_{\alpha\in\Delta}(F_\alpha, A)$ is a soft closed set, $\tilde{\bigcap}_{\alpha\in\Delta}(B_\alpha, A)$ is a soft semi-closed set. □

Theorem 3.1.14. [16] Let (B, A) be a soft semi-closed set in a soft topological space (X, τ, A) and suppose $(B, A)^\circ \tilde{\subseteq} (C, A) \tilde{\subseteq} (B, A)$. Then (C, A) is a soft semi-closed set.

Proof. There exists a soft closed set (F, A) such that $(F, A)^\circ \tilde{\subseteq} (B, A) \tilde{\subseteq} (F, A)$. Then $(C, A) \tilde{\subseteq} (F, A)$. But $((F, A)^\circ)^\circ = (F, A)^\circ \tilde{\subseteq} (B, A)^\circ$ and thus $(F, A)^\circ \tilde{\subseteq} (C, A)$. Hence $(F, A)^\circ$ $\tilde{\subseteq} (C, A) \tilde{\subseteq} (F, A)$. Hence (C, A) is a soft semi-closed set. □

Definition 3.1.15. [32] A soft set (F, A) in a soft topological space (X, τ, A) is called *soft pre-open* if and only if there exists a soft open set (O, A) such that $(F, A) \tilde{\subseteq} (O, A) \tilde{\subseteq} \overline{(F, A)}$.

Theorem 3.1.16. [32] A soft set (F, A) in a soft topological space (X, τ, A) is a soft pre-open if and only if $(F, A) \tilde{\subseteq} \left(\overline{(F, A)}\right)^\circ$.

Proof. ($\Rightarrow$) Let (F, A) be a soft pre-open set. Then $(F, A) \tilde{\subseteq} (O, A) \tilde{\subseteq} \overline{(F, A)}$, for some soft open set (O, A). Then $(O, A) \tilde{\subseteq} \left(\overline{(F, A)}\right)^\circ$. Hence $(F, A) \tilde{\subseteq} \left(\overline{(F, A)}\right)^\circ$.
($\Leftarrow$) Let $(F, A) \tilde{\subseteq} \left(\overline{(F, A)}\right)^\circ$. Then for $(O, A) = \left(\overline{(F, A)}\right)^\circ$, we have $(F, A) \tilde{\subseteq} (O, A) \tilde{\subseteq} \overline{(F, A)}$. □

Remark 3.1.17. [2] Every soft open set in a soft topological space (X, τ, A) is a soft pre-open set. But the converse need not be true in general as illustrated in the following example:

Example 3.1.18. [36] Let $X = \{x_1, x_2, x_3\}$, $A = \{e_1, e_2\}$ and $\tau = \{\phi_A, X_A, (F_1, A), (F_2, A), \ldots, (F_7, A)\}$ where

$$\begin{aligned}
(F_1, A) &= \{(e_1, \{x_1, x_2\}), (e_2, \{x_1, x_2\})\}, \\
(F_2, A) &= \{(e_1, \{x_2\}), (e_2, \{x_1, x_3\})\}, \\
(F_3, A) &= \{(e_1, \{x_2, x_3\}), (e_2, \{x_1\})\}, \\
(F_4, A) &= \{(e_1, \{x_2\}), (e_2, \{x_1\})\}, \\
(F_5, A) &= \{(e_1, \{x_1, x_2\}), (e_2, X)\}, \\
(F_6, A) &= \{(e_1, X), (e_2, \{x_1, x_2\})\}, \\
(F_7, A) &= \{(e_1, \{x_2, x_3\}), (e_2, \{x_1, x_3\})\}.
\end{aligned}$$

Then τ defines a soft topology on X and thus (X, τ, A) is a soft topological space over X. Let $(G, A) = \{(e_1, \{\emptyset\}), (e_2, \{x_1\})\}$. Then $\overline{(G, A)} = X_A$, $\left(\overline{(G, A)}\right)^\circ = X_A$, and so $(G, A)\tilde{\subseteq}\left(\overline{(G, A)}\right)^\circ$. Hence (G, A) is a soft pre-open set, which is not soft open set.

Theorem 3.1.19. [33] Let $\{(F_\alpha, A) : \alpha \in \Delta\}$ be a collection of soft pre-open sets in (X, τ, A). Then $\tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A)$ is a soft pre-open set.

Proof. For each $\alpha \in \Delta$, we have a soft pre-open set (O_α, A) such that $(F_\alpha, A)\tilde{\subseteq}(O_\alpha, A)\tilde{\subseteq}\overline{(F_\alpha, A)}$. Then $\tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A) \tilde{\subseteq} \tilde{\bigcup}_{\alpha\in\Delta}(O_\alpha, A) \tilde{\subseteq} \tilde{\bigcup}_{\alpha\in\Delta}\overline{(F_\alpha, A)} \tilde{\subseteq} \overline{\tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A)}$. □

Remark 3.1.20. [21] The intersection of even two soft pre-open sets need not be a soft pre-open set as illustrated in the following example:

Example 3.1.21. [21] Let $X = \{x_1, x_2, x_3\}$, $A = \{e_1, e_2\}$ and $\tau = \{X_A, \phi_A, (F, A)\}$ where $F(e_1) = \{x_1, x_3\}$ and $F(e_2) = \{x_2, x_3\}$. Then τ defines a soft topology over X. So, the soft sets (G, A) and (H, A) which defines as follows: $G(e_1) = \{x_1, x_2\}$, $G(e_2) = \{x_1\}$ and $H(e_1) = \{x_2, x_3\}$, $H(e_2) = \{x_1\}$ are soft pre-open sets over X, but their intersection $(G, A)\tilde{\cap}(H, A) = (M, A)$ where $M(e_1) = \{x_2\}$ and $M(e_2) = \{x_1\}$ is not a soft pre-open set.

Theorem 3.1.22. [32] Let (M, A) be a soft pre-open set in a soft topological space (X, τ, A) and suppose $(F, A)\tilde{\subseteq}(M, A)\tilde{\subseteq}\overline{(F, A)}$. Then (F, A) is a soft pre-open set.

Proof. There exists a soft open set (O, A) such that $(M, A)\tilde{\subseteq}(O, A)\tilde{\subseteq}\overline{(M, A)}$. Then $(F, A)\tilde{\subseteq}(O, A)\tilde{\subseteq}\overline{(M, A)}$. But $\overline{(M, A)}\tilde{\subseteq}\overline{(F, A)}$. Hence $(F, A)\tilde{\subseteq}(O, A)\tilde{\subseteq}\overline{(F, A)}$ and (F, A) is a soft pre-open set. □

Definition 3.1.23. [32] A soft set (B, A) in a soft topological space (X, τ, A) is called *soft pre-closed* set if its complement is soft pre-open set. Equivalently, there exists a soft closed set (C, A) such that $(B, A)^\circ\tilde{\subseteq}(C, A)\tilde{\subseteq}(B, A)$.

Theorem 3.1.24. [32] A soft set (B, A) in a soft topological space (X, τ, A) is a soft pre-closed set if and only if $\overline{(B, A)^\circ}\tilde{\subseteq}(B, A)$.

Proof. ($\Rightarrow$) Let (B, A) be a soft pre-closed set. Then $(B, A)^\circ\tilde{\subseteq}(C, A)\tilde{\subseteq}(B, A)$, for some soft closed set (C, A). But $\overline{(B, A)^\circ}\tilde{\subseteq}(C, A)$. Hence $\overline{(B, A)^\circ}\tilde{\subseteq}(B, A)$.
($\Leftarrow$) Let $\overline{(B, A)^\circ}\tilde{\subseteq}(B, A)$. Then for $(C, A) = \overline{(B, A)^\circ}$, we have $(B, A)^\circ\tilde{\subseteq}(C, A)\tilde{\subseteq}(B, A)$. □

Remark 3.1.25. [2] Every soft closed set in a soft topological space (X, τ, A) is a soft pre-closed set. But the converse need not be true in general as illustrated in the following example:

Example 3.1.26. [2] Consider the soft topological space (X, τ, A) in Example 3.1.18. Then $(G, A)^c = \{(e_1, X), (e_2, \{x_2, x_3\})\}$ is a soft pre-closed set, which is not soft closed set.

Theorem 3.1.27. [33] Let $\{(B_\alpha, A) : \alpha \in \Delta\}$ be a collection of soft pre-closed sets in (X, τ, A). Then $\tilde{\bigcap}_{\alpha\in\Delta}(B_\alpha, A)$ is a soft pre-closed set.

Proof. For each $\alpha \in \Delta$, we have a soft closed set (C_α, A) such that $(B_\alpha, A)^\circ \tilde{\subseteq} (C_\alpha, A) \tilde{\subseteq} (B_\alpha, A)$. Then $\left(\tilde{\bigcap}_{\alpha\in\Delta}(B_\alpha, A)\right)^\circ \tilde{\subseteq} \tilde{\bigcap}_{\alpha\in\Delta}(B_\alpha, A)^\circ \tilde{\subseteq} \tilde{\bigcap}_{\alpha\in\Delta}(C_\alpha, A) \tilde{\subseteq} \tilde{\bigcap}_{\alpha\in\Delta}(B_\alpha, A)$. Since $\tilde{\bigcap}_{\alpha\in\Delta}(C_\alpha, A)$ is a soft closed set, $\tilde{\bigcap}_{\alpha\in\Delta}(B_\alpha, A)$ is soft pre-closed set. □

Theorem 3.1.28. [32] Let (C, A) be a soft pre-closed set in a soft topological space (X, τ, A) and suppose $(B, A)^\circ \tilde{\subseteq} (C, A) \tilde{\subseteq} (B, A)$. Then (B, A) is a soft pre-closed set.

Proof. There exists a soft closed set (F, A) such that $(C, A)^\circ \tilde{\subseteq} (F, A) \tilde{\subseteq} (C, A)$. Then $(F, A) \tilde{\subseteq} (B, A)$. But $(B, A)^\circ \tilde{\subseteq} (C, A)^\circ \tilde{\subseteq} (F, A)$. So, $(B, A)^\circ \tilde{\subseteq} (F, A) \tilde{\subseteq} (B, A)$. Hence (B, A) is soft pre-closed set. □

3.2 Soft α-open and Soft β-open Sets

Definition 3.2.1. [19] In a soft topological space (X, τ, A), a soft set (G, A) is called *soft α-open* set if $(G, A) \tilde{\subseteq} \left(\overline{(G, A)^\circ}\right)^\circ$.

Theorem 3.2.2. [19] (G, A) is a soft α-open set if and only if there exists a soft open set (H, A) such that $(H, A) \tilde{\subseteq} (G, A) \tilde{\subseteq} \left(\overline{(H, A)}\right)^\circ$.

Proof. $(\Rightarrow)$ Let (G, A) be a soft α-open set. Then $(G, A) \tilde{\subseteq} \left(\overline{(G, A)^\circ}\right)^\circ$. Let $(G, A)^\circ = (H, A)$. Since $(G, A)^\circ \tilde{\subseteq} (G, A)$, $(H, A) \tilde{\subseteq} (G, A)$ and also, $(G, A) \tilde{\subseteq} \left(\overline{(H, A)}\right)^\circ$. Hence there exists a soft open set (H, A) such that $(H, A) \tilde{\subseteq} (G, A) \tilde{\subseteq} \left(\overline{(H, A)}\right)^\circ$.

($\Leftarrow$) Suppose there exists a soft open set (H,A) such that $(H,A)\tilde{\subseteq}(G,A)\tilde{\subseteq}\left(\overline{(H,A)}\right)^{\circ}$. Since $(G,A)\tilde{\subseteq}\left(\overline{(H,A)}\right)^{\circ} = \left(\overline{(H,A)^{\circ}}\right)^{\circ} \tilde{\subseteq}\left(\overline{(G,A)^{\circ}}\right)^{\circ}$, (G,A) is a soft α-open set. □

Remark 3.2.3. Every soft open set in a soft topological space (X,τ,A) is a soft α-open set. But the converse need not be true in general as illustrated in the following example:

Example 3.2.4. [19] Let $X=\{x_1,x_2,x_3\}$, $A=\{e_1,e_2,e_3\}$ and $\tau=\{\phi_A, X_A, (F_1,A), (F_2,A), (F_3,A), (F_4,A)\}$ where

$$\begin{aligned}(F_1,A) &= \{(e_1,X),(e_2,\{x_2,x_3\}),(e_3,\{x_1,x_2\})\},\\ (F_2,A) &= \{(e_2,\{x_1\}),(e_3,\{x_3\})\},\\ (F_3,A) &= \{(e_1,\{x_2\}),(e_2,\{x_1,x_3\}),(e_3,\{x_3\})\},\\ (F_4,A) &= \{(e_1,\{x_2\}),(e_2,\{x_3\})\}.\end{aligned}$$

Then τ defines a soft topology on X and thus (X,τ,A) is a soft topological space over X. Let $(G,A)=\{(e_1,\{x_1,x_2\}),(e_2,\{x_3\}),(e_3,\{x_2\})\}$. Then $(G,A)^{\circ}=(F_4,A)$, $\overline{(G,A)^{\circ}} = (F_1,A)$ and $\left(\overline{(G,A)^{\circ}}\right)^{\circ} = (F_1,A)$, and so $(G,A)\tilde{\subseteq}\left(\overline{(G,A)^{\circ}}\right)^{\circ}$. Hence (G,A) is a soft α-open set. On the other hand (G,A) is not a soft open set.

Theorem 3.2.5. [19] An arbitrary union of soft α-open sets of a soft topological space (X,τ,A) is a soft α-open set.

