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Comparative Study of Fuzzy Topology

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وبعد المداولة أوصت اللجنة بمنح الباحثة درجة الماجستير في **كلية *العلوم*/** قسم **الرياضيات**.

واللجنة إذ تمنحها هذه الدرجة فإنها توصيها بتقوى الله ولزوم طاعته وأن تسخر علمها في خدمة دينها ووطنها.

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Contents

Abstract

In this thesis, we will study Chang, Lowen and Sostak fuzzy topologies which are the most important and usefully between definitions of fuzzy topologies. Then we give some functions which convert one of them to another or to the ordinary topology. We study the fuzzy continuity, separations axioms and compactness, each of them have several definitions whith respect to what definition of fuzzy topology we use. We give a comparative and preferring between them. Both of Chang and Lowen define the fuzzy continuity. We prefer the first one since it is parallel with the definition and theorems in the ordinary topology. When we studied the separation axioms the main attention paid for T_1-ness as an important properties, we give many definitions of T_1-space and prefer some of them depending on the good extensions properity (i.e. if the ordinary topology is T_1-space then the extension fuzzy space is also fuzzy T_1-space). Then we redefine these T_1 definitions to define fuzzy T_1, strong T_1 and weak T_1-spaces. Moreover many authors define the fuzzy compactness. We compare between them especially Chang and Lowen definitions. Finally we study Sostak structures which is a generalization to Chang definitions. We completed and rewrite some proofs of many theorems.

Introduction

Fuzzy sets were introduced by Zadeh [22] in 1965 as an extension of the classical notion of sets. In classical set theory, an element either belongs or dose not belong to the set (i.e. have a degree of membership equal to 1 or 0 respectively), but the elements of fuzzy set have degree of membership belongs to the unit interval $[0, 1]$. It can be used in a wide range of some structures such as functional analysis, loop, group, ring, algebras, ideals, vector spaces, topological spaces, quantum particle physics and control. Usually a fuzzification of a mathematical concept is based on a generalization of this concept from an ordinary set to a fuzzy set.

Fuzzy topology forms a new branch of mathematics based on fuzzy sets, which is used to make applications to image processing which is prefered more than the digital topology, since the digital topology is mainly suited to black\white images or to segmented objects in images. But fuzzy topology generalize the known concepts from black\white topology to gray value topology to get gray images. Rosenfeld presented a theory of fuzzy topology for gray images. He was able to generalize such concepts as compactness of an object in an image. A large number of papers dealing with this subject were published.

Fuzzy topology started in 1968 by Chang in [7] when he replaced sets by fuzzy sets in the ordinary definition of topology.

Later, Several definitions of fuzzy topology were introduced. In 1976, Lowen gave a modification of the definition of Chang fuzzy topological space. He introduced two functions ω and ι which connect between his new structure of fuzzy topological spaces and the ordinary topological spaces. In 1985 Sostak, introduced the fundamental concept of a fuzzy topological structure as an extension of both ordinary topology and Chang's fuzzy topology, in the sense that not only the sets were fuzzifed, but also the topology. There are several reasons why one definition is preferred to another. For example, in Chang's definition the constant functions between fuzzy topological spaces are not necessarily continuous. But evidently, they are continuous in the modified definition of Lowen. Many other authors like Chattopadhyay and Samanta definitions of fuzzy topology.

In Chapter 1, we will study the concepts of fuzzy sets in two sections, in the first one we mentions some of definitions and theorems of the ordinary topology which will be needed in our study. In the second section, we study the fuzzy sets which defined by Zadah as a generalization of the ordinary sets giving some examples on it. In 1972 Zadeh define the fuzzy singleton, then in 1974 Wong gave a definition of fuzzy point, after that Pao-Ming and Liu Ming redefined the fuzzy point. In this thesis we redefine fuzzy point\singleton and belonging of\subsets to be parallel with the ordinary topology. We give a definition of fuzzy functions and their inverse images with some results and exampels.

In Chapter 2, we gather Chang and Lowen definitions, since they are more closed to each other than Sostak definition. In the first section, we define Chang fuzzy topology denoted by (C-fts) which is a naturally extensions to the ordinary topology using fuzzy sets instead of fuzzy topology. Lowen fuzzy topology, denoted by (L-fts) and it's basic properties were studied in section 2 which add one condition to the Chang conditions. On the third section, we give ways of converting between Chang fuzzy topology, Lowen topology and the usual topology. A summarize will be given at the end of this section.

In the senses of Change and Lowen topology, the third chapter study some topological concepts such as continuity, compactness and separation axioms. Chapter three including four sections. In the first one, we mention an important properties which we need in the later sections such as the concepts of hereditary, productive, good extension which we use to examine which definitions (in the late sections) is more preferred than the others. In the second section, we give a comparative between Chang and Lowen continuity and we give some parallel results to the ordinary continuity, then we show some reasons to ignore Lowen definition of fuzzy continuity and depends the Chang definition. In the third section, we study separation axioms which have many different definitions. The deviations of these definitions of separation axioms depends on the definition of the fuzzy point which the author followed. We concentrate on T_1-ness and give some details, relations and implications between its several definitions. Then we define in fast the fuzzy T_i-ness definition for $i = 0, 1, 2, 2\frac{1}{2}, 3, 4$.

In Section 4, we focus on two definitions of compactness which defined by Chang and Lowen, then we will mention the other definitions of compactness and its prosperities, then we discuss which definition be more stronger to another.

Chaper 4 consists of two sections. In section one, we give Sostak definition, which denoted by (S-ft), with some examples. Then we give ways to convert Sostak topology to ordinary topology, C-ft and L-ft and conversely. In second section, we discuss some topological concept on S-ft as continuity, separation axioms and compactness. Finally, we give the conclusion including what we look forward to study in next research.

Chapter 1

Fuzzy Sets and Main Structures in Fuzzy Topological Spaces

In this chapter, we will speak about the definition of fuzzy set and some of fuzzy properties. Then we will discuss the different definitions of fuzzy point.

1.1 Introduction to Topological Space

In this section, we give definitions, properties and theorems of the ordinary topology which will be needed in this thesis. In this section we will ignore the proofs and these proofs can be found in any elementary book of topology for example see [40], [41].

Definition 1.1.1. [40] Let X be a non-empty set, a collection δ of subsets of X is said to be a *topology* on X if

i X and the empty set ϕ belong to δ,

ii the union of any members of elements in δ belongs to δ, iii the intersection of any two sets in δ belongs to δ.

The members of δ are said to be *open* sets. The complement of members of δ are said to be *closed* sets. A subset of a topology δ is said to be *clopen* if it is both open and closed in δ. The pair (X,δ) is called a *topological space*.

Definition 1.1.2. [41] Let X be a non-empty set and δ the collection of all subsets of X. Then δ is called *the discrete topology* on X and the topological space (X, δ) is called the *discrete topology space*. If $\delta = \{X, \phi\}$, then δ is called the *indiscrete topology* and (X, δ) is said to be the *indiscrete topology space*.

Proposition 1.1.3. *[41] If (X, δ) is a topological space such that for every $x \in X$, the singleton set $\{x\}$ is in δ, then δ is the discrete topology.*

Definition 1.1.4. [41] Let (X, δ), (Y, γ) be two topological spaces, and f a function from X to Y. Then we say that:

a) f *is continuous* iff for every open set B in γ, we have $f^{-1}(B)$ is open set in δ.

b) f *is open* iff for any open set A in δ, the set $f(A)$ is open set in γ.

Definition 1.1.5. [41] Let (X, δ) and (Y, γ) be topological spaces, Then X and Y are said to be *homeomorphic* if there exists a function $f : X \longrightarrow Y$ which is one-to-one, onto, continuous, and open. The map f is said to be a *homeomorphism* between (X, δ) and (Y, γ), We write $(X, \delta) \cong (Y, \gamma)$.

Recall that for a function f from a set X into a set Y, and for any subsets B_α of Y, $\alpha \in \Delta$, we have that

$$f^{-1}(\bigcup_{\alpha\in\Delta} B_\alpha) = \bigcup_{\alpha\in\Delta} f^{-1}(B_\alpha), \text{ and } f^{-1}(\bigcap_{\alpha\in\Delta} B_\alpha) = \bigcap_{\alpha\in\Delta} f^{-1}(B_\alpha).$$

Proposition 1.1.6. *[41] Let f be a function from a topological space (X, δ) into a topological space (Y, γ). Then the following statements are equivalent:*

i f *is continuous function.*

ii *For every closed subset* S *of* Y*, we have* $f^{-1}(S)$ *is a closed subset of* X*.*

iii *For each* $a \in X$ *and each* $V \in \gamma$ *with* $f(a) \in V$ *there exists* $U \in \delta$ *such that* $a \in U$ *and* $f(U) \subseteq V$*.*

Proposition 1.1.7. *[41] Let* (X, δ_1)*,* (Y, δ_2) *and* (Z, δ_3) *be topological spaces. If* $f : (X, \delta_1) \longrightarrow (Y, \delta_2)$ *and* $g : (Y, \delta_2)) \longrightarrow (Z, \delta_3)$ *are continuous functions, then the* composite function $g \circ f : (X, \delta_1) \longrightarrow (Z, \delta_3)$ *is continuous.*

Example 1.1.8. Every constant function is continuous.

Definition 1.1.9. [41] Let δ be a topology on X and $\mathcal{B}$ a collection of open subsets of X such that any open set is a union of sets from $\mathcal{B}$. Then $\mathcal{B}$ is called *a base* for the topology δ.

Example 1.1.10. The set of all open intervals on the real line is a base for a topological space called the *usual topology* on $\mathbb{R}$.

Definition 1.1.11. [41] Let (X, δ) be a topological space. Then X is said to be *compact* if for every open collection $\{O_\beta : \beta \in \Delta\}$ such that $X = \bigcup_{\beta \in \Delta} O_\beta$ there exists a finite subcollection $\{O_{\beta_1}, O_{\beta_2} ...., O_{\beta_n}\}$ such that $X = O_{\beta_1} \cup O_{\beta_2} \cup .... \cup O_{\beta_n}$. If $A \subseteq X$, then A is called compact if it is compact as a subspace.

Definition 1.1.12. [41] Let $(X_1, \delta_1), (X_2, \delta_2), ..., (X_n, \delta_n)$ be topological spaces. Then the *product topology* δ_p on the cartesian product $X_1 \times X_2 ... \times X_n$ is the topology having the family $\{O_1 \times O_2 \times ... \times O_n, O_i \in \delta_i, i = 1, ..., n\}$ as a base. The set $X_1 \times X_2 \times ... \times X_n$ with the topology δ_p is called *the product space* of the spaces $(X_1, \delta_1), (X_2, \delta_2), ..., (X_n, \delta_n)$ and it is denoted by $(\prod_{k=1}^{n} X_k, \delta_p)$.

Definition 1.1.13. [41] Let $(X_i, \delta_i), i \in \Delta$ be an infinite family of topological spaces. Then *the product* $\Pi_{i\in\Delta}X_i$ of the sets $X_i, i \in \Delta$ consists of all the infinite sequences $\Pi_{i\in\Delta}x_i$, where $x_i \in X_i, \forall i$. The product space $\Pi_{i\in\Delta}(X_i, \tau_i)$ consists of the product $\Pi_{i\in\Delta}X_i$ with the topology τ having as its basis the family

$B = \{\Pi_{i\in\Delta}O_i : O_i \in \tau_i$ and $O_i = X_i$ for all but a finite number of $i\}$. The topology τ is called the *product topology.(or the Tychonoff topology).*

Theorem 1.1.14. *[41] (*Tychonoff's Theorem*)*

A countable product of compact topological spaces is compact.

Theorem 1.1.15. *[41] The continuous image of a compact space is compact.*

Definition 1.1.16. [41] A topological space (X, δ) is said to be:

a. *T_0-space* if for any pair of distinct points a, b in X, there exists open set U such that $a \in U, b \notin U$, or $b \in U, a \notin U$.

b. *T_1-space* if for any pair of distinct points a, b in X, there exist open sets U and V such that $a \in U, b \notin U$, and $b \in V, a \notin V$.

c. *Hausdorff*(T_2) *space* iff for any pair of distinct points a, b in X, there exist disjoint open sets U and V such that $a \in U$ and $b \in V$.

d. *regular space* if for any closed subset A of X and any point $x \notin A$, there exist disjoint open sets U and V such that $x \in U$ and $A \subseteq V$.

e. *T_3-space* if (X, δ) is regular T_1-space.

f. *Normal space* if for each pair of disjoint closed sets A and B, there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

g. *T_4-space* if (X, δ) is normal T_1 space.

Clearly the following implications hold for the separation axioms:

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0.$$

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1.2 Fuzzy Set

In Mathematics, a subset A of a set X can be equivalently represented as a *characteristic* function $\mathcal{X}_A$, which is defined by $\mathcal{X}_{\mathcal{A}} : X \longrightarrow \{0, 1\}$, where

$$\mathcal{X}_{\mathcal{A}}(x) = \begin{cases} 1, & x \in A; \\ \\ 0, & \text{otherwise.} \end{cases}$$

That is, we say that x belongs to A if and only if $\mathcal{X}_A(x) = 1$. A generalization of the meaning "$x \in A$" is given in [22]. The *fuzzy* case of "belonging" between x and A is no longer "either 0 or 1" on a relation $\mathcal{X}_A(x)$, but it has a membership degree belonging to $[0, 1]$ instead of $\{0, 1\}$. See the following definition.

Definition 1.2.1. [22] A *fuzzy set* in X is a function A from a non empty set X to the unit interval $I = [0, 1]$. The image $A(x)$ of the element $x \in X$ is called a *"grade of membership "* of x in A.

Example 1.2.2. Let X to be any non-empty set and A be any subset of X. Then the characteristic function $\mathcal{X}_{\mathcal{A}}$ is a fuzzy set. This fuzzy set is called a *crisp set*, or *ordinary set*. So, an ordinary set is a fuzzy set with grade of membership equal 0 or 1 for all $x \in X$.

Example 1.2.3. Let $A : \mathbb{R} \longrightarrow [0, 1]$ be a function such that $A(x) = |sin\ x| \ \ \forall x \in \mathbb{R}$. Then A is a fuzzy set.

Definition 1.2.4. [9] Let A and B be fuzzy sets in X. Then $\forall\, x \in X$:

1. We define the *empty fuzzy set* Φ, the *universal fuzzy* set $\boldsymbol{X}$, A^c for the complement of A on X, and $A\backslash B$ for the complement of A on B as follows:

$$\Phi(x) = 0 \;\; \textit{and} \;\; \boldsymbol{X}(x) = 1,$$

$$(A^c)(x) = 1 - A(x),$$

Note that some authors write $\boldsymbol{X} \setminus A$ instead of A^c.

$$(A \setminus B)(x) = \begin{cases} A(x), & A(x) > B(x); \\ 0, & \text{otherwise.} \end{cases}$$

2. $A = B \Longleftrightarrow A(x) = B(x)$,

3. $A \subseteq B \Longleftrightarrow A(x) \leq B(x)$,

4. $(A \bigvee B)(x) = \max\{A(x), B(x)\}$,

5. $(A \bigwedge B)(x) = \min\{A(x), B(x)\}$.

   More generally, for a family of fuzzy sets $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$, the union and the intersection are defined by:

6. $(\bigvee\limits_{\alpha} A_\alpha)(x) = \sup\{A_\alpha(x) : \alpha \in I\}$.

7. $(\bigwedge\limits_{\alpha} A_\alpha)(x) = \inf\{A_\alpha(x) : \alpha \in I\}$.

Lemma 1.2.5. *[22] De Morgan's Laws and distributive laws are hold for the fuzzy sets.*

Remark 1.2.6. From now, we will denote the universal fuzzy set by $\boldsymbol{X}$, the ordinary universal set by X, the empty fuzzy set by Φ and the empty ordinary set by ϕ.
We will use symbols as A, B and C to denote the fuzzy sets and any ordinary set can be referred as a fuzzy set with grade of membership 0 or 1.

Definitions 1.2.7. [29] Two fuzzy sets A, B in X are said to be *intersecting* iff there exists a point $x \in X$ such that $(A \bigwedge B)(x) \neq 0$. For such a case, we say that A and B intersect at x. If $(A \wedge B)(x) = 0 \ \forall x \in X$, A and B are disjoint.

Definitions 1.2.8.

1. (*Zadeh Definition of a fuzzy singleton set.*)[22] or (*Pu-Ming and Liu-Ming definition of a fuzzy point p*)[29]

   Let X be a non-empty set. A *fuzzy singleton* in X is a fuzzy set taking value $\lambda \in \underline{\mathbf{(0,\ 1]}}$ at only one point say z and zero elsewhere.

2. (*Wong definition of a fuzzy point p*)[9]

   Let X be any non-empty set. A *fuzzy point* p in X is a fuzzy set taking value $\lambda \in \underline{\mathbf{(0,\ 1)}}$ at only one point say z and zero elsewhere.

With respect to any definition; we denote the fuzzy point\ singleton by z_λ where the point z is called its *support*, and λ is called *the value of z*. So, we have that

$$z_\lambda(x) = \begin{cases} \lambda, & x = z; \\ & \\ 0, & x \neq z. \end{cases}$$

Definitions 1.2.9. There is two definitions of *belonging:*

1. [29] We say that a fuzzy point z_λ belongs to a fuzzy set A if $\lambda \leq A(z)$.

2. [27] We say that a fuzzy point z_λ belongs to a fuzzy set A if $\lambda < A(z)$.

Remark 1.2.10. Both of the two definitions of the fuzzy point and also both of the tow definitions of belonging have some parallel and also some deviation of the usual topology. Although many of research use one of them rather than the other for some reasons, we will use both of them. So, to avoid this problem we suggest the following:

1. We want to denote to the first definition by $z_\lambda \subseteq A$, (i.e $z_\lambda \subseteq A$ if and only if $\lambda \leq A(x)$). This is because every fuzzy point (or fuzzy singleton) is in fact a fuzzy set and this is parallel with the Definition 1.2.4(3) of "subset".

2. We denote to the second definition by $z_\lambda \in A$, (i.e $z_\lambda \in A$ if and only if $\lambda < A(x)$) because this makes the following important theorem true, and this is parallel with the ordinary set theory. Note that if we say $z_\lambda \in A$ if and only if $\lambda \leq A(x)$, then the following theorem will be no longer true, and we give a counter example illustrates this fact.

Theorem 1.2.11. *[9] Let Δ be any index set, then*

$$z_\lambda \in \bigvee_{i\in\Delta} A_i \Leftrightarrow \exists i \in \Delta \text{ such that } z_\lambda \in A_i.$$

Proof. ($\Rightarrow$) Let $z_\lambda \in \bigvee_{i\in\Delta} A_i$, then $\lambda < (\bigvee_{i\in\Delta} A_i)(z) = \sup\{A_i(z) : i \in \Delta\}$. So, $\exists k \in \Delta$ such that $\lambda < A_k(z)$, thus $z_\lambda \in A_k$.

($\Leftarrow$) Let $z_\lambda \in A_j$ for some $j \in \Delta$, then $\lambda < A_j(z) \leq \sup\{A_i(z) : i \in \Delta\} = \bigvee_{i\in\Delta} A_i(z)$. So, $z_\lambda \in \bigvee_{i\in\Delta} A_i$.

□

The following example shows that the above Theorem is not true if we take the definition of " $\in$ " with equality in [29].

Example 1.2.12. For $i \in (0, \frac{1}{2})$, define $A_i(x) = i$ for all $x \in X$.
Then $(\bigvee_{i\in(0,\frac{1}{2})} A_i)(x) = \dfrac{1}{2}$. Let $z \in X$, and suppose that $z_\lambda \in A$ if and only if $\lambda \leq A(z)$. (The definition of $\in$ in [29]). Then $z_{\frac{1}{2}} \in (\bigvee_{i\in(0,\frac{1}{2})} A_i)$. Now for $i \in (0, \frac{1}{2})$, $A_i(z) = i < \frac{1}{2} = z_{\frac{1}{2}}(z)$. So, $z_{\frac{1}{2}} \notin A_i$ for all i, and this illustrate that under the definition of $\in$ in [29], Theorem 1.2.11 dose not hold.

Conclusion In this thesis and unless otherwise statement, we will contact with the following:

1. The Wong's definition for fuzzy points.

2. The Zadeh's definition for the fuzzy singletons. Note that the fuzzy singleton is a fuzzy set and this agree with the singleton $\{x\}$ on the usual topology which is also a usual set.

3. We will use the symbol "$\in$" for the fuzzy points. And the symbol "$\subseteq$" for the fuzzy singletons and use them as fuzzy sets. This relations agree with the definition: "if A and B are any fuzzy sets in X, then $A \subseteq B$ if and only if $A(x) \leq B(x)$", since any fuzzy singleton is a fuzzy set.

Note that if $\lambda \in (0, 1]$ in Zadeh's definition, then we have a conflict with the usual set theory as we see in the following example.

Example 1.2.13. Let $X = \{a, b, c\}$, and let S be the fuzzy set $\{(a, 1), (b, 1), (c, 0)\}$. Take a_1 to be the fuzzy set $\{(a, 1), (b, 0), (c, 0)\}$. Then in the usual set theory, S and a_1 redefined as $S = \{a, b\},\ a_1 = \{a\} \subseteq S$. Now, if we use the definition of belonging $(\equiv z_\lambda \in A \Longleftrightarrow \lambda < A(x))$, we get that $a_1 \notin S$, and if we use the definition of subset $(\equiv z_\lambda \subseteq A \Longleftrightarrow \lambda \leq A(x))$ we get that $a_1 \subseteq S$.