Proof. Let $\{(G_\alpha,A):\alpha\in\Delta\}$ be a collection of soft α-open sets of a soft topological space (X,τ,A). Then for each $\alpha\in\Delta$ there exists a soft open set (H_α,A) such that $(H_\alpha,A)\tilde{\subseteq}(G_\alpha,A)\tilde{\subseteq}\left(\overline{(H_\alpha,A)}\right)^{\circ}$. Hence $\tilde{\bigcup}_{\alpha\in\Delta}(H_\alpha,A)\tilde{\subseteq}\tilde{\bigcup}_{\alpha\in\Delta}(G_\alpha,A)\tilde{\subseteq}\tilde{\bigcup}_{\alpha\in\Delta}\left(\overline{(H_\alpha,A)}\right)^{\circ}\tilde{\subseteq}\left(\tilde{\bigcup}_{\alpha\in\Delta}\overline{(H_\alpha,A)}\right)^{\circ}\tilde{\subseteq}\left(\overline{\tilde{\bigcup}_{\alpha\in\Delta}(H_\alpha,A)}\right)^{\circ}$. Sine $\tilde{\bigcup}_{\alpha\in\Delta}(H_\alpha,A)$ is soft open set, $\tilde{\bigcup}_{\alpha\in\Delta}(G_\alpha,A)$ is a soft α-open set. □

Remark 3.2.6. [21] The intersection of even two soft α-open sets need not be a soft α-open set as illustrated in the following example:

Example 3.2.7. Consider the soft topological space (X,τ,A) in Example 3.1.7. The soft sets (G,A) and (H,A) which defines as follows: $G(e_1)=\{x_1,x_3\}$, $G(e_2)=\{x_1\}$ and $H(e_1)=\{x_2,x_3\}$, $H(e_2)=\{x_2\}$ are soft α-open sets over X, but their intersection $(G,A)\tilde{\cap}(H,A)=(M,A)$ where $M(e_1)=\{x_3\}$ and $M(e_2)=\emptyset$ is not soft α-open set.

Theorem 3.2.8. [19] Every soft α-open set is a soft semi-open set.

Proof. Let (G, A) be a soft α-open set. Then $(G, A)\tilde{\subseteq}\left(\overline{(G, A)^{\circ}}\right)^{\circ}\tilde{\subseteq}\overline{(G, A)^{\circ}}$ which implies that (G, A) is a soft semi-open set. □

Remark 3.2.9. [19] The converse of the above theorem need not be true in general as illustrated in the following example:

Example 3.2.10. Let $X = \{x_1, x_2, x_3\}$, $A = \{e_1\}$ and $\tau = \{\phi_A, X_A, (F_1, A), (F_2, A), (F_3, A)\}$ where $(F_1, A) = \{(e_1, \{x_1, x_2\})\}$, $(F_2, A) = \{(e_1, \{x_1\})\}$ and $(F_3, A) = \{(e_1, \{x_2\})\}$.
Then τ defines a soft topology on X and thus (X, τ, A) is a soft topological space over X. Let $(G, A) = \{(e_1, \{x_1, x_3\})\}$. Then $(G, A)^{\circ} = (F_2, A)$, $\overline{(G, A)^{\circ}} = \{(e_1, \{x_1, x_3\})\}$, and so $(G, A)\tilde{\subseteq}\overline{(G, A)^{\circ}}$. Hence (G, A) is a soft semi-open set. But since $\left(\overline{(G, A)^{\circ}}\right)^{\circ} = \{(e_1, \{x_1\})\}$, (G, A) is not soft α-open set.

Theorem 3.2.11. Every soft α-open set is a soft pre-open set.

Proof. Let (G, A) be a soft α-open set. Then $(G, A)\tilde{\subseteq}\left(\overline{(G, A)^{\circ}}\right)^{\circ}\tilde{\subseteq}\left(\overline{(G, A)}\right)^{\circ}$ which implies that (G, A) is a soft pre-open set. □

Remark 3.2.12. The converse of the above theorem need not be true in general as illustrated in the following example:

Example 3.2.13. Let $X = \{x_1, x_2, x_3\}$, $A = \{e_1\}$ and $\tau = \{\phi_A, X_A, (F_1, A), (F_2, A)\}$ where $(F_1, A) = \{(e_1, \{x_2, x_3\})\}$ and $(F_2, A) = \{(e_1, \{x_1\})\}$.
Then τ defines a soft topology over X and thus (X, τ, A) is a soft topological space. Let $(G, A) = \{(e_1, \{x_3\})\}$. Then $\overline{(G, A)} = (F_1, A)$ and $\left(\overline{(G, A)}\right)^{\circ} = (F_1, A)$, and so $(G, A)\tilde{\subseteq}\left(\overline{(G, A)}\right)^{\circ}$. Hence (G, A) is a soft pre-open set. Since $(G, A)^{\circ} = \phi_A$, (G, A) is not soft α-open set.

Definition 3.2.14. [19] A soft set (F, A) in a soft topological space (X, τ, A) is called *soft α-closed* set if its complement is a soft α-open set. Equivalently, if $\overline{\left(\overline{(F, A)}\right)^{\circ}}\tilde{\subseteq}(F, A)$.

Theorem 3.2.15. [19] (F, A) is a soft α-closed set if and only if there exists a soft closed set (K, A) such that $\overline{(K, A)^{\circ}}\tilde{\subseteq}(F, A)\tilde{\subseteq}(K, A)$.

Proof. (⇒) Let (F, A) be a soft α-closed set. Then $\overline{\left(\overline{(F, A)}\right)^\circ} \tilde{\subseteq} (F, A)$. Let $\overline{(F, A)} = (K, A)$. Then (K, A) is soft closed set. Since $(F, A) \tilde{\subseteq} \overline{(F, A)}$, $(F, A) \tilde{\subseteq} (K, A)$ and $\overline{\left(\overline{(F, A)}\right)^\circ} \tilde{\subseteq} (F, A)$ and so $\overline{(K, A)^\circ} \tilde{\subseteq} (F, A)$. Hence $\overline{(K, A)^\circ} \tilde{\subseteq} (F, A) \tilde{\subseteq} (K, A)$.

(⇐) Suppose there exists a soft closed set (K, A) such that $\overline{(K, A)^\circ} \tilde{\subseteq} (F, A) \tilde{\subseteq} (K, A)$. Since (K, A) is soft closed. $\overline{(K, A)} = (K, A)$. By hypothesis, $\overline{(K, A)^\circ} \tilde{\subseteq} (F, A)$. This implies that $\overline{\left(\overline{(K, A)}\right)^\circ} \tilde{\subseteq} (F, A)$. Since $(F, A) \tilde{\subseteq} (K, A)$, $\overline{(F, A)} \tilde{\subseteq} (K, A)$, and so $\overline{\left(\overline{(F, A)}\right)^\circ} \tilde{\subseteq} \overline{\left(\overline{(K, A)}\right)^\circ} \tilde{\subseteq} (F, A)$. Hence (F, A) is a soft α-closed set. □

Remark 3.2.16. Every soft closed set in a soft topological space (X, τ, A) is a soft α-closed set. But the converse need not be true in general as illustrated in the following example:

Example 3.2.17. Consider the soft topological space (X, τ, A) in Example 3.2.4. Then $(G, A) = \{(e_1, \{x_3\}), (e_2, \{x_1, x_3\}), (e_3, \{x_1, x_3\})\}$ is a soft α-closed set, which is not soft closed set.

Theorem 3.2.18. [19] An arbitrary intersection of soft α-closed sets is a soft α-closed set.

Proof. Let $\{(C_\alpha, A) : \alpha \in \Delta\}$ be a collection of soft α-closed set of a soft topological space (X, τ, A). Then $\{(C_\alpha, A)^c : \alpha \in \Delta\}$ is a collection of soft α-open sets. By Theorem 3.2.5, $\tilde{\bigcup}_{\alpha \in \Delta} (C_\alpha, A)^c = \left(\tilde{\bigcap}_{\alpha \in \Delta} (C_\alpha, A)\right)^c$ is soft α-open set. Hence $\tilde{\bigcap}_{\alpha \in \Delta} (C_\alpha, A)$ is soft α-closed set. □

Theorem 3.2.19. [19] Every soft α-closed set is a soft semi-closed set.

Proof. Let (F, A) be a soft α-closed set. Then $\left(\overline{(F, A)}\right)^\circ \tilde{\subseteq} \overline{\left(\overline{(F, A)}\right)^\circ} \tilde{\subseteq} (F, A)$. This implies that (F, A) is a soft semi-closed set. □

Remark 3.2.20. [19] The converse of the above theorem need not be true in general as illustrated in the following example:

Example 3.2.21. Consider the soft topological space (X, τ, A) in Example 3.2.13. Then $(G, A) = \{(e_1, \{x_2\})\}$ is a soft semi-closed set, which is not soft α-closed set.

Theorem 3.2.22. Every soft α-closed set is a soft pre-closed set.

Proof. Let (C, A) be a soft α-closed set. Then $\overline{((C, A)^{\circ})}\tilde{\subseteq}\overline{\left(\overline{(C, A)}\right)^{\circ}}\tilde{\subseteq}(C, A)$. This implies that (C, A) is a soft pre-closed set. □

Remark 3.2.23. The converse of the above theorem need not be true in general as illustrated in the following example:

Example 3.2.24. Consider the soft topological space (X, τ, A) in Example 3.2.13. Then $(C, A) = \{(e_1, \{x_1, x_2\})\}$ is a soft pre-closed set, which is not soft α-closed set.

Definition 3.2.25. [36] A soft set (F, A) in a soft topological space (X, τ, A) is called *soft β-open* set if $(F, A)\tilde{\subseteq}\overline{\left(\overline{(F, A)}\right)^{\circ}}$.

Definition 3.2.26. [36] A soft set (F, A) in a soft topological space (X, τ, A) is called soft β-closed set if $\left(\overline{(F, A)^{\circ}}\right)^{\circ}\tilde{\subseteq}(F, A)$.

Remark 3.2.27. [36] It is clear that the class of soft β-open sets contains each of classes of soft semi-open sets and soft pre-open sets. The implication between them and other types of soft open sets are given by Figure 3.1 below:

soft open set $\longrightarrow$ soft α-open set $\longrightarrow$ soft semi-open set
$\downarrow$ (from soft α-open set) $\downarrow$ (from soft semi-open set)
soft *pre*-open set $\longrightarrow$ soft β-open set

Figure 3.1: Relations between types of soft open sets

The converse of these implications do not hold, in general, as shown in the following example:

Example 3.2.28. [36] Let $X = \{x_1, x_2, x_3\}$, $A = \{e_1, e_2\}$ and $\tau = \{\phi_A, X_A, (F_1, A), (F_2, A),$

$\ldots(F_7, A)\}$ where

$$\begin{aligned}
(F_1, A) &= \{(e_1, \{x_1, x_2\}), (e_2, \{x_1, x_2\})\}, \\
(F_2, A) &= \{(e_1, \{x_2\}), (e_2, \{x_1, x_3\})\}, \\
(F_3, A) &= \{(e_1, \{x_2, x_3\}), (e_2, \{x_1\})\}, \\
(F_4, A) &= \{(e_1, \{x_2\}), (e_2, \{x_1\})\}, \\
(F_5, A) &= \{(e_1, \{x_1, x_2\}), (e_2, X)\}, \\
(F_6, A) &= \{(e_1, X), (e_2, \{x_1, x_2\})\}, \\
(F_7, A) &= \{(e_1, \{x_2, x_3\}), (e_2, \{x_1, x_3\})\}.
\end{aligned}$$

Then τ defines a soft topology on X and thus (X, τ, A) is a soft topological space over X. Let $(H, A) = \{(e_2, \{x_1\})\}$. Then, (H, A) is a soft pre-open set, but is not a soft α-open set, also it is a soft β-open set, which is not soft semi-open set.

Theorem 3.2.29. [36] An arbitrary union of soft β-open sets is a soft β-open set.

Proof. Let $\{(F_\alpha, A) : \alpha \in \Delta\}$ be a collection of soft β-open sets of a soft topological space (X, τ, A). Then for each $\alpha \in \Delta$ we have, $(F_\alpha, A)\tilde{\subseteq}\overline{\left(\overline{(F_\alpha, A)}\right)^\circ}$. Then $\tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A)\tilde{\subseteq}\tilde{\bigcup}_{\alpha\in\Delta}\overline{\left(\overline{(F_\alpha, A)}\right)^\circ}$ $\tilde{\subseteq}\overline{\left(\tilde{\bigcup}_{\alpha\in\Delta}\left(\overline{(F_\alpha, A)}\right)^\circ\right)}\tilde{\subseteq}\overline{\left(\tilde{\bigcup}_{\alpha\in\Delta}\overline{(F_\alpha, A)}\right)^\circ}\tilde{\subseteq}\overline{\left(\overline{\tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A)}\right)^\circ}$. Hence $\tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A)$ is a soft β-open set. □

Remark 3.2.30. [36] The intersection of even two soft β-open sets need not be a soft β-open set as illustrated in the following example:

Example 3.2.31. [36] Let $X = \{x_1, x_2\}$, $A = \{e_1, e_2\}$ and $\tau = \{\phi_A, X_A, (F_1, A), (F_1, A), (F_3, A)\}$ where

$$\begin{aligned}
(F_1, A) &= \{(e_1, \{x_1\}), (e_2, \{x_2\})\}, \\
(F_2, A) &= \{(e_1, X), (e_2, \{x_2\})\}, \\
(F_3, A) &= \{(e_1, \{x_1\}), (e_2, X)\}.
\end{aligned}$$

Then τ defines a soft topology over X and thus (X, τ, A) is a soft topological space over X. Consider the two soft sets (G, A) and (H, A) over X where$(G, A) = \{(e_1, x_2\}, (e_2, \{x_1\})\}$ and $(H, A) = \{(e_1, \{x_1, x_2\}), (e_2, \{x_2\})\}$. Then (G, A) and (H, A) are soft β-open sets over X. But $(G, A)\tilde{\cap}(H, A) = \{(e_1, \{x_2\})\} = (K, A)$ and $\overline{\left(\overline{(K, A)}\right)^{\circ}} = \phi_A$. Hence (K, A) is not soft β-open set.

Theorem 3.2.32. [36] An arbitrary intersection of soft β-closed sets is a soft β-closed set.