Definition 1.2.14. Two fuzzy points (or fuzzy singletons) are said to be *distinct* if their supports are different.
Note that if they have the same support with different value then the one with minimum value will be belong to (or a subset of) the second.

Definitions 1.2.15. [26] A fuzzy point z_λ is said to be *quasi-coincident* with a fuzzy set A, which is denoted by $z_\lambda\ q\ A$ if and only if $\lambda > A^c(z)$, or equivalently $\lambda + A(z) > 1$.

Definitions 1.2.16. [26] A fuzzy set A is said to be *quasi-coincident* with B, which is denoted by $A\ q\ B$, if and only if there exists $x \in X$ such that $A(x) > B^c(x)$, or equivalently $A(x)+B(x) > 1$. In this case, we say that A and B are quasi-coincident (with each other) at x. We will use the notation $A \; \overline{q} \; B^c$ if A and B are not quasi-coincident.

It is clearly that if A is quasi-coincident with B (i.e $A \; q \; B$), then $B \; q \; A$.

Also if A and B are quasi-coincident at x, then both $A(x)$ and $B(x)$ are not zero and hence A and B intersect at x.

Remark 1.2.17. We are summarizing some of previous facts about fuzzy points and fuzzy singletons as follows:

1. If z_λ is a fuzzy singleton, then $z_\lambda \subseteq A$ if and only if $\lambda \leq A(x)$.
2. If z_λ is a fuzzy point, then $z_\lambda \in A$ if and only if $\lambda < A(x)$.
3. If z_λ is a fuzzy singleton or a fuzzy point, then $A^c(z) < \lambda$ if and only if $z_\lambda \; q \; A$.
4. It's clearly that if A and B are a fuzzy set on X, then $A \subseteq B$ if and only if $A \; \overline{q} \; B^c$ [29].
5. Any fuzzy set can be express as the union of it's fuzzy point or it's fuzzy singleton as follows:

   $A = \bigvee\{p \in A, p \text{ is a fuzzy point}\} = \bigvee\{t \subseteq A : t \text{ is a fuzzy singleton}\}$..

Definition 1.2.18. The product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$, of mappings $f_i : X_i \to Y_i$, $i = 1, 2$, is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$, for each $(x_1, x_2) \in X_1 \times X_2$.

Definition 1.2.19. [34] Let A and B be fuzzy sets in X and Y, respectively. The Cartesian product $A \times B$ of A and B is a fuzzy set in $X \times Y$ defined by:

$$(A \times B)(x, y) = inf\{A(x), B(y)\} \quad \forall (x, y) \in X \times Y.$$

This definition can be easily extended to arbitrary Cartesian products (we use Π the usual notation of product).

Now, we will define the image and inverse image for any fuzzy set, then we give an example showing how to find them.

Definition 1.2.20. [9] Let $f : X \to Y$ be a function from X to Y. The *fuzzy function* $f : [0,1]^X \to [0,1]^Y$ is defined by the following:
for a fuzzy set A in X, $f[A]$ is a fuzzy set in Y whose membership function is given by:

$$f[A](y) = \begin{cases} \sup\limits_{z \in f^{-1}[y]} \{A(z)\}, & f^{-1}[y] \text{ is not empty;} \\ 0, & f^{-1}[y] \text{ is empty.} \end{cases}$$

$\forall y \in Y$.

Definition 1.2.21. [9] Let f be a function from X to Y, B is a fuzzy set in Y. Then *the inverse image* of B, written as $f^{-1}[B]$, is a fuzzy set in X whose membership function is defined by:

$$f^{-1}[B](x) = B(f(x)) \qquad \forall x \in X.$$

The following example show how to find the image and inverse image for the fuzzy sets.

Example 1.2.22. .
Let $X = \{a, b, c\}$, $Y = \{d, e, g\}$ Let A, B be two fuzzy sets on X.
$A = \{(a, 0.5), (b, 0.3), (c, 0.9)\} \subseteq X$, $B = \{(d, 0.2), (e, 0.7), (g, 0.11)\} \subseteq Y$.
Let $f : X \longrightarrow Y$ be a function defined by $f(a) = d$, $f(b) = d$, $f(c) = e$, then:

(1) Since $f^{-1}(d) = \{a, b\} \neq \Phi$, we have that:
$f[A](d) = \sup\{A(x) : x \in f^{-1}(d)\} = \sup\{A(a), A(b)\} = \sup\{0.5, 0.3\} = 0.5$.
Since $f^{-1}(e) = \{c\} \neq \Phi$, we have that:
$f[A](e) = \sup\{A(x) : x \in f^{-1}(e)\} = \sup\{A(c)\} = 0.9$.
Since $f^{-1}(g) = \Phi$, then $f[A](g) = 0$. So, $f[A] = \{(d, 0.5), (e, 0.9), (g, 0)\}$.

(2) Since $f^{-1}[B](a) = B(f(a)) = B(d) = .2$,

$f^{-1}[B](b) = B(f(b)) = B(d) = .2$,

$f^{-1}[B](c) = B(f(c)) = B(e) = .7$. So, $f^{-1}[B] = \{(a, 0.2), (b, 0.2), (c, .7)\}$.

Images and inverse images of fuzzy sets have properties like those of images and inverse images of ordinary sets. As an example, see the following lemma:

Lemma 1.2.23. *[19] Let $f : X \to Y$ be a mapping and let $B_\alpha, \alpha \in \Delta$ be a fuzzy sets of Y. Then:*

$$f^{-1}[\bigvee_{\alpha\in\Delta} B_\alpha] = \bigvee_{\alpha\in\Delta} f^{-1}[B_\alpha]$$

and

$$f^{-1}[\bigwedge_{\alpha\in\Delta} B_\alpha] = \bigwedge_{\alpha\in\Delta} f^{-1}[B_\alpha].$$

Proof. $f^{-1}[\bigvee_{\alpha\in\Delta} B_\alpha](x) = (\bigvee_{\alpha\in\Delta} B_\alpha)f(x) = \bigvee_{\alpha\in\Delta} B_\alpha(f(x)) = \bigvee_{\alpha\in\Delta} f^{-1}[B_\alpha](x) = (\bigvee_{\alpha\in\Delta} f^{-1}[B_\alpha])(x)$.
The same for the second part. □

Definition 1.2.24. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, then the *composition* of f and g written as $g \circ f$ is defined by $(g \circ f)[A] = g[f[A]]$ for any fuzzy set A in X.

Theorem 1.2.25. *[7] Let f be a function from X to Y, then:*

a. $f^{-1}[\boldsymbol{Y} \setminus B] = \boldsymbol{X} \setminus f^{-1}[B]$ *for any fuzzy set B in Y.*

b. $\boldsymbol{Y} \setminus f[A] \subseteq f[\boldsymbol{X} \setminus A]$ *for any fuzzy set A in X, f is onto function.*

c. if $B_1 \subseteq B_2$, *then* $f^{-1}[B_1] \subseteq f^{-1}[B_2]$, *where* B_1, B_2 *are fuzzy sets in Y.*

d. if $A_1 \subseteq A_2$, *then* $f[A_1] \subseteq f[A_2]$, *where* A_1, A_2 *are fuzzy sets in X.*

e. $f[f^{-1}[B]] \subseteq B$ *for any fuzzy set B in Y.*

f. $A \subseteq f^{-1}[f[A]]$ *for any fuzzy set A in X.*

g. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, then $(g \circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$ for any fuzzy set C in Z.

Proof. We proof a, b, and f.

a) Let $x \in X$

$f^{-1}[\boldsymbol{Y} \setminus B](x) = (\boldsymbol{Y} \setminus B)[f(x)] = 1 - B[f(x)] = 1 - f^{-1}[B(x)] = \boldsymbol{X} \setminus f^{-1}[B(x)]$.

b) Let $y \in Y$ Since f is onto, then $f^{-1}[y]$ is not empty, so $\boldsymbol{Y} \setminus f[A](y) = 1 - f[A](y) = 1 - \sup\limits_{z \in f^{-1}[y]} \{A(z)\} = \inf\limits_{z \in f^{-1}[y]} \{1 - A(z)\} \leq \sup\limits_{z \in f^{-1}[y]} \{1 - A(z)\} = \sup\limits_{z \in f^{-1}[y]} \{\boldsymbol{X} \setminus A(z)\} = f[\boldsymbol{X} \setminus A](y). \Longrightarrow \boldsymbol{Y} \setminus f[A] \subseteq f[\boldsymbol{X} \setminus A]$

f) Let $x \in X$

since $x \in f^{-1}[f(x)]$, we have $A(x) \leq \sup\limits_{z \in f^{-1}[f(x)]} \{A(z)\} = f[A](f(x)) = f^{-1}[f[A]](x)$

For the other parts see[7]

□

Remark 1.2.26. In the above theorem (part b), we add the condition that f is onto which was not added in [7]. The necessity of this condition come from if f is not onto, choose $z \in Y$ such that $f^{-1}[z]$ is empty, so $f[A](z) = 0$ for any set $A \in X$,then we have that $\boldsymbol{Y} \setminus f[A](z) = 1 - f[A](z) = 1 - 0 = 1 \geq 0 = f[\boldsymbol{X} \setminus A](z)$ which contradict with $\boldsymbol{Y} \setminus f[A] \subseteq f[\boldsymbol{X} \setminus A]$.

Example 1.2.27. Let f be a function $f : \mathbb{Z} \longrightarrow \mathbb{N} \cup \{0\}$ such that $f(a) = |x|$, let A be a fuzzy set in $\mathbb{Z}$. Then $\forall n \in \mathbb{N}, f^{-1}(n) = \{n, -n\} \neq \phi$.

So, $f[A](n) = \sup\{A(x) : x \in f^{-1}(n)\} = \sup\{A(x) : x = \pm n\} = \sup\{A(n), A(-n)\}$.

And $f^{-1}(0) = \{0\} \neq \phi$. So, $f[A](0) = \sup\{A(x) : x \in f^{-1}(0)\} = A(0)$.

In particular, $\forall n \in \mathbb{N}$ if $A = \Phi$, then $f[\Phi](n) = \sup\{\Phi(n), \Phi(-n)\} = \sup\{0, 0\} = 0$.

And if $A = \boldsymbol{Z}$, then $f[\boldsymbol{Z}](n) = \sup\{\boldsymbol{Z}(n), \ \boldsymbol{Z}(-n)\} = \sup\{1, 1\} = 1$.

Hence $f[\Phi] = \{(n, 0) : n \in \mathbb{N}\} \cup \{(0, 0)\} = \Phi$, and $f[\boldsymbol{Z}] = \{(n, 1) : n \in \mathbb{N}\} \cup \{(0, 1)\} = \boldsymbol{N} \cup \{(0, 1)\}$.

In this chapter we state Example 1.2.3, Remark 1.2.10, the proof of Theorem 1.2.11, Example 1.2.12, Example 1.2.13, Remark 1.2.17, Example 1.2.22, the proof of Lemma 1.2.23, correct Theorem 1.2.5 (part b), Example 1.2.27.

Chapter 2

Chang and Lowen Fuzzy Topological spaces

Fuzzy sets have many applications in many fields such as information [7] and control [11]. Chang [1] was the first one who introduced the notion of fuzzy topology as an application of fuzzy sets. Later Lowen [3] redefined what is now known as stratified fuzzy topology. Sostak [8] introduced the notion of fuzzy topology as an extension of Chang and Lowens fuzzy topology.

We will gather Chang and Lowen definitions in this chapter, since they are more closed to each other than Sostak definition. Then in the last section we will give a methods to convert between Chang fuzzy topology, Lowen fuzzy topology and ordinary topology.

We will denote Chang fuzzy topology by C-ft, Lowen fuzzy topology by L-ft, Sostak fuzzy topology by S-ft and the unit interval $[0, 1]$ by I. Also we use the symbol δ for ordinary topology and the symbol τ for any structure of fuzzy topology.

2.1 Chang's Definition of Fuzzy Topology

Chang use the same definition of ordinary topology to define the fuzzy topology but by using fuzzy sets not ordinary sets.

Definition 2.1.1. (Chang Diffinition)[7]

A *fuzzy topology* is a family τ of fuzzy sets in X which satisfies the following conditions:

a. $\Phi, \boldsymbol{X} \in \tau$,

b. If $A, B \in \tau$, then $A \wedge B \in \tau$.

c. If $A_\gamma \in \tau$ for each $\gamma \in \Delta$, then $\bigvee_{\gamma \in \Delta} A_\gamma \in \tau$.

The pair (X, τ) is called a *Chang fuzzy topological space*, or *C-ft* for short.

Definition 2.1.2. [7] Let τ be a C-ft on X, and A be a fuzzy set in X. Then we call

1. A *open fuzzy* set, if $A \in \tau$.

2. A *closed fuzzy* set, if $A^c \in \tau$.

3. A *cloben fuzzy set*, if A and $A^c \in \tau$.

Example 2.1.3. Let $X = \{x, y\}$ and $A = \{(x, 0.6), (y, 0.3)\}$, $B = \{(x, 0.2), (y, 0.7)\}$
$(A \wedge B)(x) = \inf\{0.6, 0.2\} = 0.2 \quad (A \wedge B)(y) = \inf\{0.3, 0.7\} = 0.3$.
So, $A \wedge B = \{(x, 0.2), (y, .03)\}$. Moreover, $(A \vee B)(x) = \sup\{0.6, 0.2\} = 0.6$.
$\{A \vee B\}(y) = \sup\{0.3, 0.7\} = 0.7$. So, $A \vee B = \{(x, 0.6), (y, 0.7)\}$.
Note that $\{\mathbf{X} \vee A\}(x) = \sup\{1, 0.6\} = 1 \qquad \{\boldsymbol{X} \vee A\}(y) = \sup\{0.3, 1\} = 1$. So $\boldsymbol{X} \vee A = \boldsymbol{X}$

Similarly $\quad A \wedge \mathbf{X} = A$, $\Phi \wedge A = \Phi$ and $\Phi \vee A = A$

Now, the collection $\tau = \{\Phi,\ \mathbf{X},\ A,\ B,\ A \wedge B,\ A \vee B\}$ is a Chang fuzzy topology. So, we have that Φ, $\boldsymbol{X}$, A, B, $A \vee B$, $A \wedge B$ are open fuzzy sets in C-ft. Moreover, the fuzzy sets $A^c = \{(x, 0.4),\ (y, 0.7)\}$, $B^c = \{(x, 0.8),\ (y, 0.3)\}$, Φ, and $\boldsymbol{X}$ are closed fuzzy sets in τ.

$\Phi, \mathbf{X}$ are clopen fuzzy sets in C-ft. The fuzzy set $C = \{(x, 0.1),\ (y, 0.1257)\}$ is neither closed nor open.

The following propositions and definitions are a natural extension to the ordinary topology

Proposition 2.1.4. *Let $(X,\ \tau)$ be a C-fts, Then:*

a. $\Phi, \boldsymbol{X}$ *are clopen fuzzy sets.*

b. Finite union of closed fuzzy sets is closed.

c. Arbitrary intersection of closed fuzzy sets is closed.

d. If I^X denote the family of all fuzzy sets on X, then I^X is a C-fts. Also, the set $\{\Phi, \boldsymbol{X}\}$ is a C-fts.

Proof. a. For $x \in X$, $\Phi^c(x) = 1 - \Phi(x) = 1 - 0 = 1 = \boldsymbol{X}(x)$, so $\Phi^c = \boldsymbol{X} \in \tau$ similarly $\boldsymbol{X}^c = \Phi \in \tau$.

b. Let A, B be two closed fuzzy sets. So, A^c, B^c and $A^c \wedge B^c$ are in τ. Using De Morgan lows we have that $(A \vee B)^c = A^c \wedge B^c \in \tau$, i.e $A \vee B$ is a closed fuzzy sets. By inductions finite union of closed fuzzy sets is closed.

c. For $i \in \Delta$, let A_i be closed fuzzy sets. So, A_i^c and $\bigvee A_i^c$ are in τ.
Now, $\forall i \in \Delta$, $(\bigwedge A_i)^c(x) = 1 - \bigwedge A_i(x) = 1 - inf\{A_i(x)\} = sup\{1 - A_i(x)\} = sup\{A_i^c(x)\} = \bigvee A_i^c(x) \in \tau$. So arbitrary intersection of closed fuzzy sets is closed.

d. $\Phi, \mathbf{X}$ and all fuzzy sets and their intersections and unions are in I^X, then I^X is a C-fts. also, the set $\{\Phi, \mathbf{X}\}$ is a C-fts.

□

Definition 2.1.5. [15] Let X be a non empty set, then the fuzzy topology I^X is called the *discrete fuzzy topology*, and the fuzzy topology $\{\Phi, \mathbf{X}\}$ is called the *indiscrete fuzzy topology.*

As in ordinary topology, the following proposition holds in fuzzy topology.

Proposition 2.1.6. *Let τ be a C-ft on a non-empty set X contains all fuzzy singletons on X. Then $\tau = I^X$.*

Proof. Take any fuzzy set in $A \in I^X$, as in Remark 1.2.17 (part 4), this fuzzy set can be expressed as the union of it's fuzzy singleton. Since, τ fuzzy topology contains all fuzzy singleton on X. So, $A \in \tau$. Hence $\tau = I^X$. □

2.2 Lowen's Definition of Fuzzy Topology

One ask: Why Lowen change the definition of Chang? This because some of intuitive and well known results in ordinary topology do not satisfy in the case of Chang definition. For examples:

1) Some products of countably many compact C-fts fail to be compact which contradicts the Tychonoff's Theorem in ordinary topology.

2) Some of constant functions from one C-ft to another fail to be continuous.

Lowen's definition adds to Chang definition one condition. For Lowen's definition, all constant fuzzy sets are open.

Definition 2.2.1. .[30](Lowen Definition)

A *Lowen fuzzy topology*, or simply *L-ft.* is a family τ of fuzzy sets in X which satisfies the following conditions:

a.) τ contains all constant fuzzy sets in X.

b.) If $A, B \in \tau$, then $A \wedge B \in \tau$.

c.) If $A_i \in \tau$ for each $i \in \Delta$, then $\bigvee_{i\in\Delta} A_i \in \tau$

Since $\boldsymbol{X}, \Phi$ are constant fuzzy sets, the L-ft definition generalize the C-ft definition. So, all results of C-ft definition still true with respect to the new definition.

Example 2.2.2. a. In example 2.1.3, τ is a C-ft but not L-ft. So, we can add to τ all the constant fuzzy set and it's intersections and unions with the fuzzy sets in τ to make it a fuzzy topology in the sense of Lowen.

b. Consider the following example stated by Lowen. Let X be any set. Consider the L-ft τ on X whose open fuzzy sets are given by the membership functions from $[\frac{1}{2},1]^X \bigcup \{\alpha : \alpha$ is a constant function with value $\alpha \le \frac{1}{2}\}$[34].

To see this, note that all constant function belong to τ. Now, let $A, B \in \tau$. If both $A, B \in [\frac{1}{2},1]^X$, then $A \wedge B \in [\frac{1}{2},1]^X$. And if both A, B are constant, then $A \wedge B = min\{A, B\} = A$ or $B \in \tau$.

Otherwise if $A = \alpha \le \frac{1}{2}$, $B \in [\frac{1}{2},1]^X$, so $B(x) \ge \frac{1}{2}$, then $A \wedge B = A \in \tau$.

Now, assume $\forall i \in \Delta, A_i \in \tau$. If $\forall i$, A_i's is a constant fuzzy set $\le \frac{1}{2}$, then $\bigvee A_i$ is a constant $\in \tau$. If $\exists A_j, A_j \in [\frac{1}{2},1]^X$, then $\bigvee A_i = \sup\ \{A_i : i \in \Delta\} = \sup\ \{A_j : A_j \in [\frac{1}{2},1]^X\} \in [\frac{1}{2},1]^X$. So, τ is L-fts.

2.3 Between C-fts, L-fts and Ordinary Topology.

In this section, we will give some ways convert between Chang fuzzy topology, Lowen topology and the usual topology. Then we will give a summary at the end of this section.

This theorem give us a ways to get a C-ft from ordinary topology

Theorem 2.3.1. *Let (X, δ) be a topological space. Define:*

$$\tau = \{\mathcal{X}_{\mathcal{A}} : A \text{ is ordinary open set in } (X, \delta)\ \} .$$

Then (X, τ) is a C-fts.

We can generalize this method by using $\alpha\mathcal{X}_{\mathcal{A}}$ instead of $\mathcal{X}_{\mathcal{A}}$, where $\alpha\mathcal{X}_{\mathcal{A}}(x) = \alpha$ if $x \in A$ and $\alpha\mathcal{X}_{\mathcal{A}}(x) = 0$ otherwise, $\alpha \in (0, 1]$.

Proof. Clearly $\mathbf{X} = \mathcal{X}_X$, $\Phi = \mathcal{X}_\phi \in \tau$. Now, let $\mathcal{X}_{\mathcal{A}}$, $\mathcal{X}_{\mathcal{B}} \in \tau$. Then A, B are open in the ordinary topological space (X, δ). So, $A \cap B$ open in (X, δ). Hence $\mathcal{X}_{\mathcal{A}\cap\mathcal{B}} \in \tau$. But

$$\mathcal{X}_{\mathcal{A}\cap\mathcal{B}}(x) = \begin{cases} 1 & x \in A \cap B \\ 0 & elsewhere \end{cases}, = \begin{cases} 1 & (x \in A), (x \in B) \\ 0 & elsewhere \end{cases},$$

$$= min\{\mathcal{X}_{\mathcal{A}}(x), \mathcal{X}_{\mathcal{B}}(x)\} \ = \mathcal{X}_{\mathcal{A}}(x) \wedge \mathcal{X}_{\mathcal{B}}(x).$$

Thus $\mathcal{X}_{\mathcal{A}} \wedge \mathcal{X}_{\mathcal{B}} = \mathcal{X}_{\mathcal{A}\cap\mathcal{B}} \in \tau$.