Proof. Let $\{(F_\alpha, A) : \alpha \in \Delta\}$ be a collection of soft β-closed sets. Then for each $\alpha \in \Delta$, we have $(F_\alpha, A)\tilde{\supseteq}\left(\overline{(F_\alpha, A)^{\circ}}\right)^{\circ}$. Then $\tilde{\bigcap}_{\alpha\in\Delta}(F_\alpha, A)\tilde{\supseteq}\tilde{\bigcap}_{\alpha\in\Delta}\left(\overline{(F_\alpha, A)^{\circ}}\right)^{\circ} \tilde{\supseteq} \left(\tilde{\bigcap}_{\alpha\in\Delta}\overline{(F_\alpha, A)^{\circ}}\right)^{\circ} \tilde{\supseteq} \left(\overline{\tilde{\bigcap}_{\alpha\in\Delta}(F_\alpha, A)^{\circ}}\right)^{\circ} \tilde{\supseteq} \left(\overline{\left(\tilde{\bigcap}_{\alpha\in\Delta}(F_\alpha, A)\right)^{\circ}}\right)^{\circ}$. Hence $\tilde{\bigcap}_{\alpha\in\Delta}(F_\alpha, A)$ is a soft β-closed set. □

Remark 3.2.33. [36] The union of even two soft β-closed sets need not be a soft β-closed set as illustrated in the following example:

Example 3.2.34. [36] Consider the soft topological space (X, τ, A) in Example 3.2.31. Define (G, A) and (H, A) as follows: $(G, A) = \{(e_1, \{x_1\}), (e_2, \{x_1\})\}$ and $(H, A) = \{(e_2, \{x_2\})\}$. Then, (G, A) and (H, A) are soft β-closed sets over X. Now $(G, A)\tilde{\cup}(H, A) = \{(e_1, \{x_1\}), (e_2, \{x_1, x_2\})\} = (K, A)$. But $\left(\overline{(K, A)^{\circ}}\right)^{\circ} = X_A$. Hence (K, A) is not soft β-closed set.

Theorem 3.2.35. [36] Each soft β-open set which is a soft semi-closed set is a soft semi-open set.

Proof. Let (F, A) be a soft β-open set and soft semi-closed set. Then $(F, A)\tilde{\subseteq}\overline{\left(\overline{(F, A)}\right)^{\circ}}$ and $\left(\overline{(F, A)}\right)^{\circ}\tilde{\subseteq}(F, A)$. Then $\left(\overline{(F, A)}\right)^{\circ}\tilde{\subseteq}(F, A)\tilde{\subseteq}\overline{\left(\overline{(F, A)}\right)^{\circ}}$. Since $\left(\overline{(F, A)}\right)^{\circ} = (U, A)$ is a soft open set, we can write $(U, A)\tilde{\subseteq}(F, A)\tilde{\subseteq}\overline{(U, A)}$. Hence (F, A) is a soft semi-open set. □

Corollary 3.2.36. [36] If a soft set (F, A) in a soft topological space (X, τ, A) is soft β-closed set and soft semi-open set, then (F, A) is a soft semi-closed set.

Proof. Direct from Theorem 3.2.35. □

Theorem 3.2.37. [36] Each soft β-open and soft α-closed set is a soft closed set.

Proof. Let (F,A) be a soft β-open and a soft α-closed set. Then $(F,A)\tilde{\subseteq}\overline{\left(\overline{(F,A)}\right)^{\circ}}$ and $\overline{\left(\overline{(F,A)}\right)^{\circ}}\tilde{\subseteq}(F,A)$. Then $\overline{\left(\overline{(F,A)}\right)^{\circ}}\tilde{\subseteq}(F,A)\tilde{\subseteq}\overline{\left(\overline{(F,A)}\right)^{\circ}}$, $(F,A)=\overline{\left(\overline{(F,A)}\right)^{\circ}}$ which is a soft closed set.

Corollary 3.2.38. [36] Each soft β-closed and soft α-open set is a soft open set.

Proof. Direct from Theorem 3.2.37. □

Theorem 3.2.39. Let (X,τ,A) be a soft topological space, (F,A) and (G,A) are soft sets over X. If (F,A) is a soft open set and (G,A) is a soft β-open set, then $(F,A)\tilde{\cap}(G,A)$ is a soft β-open set.

Proof. $(F,A)\tilde{\cap}(G,A)\tilde{\subseteq}(F,A)\tilde{\cap}\overline{\left(\overline{(G,A)}\right)^{\circ}}\tilde{\subseteq}\overline{\left((F,A)\tilde{\cap}\left(\overline{(G,A)}\right)^{\circ}\right)}=\overline{\left((F,A)\tilde{\cap}\overline{(G,A)}\right)^{\circ}}$ $\tilde{\subseteq}\overline{\left(\overline{(F,A)\tilde{\cap}(G,A)}\right)^{\circ}}$. Hence, $(F,A)\tilde{\cap}(G,A)$ is a soft β-open set. □

Theorem 3.2.40. Let (Y,τ_Y,A) be a soft subspace of a soft topological space (X,τ,A) and $(F,A)\tilde{\subseteq}Y_A$ such that Y_A is a soft β-open set over X. Then, if (F,A) is a soft β-open set over X, then (F,A) is a soft β-open set over Y.

Proof. Since (F,A) is a soft β-open set X, $(F,A)\tilde{\subseteq}\overline{\left(\overline{(F,A)}\right)^{\circ}}$. Since $(F,A)\tilde{\cap}Y_A=(F,A)$, $((F,A)\tilde{\cap}Y_A)\tilde{\subseteq}\overline{\left(\overline{(F,A)\tilde{\cap}Y_A}\right)^{\circ}}$. Hence (F,A) is a soft β-open set over Y. □

3.3 Soft Generalized Interior and Soft Generalized Closure

Definition 3.3.1. Let (X,τ,A) be a soft topological space over X and (F,A) a soft set over X. Then the *soft semi-interior* [16] (resp. the *soft pre-interior* [2], the *soft α-interior* [19], the *soft β-interior* [3]) denoted by $Int_s(F,A)$(resp. $Int_p(F,A)$, $Int_\alpha(F,A)$, $Int_\beta(F,A)$) is the union of all soft semi-open sets (resp. soft pre-open sets, soft α-open sets, soft β-open sets) contained in (F,A). Clearly $Int_s(F,A)$ (resp. $Int_p(F,A)$, $Int_\alpha(F,A)$, $Int_\beta(F,A)$) is the largest soft semi-open set (resp.soft pre-open set, soft α-open set, soft β-open set) contained in (F,A).

Definition 3.3.2. Let (X, τ, A) be a soft topological space over X and (B, A) a soft set over X. Then the *soft semi-closure* [16] (resp. the *soft pre-closure* [2], the *soft α-closure* [19], the *soft β-closure* [3]) denoted by $Cl_s(F, A)$(resp. $Cl_p(F, A)$, $Cl_\alpha(F, A)$, $Cl_\beta(F, A)$) is the intersection of all soft semi-closed sets (resp. soft pre-closed sets, soft α-closed sets, soft β-closed sets) supersets of (B, A). Clearly $Cl_s(F, A)$ (resp. $Cl_p(F, A)$, $Cl_\alpha(F, A)$, $Cl_\beta(F, A)$)is the smallest soft semi-closed set (resp. soft pre-closed set, soft α-closed set, soft β-closed set) which contains (B, A).

Because of the similarity in the dealing with the topics, we will integrate and treatment topic to the different concepts, taking into consideration that the theories relating to the soft semi-open sets (resp. soft pre-open sets, soft α-open sets, soft β-open sets) found in [16] (resp. [2], [19], [3]).

Theorem 3.3.3. Let (X, τ, A) be a soft topological space and (M, A) be a soft over X. Then for $t \in \{s, p, \alpha, \beta\}$, $(M, A)^\circ \tilde{\subseteq} Int_t(M, A) \tilde{\subseteq} (M, A) \tilde{\subseteq} Cl_t(M, A) \tilde{\subseteq} \overline{(M, A)}$.

Proof. Using Definition 3.3.1, Definition 3.3.2 and the facts that every soft open (resp. soft closed) set is a soft semi-open (resp. soft semi-closed) set.
The other cases can be proved in similar way. □

Theorem 3.3.4. Let (X, τ, A) be a soft topological space over X and (M, A) be a soft set over X. Then for $t \in \{s, p, \alpha, \beta\}$,

(1) $(Cl_t(M, A))^c = Int_t((M, A)^c)$.

(2) $(Int_t(M, A))^c = Cl_t((M, A)^c)$.

(3) $Int_t(M, A) = (Cl_t((M, A)^c))^c$.

Proof. $(Cl_s(M, A))^c = \left(\tilde{\bigcap}\{(F, A) : (F, A) \text{ } is \text{ } soft \text{ } semi-closed \text{ } and \text{ } (M, A) \tilde{\subseteq} (F, A)\}\right)^c =$ $\tilde{\bigcup}\{(F, A)^c : (F, A) \text{ } is \text{ } soft \text{ } semi{-}closed \text{ } and \text{ } (M, A) \tilde{\subseteq} (F, A)\} = \tilde{\bigcup}\{(F, A)^c : (F, A)^c \text{ } is \text{ } soft \text{ } semi{-}$ $open \text{ } and \text{ } (F, A)^c \tilde{\subseteq} (M, A)^c\} = Int_s((M, A)^c)$.
The other cases and parts can be proved in similar way. □

Theorem 3.3.5. Let (X, τ, A) be a soft topological space and let (F, A) and (G, A) be soft sets over X. Then for $t \in \{s, p, \alpha, \beta\}$,

(1) (F, A) is soft semi-closed (resp. pre-closed, α-closed, β-closed) set if and only if $(F, A) = Cl_t(F, A)$, for $t = s$ (resp. p, α, β).

(2) $Cl_t(\phi_A) = \phi_A$ and $Cl_t(X_A) = X_A$.

(3) $Cl_t(Cl_t(F, A)) = Cl_t(F, A)$.

(4) $(F, A)\tilde{\subseteq}(G, A)$ implies $Cl_t(F, A)\tilde{\subseteq}Cl_t(G, A)$.

(5) $Cl_t\left((F, A)\tilde{\cap}(G, A)\right)\tilde{\subseteq}Cl_t(F, A)\tilde{\cap}Cl_t(G, A)$.

(6) $Cl_t(F, A)\tilde{\cup}Cl_t(G, A)\tilde{\subseteq}Cl_t\left((F, A)\tilde{\cup}(G, A)\right)$

Proof. (1) ($\Rightarrow$) If (F, A) is a soft semi-closed set, then (F, A) is itself a soft semi-closed set over X which contains (F, A). So (F, A) is the smallest soft semi-closed set containing (F, A) and $(F, A) = Cl_s(F, A)$.
($\Leftarrow$) Suppose that $(F, A) = Cl_s(F, A)$. Since $Cl_s(F, A)$ is a soft semi-closed set, so (F, A) is soft semi-closed set.

(2) Since ϕ_A and X_A are soft semi-closed set, by part (1) $Cl_s(\phi_A) = \phi_A$ and $Cl_s(X_A) = X_A$.

(3) Since $Cl_s(F, A)$ is a soft semi-closed set therefore by part (2) we have $Cl_s(Cl_s(F, A)) = Cl_s(F, A)$.

(4) Suppose that $(F, A)\tilde{\subseteq}(G, A)$. Then every soft semi-closed super set (K, A) of (G, A) will also contains (F, A), and hence contains $Cl_s(F, A)$. Take $Cl_s(F, A) = (K, A)$. Then $Cl_s(F, A)\tilde{\subseteq}Cl_s(G, A)$.

(5) Since $((F, A)\tilde{\cap}(G, A))\tilde{\subseteq}(F, A)$ and $((F, A)\tilde{\cap}(G, A))\tilde{\subseteq}(G, A)$. So by part (4) $Cl_s((F, A)$ $\tilde{\cap}(G, A))\tilde{\subseteq}Cl_s(F, A)$ and $Cl_s((F, A)\tilde{\cap}(G, A))\tilde{\subseteq}Cl_s(G, A)$. Thus $Cl_s((F, A)\tilde{\cap}(G, A))\tilde{\subseteq}Cl_s(F, A)$ $\tilde{\cap}Cl_s(G, A)$.

(6) Since $(F, A)\tilde{\subseteq}(F, A)\tilde{\cup}(G, A)$ and $(G, A)\tilde{\subseteq}(F, A)\tilde{\cup}(G, A)$, by part (4) $Cl_s(F, A)$ $\tilde{\subseteq}Cl_s((F, A)\tilde{\cup}(G, A))$ and $Cl_s(G, A)\tilde{\subseteq}Cl_s((F, A)\tilde{\cup}(G, A))$. Hence, $Cl_s(F, A)\tilde{\cup}Cl_s(G, A)$ $\tilde{\subseteq}Cl_s((F, A)\tilde{\cup}(G, A))$.

The other cases can be proved in similar way. □

Theorem 3.3.6. Let (X, τ, A) be a soft topological space and let (F, A) and (G, A) be a soft sets over X. Then for $t \in \{s, p, \alpha, \beta\}$,

(1) (F, A) is soft semi-open (resp. pre-open, α-open, β-open) set if and only if $(F, A) = Int_t(F, A)$, for $t = s$ (resp. p, α, β).

(2) $Int_t(\phi_A) = \phi_A$ and $Int_t(X_A) = X_A$.

(3) $Int_t(Int_t(F, A)) = Int_t(F, A)$.

(4) $(F, A)\tilde{\subseteq}(G, A)$ implies $Int_t(F, A)\tilde{\subseteq}Int_t(G, A)$.

(5) $Int_t(F, A)\tilde{\cup}Int_t(G, A)\tilde{\subseteq}Int_t((F, A)\tilde{\cup}(G, A))$.

(6) $Int_t((F, A)\tilde{\cap}(G, A))\tilde{\subseteq}Int_t(F, A)\tilde{\cap}Int_t(G, A)$.

Proof. (1) ($\Rightarrow$) If (F, A) is soft semi-open set over X, then (F, A) is itself a soft semi-open set over X which contains (F, A). So $Int_s(F, A)$ is the largest soft semi-open set contained in (F, A) and $(F, A) = Int_s(F, A)$.
($\Leftarrow$) Suppose that $(F, A) = Int_s(F, A)$. Since $Int_s(F, A)$ is a soft semi-open set, (F, A) is a soft semi-open set over X.