Similarly if $\mathcal{X}_{\mathcal{A}_\alpha} \in \tau$ for each $\alpha \in \Delta$, then $\bigvee_{\alpha\in\Delta} \mathcal{X}_{\mathcal{A}_\alpha} \in \tau$. □

To make a reverse, we need this definition:

Definition 2.3.2. [37] If A is a fuzzy set in a set X and $0 \leq \alpha < 1$, $(0 < \alpha \leq 1)$, then $\alpha(A) = \{x \in X : A(x) > \alpha\}$ (resp. $\alpha^*(A) = \{x \in X : A(x) \geq \alpha\}$) is called an α-*level* (resp. α^*-*level*), or α-*cut* set in X.

Using the definition of α-cut, in this theorem we can convert any C-ft or L-ft to an ordinary topology. In the next proposition we will give a generalization of this method.

Theorem 2.3.3. *[37] Let (X, τ) be a C-fts. For each α such that $0 \leq \alpha < 1$, the family $\iota_\alpha = \{\alpha(A) : A \in \tau\}$ forms a topology on X.*

Proof. Since $\boldsymbol{X} \in \tau$, we have that $\alpha(\boldsymbol{X}) \in \iota_\alpha$. But $\alpha(\boldsymbol{X}) = \{x \in X : \boldsymbol{X}(x) > \alpha\} = \boldsymbol{X}$. Hence $\boldsymbol{X} \in \iota_\alpha$. Moreover, $\Phi \in \tau$, so $\alpha(\Phi) \in \iota_\alpha$. But $\alpha(\Phi) = \phi$. So, $\phi \in \iota_\alpha$.

Suppose that $\alpha(A)$, $\alpha(B) \in \iota_\alpha$. Then A, $B \in \tau$. So, $A \wedge B \in \tau$ and so, $\alpha(A \wedge B) \in \iota_\alpha$. But

$$\begin{aligned}\alpha(A \wedge B) &= \{x : (A \wedge B)(x) > \alpha\} \\ &= \{x : min\{A(x), B(x)\} > \alpha\} \\ &= \{x : A(x) > \alpha, B(x) > \alpha\} \\ &= \{x : A(x) > \alpha\} \cap \{x : B(x) > \alpha\} \\ &= \alpha(A) \cap \alpha(B).\end{aligned}$$

Therefore $\alpha(A) \cap \alpha(B) \in \iota_\alpha$.

Finally, Let $\alpha(A_\beta) \in \iota_\alpha$ for $\beta \in \Delta$. So, $A_\beta \in \tau, \forall \beta \in \Delta$ and hence $\bigvee_{\beta\in\Delta} A_\beta \in \tau$ which implies that $\alpha(\bigvee_{\beta\in\Delta} A_\beta) \in \iota_\alpha$. Consider

$$\begin{aligned}
\alpha(\bigvee_{\beta\in\Delta} A_\beta) &= \{x : \bigvee_{\beta\in\Delta} A_\beta(x) > \alpha\} \\
&= \{x : sup\{A_\beta(x), \beta \in \Delta\} > \alpha\} \\
&= \{x : A_\beta(x) > \alpha \text{ , for some } \beta \in \Delta\} \\
&= \bigcup_{\beta\in\Delta} \{x : A_\beta(x) > \alpha\} \\
&= \bigcup_{\beta\in\Delta} \alpha(A_\beta)
\end{aligned}$$

That is, $\bigcup_{\beta\in\Delta} \alpha(A_{\beta)} \in \iota_\alpha$, and hence the collection ι_α is a topological space on X.

□

Definition 2.3.4. [37] Let (X, τ) be a C-fts. For each α such that $0 \leq \alpha < 1$, the topology ι_α called the *α-level topology* on X.

Proposition 2.3.5. *[28]Let (X, τ) be a fts (L-ft or C-ft). Define the topological space $(X, \iota(\tau))$ such that: For $A \in \tau$,*

$$\iota(A) = \{\alpha(A) : \alpha \in [0, 1)\} = \bigcup_{\alpha\in[0,1)} \alpha(A).$$

And $\iota(\tau) = \{\iota(A) : A \in \tau\}$. In fact, $\iota(\tau)$ is precisely the topology generated by the collection $\{\iota_\alpha : \alpha \in [0, 1)\}$.

Proof. Since $\iota(\tau)$ is generated by the collection of the topologies $\{\iota_\alpha : \alpha \in [0, 1)\}$, and since the arbitrary union of ordinary topologies is an ordinary topology. So, $\iota(\tau)$ is a topology. □

In the next theorem Lowen defined a function to convert an ordinary topology to L-ft.

Theorem 2.3.6. *[30] Let (X, δ) be an ordinary topological space. The following collection is an L-ft.*

$$\omega(\delta) = \{A \in I^X : \ \alpha(A) \in \delta, \ \ \forall \alpha \in [0, \ 1)\}.$$

Proof. Let $C \in [0,1]^X$ be any constant fuzzy set with grade $t \in [0,1]$. Then for all $\alpha \in [0,\ 1)$ $\alpha(C) = X$ if $t > \alpha$ and $\alpha(C) = \phi$ if $t \leq \alpha$. In both cases, $\alpha(t) \in \delta$ and so $C \in \omega(\delta)$. Thus $\omega(\delta)$ contains all constant fuzzy set. Now, let A,$B \in \omega(\delta)$, so $\alpha(A), \alpha(B) \in \delta$, $\forall \alpha \in [0,\ 1)$. Since $\alpha(A \wedge B) = \alpha(A) \cap \alpha(B) \in \delta$, $A \wedge B \in \omega(\delta)$ $\forall \alpha \in [0,\ 1)$. Finally if $A_\beta \in \omega(\delta)$, $\beta \in \Delta$, then $\forall \alpha \in [0,\ 1)$, and $\beta \in \Delta$ $\alpha(A_\beta) \in \delta$.
So, as we showed above $\alpha(\bigvee_{\beta \in \Delta} A_\beta) = \bigcup_{\beta \in \Delta} \alpha(A_\beta) \in \delta$. This implies that $\bigvee_{\beta \in \Delta} A_\beta \in \omega(\delta)$ and so $(X, \omega(\delta))$ is a L-fts. □

Definitions 2.3.7. The L-fts $(X, \omega(\delta))$ of a topological space (X, δ) is called the *induced fuzzy topology*, or *natural* fuzzy topology.
A fuzzy topology (X, τ) is called *topologically generated* if it equals $\omega(\delta)$ for some topology δ on X.

Note that: A is fuzzy open in $(X, \omega(\delta))$ iff all the components $\alpha(A)$ (the set of cuts of A) are open in (X, δ).

Remark 2.3.8. [39] To link between ι, ω: for a topology δ on X, $\iota(\omega(\delta)) = \delta$. However, for a fuzzy topology τ on X, $\omega(\iota(\tau))$ may not equal τ; it is in fact the smallest topologically generated fuzzy topology containing τ.
To show that $\iota(\omega(\delta)) = \delta$ recall that:
$\omega(\delta) = \{A \in I^X :\ \alpha(A) \in \delta,\ \ \forall \alpha \in [0,\ 1)\}$ and $\iota(\tau) = \{\alpha(A) : A \in \tau, \alpha \in [0,1)\}$.
So, $\iota(\omega(\delta)) = \{\alpha(A) : A \in \omega(\delta), \alpha \in [0,1)\} = \{\alpha(A) : \alpha(A) \in \delta, \alpha \in [0,1)\} \subseteq \delta$.
Conversely, let $A \in \delta$ the characteristic function of A is given by:

$$\mathcal{X}_{\mathcal{A}}(x) = \begin{cases} 1, & x \in A; \\ \\ 0, & \text{otherwise.} \end{cases}$$

and $\forall \alpha \in [0,1)$, $\alpha(\mathcal{X}_{\mathcal{A}}) = \{x : \mathcal{X}_{\mathcal{A}}(x) \geq \alpha\} = A$.
Now, $\forall \alpha \in [0,1), \alpha(\mathcal{X}_{\mathcal{A}}) = A \in \delta$, then $\mathcal{X}_{\mathcal{A}} \in \omega(\delta)$, but by definition of $(\iota(\tau))$ $\alpha(\mathcal{X}_{\mathcal{A}}) \in \iota(\omega(\delta))$, so $A \in \iota(\omega(\delta))$. Hence $\iota(\omega(\delta)) = \delta$.

The following counter example show that $\omega(\iota(\tau)) \neq \tau$, so let $X = \{a, b\}$ and $\tau = \{\Phi, \boldsymbol{X}, \{(a, \frac{1}{3}), (b, \frac{1}{2})\}\ \}$, we have that $\iota(\tau) = \{\phi, X, \{b\}\}$ and hence $\omega(\iota(\tau)) = \{\Phi, \boldsymbol{X}, \{(a, r), (b, s) : r, s \in [0, 1], r \leq s\}\ \}$ which contains τ but not equal it.

conclusions

1. From the Lowen Definition, we have that every L-fts is C-fts and the converse is not necesserly true since the C-ft not necessarily contains the constants fuzzy sets. In general we can convert a C-ft to L-fts by adding to the C-fts all the constant fuzzy sets and it's intersections and unions with the open fuzzy sets to make it a fuzzy topology in the sense of Lowen.

2.We can use theorem 2.3.3, and proposition 2.3.5 to get an ordinary topological space from a C-fts or L-fts which called the α-level topology.

3. Note that if (X, δ) is a topological space then by using the characteristic function see Theorem 2.3.1, we can get C-fts, which is surely not L-fts. For this end, to solve this problem:

4. Lowen used the $\omega(\delta)$ function to convert an ordinary topology to L-ft, and clearly we can use it to convert an ordinary topology to C-ft .

5. Note that $\{A \in \underline{\{0, 1\}}^X : \ \alpha(A) \in \delta, \ \ \forall \alpha \in [0, \ 1)\}$, are exactly the characteristic functions of the open sets of the topological space X [15].

From now, and unless otherwise, if we write fts we mean both L-fts and C-fts. In this chapter we state the proof of the following:

Proposition 2.1.4, proposition 2.1.6, Example 2.2.2, Theorem 2.3.1, Theorem 2.3.3, Proposition 2.3.5, Theorem 2.3.6, Remark 2.3.8. Also we put Example 2.1.3, and the counterexample in Remark 2.3.8.