(2) Since ϕ_A and X_A are soft semi-open sets, by part (1) $Int_s(\phi_A) = \phi_A$ and $Int_s(X_A) = X_A$.

(3) Since $Int_s(F, A)$ is a soft semi-open set therefore by part (2) we have $Int_s(Int_s(F, A)) = Int_s(F, A)$.

(4) Suppose that $(F, A)\tilde{\subseteq}(G, A)$. Since $Int_s(F, A)\tilde{\subseteq}(F, A)\tilde{\subseteq}(G, A)$. $Int_S(F, A)$ is a soft semi-open subset of (G, A), so by definition of $Int_s(G, A)$, $Int_s(F, A)\tilde{\subseteq}Int_s(G, A)$.

(5) Since $(F, A)\tilde{\subseteq}((F, A)\tilde{\cup}(G, A))$ and $(G, A)\tilde{\subseteq}((F, A)\tilde{\cup}(G, A))$. So by part (4), $Int_s(F, A)\tilde{\subseteq} Int_s((F, A)\tilde{\cup}(G, A))$ and $Int_s(G, A)\tilde{\subseteq}Int_s((F, A)\tilde{\cup}(G, A))$. Thus $Int_s(F, A)\tilde{\cup}Int_s(G, A)\tilde{\subseteq} Int_s((F, A)\tilde{\cup}(G, A))$.

(6) Since $(F, A)\tilde{\cap}(G, A)\tilde{\subseteq}(F, A)$ and $(F, A)\tilde{\cap}(G, A)\tilde{\subseteq}(G, A)$, by part (4) $Int_s((F, A) \tilde{\cap}(G, A)\tilde{\subseteq}Int_s(F, A)$ and $Int_s((F, A)\tilde{\cap}(G, A)\tilde{\subseteq}Int_s(G, A)$. Hence $Int_s((F, A)\tilde{\cap}(G, A)) \tilde{\subseteq}Int_s(F, A)\tilde{\cap}Int_s(G, A)$.

The other cases can be proved in similar way. □

Theorem 3.3.7. Let (X, τ, A) be a soft topological space and let (F, A) be a soft set over X. Then $\forall\ t \in \{s, p, \alpha, \beta\}$.

(1) $Int_t((F,A)^\circ) = \overline{(Int_t(F,A))}^{\circ} = (F,A)^\circ$.

(2) $Cl_t(\ \overline{(F,A)}\ ) = \overline{(Cl_t(F,A))} = \overline{(F,A)}$.

Proof. (1) Since $(F,A)^\circ$ is a soft open set, by Remark 3.1.3 we have $(F,A)^\circ$ is a soft semi-open set. So, by Theorem 3.3.6(1), $Int_s((F,A)^\circ) = (F,A)^\circ$. By Theorem 3.3.3, we have $(F,A)^\circ \tilde{\subseteq} Int_s(F,A) \tilde{\subseteq} (F,A)$, then we can get $(F,A)^\circ \tilde{\subseteq} (Int_s(F,A))^\circ \tilde{\subseteq} (F,A)^\circ$ and so $(Int_s(F,A))^{\circ} = (F,A)^\circ$.

(2) Since $\overline{(F,A)}$ is a soft closed set, by Remark 3.1.10 we have $\overline{(F,A)}$ is a soft semi-closed set. So, by Theorem 3.3.5(1) $Cl_s(\ \overline{(F,A)}\ ) = \overline{(F,A)}$. By Theorem 3.3.3, we have $(F,A) \tilde{\subseteq} Cl_s(F,A) \tilde{\subseteq} \overline{(F,A)}$, then we can get $\overline{(F,A)} \tilde{\subseteq} \overline{Cl_s(F,A)} \tilde{\subseteq} \overline{(F,A)}$ and so $\overline{(Cl_s(F,A))} = \overline{(F,A)}$.

The other cases can be proved in similar way. □

3.4 Several Types of Soft Continuity

Definition 3.4.1. Let (X, τ_1, A) and (Y, τ_2, B) be two soft topological spaces. A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is called:

(1) *soft semi-continuous* [23] (resp. *soft pre-continuous* [5], *soft α-continuous* [5], *soft β-continuous* [36]) function if for each soft open set (F,B) over Y, the inverse image $f_{pu}^{-1}(F,B)$ is a soft semi-open (resp. soft pre-open, soft α-open, soft β-open) set over X.

(2) *soft semi-irresolute* [23] (resp. *soft pre-irresolute*, *soft α-irresolute*, *soft β-irresolute* [36]) function if for each soft semi-open (resp. soft pre-open, soft α-open, soft β-open) set (G,B) over Y, the inverse image $f_{pu}^{-1}(G,B)$ is a soft semi-open (resp. soft pre-open, soft α-open, soft β-open) set over X.

(3) *soft semi-open* [23](resp. *soft pre-open* [5], *soft α-open* [5], *soft β-open* [3]) function if for each soft open set (K,A) over X, the image $f_{pu}(K,A)$ is a soft semi-open (resp. soft pre-open, soft α-open, soft β-open) set over X.

(4) *soft semi-closed* [23] (resp. *soft pre-closed*, *soft α-closed*, *soft β-closed*) function if for each soft closed set (C, A) over X, the image $f_{pu}(C, A)$ is a soft semi-closed (resp. soft pre-closed, soft α-closed, soft β-closed) set over X.

Theorem 3.4.2. A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is a soft semi-continuous [23](resp. a soft pre-continuous, a soft α-continuous, a soft β-continuous) if for each soft closed set (F, B) over Y, the inverse image $f_{pu}^{-1}(F, B)$ is a soft semi-closed [23] (resp. soft pre-closed, soft α-closed [5], soft β-closed) set over X.

Proof. Let (G, B) is a soft open set over Y. Then $(G, B)^c$ is a soft closed set over Y, by assumption, $f_{pu}^{-1}((G, B)^c) = \left(f_{pu}^{-1}(G, B)\right)^c$ is a soft semi-closed set over X. Hence, $f_{pu}^{-1}(G, B)$ is a soft semi-open set over X. Therefore, f_{pu} is a soft semi-continuous function. The other cases can be proved in similar way. □

Remark 3.4.3. A soft semi-continuous (resp. a soft pre-continuous, a soft α-continuous, a soft β-continuous) function is a soft semi-irresolute [23] (resp. a soft pre-irresolute, a soft α-irresolute, a soft β-irresolute) function.

Theorem 3.4.4. A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is a soft semi-continuous [23](resp. a soft pre-continuous, a soft α-continuous [5], a soft β-continuous) if and only if for $t = s$ (resp. p, α, β), $f_{pu}(Cl_t(F, A))\tilde{\subseteq}\overline{f_{pu}(F, A)}$ for every soft set (F, A) over X.

Proof. ($\Rightarrow$) Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft semi-continuous function. Now $\overline{f_{pu}(F, A)}$ is a soft closed set over Y, so by soft semi-continuity of f_{pu}, $f_{pu}^{-1}(\overline{f_{pu}(F, A)})$ is a soft semi-closed set over X and $(F, A)\tilde{\subseteq}f_{pu}^{-1}(\overline{f_{pu}(F, A)})$. But $Cl_s(F, A)$ is the smallest soft semi-closed set containing (F, A), so $Cl_s(F, A)\tilde{\subseteq}f_{pu}^{-1}(\overline{f_{pu}(F, A)})$ and so $f_{pu}(Cl_s(F, A))\tilde{\subseteq}\overline{f_{pu}(F, A)}$.
($\Leftarrow$) Let (F, B) be any soft closed set over Y. Then $f_{pu}^{-1}(F, B)$ is a soft set over X. Now $f_{pu}(Cl_s(f_{pu}^{-1}(F, B)))\tilde{\subseteq}\overline{f_{pu}(f_{pu}^{-1}(F, B))}$ implies $f_{pu}(Cl_s(f_{pu}^{-1}(F, B)))\tilde{\subseteq}\overline{(F, B)} = (F, B)$, and so $Cl_s(f_{pu}^{-1}(F, B)) = f_{pu}^{-1}(F, B)$. Hence $f_{pu}^{-1}(F, B)$ is a soft semi-closed set. Thus f_{pu} is a soft semi-continuous function.
The other cases can be proved in similar way. □

Theorem 3.4.5. A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is a soft semi-continuous [23](resp. a soft pre-continuous, a soft α-continuous [5], a soft β-continuous) if and only if for $t = s$ (resp. p, α, β), $f_{pu}^{-1}((H,B)^\circ)\tilde{\subseteq}Int_t(f_{pu}^{-1}(H,B))$ for every soft set (H,B) over Y.

Proof. ($\Rightarrow$) Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft semi-continuous function. Now $(H,B)^\circ$ is a soft open set over Y, so by soft semi-continuity of f_{pu}, $f_{pu}^{-1}((H,B)^\circ)$ is a soft semi-open set over X and $f_{pu}^{-1}((H,B)^\circ)\tilde{\subseteq}f_{pu}^{-1}(H,B)$. As $Int_s f_{pu}^{-1}(H,B)$ is the largest soft semi-open set contained in $f_{pu}^{-1}(H,B)$, $f_{pu}^{-1}((H,B)^\circ)\tilde{\subseteq}Int_s(f_{pu}^{-1}(H,B))$.
($\Leftarrow$) Let (H,B) be a soft open set over Y. Then $f_{pu}^{-1}((H,B)^\circ)\tilde{\subseteq}Int_s(f_{pu}^{-1}(H,B))$, and so $f_{pu}^{-1}(H,B)\tilde{\subseteq}Int_s(f_{pu}^{-1}(H,B))$. Hence $f_{pu}^{-1}(H,B) = Int_s(f_{pu}^{-1}(H,B))$. Thus $f_{pu}^{-1}(H,B)$ is a soft semi-open set over X.
The other cases can be proved in similar way. □

Theorem 3.4.6. A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is a soft semi-open [23](resp. a soft pre-open, a soft α-open [5], a soft β-open) if and only if for $t = s$ (resp. p, α, β) $f_{pu}((F,A)^\circ)\tilde{\subseteq}Int_t(f_{pu}(F,A))$ for every soft set (F,A) over X.

Proof. ($\Rightarrow$) If $f_{pu} : SS(X)_A \to SS(Y)_B$ is a soft semi-open function, then $f_{pu}((F,A)^\circ) = Int_s(f_{pu}(F,A)^\circ)\tilde{\subseteq}Int_s(f_{pu}(F,A))$.
($\Leftarrow$) Let (F,A) be a soft open set over X. Then $f_{pu}(F,A) = f_{pu}((F,A)^\circ)\tilde{\subseteq}Int_s(f_{pu}(F,A))$, and so $f_{pu}(F,A) = Int_s(f_{pu}(F,A))$. Hence $f_{pu}(F,A)$ is a soft semi-open set over Y.
The other cases can be proved in similar way. □

Theorem 3.4.7. A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is a soft semi-closed [23](resp. a soft pre-closed, a soft α-closed [5], a soft β-closed) if and only if for $t = s$ (resp. p, α, β), $Cl_t(f_{pu}(F,A))\tilde{\subseteq}f_{pu}(\overline{(F,A)})$ for every soft set (F,A) over X.

Proof. ($\Rightarrow$) If $f_{pu} : SS(X)_A \to SS(Y)_B$ is a soft semi-closed function, then $Cl_s(f_{pu}(F,A))$ $\tilde{\subseteq}Cl_s(f_{pu}\overline{(F,A)}) = f_{pu}(\overline{(F,A)})$.
($\Leftarrow$) Let (F,A) be a soft closed set over X. Then by hypothesis $Cl_s(f_{pu}(F,A))\tilde{\subseteq}f_{pu}(\overline{(F,A)}) = f_{pu}(F,A)$, and so $f_{pu}(F,A) = Cl_s(f_{pu}(F,A))$. Hence $f_{pu}(F,A)$ is a soft semi-closed set over Y.
The other cases can be proved in similar way. □

Remark 3.4.8. [36] It is clears that the class of soft β-continuity contains each of classes of soft semi-continuous and soft pre-continuous. The implications between them and other types of soft continuities are given by Figure 3.2:

$$\begin{array}{ccccc} \text{soft continuity} & \longrightarrow & \text{soft } \alpha\text{-continuity} & \longrightarrow & \text{soft semi-continuity} \\ & & \downarrow & & \downarrow \\ & & \text{soft } pre\text{-continuity} & \longrightarrow & \text{soft } \beta\text{-continuity} \end{array}$$

Figure 3.2: Relations between types of soft continuity.

The converse of these implications do not hold, in general, as shown in the following examples:

Example 3.4.9. Let $X = Y = \{x_1, x_2, x_3\}$, $A = B = \{e_1, e_2\}$ and let the soft topology on X be the soft indiscrete and on Y be the soft discrete. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft function such that $u : X \to Y$ and $p : A \to B$ are the identity functions. Let $(G, B) \neq \phi_B, Y_B$ be a soft open set over Y. Since $\overline{\left(\overline{f_{pu}^{-1}(G, B)}\right)^{\circ}} = f_{pu}^{-1}(G, B)$, $f_{pu}^{-1}(G, B)$ is a soft β-open set. Also, since $\left(f_{pu}^{-1}(G, B)\right)^{\circ} = \phi_A$, $f_{pu}^{-1}(G, B)$ is not a soft semi-open over X. Thus, f_{pu} is a soft β-continuous function, but is not a soft semi-continuous function.