Chapter 3

Chang and Lowen Properties

In this chapter, we will study the generalization of topological concepts of the ordinary topology such as continuity, compactness and separation axioms. From now, if we write fts we mean both L-fts and C-fts.

3.1 Introduction

In this section we give important definitions such as hereditary, productive, good extension properties, and definition of the base of an fts. Firstly, we introduced a theorem and define the subspace of a given fts.

Theorem 3.1.1. *[3] Let (X, τ) be fts (L-ft or C-ft), and M an arbitrary non-empty ordinary subset of X. Define $\tau_M = \{A \wedge \mathcal{X}_M : A \in \tau\}$. Then τ_M is a fts (L-ft or C-ft) on M.*

Proof. $\Phi, \boldsymbol{X} \in \tau$. So, $\Phi \wedge \mathcal{X}_M, \boldsymbol{X} \wedge \mathcal{X}_M \in \tau_M$. But $(\Phi \wedge \mathcal{X}_M)(x) = \min\{\mathcal{X}_M(x), \Phi(x)\} = 0 = \Phi(x)$, and $(\boldsymbol{X} \wedge \mathcal{X}_M)(x) = \min\{\mathcal{X}_M(x), \boldsymbol{X}(x)\} = \mathcal{X}_M(x)$, then we have that $\Phi, \mathcal{X}_M \in \tau_M$

Choose $A \wedge \mathcal{X}_M, B \wedge \mathcal{X}_M \in \tau_M$ where $A, B \in \tau$, so $A \wedge B \in \tau$ and hence $(A \wedge B) \wedge \mathcal{X}_M \in \tau_M$.

Now, $(A\wedge\mathcal{X}_M)\wedge(B\wedge\mathcal{X}_M)(x) = \min\{(A\wedge\mathcal{X}_M)(x), (B\wedge\mathcal{X}_M)(x)\} = \min\{A(x), \mathcal{X}_M(x), B(x)\} = \min\{\min\{A(x), B(x)\}, \mathcal{X}_M)(x)\} = \min\{(A\wedge B)(x), \mathcal{X}_M)(x)\} = ((A\wedge B)\wedge\mathcal{X}_M))(x) \in \tau_M$.

Let $(A_i \wedge \mathcal{X}_M) \in \tau_M$, where $i \in \Delta$ and $A_i \in \tau$, so $\bigvee A_i \in \tau$. By using distributive Low $\bigvee_{i\in\Delta}(A_i \wedge \mathcal{X}_M) = (\bigvee_{i\in\Delta} A_i) \wedge \mathcal{X}_M \in \tau_M$. □

Definition 3.1.2. [3] Let (X, τ) be fts (L-ft or C-ft), and M arbitrary non-empty ordinary subset of X. The collection (M, τ_M) is called a *subspace* of (X, τ), or the *relative fuzzy topology* for M. A subspace (M, τ_M) of a fts (X, τ) is called an *open (closed) subspace* iff the set M is open (closed) in τ.

There are many authors and research which given a different definition of fuzzy continuity, separation axioms and compactness. The following definitions and proprieties (especially the hereditary, productive and good extension proprieties) are very important to decide wether one of these definition is more stronger than others

Definition 3.1.3. [9] Let (X, τ) be a fuzzy topology. A subfamily $\mathfrak{B}$ of τ is a *base* for τ if each member of τ can be expressed as a union of some members of $\mathfrak{B}$.

Definition 3.1.4. A subbase for (X, τ) is a subcollection S of τ such that the collection of infimum of finite subfamilies of S forms a base for (X, τ).

Definition 3.1.5. Let $(X_i, \tau_i), i \in \Delta$ be a familiy of fts, the *product fuzzy topology* denoted by $(X, \tau) = (\Pi_{i\in\Delta}X_i, \Pi_{i\in\Delta}\tau_i)$ is given by using the set S as it's subbase where $S = \{\pi_i^{-1}A_i : A_i \in \tau_i, i \in \Delta\}$

So, that a base of the product fuzzy topology (X, τ) can be taken to be

$$B = \{\bigwedge_{j=1}^{n} \pi_{i_j}^{-1}A_{i_j} : A_{i_j} \in \tau_{i_j}, i_j \in \Delta, j = 1...n, n \in \mathbb{N}\}$$

Definitions 3.1.6. [25]

1. A fuzzy topological property $\mathcal{P}$ is *called hereditary* (*hereditary with respect to closed subspaces, hereditary with respect to open subspaces),* iff each subspace (closed subspace, open subspace) of a fts with property $\mathcal{P}$ also has property $\mathcal{P}$.

2. A fuzzy topological property $\mathcal{P}$ is *productive (finitely productive, countably productive)* iff the product $\Pi_{i\in\Delta}X_i$ of every (every finitely, every countably) family $(X_i)_{i\in\Delta}$ is of pairwise disjoint fts enjoying property $\mathcal{P}$ also has property $\mathcal{P}$.

3. For a property $\mathcal{P}$ in ordinary topological spaces, a property $\mathcal{P}^*$ of fuzzy topological spaces is *called a good extension* of $\mathcal{P}$, if for every ordinary topological space $(X,\delta),(X,\delta)$ has $\mathcal{P}$ if and only if $(X,\omega(\delta))$ has $\mathcal{P}^*$.

3.2 F-Continuous Functions

The following definition which is defined by Chang is the basic definition of the fuzzy continuity, and it is used by the most of researcher except Lowen.

Definition 3.2.1. [14] Let f be a function from a C-fts (X,T) to a C-fts (Y,S). The function f is called *fuzzy continuous,* (in short *F-continuous*) if the inverse of each S-open fuzzy set is T-open. Equivalently, the inverse of each S-closed, (i.e., a complement of an S-open fuzzy set), is T-closed. The function f is called *F-open* (*F-closed*) if the image of each T-open (T-closed) fuzzy set is S-open (S-closed). The function f is a *F-homeomorphism* if it is F-continuous, one-to-one, onto, and f^{-1} is F-continuous.

Definition 3.2.2. Let (X,τ_1)$,(X,\tau_2)$ be two fts, then (τ_1 is said to be *finer* than τ_2) or (τ_2 is said to be *coarser* than τ_1) iff $\tau_2\subseteq\tau_1$.

Example 3.2.3. Let $(X,\tau),(X,\gamma)$ be two C-fts and, let $id_X : (X,\tau) \longrightarrow (X,\gamma)$ where id_X is the identity function $id_X(x) = x$, then τ is finer than γ if and only if id_X is F-continuous. To see this, if τ is finer than γ, let B be open fuzzy set in γ. Then $id_X^{-1}[B](x) = B[id_X(x)] = B(x)$. So, $id_X^{-1}[B] = B$. Since $\gamma \subseteq \tau$, $id_X^{-1}[B] = B \in \tau$ and hence id_X is F-continuous.
Conversely, if id_X is F-continuous, and $B \in \gamma$, then $B = id_X^{-1}[B] \in \tau$. So $\gamma \subseteq \tau$, and τ is finer than γ.

Example 3.2.4. Let $(X,\mathbf{I^X})$ be the discrete fts, and (Y,τ) any fts.
Let $f : (X,\mathbf{I^X}) \longrightarrow (X,\tau)$, then f is F-continuous. To see this, let B be an open fuzzy set in τ. Note that $\mathbf{I^X}$ is the discrete fuzzy topology, so it contains all fuzzy set in X. Thus $f^{-1}[B]$ is open fuzzy set in $\mathbf{I^X}$ and hence f is F-continuous.

Remark 3.2.5. Let (X,τ) and (Y,γ) be two fts, and let $f : (X,\tau) \longrightarrow (Y,\gamma)$. Then for all $x \in X$,
$f^{-1}[\mathbf{Y}](x) = \mathbf{Y}[f(x)] = 1 = \mathbf{X}(x)$. So, $f^{-1}[\mathbf{Y}] = \mathbf{X}$. And $f^{-1}[\Phi](x) = \Phi[f(x)] = 0 = \Phi(x)$. So, $f^{-1}[\Phi] = \Phi$.

Example 3.2.6. Let (X,τ) be any fts and (Y,γ) the indiscrete fts. Let $f : (X,\tau) \longrightarrow (Y,\gamma)$, then for any open fuzzy set B in the indiscrete fuzzy topology γ, either $B = \mathbf{Y}$ or $B = \Phi$. So, $f^{-1}[B] = f^{-1}[\mathbf{Y}] = \mathbf{X} \in \tau$, or $f^{-1}[B] = f^{-1}[\Phi] = \Phi \in \tau$. Therefore f is F-continuous.

Proposition 3.2.7.

1. *If f is an F-continuous function from X to Y and g is an F-continuous function from Y to Z, then the composition $g \circ f$ is an F-continuous function from X to Z[7].*

2. *If f is an F-open (F-closed) function from X to Y and g is an F-open (F-closed) function from Y to Z, then the composition $g \circ f$ is an F-open (F-closed) function from X to Z[19].*

Proof. (1) Let V be open on Z, By Theorem 1.2.25.g we have $(g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]]$, and using the F-continuity of g and f it follows that $(g \circ f)^{-1}[V]$ is open on X.
(2) Let A be open on X, since $(g \circ f)[A] = g[f[A]]$, and since g and f are F-open (F-closed), it follows that $(g \circ f)[A]$ is open (closed) on Z. □

Remark 3.2.8. In the sense of Chang, some of constant functions from one fuzzy topological space to another fail to be continuous, which makes a big different from the ordinary topology.
This is one resons for why Lowen fuzzy topology contains all constant fuzzy sets, so every constant function is F-continuous in the sence of Lowen.

In section 2.3, we give a methods to convert ordinary topology to C-ft by using ω function, and we use ι function to convert a C-fts to ordinary topology. The following theorem shows that if τ is topologically generated, then we can determine the fuzzy continuity of f by examine the ordinary continuity.

Theorem 3.2.9. *[15] Let (X, δ) and (Y, φ) be two topological spaces, and let $f : X \to Y$. For the induced topologies $\omega(\delta)$ and $\omega(\varphi)$, we have the following prosperities:*

1. *f is F-continuous if and only if f is continuous.*

2. *f is F-open if and only if f is open.*

Proof. Our claim is $f^{-1}(\alpha(B)) = \alpha(f^{-1}[B])$. To see this, note that $\forall \alpha \in [0, 1)$ and $x \in X$, $x \in f^{-1}[\alpha(B)] \iff f(x) \in \alpha(B) \iff B(f(x)) > \alpha \iff f^{-1}[B](x) > \alpha \iff x \in \alpha(f^{-1}[B])$. Thus, $f^{-1}[\alpha(B)] = \alpha(f^{-1}[B])$.
Also to prove part 2. we claim that $\alpha(f[A]) = f[\alpha(A)]$, so let $y \in f(\alpha(A))$, so $\exists x \in \alpha(A), y = f(x)$ hence $A(x) > \alpha$ and so $\sup\{A(z) : z \in f^{-1}(y)\} > \alpha$,
so $f(A)(y) > \alpha$ that is $y \in \alpha(f(A))$.