Example 3.4.10. [36] Let $X = Y = \{x_1, x_2, x_3\}$, $A = B = \{e_1, e_2\}$, and let $\tau_1 = \{\phi_A, X_A, (F_1, A), (F_2, A), (F_3, A)\}$ be a soft topological space over X and $\tau_2 = \{\phi_B, Y_B, (G_1, B), (G_2, B)\}$ a soft topological space over Y, where (F_1, A), (F_2, A), (F_3, A) are soft sets over X and (G_1, B), (G_2, B) are soft sets over Y, defined by the following: $(F_1, A) = \{(e_1, \{x_1\}), (e_2, \{x_1\})\}$, $(F_2, A) = \{(e_1, \{x_2\}), (e_2, \{x_2\})\}$, $(F_3, A) = \{(e_1, \{x_1, x_2\}), (e_2, \{\emptyset\})\}$, and $(G_1, B) = \{(e_1, \{x_1\}), (e_2, \{x_1\})\}$, $(G_2, B) = \{(e_1, \{x_1, x_2\}), (e_2, \{x_1, x_2\})\}$. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ such that $u : X \to Y$ defines as $u(x_1) = x_1$, $u(x_2) = x_3$, $u(x_3) = x_2$, and $p : A \to B$ is the identity function. Since $f_{pu}^{-1}(G_2, B) = \{(e_1, \{x_1, x_3\}), (e_2, \{x_1, x_3\})\}$ is not a soft pre-open set over X, f_{pu} is not a soft pre-continuous function. But, since $f_{pu}^{-1}(G_1, B) = (F_1, A)$ and $f_{pu}^{-1}(G_2, B) = \{(e_1, \{x_1, x_3\}), (e_2, \{x_1, x_3\})\}$ are soft β-open sets over X, f_{pu} is a soft β-continuous function.

Chapter 4

New Type of Soft β-open Sets

In [6], A. EL-Mabhouh and A. Mizyed introduced the notion of βc-open sets in a topological spaces and studied some of its properties. In this chapter, we introduce and study a new type of soft β-open sets called soft βc-open sets. Also, we introduce the concepts of soft βc-interior and soft βc-closure and study some of their properties in detail. Finally, we introduce the concepts of soft βc-continuous function, soft βc-open function, soft βc-closed function and soft βc-irresolute function, and study some of their properties in details.

4.1 Soft βc-open Sets

Definition 4.1.1. Let (X, τ, A) be a soft topological space and $(F, A)\tilde{\in}SS(X)_A$. Then (F, A) is called a *soft βc-open* if (F, A) is a soft β-open set and for each $x_e\tilde{\in}(F, A)$ there is a soft closed set (H, A) in $SS(X)_A$ such that $x_e\tilde{\in}(H, A)\tilde{\subseteq}(F, A)$. The complement of a soft βc-open set is said to be *soft βc-closed set*.

Theorem 4.1.2. A soft set (F, A) in a soft topological space (X, τ, A) is soft βc-open set if and only if (F, A) is soft β-open set and it is a union of soft closed set.

Proof. $(\Rightarrow)$ Let (F, A) be a soft βc-open set. Then (F, A) is soft β-open set and for each $x_e\tilde{\in}(F, A)$ there exists soft closed set (H_{x_e}, A) such that $x_e\tilde{\in}(H_{x_e}, A)\tilde{\subseteq}(F, A)$. This implies that $\tilde{\bigcup}_{x_e\tilde{\in}(F,A)}(H_{x_e}, A)\tilde{\subseteq}(F, A)\tilde{\subseteq}\tilde{\bigcup}_{x_e\tilde{\in}(F,A)}(H_{x_e}, A)$. Thus, $(F, A) = \tilde{\bigcup}_{x_e\tilde{\in}(F,A)}(H_{x_e}, A)$, where (H_{x_e}, A) is soft closed set for each $x_e \tilde{\in} (F, A)$.

($\Leftarrow$) Direct from the definition of soft βc-open sets. $\square$

Corollary 4.1.3. Let (X, τ, A) be a soft topological space and (F, A) a soft β-open set over X. If (F, A) is a soft closed set, then (F, A) is a soft βc-open set.

Remark 4.1.4. (1) Every soft βc-open set is soft β-open set. But the converse need not be true in general.

(2) The family of soft open sets is incomparable with the family of soft βc-open sets.

(3) The family of soft closed sets is incomparable with the family of soft βc-open sets.

(4) The family of soft semi-open sets is incomparable with the family of soft βc-open sets.

(5) The family of soft pre-open sets is incomparable with the family of soft βc-open sets.

The following examples illustrate the previous remark:

Example 4.1.5. Let $X = \{a, b, c\}$ and $A = \{e\}$, with a soft topology $\tau = \{X_A, \phi_A, (e, \{a\}), (e, \{b\}), (e, \{a, b\})\}$. Then we have the following:

(1) The family of soft closed sets is $\{X_A, \phi_A, (e, \{c\}), (e, \{b, c\}), (e, \{a, c\})\}$.

(2) The family of β-open soft sets is $\{X_A, \phi_A, (e, \{a\}), (e, \{b\}), (e, \{a, b\}), (e, \{a, c\}), (e, \{b, c\})\}$.

(3) The family of soft βc-open sets is $\{X_A, \phi_A, (e, \{a, c\}), (e, \{b, c\})\}$.

Note that $(e, \{a\})$ is a soft β-open set, but is not a soft βc-open set. Also $(e, \{b\})$ is a soft open set but is not soft βc-open set and $(e, \{a, c\})$ is soft βc-open set but is not soft open set.

Example 4.1.6. Consider the soft topological space in Example 3.2.4. Then $(G, A) = \{(e_1, \{x_1, x_3\}), (e_2, \{x_2, x_3\}), (e_3, \{x_1, x_3\})\}$ is a soft βc-open set, but is not a soft closed set. Also $(F, A) = \{(e_1, \{x_1, x_3\}), (e_2, \{x_2\}), (e_3, \{x_1, x_2\})\}$ is a soft closed set, but is not a soft βc-open set.

Example 4.1.7. In Example 4.1.5 $(e, \{b, c\})$ is a soft βc-open set, but is not a soft pre-open set. Also $(e, \{b\})$ is a soft pre-open set, but is not a soft βc-open set

Example 4.1.8. Consider the soft topological (X, τ, A) space in Example 3.2.4. Then $(G, A) = \{(e_1, \{x_1, x_3\}, (e_2, \{x_2, x_3\}), (e_3, \{x_1, x_2\})\}$ is a soft βc-open set, but is not a soft semi-open set. Also $(H, A) = \{(e_1, \{x_1, x_2\}, (e_2, \{x_3\}), (e_3, \{x_2\})\}$ is a soft semi-open set, but is not a soft βc-open set

Theorem 4.1.9. An arbitrary union of soft βc-open sets is a soft βc-open set.

Proof. Let $\{(F_\alpha, A) : \alpha \in \Delta\}$ be a family of soft βc-open sets in a soft topological space (X, τ, A). Then (F_α, A) is a soft β-open set for each $\alpha \in \Delta$ and by lemma 1.13, $\tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A)$ is a soft β-open set. If $x_e \tilde{\in} \tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A)$, then there exists $\gamma \in \Delta$ such that $x_e \tilde{\in} (F_\gamma, A)$. Since (F_γ, A) is soft βc-open set, there exists a soft closed set (H_{x_e}, A) such that $x_e \tilde{\in} (H_{x_e}, A) \tilde{\subseteq} (F_\gamma, A) \tilde{\subseteq} \tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A)$. Hence, $\tilde{\bigcup}_{\alpha\in\Delta}(F_\alpha, A)$ is soft βc-open set. □

Remark 4.1.10. The intersection of even two soft βc-open sets need not be a soft βc-open set as is illustrated in the following example:

Example 4.1.11. In Example 4.1.5, $(e, \{a, c\})$ and $(e, \{b, c\})$ are soft βc-open sets but $(e, \{a, c\}) \tilde{\cap} (e, \{b, c\}) = (e, \{c\})$ is not a soft βc-open set.

Corollary 4.1.12. An arbitrary intersection of soft βc-closed sets is soft βc-closed set.

Proof. Using Theorem 4.1.9 and De Morgen's law. □

Remark 4.1.13. The union of even two soft βc-closed sets need not be a soft βc-closed set as is illustrated in the following example:

Example 4.1.14. In Example 4.1.5, $(e, \{b\})$ and $(e, \{a\})$ are soft βc-closed set. But $(e, \{b\}) \tilde{\cup} (e, \{a\}) = (e, \{a, b\})$ is not a soft βc-closed set.

Theorem 4.1.15. Every soft clopen set is a soft βc-open set.

Proof. Let (F, A) be a soft clopen set in a soft topological space (X, τ, A). Then (F, A) is a soft open set and so (F, A) is a soft β-open set. Now since (F, A) is a soft closed set, (F, A) is a soft βc-open set by Corollary 4.1.3. □

Theorem 4.1.16. A soft set (F, A) in a soft topological space (X, τ, A) is a soft βc-open set if and only if for each $x_e \tilde{\in} (F, A)$, there exists a soft βc-open set (H_{x_e}, A) such that $x_e \tilde{\in} (H_{x_e}, A) \tilde{\subseteq} (F, A)$.

Proof. Direct from Theorem 4.1.9 and Definition 4.1.1. □

Theorem 4.1.17. Let (X, τ, A) be a soft topological space and (F, A) a soft set over X. If (X, τ, A) is a soft T_1-space and if (F, A) is soft a β-open set, then (F, A) is a soft βc-open set.

Proof. Let (X, τ, A) be a soft T_1-space and (F, A) a soft β-open set. If $(F, A) = \phi_A$, then ϕ_A is a soft βc-open set. If not, then for each $x_e \tilde{\in} (F, A)$, x_e is a soft closed set by Theorem 2.4.4. But $x_e \tilde{\in} \{x_e\} \tilde{\subseteq} (F, A)$. Therefore, (F, A) is a soft βc-open set. □

Theorem 4.1.18. Let (X, τ, A) be a soft topological space, (F, A) a soft open set and (G, A) a soft βc-open set. If (F, A) is a union of soft closed sets, then $(F, A) \tilde{\cap} (G, A)$ is a soft βc-open set.

Proof. Let (F, A) be a soft open set and (G, A) be a soft βc-open set then $(F, A) \tilde{\cap} (G, A) \tilde{\subseteq} (F, A) \tilde{\cap} \overline{\left(\overline{(G, A)}\right)^{\circ}} \tilde{\subseteq} \overline{\left((F, A) \tilde{\cap} \left(\overline{(G, A)}\right)^{\circ}\right)} = \overline{\left((F, A) \tilde{\cap} \overline{(G, A)}\right)^{\circ}} \tilde{\subseteq} \overline{\left(\overline{(F, A) \tilde{\cap} (G, A)}\right)^{\circ}}$ which implies $(F, A) \tilde{\cap} (G, A)$ is a soft β-open set. If $x_e \tilde{\in} (F, A) \tilde{\cap} (G, A)$, then since (F, A) is a union of soft closed sets, there exists a soft closed set (C, A) over X such that $x_e \tilde{\in} (C, A) \tilde{\subseteq} (F, A)$ and since (G, A) is a soft βc-open set, there exists a soft closed set (M, A) over X such that $x_e \tilde{\in} (M, A) \tilde{\subseteq} (G, A)$. Hence, $x_e \tilde{\in} (C, A) \tilde{\cap} (M, A) \tilde{\subseteq} (F, A) \tilde{\cap} (G, A)$ where $(C, A) \tilde{\cap} (M, A)$ is a soft closed set over X. Therefore, $(F, A) \tilde{\cap} (G, A)$ is a soft βc-open set. □

Corollary 4.1.19. Let (X, τ, A) be a soft topological space, (F, A) a soft clopen set and (G, A) a soft βc-open set, then $(F, A) \tilde{\cap} (G, A)$ is a soft βc-open set.

Remark 4.1.20. The condition of being (F, A) a soft clopen set is necessary in Corollary 4.1.19 as illustrated in the following example:

Example 4.1.21. In Example 4.1.5 $(F, A) = (e, \{a, b\})$ is not a soft clopen set and $(G, A) = (e, \{a, c\})$ is a soft βc-open set. But $(F, A) \tilde{\cap} (G, A) = (e, \{a\})$ is not a soft βc-open set.

Definition 4.1.22. [35] Let (X, τ, A) be a soft topological space. A soft set (F, A) is called *soft regular open* (resp. *soft regular closed*) over X if $(F, A) = \left(\overline{(F, A)}\right)^{\circ}$ (resp. $(F, A) = \overline{(F, A)^{\circ}}$).

Theorem 4.1.23. Every soft regular closed set is a soft βc-open set.

Proof. Let (F,A) be a soft regular closed set. Then $(F,A) = \overline{(F,A)^\circ}$ but $\overline{(F,A)^\circ} \tilde{\subseteq} \overline{\left(\overline{(F,A)}\right)^\circ}$ and so, (F,A) is a soft β-open set. Since (F,A) is a soft closed set, by Corollary 4.1.3 (F,A) is a soft βc-open set. □

Definition 4.1.24. [30] A soft topological space (X,τ,A) is called a *soft locally indiscrete*, if every soft open set over X is a soft closed set over X.

Theorem 4.1.25. If the soft topological space (X,τ,A) is a soft locally indiscrete, then every soft semi-open set is a soft βc-open set.

Proof. Let (F,A) be a soft semi-open set. Then $(F,A) \tilde{\subseteq} \overline{(F,A)^\circ} \tilde{\subseteq} \overline{\left(\overline{(F,A)}\right)^\circ}$. So (F,A) is a soft β-open set. Since (X,τ,A) is a soft locally indiscrete, $(F,A)^\circ$ is a soft closed set and $(F,A) \tilde{\subseteq} \overline{(F,A)^\circ} = (F,A)^\circ$ which implies, (F,A) is a soft open set and for any $x_e \tilde{\in} (F,A)$, $x_e \tilde{\in} (F,A)^\circ \tilde{\subseteq} (F,A)$. Hence (F,A) is a soft βc-open set. □

Corollary 4.1.26. Let (X,τ,A) be a soft locally indiscrete space. Then:

(1) every soft open set is a soft βc-open set .

(2) Every soft α-open set is a soft βc-open set .

Proof. (1) Direct from Remark 3.1.3 and Theorem 4.1.25.
(2) Direct from Theorem 3.2.8 and Theorem 4.1.25. □

Theorem 4.1.27. Let (Y,τ_Y,A) be a soft subspace of a soft topological space (X,τ,A). If (F,A) is a soft βc-open set over X and $(F,A) \tilde{\subseteq} Y_A$ such that Y_A is a soft β-open set over X, then (F,A) is a soft βc-open set over Y.