Conversely, let $y \in \alpha(f(A))$ so $f(A)(y) > \alpha$ that is $\sup\{A(z) : f(z) = y\} > \alpha$. But α is not a spermium, so $\exists x$ such that $A(x) > \alpha, f(x) = y$ i.e $x \in \alpha(A)$. Hence $y = f(x) \in f(\alpha(A))$. So, $\alpha(f[A]]) = f[\alpha(A)]$.

1. Let f be continuous and B fuzzy open set in $(Y, \omega(\varphi))$. So, all the cuts $\alpha(B), \alpha \in [0, 1)$ are ordinary open sets in (Y, φ). Then $\forall \alpha \in [0, 1)$, $f^{-1}[\alpha(B)]$ ordinary open sets in (X, δ). Thus, $\alpha(f^{-1}[B]) = f^{-1}[\alpha(B)]$ ordinary open sets in (X, δ), and $f^{-1}[B]$ is a fuzzy open in $(X, \omega(\delta))$ and hence f is F-continuous. Conversely if f is F-continuous and B ordinary open set in (Y, φ), then B is fuzzy open in $(Y, \omega(\varphi))$. Then $f^{-1}[B]$ is fuzzy open in $(X, \omega(\delta))$. So, the cut of $f^{-1}[B]$ when $\alpha = 0$ is open in (X, δ). Thus $f^{-1}[B]$ is ordinary open in (X, δ), and hence f is continuous.

2. Now, let f be an open function and A a fuzzy open set in $(X, \omega(\delta))$. So, all the cuts $\alpha(A), \alpha \in [0, 1)$ are ordinary open in (X, δ). But f is open, so $f[\alpha(A)]$ open in (Y, φ), for all α. Then we have that $\alpha(f[A]) = f[\alpha(A)]$ open in (Y, φ), for all α. So, $f[A]$ is a fuzzy open in $(X, \omega(\delta))$ and hence f is F-open. Conversely if f is F-open, and A is an ordinary open set in (X, δ), thus it is a fuzzy open set in $(X, \omega(\delta))$. Then $f[A]$ is a fuzzy open in $(Y, \omega(\varphi))$. So, the cut of $f[A]$ when $\alpha = 0$ is open in (Y, φ). Thus $f[A]$ is ordinary open in (Y, φ), and hence f is open.

$\square$

If a function f is F-open, it need not to be F-closed. To see this we give the following counterexample:

Example 3.2.10. Let (X, τ) and (Y, ψ) be two ftss, such that:
$X = \{x, y, z\}, \tau = \{\Phi, \boldsymbol{X}, A\}$, where $A = \{(x, 0.5), (y, 0), (z, 0.2)\}$
$Y = \{a, b\}, \psi = \{\Phi, \boldsymbol{Y}, B\}$, where $B = \{(a, 0.5), (b, 0.2)\}$.

and let $f : X \to Y$, given by: $f(x) = a, f(y) = a, f(z) = b$. Then f is F-open, to see this, $f[A](a) = sup\{A(x), A(y)\} = 0.5$ and $f[A](b) = A(z) = 0.2$. So, $f(\boldsymbol{X}) = \boldsymbol{Y} \in \psi$, $f(\Phi) = \Phi \in \psi$ and $f(A) = \{(a, 0.5), (b, 0.2)\} = B \in \psi$, so f is F-open

Now, take the closed fuzzy set in τ $A^c = \{(x, 0.5), (y, 1), (z, 0.8)\}$, but its image $f(A^c) = \{(a, 1), (b, 0.8)\}$ is not closed fuzzy set in ψ, so f is not F-close.

Proposition 3.2.11. *[39]*

1. *If $f : (X, \tau_1) \to (Y, \tau_2)$ is a F-continuous function of fuzzy topological spaces, then $f : (X, \iota(\tau_1)) \to (Y, \iota(\tau_2))$ is continuous.*

2. *If τ is a fuzzy topology which is topologically generated by δ for some topology δ on X, then for any fuzzy topological space (Y, ψ) if $f : (X, \tau) \to (Y, \psi)$ is F-continuous function then $f : (X, \delta) \to (Y, \iota(\psi))$ is a continuous function.*

Proof. 1. Let $\alpha(B) \in \iota(\tau_2)$, where $\alpha \in (0, 1], B \in \tau_2$. By the F-continuity of f, $f^{-1}[B] \in \tau_1$, then $\alpha(f^{-1}[B]) \in \iota(\tau_1)$, and we show above that $f^{-1}(\alpha(B)) = \alpha(f^{-1}[B])$, which is in $\iota(\tau_1)$.

2. By Remark 2.3.8 we have that $\iota(\tau) = \iota(\omega(\delta)) = \delta$, and using (Part 1) the proof is complete.

□

Now, we give Lowen definition of continuity, then we will give some reasons to prefer Chang continuity more than this definition.

Definition 3.2.12. [30] Lowen definition of continuity

Let f be a function from an L-fts (X, T) to an L-fts (Y, S). Then f is *Lowen fuzzy continuous,* in (short LF-continuous) iff $f : (X, \iota(T)) \to (Y, \iota(S))$ is continuous.

Proposition 3.2.13. *[30] Give a function $f : (X, \tau) \to (Y, \psi)$, where X, Y are L-fts If f is F-continuous then f is LF-continuous.*

Proof. By Proposition 3.2.11. we have that: if $f : (X, \tau_1) \to (Y, \tau_2)$ is a F-continuous function of fuzzy topological spaces, then $f : (X, \iota(\tau_1)) \to (Y, \iota(\tau_2))$ is continuous, i.e. f is LF-continuous. □

conclusion From now, and unless otherwise, we will ignore the last definition (Lowen definition) of fuzzy continuity and depends the Chang definition for many reasons:

1. Chang definition is a natural extension of the ordinary continuity. (i.e. both of them have the same definition.)

2. There exists equivalency between ordinary continuity of a topological space (X, δ) and Chang fuzzy continuity with the fuzzy topological space $(X, \omega(\delta))$.

3. Lowen definition of continuity have a rarely use, and even the research witch depend on Lowen fuzzy topology use the Chang continuity

4. If any function is continuous in Chang sense then it is continuous in Lowen sense.

3.3 Separation Axioms

We study here fuzzy T_i for $i = 0, 1, 2, 3$. The fuzzy separation axioms are said to be good extensions if (the induced fuzzy topological space $(X, \omega(\delta))$ is fuzzy T_i if and only if the ordinary topological space (X, δ) is $T_i, i = 0, 1, 2, 3)$. The implications between these axioms goes well, that is, each T_i-space is T_{i-1} for $i = 1, 2, 3, 4$.

Here we shall discuss properties of separation axioms $T_0 - T_4$ in fuzzy spaces. **The main attention will be paid to T_1-separation as an important property of this kind, and the other separation axioms will be mention without details.**

Comparison of Fuzzy T_1-ness Definitions

There is many definitions of fuzzy T_1-space, (FT_1), we will mention them and the relations between them in the next proposition:

Definition 3.3.1. [35],[34]

Consider the followings definitions of *fuzzy* T_1 for an fts (X, τ):

I1. $\forall x \in X$, the fuzzy singleton $\{x\} = x_1$ is fuzzy closed.

I2. For any two distinct points $x, y \in X$, $\exists U, V \in \tau$ such that $U(x) = 1,\ U(y) = 0$ and $V(y) = 1,\ V(x) = 0$.

II. $\forall x \in X$ each fuzzy point $x_t, t \in (0, 1)$ and each $\{x\} = x_1$, is fuzzy closed. (Equivalently each fuzzy singleton $x_t, t \in (0, 1]$ is a fuzzy closed.)

III. Each fuzzy point in X is fuzzy closed.

IV. For any two distinct fuzzy points $x_t, y_s \in X$, $\exists U, V \in \tau$ with $x_t \in U, x_t \notin V$ and $y_s \in V, y_s \notin U$.

Proposition 3.3.2. *[35],[34]*

We have the followings relations between the previous definitions: If (X, τ) *is a topologically generated fts, then*

$$(I1) \Leftrightarrow (I2) \Leftrightarrow (II) \Leftrightarrow (III) \Leftrightarrow (IV)$$

.

If τ *is L-fts :*

$$\begin{array}{c} (I1) \Leftrightarrow (I2) \Leftrightarrow (II) \\ \Downarrow \\ (III) \Leftrightarrow (IV) \end{array}$$

If τ *is C-fts:*

$$\begin{array}{ccc} & (II) & \\ \swarrow & & \searrow \\ (III) & & (I1) \Leftrightarrow (I2) \\ \searrow & & \swarrow \\ & (IV) & \end{array}$$

Proof. Now, in any sense of fts, we have $(I1) \Leftrightarrow (I2)$. To see this, we show that $(I1) \Rightarrow (I2)$. For $x, y \in X$, $x \neq y$, $X - \{x\}$ and $X - \{y\}$ are fuzzy open in view of (I1) and, clearly, $(X - \{x\})(x) = 0$, $(X - \{x\})(y) = 1$, $(X - \{Y\})(y) = 0$, $(X - \{y\})(x) = 1$. so we have (I2). For the other inclusion, assume (I2) holds. For $x \in X$, and for $t \in (0, 1)$, choose $y_t \in X - \{x\}$ any fuzzy point such that $y \neq x$. Then $\exists U, V \in X$ such that $U(x) = 1$, $U(y) = 0$ and $V(y) = 1$, $V(x) = 0$. Clearly, $y_t \in V \subseteq X - \{x\}$.

recall that any fuzzy set can be express as the union of its fuzzy point, and since y_t is arbitrary, it follows that $X - \{x\}$ is fuzzy open. That is $\{x\}$ is fuzzy closed and (I1) holds.

Now, suppose that τ is Lowen fuzzy topology. We want to prove that (I2) $\Leftrightarrow$ (II) It suffices to show that (I1) $\Rightarrow$ (II). Since τ contains all the constant fuzzy sets, then each constant fuzzy set t is closed, since it is the complement of the open constant fuzzy set $1 - t$. If $x_t \in X$ is a fuzzy point then clearly, $x_t = \{x\} \wedge t$, where t is the t-valued constant fuzzy set. Thus, x_t, being an intersection of fuzzy closed sets, so it is fuzzy closed. Thus (I1) $\Rightarrow$ (II) (i.e.(I2) $\Rightarrow$ (II) in the sense of Lowen but this not true in the sense of Chang)

The same proof do in similar way to show that (II) $\Rightarrow$ (I2).

(II $\Rightarrow$ III) is clear.

(III) $\Leftrightarrow$ (IV) (Consider τ in the sense of Lowen).