Proof. Suppose that (F,A) is a soft βc-open set over X, then (F,A) is a soft β-open set over X but $(F,A) \tilde{\subseteq} Y_A$ and Y_A is a soft β-open set over X. So that, by Theorem 3.2.40, (F,A) is a soft β-open set over Y. Also, for each $x_e \tilde{\in} (F,A)$ there exists a soft closed set (C,A) over X such that $x_e \tilde{\in} (C,A) \tilde{\subseteq} (F,A)$. Since (C,A) is a soft closed set over X and $(F,A) \tilde{\subseteq} Y_A$, by Theorem 2.2.11 (C,A) is a soft closed set over Y. Hence (F,A) is a soft βc-open set over Y. □

Remark 4.1.28. The condition of being Y_A a soft β-open set over X is necessary in Theorem 4.1.27 as illustrated in the following example:

Example 4.1.29. Let $X = \{x_1, x_2, x_3\}$, $A = \{e_1, e_2\}$ and $\tau = \{\phi_A, X_A, (F_1, A), (F_2, A), (F_3, A)\}$ where

$$\begin{aligned}(F_1, A) &= \{(e_1, X), (e_2, \{x_1, x_2\}), (e_3, X)\},\\ (F_2, A) &= \{(e_1, \{x_1, x_3\}), (e_2, X), (e_3, \{x_2, x_3\})\},\\ (F_3, A) &= \{(e_1, \{x_1, x_3\}), (e_2, \{x_1, x_2\}), (e_3, \{x_2, x_3\})\}.\end{aligned}$$

Then τ defines a soft topology on X and thus (X, τ, A) is a soft topological space over X. If $(F, A) = \{(e_1, \{x_2\}), (e_2, \{x_3\})\}$, then (F, A) is a soft βc-open set over X. If $Y_A = \{(e_1, \{x_2\}), (e_2, \{x_3\}), (e_3, \{x_1\})\}$, then Y_A is not a soft β-open set over X. Clearly $(F, A)\tilde{\subseteq}Y_A$. Now $\tau_Y = \{\phi_A, Y_A, (F_Y, A), (G_Y, A)\}$ where $(F_Y, A) = \{(e_1, \{x_2\}), (e_3, \{x_1\})\}$ and $(G_Y, A) = \{(e_2, \{x_3\})\}$. Since $x_{2_{e_1}}\tilde{\in}(F, A)$ and there is no soft closed set (C_Y, A) over Y such that $x_{2_{e_1}}\tilde{\in}(C_Y, A)\tilde{\subseteq}(F, A)$, (F, A) is not a soft βc-open set over Y.

Theorem 4.1.30. Let (Y, τ_Y, A) be a soft subspace of a soft topological space (X, τ, A). If (F, A) is a soft βc-open set over X and Y_A is a soft clopen set over X, then $(F, A)\tilde{\cap}Y_A$ is a soft βc-open set over Y.

Proof. Let (F, A) be a soft βc-open set over X, then (F, A) is a soft β-open set over X. Since Y_A is a soft open set over X, by Theorem 3.2.39 $(F, A)\tilde{\cap}Y_A$ is a soft β-open set over X. Since (F, A) is a soft βc-open set over X, then for each $x_e\tilde{\in}(F, A)$, there exists a soft closed set (C, A) over X such that $x_e\tilde{\in}(C, A)\tilde{\subseteq}(F, A)$. Hence $x_e\tilde{\in}(C, A)\tilde{\cap}Y_A\tilde{\subseteq}(F, A)\tilde{\cap}Y_A$ and therefore, $(F, A)\tilde{\cap}Y_A$ is a soft βc-open set over X such that $(F, A)\tilde{\cap}Y_A\tilde{\subseteq}Y_A$. Thus, by Theorem 4.1.27, $(F, A)\tilde{\cap}Y_A$ is a soft βc-open set over Y. □

Remark 4.1.31. The condition of being Y_A a soft clopen set over X is necessary in Theorem 4.1.30 as illustrated in the following example:

Example 4.1.32. In Example 4.1.29 (F, A) is a soft βc-open set over X and Y_A is not a soft clopen set over X. But $(F, A)\tilde{\cap}Y_A = (F, A)$ is not a soft βc-open set over Y.

4.2 Soft βc-Interior and Soft βc-Closure

Definition 4.2.1. Let $\widetilde{X}$ be a soft topological space and (F, A) a soft set over X. Then the *soft βc-interior* of (F, A), denoted by $Int_{\beta c}(F, A)$, is the union of all soft βc-open sets contained in (F, A). (F, A) is called a *soft βc-neighborhood* (briefly, soft βc-nhood) of a soft point $x_e \tilde{\in} X_A$ if there exists a soft open set (G, A) over X such that $x_e \tilde{\in} (G, A) \tilde{\subseteq} (F, A)$. A soft point $x_e \tilde{\in} X_A$ is called a *soft βc-interior point* of (F, A) if there exists a soft βc-open set (G, A) over X such that $x_e \tilde{\in} (G, A) \tilde{\subseteq} (F, A)$. Clearly $Int_{\beta c}(F, A)$ is the largest soft βc-open set contained in (F, A). Moreover, (F, A) is βc-nhood of x_e if $x_e \tilde{\in} Int_{\beta c}(F, A)$.

Remark 4.2.2. There is no relations between the soft βc-interior and the soft interior as illustrated in the following example:

Example 4.2.3. Consider a soft topological space in Example 4.1.5. If $(F, A) = (e, \{a, c\})$, then $(F, A)^\circ = (e, \{a\})$ and $Int_{\beta c}(F, A) = (e, \{a, c\})$ which implies, $Int_{\beta c}(F, A) \tilde{\nsubseteq} (F, A)^\circ$. On the other hand, if $(G, A) = (e, \{a\})$, then $(G, A)^\circ = (e, \{a\})$ and $Int_{\beta c}(G, A) = \phi_A$. That is, $(G, A)^\circ \tilde{\nsubseteq} Int_{\beta c}(G, A)$.

Theorem 4.2.4. Let (X, τ, A) be a soft topological space and let (F, A) and (G, A) be a soft sets over X. Then

(1) (F, A) is soft βc-open set if and only if $(F, A) = Int_{\beta c}(F, A)$.

(2) $Int_{\beta c}(\phi_A) = \phi_A$ and $Int_{\beta c}(X_A) = X_A$.

(3) $Int_{\beta c}(Int_{\beta c}(F, A)) = Int_{\beta c}(F, A)$.

(4) $(F, A) \tilde{\subseteq} (G, A)$ implies $Int_{\beta c}(F, A) \tilde{\subseteq} Int_{\beta c}(G, A)$.

(5) If $(F, A) \tilde{\cap} (G, A) = \phi_A$, then $Int_{\beta c}(F, A) \tilde{\cap} Int_{\beta c}(G, A) = \phi_A$

(6) $Int_{\beta c}((F, A) \tilde{\cap} (G, A)) \tilde{\subseteq} Int_{\beta c}(F, A) \tilde{\cap} Int_{\beta c}(G, A)$.

(7) $Int_{\beta c}(F, A) \tilde{\cup} Int_{\beta c}(G, A) \tilde{\subseteq} Int_{\beta c}((F, A) \tilde{\cup} (G, A))$.

Proof. (1) ($\Rightarrow$) If (F, A) is soft βc-open set over X, then (F, A) is itself a soft βc-open set over X which contains (F, A). So $Int_{\beta c}(F, A)$ is the largest soft βc-open set contained in

(F, A) and $(F, A) = Int_{\beta c}(F, A)$.
$(\Leftarrow)$ Suppose that $(F, A) = Int_{\beta c}(F, A)$. Since $Int_{\beta c}(F, A)$ is a soft βc-open set, (F, A) is a soft βc-open set over X.

(2) Since ϕ_A and X_A are soft βc-open sets, by part 1 $Int_{\beta c}(\phi_A) = \phi_A$ and $Int_{\beta c}(X_A) = X_A$.

(3) Since $Int_{\beta c}(F, A)$ is a soft βc-open set therefore by part 1 we have $Int_{\beta c}(Int_{\beta c}(F, A)) = Int_{\beta c}(F, A)$.

(4) Suppose that $(F, A)\tilde{\subseteq}(G, A)$. Since $Int_{\beta c}(F, A)\tilde{\subseteq}(F, A)\tilde{\subseteq}(G, A)$. $Int_{\beta c}(F, A)$ is a soft βc-open subset of (G, A), so by definition of $Int_{\beta c}(G, A)$, $Int_{\beta c}(F, A)\tilde{\subseteq}Int_{\beta c}(G, A)$.

(5) If $Int_{\beta c}(F, A)\tilde{\cap}Int_{\beta c}(G, A) \neq \phi_A$, then there is $x_e\tilde{\in}Int_{\beta c}(F, A)\tilde{\cap}Int_{\beta c}(G, A)$. So there exist soft βc-open sets (M, A) and (V, A) such that $x_e\tilde{\in}(M, A)\tilde{\subseteq}(F, A)$ and $x_e\tilde{\in}(V, A)\tilde{\subseteq}(G, A)$ which implies, $x_e\tilde{\in}(M, A)\tilde{\cap}(V, A)\tilde{\subseteq}(M, A)\tilde{\subseteq}(F, A)$ and $x_e\tilde{\in}(M, A)\tilde{\cap}(V, A)\tilde{\subseteq}(V, A)\tilde{\subseteq}(G, A)$. Hence, $x_e\tilde{\in}(F, A)\tilde{\cap}(G, A)$ and therefore, $(F, A)\tilde{\cap}(G, A) \neq \phi_A$.

(6) Since $(F, A)\tilde{\cap}(G, A)\tilde{\subseteq}(F, A)$ and $(F, A)\tilde{\cap}(G, A)\tilde{\subseteq}(G, A)$, by part 4 $Int_{\beta c}((F, A)\tilde{\cap}(G, A)$ $\tilde{\subseteq}Int_{\beta c}(F, A)$ and $Int_{\beta c}((F, A)\tilde{\cap}(G, A)\tilde{\subseteq}Int_{\beta c}(G, A)$. Hence $Int_{\beta c}((F, A)\tilde{\cap}(G, A))$ $\tilde{\subseteq}Int_{\beta c}(F, A)\tilde{\cap}Int_{\beta c}(G, A)$.

(7) Since $(F, A)\tilde{\subseteq}(F, A)\tilde{\cup}(G, A)$ and $(G, A)\tilde{\subseteq}(F, A)\tilde{\cup}(G, A)$, by part 4 $Int_{\beta c}(F, A)\tilde{\subseteq}$ $Int_{\beta c}((F, A)\tilde{\cup}(G, A))$ and $Int_{\beta c}(G, A)\tilde{\subseteq}Int_{\beta c}((F, A)\tilde{\cup}(G, A))$. Hence, $Int_{\beta c}(F, A)\tilde{\cup}Int_{\beta c}(G, A)$ $\tilde{\subseteq}Int_{\beta c}((F, A)\tilde{\cup}(G, A))$. □

Remark 4.2.5. The converse of parts (4) and (5) and the reverse inclusions of parts (6) and (7) need not be true in general as illustrated in the following examples:

Example 4.2.6. $[Int_{\beta c}(F, A)\tilde{\subseteq}Int_{\beta c}(G, A) \nRightarrow (F, A)\tilde{\subseteq}(G, A)]$
Let $X = \{a, b, c\}$ and $A = \{e\}$ with a soft topology $\tau = \{\phi_A, X_A, (e, \{a\})\}$, then if $(F, A) = (e, \{a\})$ and $(G, A) = (e, \{b\})$, we have $Int_{\beta c}(F, A) = Int_{\beta c}(F, A) = \phi_A$ but $(F, A)\tilde{\nsubseteq}(G, A)$.

Example 4.2.7. $[Int_{\beta c}(F, A)\tilde{\cap}Int_{\beta c}(G, A) = \phi_A \nRightarrow (F, A)\tilde{\cap}(G, A) = \phi_A]$
In Example 4.2.6, if $(F, A) = (e, \{a\})$ and $(G, A) = (e, \{a, b\})$, then $Int_{\beta c}(F, A) = Int_{\beta c}(F, A) = \phi_A$ which implies, $Int_{\beta c}(F, A)\tilde{\cap}Int_{\beta c}(G, A) = \phi_A$ but $(F, A)\tilde{\cap}(G, A) = (e, \{a\}) \neq \phi_A$.

Example 4.2.8. $[Int_{\beta c}((F,A)\tilde{\cup}(G,A))\tilde{\nsubseteq}Int_{\beta c}(F,A)\tilde{\cup}Int_{\beta c}(G,A)]$
In Example 4.2.6, if $(F,A) = (e,\{a\})$ and $(G,A) = (e,\{b,c\})$, then $Int_{\beta c}((F,A) = Int_{\beta c}(G,A) = \phi_A$. But $(F,A)\tilde{\cup}(G,A) = X_A$ and $Int_{\beta c}((F,A)\tilde{\cup}(G,A)) = X_A$.

Example 4.2.9. $[Int_{\beta c}(F,A)\tilde{\cap}Int_{\beta c}(G,A)\tilde{\nsubseteq}Int_{\beta c}((F,A)\tilde{\cap}(G,A))]$
Consider a soft topological space (X,τ,A) in Example 4.1.5. Then if $(F,A) = (e,\{a,c\})$ and $(G,A) = (e,\{b,c\})$, then $Int_{\beta c}((F,A) = (e,\{a,c\})$ and $Int_{\beta c}((G,A) = (e,\{b,c\})$ which implies, $Int_{\beta c}(F,A)\tilde{\cap}Int_{\beta c}(G,A) = (e,\{c\})$. But $(F,A)\tilde{\cap}(G,A) = (e,\{c\})$ and $Int_{\beta c}((F,A)\tilde{\cap}(G,A)) = \phi_A$.

Definition 4.2.10. Let (X,τ,A) be a soft topological space over X and (B,A) a soft set over X. Then the *soft βc-closure* of (B,A), denoted by $Cl_{\beta c}(B,A)$, is the intersection of all soft βc-closed supersets of (B,A). Clearly $Cl_{\beta c}(B,A)$ is the smallest soft βc-closed set in (X,τ,A) which contains (B,A).

Remark 4.2.11. There is no relation between the soft βc-closure and the soft closure as illustrated in the following examples:

Example 4.2.12. Consider a soft topological space (X,τ,A) in Example 4.2.6. Then, if $(F,A) = (e,\{b,c\})$, then $\overline{(F,A)} = (e,\{b,c\})$ and $Cl_{\beta c}(F,A) = X_A$. Hence $Cl_{\beta c}(F,A)\tilde{\nsubseteq}\overline{(F,A)}$.

Example 4.2.13. Consider a soft topological space (X,τ,A) in Example 4.1.5. Then, if $(F,A) = (e,\{b\})$, then $\overline{(F,A)} = (e,\{b,c\})$ and $Cl_{\beta c}(F,A) = (e,\{b\})$. Hence $\overline{(F,A)}\tilde{\nsubseteq}Cl_{\beta c}(F,A)$.

Theorem 4.2.14. Let (X,τ,A) be a soft topological space, (F,A) a soft set over X and $x_e\tilde{\in}X_A$. Then, the following are equivalent:

(1) For any soft βc-open set (G,A) over X containing x_e we have, $(F,A)\tilde{\cap}(G,A) \neq \phi_A$.

(2) $x_e\tilde{\in}Cl_{\beta c}(F,A)$.

Proof. $(1 \Rightarrow 2)$ If $x_e\tilde{\notin}Cl_{\beta c}(F,A)$, then there exists a soft βc-closed set (C,A) such that $(F,A)\tilde{\subseteq}(C,A)$ and $x_e\tilde{\notin}(C,A)$. But $(C,A)^c$ is a soft βc-open set containing x_e and therefore, $(F,A)\tilde{\cap}(C,A)^c\tilde{\subseteq}(F,A)\tilde{\cap}(F,A)^c = \phi_A$ which is a contradiction. Hence, $x_e\tilde{\in}Cl_{\beta c}(G,A)$.

(2 ⇒ 1) Suppose there exists a soft βc-open set (G, A) containing x_e such that $(F, A)\tilde{\cap}(G, A) = \phi_A$. Then $(F, A)\tilde{\subseteq}(G, A)^c$. Since $(G, A)^c$ is a soft βc-closed set, $Cl_{\beta c}(F, A)\tilde{\subseteq}(G, A)^c$. Hence $x_e\tilde{\notin}Cl_{\beta c}(F, A)$. A contradiction. □

Theorem 4.2.15. Let (X, τ, A) be a soft topological space and let (F, A) and (G, A) be soft sets over X. Then

(1) (F, A) is a soft βc-closed set if and only if $(F, A) = Cl_{\beta c}(F, A)$.

(2) $Cl_{\beta c}(\phi_A) = \phi_A$ and $Cl_{\beta c}(X_A) = X_A$.

(3) $Cl_{\beta c}(Cl_{\beta c}(F, A)) = Cl_{\beta c}(F, A)$.

(4) $(F, A)\tilde{\subseteq}(G, A)$ implies $Cl_{\beta c}(F, A)\tilde{\subseteq}Cl_{\beta c}(G, A)$.

(5) If $Cl_{\beta c}(F, A)\tilde{\cap}Cl_{\beta c}(G, A) = \phi_A$, then $(F, A)\tilde{\cap}(G, A) = \phi_A$

(6) $Cl_{\beta c}(F, A)\tilde{\cup}Cl_{\beta c}(G, A)\tilde{\subseteq}Cl_{\beta c}((F, A)\tilde{\cup}(G, A))$.

(7) $Cl_{\beta c}((F, A)\tilde{\cap}(G, A))\tilde{\subseteq}Cl_{\beta c}(F, A)\tilde{\cap}Cl_{\beta c}(G, A)$.

Proof. (1) (⇒) If (F, A) is a soft βc-closed set, then (F, A) is itself a soft βc-closed set over X which contains (F, A). So (F, A) is the smallest soft βc-closed set containing (F, A) and $(F, A) = Cl_{\beta c}(F, A)$.
(⇐) Suppose that $(F, A) = Cl_{\beta c}(F, A)$. Since $Cl_{\beta}(F, A)$ is a soft βc-closed set, so (F, A) is soft βc-closed set.

(2) Since ϕ_A and X_A are soft βc-closed set, by part (1) $Cl_{\beta c}(\phi_A) = \phi_A$ and $Cl_{\beta c}(X_A) = X_A$.

(3) Since $Cl_{\beta c}(F, A)$ is a soft βc-closed set, by part (1) we have $Cl_{\beta c}(Cl_{\beta c}(F, A)) = Cl_{\beta c}(F, A)$.

(4) Suppose that $(F, A)\tilde{\subseteq}(G, A)$. Then every soft βc-closed super set (K, A) of (G, A) will also contains (F, A), and hence contains $Cl_{\beta c}(F, A)$. Take $Cl_{\beta c}(F, A) = (K, A)$. Then $Cl_{\beta c}(F, A)\tilde{\subseteq}Cl_{\beta c}(G, A)$.

(5) Let $(F, A)\tilde{\cap}(G, A) \neq \phi_A$, then there is $x_e\tilde{\in}(F, A)\tilde{\cap}(G, A)$ which implies $x_e\tilde{\in}(F, A)$ and $x_e\tilde{\in}(G, A)$. Then $x_e\tilde{\in}Cl_{\beta c}(F, A)$ and $x_e\tilde{\in}Cl_{\beta c}(G, A)$ and so $Cl_{\beta c}(F, A)\tilde{\cap}Cl_{\beta c}(G, A) \neq \phi_A$.

(6) Since $(F,A)\tilde{\subseteq}(F,A)\tilde{\cup}(G,A)$ and $(G,A)\tilde{\subseteq}(F,A)\tilde{\cup}(G,A)$, by part (4) $Cl_{\beta c}(F,A)\tilde{\subseteq}Cl_{\beta c}((F,A)$ $\tilde{\cup}(G,A))$ and $Cl_{\beta c}(G,A)\tilde{\subseteq}Cl_{\beta c}((F,A)\tilde{\cup}(G,A))$. Hence, $Cl_{\beta c}(F,A)\tilde{\cup}Cl_{\beta c}(G,A)\tilde{\subseteq}Cl_{\beta c}((F,A)$ $\tilde{\cup}(G,A))$.

(7) Since $((F,A)\tilde{\cap}(G,A))\tilde{\subseteq}(F,A)$ and $((F,A)\tilde{\cap}(G,A))\tilde{\subseteq}(G,A)$. So by part 4 $Cl_{\beta c}((F,A)$ $\tilde{\cap}(G,A))\tilde{\subseteq}Cl_{\beta c}(F,A)$ and $Cl_{\beta c}((F,A)\tilde{\cap}(G,A))\tilde{\subseteq}Cl_{\beta c}(G,A)$. Thus $Cl_{\beta c}((F,A)\tilde{\cap}(G,A))\tilde{\subseteq}$ $Cl_{\beta c}(F,A)\tilde{\cap}Cl_{\beta c}(G,A)$. □

Remark 4.2.16. The converse of parts (4) and (5) and the reverse inclusions of parts (6) and (7) need not be true in general as illustrated in the following examples:

Example 4.2.17. [$Cl_{\beta c}(F,A)\tilde{\subseteq}Cl_{\beta c}(G,A) \not\Rightarrow (F,A)\tilde{\subseteq}(G,A)$ *and* $(F,A)\tilde{\cap}(G,A)=\phi_A \not\Rightarrow$ $Cl_{\beta c}(F,A)\tilde{\cap}Cl_{\beta c}(G,A)=\phi_A$]
Consider a soft topological (X,τ,A) in Example 4.2.6. Then, if $(F,A)=(e,\{a\})$ and $(G,A)=(e,\{b\})$, then $X_A=Cl_{\beta c}(F,A)\tilde{\subseteq}Cl_{\beta c}(G,A)=X_A$ but $(F,A)\tilde{\nsubseteq}(G,A)$. Moreover, $(F,A)\tilde{\cap}(G,A)=\phi_A$ but $Cl_{\beta c}(F,A)\tilde{\cap}Cl_{\beta c}(G,A)=X_A\neq\phi_A$.

Example 4.2.18. [$Cl_{\beta c}(F,A)\tilde{\cap}Cl_{\beta c}(G,A)\tilde{\nsubseteq}Cl_{\beta c}((F,A)\tilde{\cap}(G,A))$]
In Example 4.2.17, $(F,A)\tilde{\cap}(G,A)=\phi_A$ and so, $Cl_{\beta c}((F,A)\tilde{\cap}(G,A))=\phi_A$. But $Cl_{\beta c}(F,A)=X_A$ and $Cl_{\beta c}(G,A)=X_A$ which implies, $Cl_{\beta c}(F,A)\tilde{\cap}Cl_{\beta c}(G,A)=X_A\tilde{\nsubseteq}Cl_{\beta c}((F,A)\tilde{\cap}(G,A))=\phi_A$.

Example 4.2.19. [$Cl_{\beta c}((F,A)\tilde{\cup}(G,A))\tilde{\nsubseteq}Cl_{\beta c}(F,A)\tilde{\cup}Cl_{\beta c}(G,A)$]
Consider a soft topological space (X,τ,A) in Example 4.1.5. Then, if $(F,A)=(e,\{a\})$ and $(G,A)=(e,\{b\})$, then $(F,A)\tilde{\cup}(G,A)=(e,\{a,b\})$ and so, $Cl_{\beta c}((F,A)\tilde{\cup}(G,A))=X_A$. Now $Cl_{\beta c}(F,A)=(e,\{a\})$ and $Cl_{\beta c}(G,A)=(e,\{b\})$. Hence, $Cl_{\beta c}((F,A)\tilde{\cup}(G,A))=X_A\tilde{\nsubseteq}Cl_{\beta c}(F,A)\tilde{\cup}Cl_{\beta c}(G,A)=(e,\{a,b\})$.

The relations between the soft βc-closure and soft βc-interior can be considered in the following theorem:

Theorem 4.2.20. Let (X,τ,A) be a soft topological space over X and (M,A) be a soft set over X. Then

(1) $(Cl_{\beta c}(M,A))^c = Int_{\beta c}((M,A)^c)$.

(2) $(Int_{\beta c}(M,A))^c = Cl_{\beta c}((M,A)^c)$.

(3) $Cl_{\beta c}(F,A) = (Int_{\beta c}(F,A)^c)^c$.

(4) $Int_{\beta c}(M,A) = (Cl_{\beta c}((M,A)^c))^c$.

Proof. (1) $(Cl_{\beta c}(M,A))^c = \left(\tilde{\bigcap}\{(F,A) : (F,A) \text{ is soft } \beta c\text{−closed and}(M,A)\tilde{\subseteq}(F,A)\}\right)^c$
$= \tilde{\bigcup}\{(F,A)^c : (F,A) \text{ is soft } \beta c\text{−closed and}(M,A)\tilde{\subseteq}(F,A)\} = \tilde{\bigcup}\{(F,A)^c : (F,A)^c$ is soft βc−open and$(F,A)^c\tilde{\subseteq}(M,A)^c\} = Int_{\beta c}((M,A)^c)$.
The other parts can be proved in similar way. □

Theorem 4.2.21. Let (X,τ,A) be a soft topological space, then for any soft set over X

(1) If (G,A) is a soft clopen set, then $Cl_{\beta c}(F,A)\tilde{\cap}(G,A)\tilde{\subseteq}Cl_{\beta c}((F,A)\tilde{\cap}(G,A))$.

(2) If (M,A) is a soft clopen set, then $Int_{\beta c}((F,A)\tilde{\cup}(M,A))\tilde{\subseteq}Int_{\beta c}(F,A)\tilde{\cup}(M,A)$.

Proof. (1) Let $x_e\tilde{\in}Cl_{\beta c}(F,A)\tilde{\cap}(G,A)$, then $x_e\tilde{\in}Cl_{\beta c}(F,A)$ and $x_e\tilde{\in}(G,A)$. Let (O,A) be any soft βc-open set containing x_e, then by Corollary 4.1.19, $(O,A)\tilde{\cap}(G,A)$ is a soft βc-open set containing x_e. Since $x_e\tilde{\in}Cl_{\beta c}(F,A)$, by Theorem 4.2.14, $((O,A)\tilde{\cap}(G,A))\tilde{\cap}(F,A) \neq \phi_A$ which implies $(O,A)\tilde{\cap}((G,A)\tilde{\cap}(F,A)) \neq \phi_A$. Hence, by Theorem 4.2.14, $x_e\tilde{\in}Cl_{\beta c}((F,A)\tilde{\cap}(G,A))$.

(2) Direct using part (1) and Theorem 4.2.20. □

Remark 4.2.22. The condition of being (G,A) and (M,A) are soft clopen sets is necessary in Theorem 4.2.20 as illustrated in the following example:

Example 4.2.23. Consider a soft topological space in Example 4.2.6. Let $(F,A) = (e,\{a\})$ and $(G,A) = (e,\{b\})$, then (G,A) is not a soft clopen set. Now $Cl_{\beta c}(F,A)\tilde{\cap}(G,A) = X_A\tilde{\cap}(G,A) = (G,A)\tilde{\nsubseteq}Cl_{\beta c}((F,A)\tilde{\cap}(G,A)) = Cl_{\beta c}(\phi_A) = \phi_A$. Also, if $(M,A) = (e,\{b,c\})$, then (M,A) is not a soft clopen set. Now $Int_{\beta c}((F,A)\tilde{\cup}(M,A)) = Int_{\beta c}(X_A) = X_A\tilde{\nsubseteq}Int_{\beta c}(F,A)\tilde{\cup}(M,A) = \phi_A\tilde{\cup}(M,A) = (M,A)$.

4.3 Soft βc-Continuous Functions

Definition 4.3.1. Let (X, τ_1, A) and (Y, τ_2, B) be two soft topological spaces. A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is called a *soft βc-continuous function* at a soft point $x_e \tilde{\in} X_A$ if for each soft open set (F, B) over Y containing $f_{pu}(x_e)$, there exists a soft βc-open set (G, A) over X containing x_e such that $f_{pu}(G, A) \tilde{\subseteq} (F, B)$. If f_{pu} is a soft βc-continuous at every soft point x_e over X, then it is called *soft βc-continuous* on X.

Remark 4.3.2. Every soft βc-continuous function is a soft β-continuous function.

Theorem 4.3.3. A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is a soft βc-continuous if and only if the inverse image of every soft open set over Y is a soft βc-open set over X.

Proof. ($\Rightarrow$) Let f_{pu} be a soft βc-continuous function and (F, B) be a soft open set over Y. If $f_{pu}^{-1}(F, B) = \phi_A$, then $f_{pu}^{-1}(F, B)$ is a soft βc-open set over X. If not, then for any $x_e \tilde{\in} f_{pu}^{-1}(F, B)$, $f_{pu}(x_e) \tilde{\in} (F, B)$. So by soft βc-continuity of f_{pu}, there exists a soft βc-open set (G, A) over X containing x_e such that $f_{pu}(G, A) \tilde{\subseteq} (F, B)$. Hence, $x_e \tilde{\in} (G, A) \tilde{\subseteq} f_{pu}^{-1}(F, B)$ and therefore, by Theorem 4.1.16, $f_{pu}^{-1}(F, B)$ is a soft βc-open set over X.
($\Leftarrow$) Assume that the inverse image of every soft open set over Y is a soft βc-open set over X. Let $x_e \tilde{\in} X_A$ and (F, B) be a soft open set over Y containing $f_{pu}(x_e)$. Then, $x_e \tilde{\in} f_{pu}^{-1}(F, B)$ which is a soft βc-open set. So $f_{pu}(f_{pu}^{-1}(F, B)) \tilde{\subseteq} (F, B)$. Therefore f_{pu} is a soft βc-continuous function. □

Theorem 4.3.4. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft function such that (X, τ, A) is a soft T_1-space. Then f_{pu} is a soft βc-continuous function if and only if f_{pu} is a soft β-continuous function.

Proof. ($\Rightarrow$) Assume that f_{pu} is a soft β-continuous such that (X, τ, A) is a soft T_1-space. Then for any soft open set (F, B) over Y we have, $f_{pu}^{-1}(F, B)$ is a soft β-open set, and by Theorem 2.4.4, $f_{pu}^{-1}(F, B)$ is a soft βc-open set. Therefore, f_{pu} is a soft βc-continuous function.
($\Leftarrow$) If f_{pu} is a soft βc-continuous function, then by Remark 4.3.2, f_{pu} is a soft β-continuous function. □

Theorem 4.3.5. A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is a soft βc-continuous if and only if f_{pu} is a soft β-continuous function and for each $x_e \tilde{\in} X_A$ and each soft open set (F, B) over Y such that $f_{pu}(x_e) \tilde{\in} (F, B)$, there exists a soft closed set (C, A) over X containing x_e such that $f_{pu}(C, A) \tilde{\subseteq} (F, B)$.

Proof. Let $x_e \tilde{\in} X_A$ such that $f_{pu}(x_e) \tilde{\subseteq} (F, B)$. Then, there exists a soft βc-open set (G, A) over X such that $x_e \tilde{\in} (G, A)$ and $f_{pu}(G, A) \tilde{\subseteq} (F, B)$. But (G, A) is a soft βc-open set which implies that there exists a soft closed set (C, A) over X such that $x_e \tilde{\in} (C, A) \tilde{\subseteq} (G, A)$ and hence, $f_{pu}(C, A) \tilde{\subseteq} (F, B)$.
($\Leftarrow$)If (F, B) be a soft open set over Y, then by assumption, $f_{pu}^{-1}(F, B)$ is soft β-open set and for any $x_e \tilde{\in} f_{pu}^{-1}(F, B)$ we have, $f_{pu}(x_e) \tilde{\in} (F, B)$ and so, there exists a soft closed set (C, A) over X containing x_e such that $f_{pu}(C, A) \tilde{\subseteq} (F, B)$. So, $x_e \tilde{\in} (C, A) \tilde{\subseteq} f_{pu}^{-1}(F, B)$. Therefore, $f_{pu}^{-1}(F, B)$ is a soft βc-open set over X. □

Theorem 4.3.6. Let (X, τ_1, A) and (Y, τ_2, A) be two soft topological spaces and consider $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft function. Then, the following are equivalent:

(1) f_{pu} is a soft βc-continuous function.

(2) $f_{pu}^{-1}(G, B)$ is a soft βc-open set over X for any soft open set (G, B) over Y.

(3) $f_{pu}^{-1}(C, B)$ is a soft βc-closed set over X for any soft closed set (C, B) over Y.

(4) $f_{pu}\left(Cl_{\beta c}(M, A)\right) \tilde{\subseteq} \overline{f_{pu}(M, A)}$ for any soft set (M, A) over X.

(5) $Cl_{\beta c}\left(f_{pu}^{-1}(K, B)\right) \tilde{\subseteq} f_{pu}^{-1}\left(\overline{(K, B)}\right)$ for any soft set (K, B) over Y.

(6) $f_{pu}^{-1}\left((M, B)^{\circ}\right) \tilde{\subseteq} Int_{\beta c}\left(f_{pu}^{-1}(M, B)\right)$ for any soft set (M, B) over Y.

Proof. $(1 \Rightarrow 2)$ Follows from Theorem 4.3.3.
$(2 \Rightarrow 3)$ Let (C, B) be any soft closed set over Y then $(C, B)^c$ is a soft open set. By part (2), we have, $f_{pu}^{-1}((C, B)^c) = (f_{pu}^{-1}(C, B))^c$ is a soft βc-open set over X. Hence, $f_{pu}^{-1}(C, B)$ is a soft βc-closed set over X.
$(3 \Rightarrow 4)$ Let (M, A) be a soft set over X. Then, $f_{pu}(M, A) \tilde{\subseteq} \overline{f_{pu}(M, A)}$ where $\overline{f_{pu}(M, A)}$ is a soft closed set over Y.Hence, by part (3) we have, $f_{pu}^{-1}(\overline{f_{pu}(M, A)})$ is a soft βc-closed set

over X and $(M,A)\tilde{\subseteq} f_{pu}^{-1}(\overline{f_{pu}(M,A)})$. Therefore, $Cl_{\beta c}(M,A)\tilde{\subseteq} f_{pu}^{-1}(\overline{f_{pu}(M,A)})$ and hence, $f_{pu}(Cl_{\beta c}(M,A))\tilde{\subseteq}\overline{f_{pu}(M,A)}$.

$(4 \Rightarrow 5)$ Let (K,B) be any soft set over Y. Then, $f_{pu}^{-1}(K,B)$ is a soft set over X and by using part (4) we have, $f_{pu}(Cl_{\beta c}(f_{pu}^{-1}(K,B)))\tilde{\subseteq}\overline{f_{pu}(f_{pu}^{-1}(K,B))}\tilde{\subseteq}\overline{(K,B)}$.

$(5 \Rightarrow 6)$ Let (M,B) be a soft set over Y. Then by part (5) to $(M,B)^c$ we have, $Cl_{\beta c}(f_{pu}^{-1}((M,B)^c))$
$\tilde{\subseteq} f_{pu}^{-1}(\overline{(M,B)^c})$ which implies, $Cl_{\beta c}(f_{pu}^{-1}((M,B)^c))\tilde{\subseteq} f_{pu}^{-1}\left(((M,B)^\circ)^c\right)$ and so, $\left(Int_{\beta c}(f_{pu}^{-1}(M,B))\right)^c$ $\tilde{\subseteq}\left(f_{pu}^{-1}((M,B)^\circ)\right)^c$. Hence $f_{pu}^{-1}((M,B)^\circ)\tilde{\subseteq} Int_{\beta c}(f_{pu}^{-1}(M,B))$.

$(6 \Rightarrow 1)$ Let $x_e\tilde{\in} X_A$ and let (F,B) be any soft open set over Y containing $f_{pu}(x_e)$, then $x_e\tilde{\in} f_{pu}^{-1}(F,B)$ and $f_{pu}^{-1}(F,B)$ is a soft set over X. By part (6) $f_{pu}^{-1}(F,B) = f_{pu}^{-1}((F,B)^\circ)$ $\tilde{\subseteq} Int_{\beta c}(f_{pu}^{-1}(F,B))$. Therefore, $f_{pu}^{-1}(F,B)$ is a soft βc-open set over X which contains x_e and $f_{pu}(f_{pu}^{-1}(F,B))\tilde{\subseteq}(F,B)$. Hence, f_{pu} is a soft βc-continuous function. □

Definition 4.3.7. A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is called:

(1) *soft βc-irresolute* if for every soft βc-open set (G,B) over Y, $f_u^{-1}(G,B)$ is soft βc-open set over X.

(2) *soft βc-open* if for every soft βc-open set (H,A) over X, $f_{pu}(H,A)$ is a soft βc-open set over Y.

(3) *soft βc-closed* if for every soft βc-closed set (C,A) over X, $f_{pu}(C,A)$ is a soft βc-closed set over Y.

Theorem 4.3.8. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ and $g_{qv} : SS(Y)_B \to SS(Z)_C$ be two soft functions, then the following properties hold:

(1) If f_{pu} is a soft βc-continuous and g_{qv} is a soft continuous, then $g_{qv} \circ f_{pu}$ is a soft βc-continuous.

(2) If f_{pu} is a soft βc-irresolute and g_{qv} is a soft βc-continuous, then $g_{qv} \circ f_{pu}$ is a soft βc-continuous.

Proof. (1) Let (F,C) be a soft open set over Z. Then $g_{qv}^{-1}(F,C)$ is a soft open set over Y by continuity of g_{qv}. Since f_{pu} is a soft βc-continuous, $f_{pu}^{-1}(g_{qv}^{-1}(F,C))$ is a soft βc-open set over X and hence, $(g_{qv} \circ f_{pu})^{-1}(F, C) = f_{pu}^{-1}(g_{qv}^{-1}(F, C))$ is a soft βc-open set over X. Therefore, $g_{qv} \circ f_{pu}$ is a soft βc-continuous.

(2) Let (F, C) be a soft open set over Z. Since g_{qv} is a soft βc-continuous, $g_{qv}^{-1}(F, C)$ is a soft βc-open set over Y. Since f_{pu} is a soft βc-irresolute, $f_{pu}^{-1}(g_{qv}^{-1}(F, C))$ is a soft βc-open set over X and hence, $(g_{qv} \circ f_{pu})^{-1}(F, C) = f_{pu}^{-1}(g_{qv}^{-1}(F, C))$ is a soft βc-open set over X. Therefore, $g_{qv} \circ f_{pu}$ is a soft βc-continuous. □

Theorem 4.3.9. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ and $g_{qv} : SS(Y)_B \to SS(Z)_C$ be two soft functions. If f_{pu} is a soft βc-open and surjective and $g_{qv} \circ f_{pu}$ is a soft βc-continuous, then g_{qv} is a soft βc-continuous.

Proof. Let (F, C) be a soft open set over Z. Since $g_{qv} \circ f_{pu}$ is a soft βc-continuous, $(g_{qv} \circ f_{pu})^{-1}(F, C) = f_{pu}^{-1}(g_{qv}^{-1}(F, C))$ is a soft βc-open set over X. Since f_{pu} is a soft βc-open and surjective, then $f_{pu}(f_{pu}^{-1}(g_{qv}^{-1}(F, C))) = g_{qv}^{-1}(F, C)$ is a soft βc-open set over Y. Hence, g_{qv} is a soft βc-continuous. □

Corollary 4.3.10. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft βc-open, a soft βc-irresolute and surjective function and let $g_{qv} : SS(Y)_B \to SS(Z)_C$ a soft functions. Then, $g_{qv} \circ f_{pu} : SS(X)_A \to SS(Z)_C$ is a soft βc-continuous if and only if g_{qv} is a soft βc-continuous.

Proof. Direct using Theorem 4.3.8 and Theorem 4.3.9. □

Theorem 4.3.11. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft function such that (X, τ, A) is a soft locally indiscrete space. Then, f_{pu} is a soft semi-continuous function if and only if f_{pu} is a soft βc-continuous function.

Proof. $(\Rightarrow)$ Assume that f_{pu} is a soft semi-continuous function. Then for any soft open set (V, B) over Y, $f_{pu}^{-1}(V, B)$ is a soft semi-open set over X. Since (X, τ, A) is a soft locally indiscrete space, by Theorem 4.1.25, $f_{pu}^{-1}(V, B)$ is a soft βc-open set over X. Therefore, f_{pu} is a soft βc-continuous function.

$(\Leftarrow)$ If f_{pu} is a soft βc-continuous function, then for any soft open set (V, B) over Y, $f_{pu}^{-1}(V, B)$ is a soft βc-open set over X which implies that, $f_{pu}^{-1}(V, B) = \tilde{\bigcup}_{x_e \tilde{\in} f_{pu}^{-1}(V,B)}(F_{x_e}, A)$ where (F_{x_e}, A) is a soft closed set over X for each $x_e \tilde{\in} f_{pu}^{-1}(V, B)$. Since (X, τ, A) is a soft locally indiscrete space, $f_{pu}^{-1}(V, B)$ is a soft open set over X because it is a union of soft open sets over X and so, $f_{pu}^{-1}(V, B)$ is a soft semi-open set over X. Therefore, f_{pu} is a soft semi-continuous function. □

Corollary 4.3.12. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft function such that (X, τ, A) is a soft locally indiscrete space. Then

(1) f_{pu} is a soft continuous function if and only if f_{pu} is a soft βc-continuous function.

(2) f_{pu} is a soft α-continuous function if and only if f_{pu} is a soft βc-continuous function

Proof. (1) Direct using Remark 3.1.3 and Theorem 4.3.11.

(2) Direct using Theorem 3.2.8 and Theorem 4.3.11. □

Conclusion

In this thesis, we studied soft βc-open sets in soft topological spaces. Moreover, the soft βc-interior and soft βc-closure were investigated. Also, a new class of soft continuous function with characterizations were introduced. This thesis will open a new way for other researchers to study the concepts of soft βc-boundary, soft βc-exterior and soft βc-limit points, and study the soft βc separation axioms in soft topological spaces. They can also study the applications of soft βc-open sets in the soft Alexandroff topological space.

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