First, assume (III) holds. So, given $x \in X$ and $r \in (0,1)$. But x_{1-r} which is in view of (III) is fuzzy closed, so $U = X - x_{1-r}$ is fuzzy open and $U(x) = r$, $U(y) = 1$ if $x \neq y$. Now, let $x_r, y_s \in X$ be distinct fuzzy points. Choose $r' \in (0,1)$ such that $r' > r$. Then as pointed out above, $\exists$ two fuzzy open sets U' and V' with $U'(x) = r'$ and $U'(z) = 1$ if $z \neq x$ and $V'(y) = s$ and $V'(z) = 1$ if $z \neq y$. Clearly, $U_1 = U' \wedge V'$ is a fuzzy open set with $U_1(z) = 1$ if $z \neq x, y$, $\;U_1(x) = r'$, and $U_1(y) = s$. Thus, $x_r \in U_1$, whereas $y_s \notin U_1$. Similarly, we can construct a fuzzy open set V with $y_s \in V$ and $x_r \notin V$ showing that (III) $\Rightarrow$ (IV).

Conversely, suppose (IV) holds. For $x \in X$, let x_t be any fuzzy point. Let $y \neq x$, and consider the fuzzy points $y_{1-\frac{1}{n}}, n \in \mathbb{N}$. Then in view of (IV) $\exists V_n \in \tau$ such that $y_{1-\frac{1}{n}} \in V_n,\; x_{1-t} \notin V_n$. Let $V_y = \bigvee_{n \in \mathbb{N}} V_n$. Now if $x \neq y$, then $V_y(y) = 1,\; x_{1-t} \notin V_y$. Let $V = \bigvee_{y \neq x} V_y$ with $V(y) = 1$ if $x \neq y$, $x_{1-t} \notin V$ (i.e. $V(x) < 1 - t$). Since τ is in the sense of Lowen then the constant fuzzy set $1 - t \in \tau$, so $1 - t \vee V$ is open fuzzy set with value $1-t$ at x and 1 elsewhere, which is the complement of the fuzzy point x_t. Thus, x_t is closed fuzzy set.

(If τ is in the sense of Chang) for (III) $\Rightarrow$ (IV) we have the same proof as above.

□

The following easy examples will serve the purpose of showing some counterexamples of some inverse inclusions.

Complete proofs and counterexamples of the converse implications are found in [35], [34].

Example 3.3.3. 1. $(I2) \nRightarrow (II)$ Let X be a non empty set. Consider τ the discrete topology on X and regard it as a fuzzy topology (it is not a fuzzy topology in the sense of Lowen). Then clearly for each $x \in X$, $\{x\}$ is fuzzy closed here so the C-fts (X, τ) satisfies (I1) which is equal to (I2). But if $x_t \in X$ is a fuzzy point, then its complement will have the value $1 - t$ at x which is neither 0 nor 1. Hence x_t cannot be fuzzy closed.

2. (III $\nRightarrow$ II). Consider a set X with the fuzzy topology generated by the complements of all its fuzzy points. (For illustration, if x_t is any fuzzy point , then its complement has the form $x^c_t(x) = 1 - t,\ x^c_t(y) = 1\ \forall y \neq x$) Clearly, this fts satisfies (III). Moreover, there is no fuzzy open set, rather than Φ, can take the value 0 $\forall x \in X$, so $(X - \{x\})(x) = 0$. Hence $X - \{x\}$ is not fuzzy open or equivalently $\forall x \in X, \{x\}$ is not fuzzy closed.

3. (IV) $\nRightarrow$ (III) for showing that, the same example as used in showing (I2) $\nRightarrow$ (II), above, will work.

Since (I1), (I2) are equivalent always so, we will consider all of them as (I).

Now we will consider some main properties of these definitions.

Good Extensions

Theorem 3.3.4. *[35] For a topological space (X, δ) and an fts (X, τ), we have that*

a. (i) (X, δ) is T_1 if and only if $(X, \omega(\delta))$ satisfies (I)

(ii) (X, τ) satisfies (I), then $(X, \iota(\tau))$ is T_1.

b. (i) (X, δ) is T_1 if and only if $(X, \omega(\delta))$ satisfies (IV)

(ii) (X, τ) satisfies (IV), then $(X, \iota(\tau))$ is T_1.

Recall that $\omega(\delta) = \{A \in I^X : \ \alpha(A) \in \delta, \ \ \forall \alpha \in [0, \ 1)\}$ and $\iota_\alpha = \{\alpha(A) : A \in \tau\}$, where $\alpha(A) = \{x \in X : A(x) > \alpha\}$

Proof. As an example we will give the proof of part (i), let (X, δ) be T_1. (i.e if $x, y \in X, x \neq y$ then $\{x\}$ and $\{y\}$ are closed in (X, δ)). Hence $\forall \alpha \in [0, 1)$, $\alpha(\boldsymbol{X} - \{x\}) = \{z : \boldsymbol{X} - \{x\}(z) > \alpha\} = X - \{x\}$ is open set in δ, so $\boldsymbol{X} - \{x\} \in \omega(\delta)$, similarly $\boldsymbol{X} - \{y\} \in \omega(\delta)$. Also, $(\boldsymbol{X} - \{x\})(y) = 1, (\boldsymbol{X} - \{x\})(x) = 0; (\boldsymbol{X} - \{y\}(x) = 1, (\boldsymbol{X} - \{y\})(y) = 0$. Hence $(X, \omega(\delta))$ satisfies (b).

Conversely, let $(X, \omega(\delta))$ satisfy (I). This means that $\{x\}$, $\forall x \in X$, is fuzzy closed in $(X, \omega(\delta))$, hence $\boldsymbol{X} - \{x\} \in \omega(\delta)$. The 0-cut of this fuzzy set which equal to $X - \{x\}$ belong's to δ. So, $\forall x \in X, \{x\}$ is closed in δ. Hence (X, δ) is T_1

Next, since (X, τ) satisfies (I), $\forall x \in X, \{x\}$ is fuzzy closed in (X, τ), $\boldsymbol{X} - \{x\} \in \tau$, and as above $X - \{x\} = \alpha(\boldsymbol{X} - \{x\})$ which is open in $\iota(\tau)$, so $\{x\}$ is closed sets in $\iota(\tau)$. Hence $(X, \iota(\tau))$ is T_1. □

Productive and Hereditary

Theorem 3.3.5. *(a) [35]Definition (I) is both productive and hereditary.*

(b) [25] Definition (II) is both productive and hereditary.

(c) [34] Definition (IV) is both productive and hereditary.

Concluding Remarks

Since $(I), (II)$ and (IV) are good extensions, productive, and hereditary, we will prefer them to define the Fuzzy T_1 as follows:

Definition 3.3.6. [25],[17] Let (X, τ) be an fts. Then X is called:

a. A *fuzzy* T_1 (FT_1) if X satisfies (I) that is, for each $x \in X$, $\{x\}$ is fuzzy closed. Using Proposition 3.3.1, X is FT_1 iff any two distinct points $x, y \in X$, $\exists U, V \in \tau$ such that $U(x) = 1, U(y) = 0$ and $V(y) = 1, V(x) = 0$.

b. A *fuzzy stronger* T_1 (FT_{s1}) if X satisfies (II) that is, every fuzzy singleton, point is a fuzzy closed set.

c. A *fuzzy weak* T_1 (FT_{w1}) if X satisfies (IV) that is, for any two distinct fuzzy points $p, q \in X$, $\exists U, V \in \tau$ with $p \in U, p \notin V$ and $q \in V, q \notin U$.

Implications:

Remark 3.3.7. Using Proposition 3.3.1, we have that:

a. In Chang sense: $FT_{s1} \Rightarrow FT_1 \Rightarrow FT_{w1}$.

b. In Lowen sense: $FT_{s1} \Leftrightarrow FT_1 \Rightarrow FT_{w1}$.

We wish to point out that the definition of a fuzzy T_1 space suggested here retains all the features of the one suggested earlier in ([27], [34]) which we refer as (FT_{w1}). In addition, it is somewhat more tangible and easier to work with, as it is described in terms of crisp rather than fuzzy points.
Finally, the equivalence in Remark 3.3.6 (b.) is a natural extension of ordinary definitions of T_1ness.

Fuzzy T_i-ness Definition $i = 0, 1, 2, 2\frac{1}{2}, 3, 4$

After studying the fuzzy T_1-ness definition in some details, now we will define in fast the fuzzy T_i-ness Definition for $i = 0, 1, 2, 2\frac{1}{2}, 3, 4$ and we will mention some theorems without proofs where their proofs are similar to the proof of FT_1-Theorems.

Definitions 3.3.8. [25] A fuzzy topological space (X, τ) is said to be:

1. *fuzzy* T_0 (FT_0) if for every pair of fuzzy singleton p, q with different supports there exists a fuzzy open set u such that either $p \leq u \leq q^c$ or $q \leq u \leq p^c$.

2. *fuzzy Housdorff* (FT_2) if for every pair of fuzzy singleton p, q with different supports, there exist two fuzzy open sets u and v such that $p \leq u \leq q^c$,$q \leq v \leq p^c$ and $u \leq v^c$.

3. *fuzzy Uryshon* ($FT_{2\frac{1}{2}}$) if for every pair of fuzzy singleton p, q with different supports, there exist two fuzzy open sets u and v such that $p \leq u \leq q^c$, $q \leq v \leq p^c$ and $\bar{u} \leq (\bar{v})^c$.

4. *fuzzy regular space* (FR) if for a fuzzy singleton p and a fuzzy closed set v, there exist two fuzzy open sets u_1 and u_2 such that $v \leq u_2, p \leq u_1$ and $u_1 \leq u_2^c$.

5. *fuzzy* $T_3(FT_3)$ if it is FR and (FT_s).

6. *fuzzy normal space (FN)* if for every pair of fuzzy closed sets v_1 and v_2 such that $v_1 \leq v_2^c$, there exist two fuzzy open sets u_1 and u_2 such that $v_1 \leq u_1, v_2 \leq u_2$ and $u_1 \leq u_2^c$.

7. *fuzzy* $T_4(FT_4)$ if it is (FN) and FT_s.

As in FT_1 definition, there is a quite different definitions for some of these separation axioms. such as the definitions given above for regularity and normality can also be modified in several ways. For example if the condition $v_1 \leq v_2^c$ is changed into $v_1 \wedge v_2 = \Phi$, then we obtain another form of normality. The investigation of this and other forms of regularity and normality will be a topic of further research. Some research use similar definitions by using the definition of quasi-coincident, instead of the definition of subset(see [2]). Some differentiations come from the differentiations of the definition of the fuzzy point and fuzzy singleton. Some authors use the neighborhood instead of the open fuzzy set to define the separation axioms. Others use the closure and interior. But in our study we choose the most important definition for each FT_i's, $i = 0, 1, s1, 2, 2\frac{1}{2}, 3, 4$, which have the following properties.

Implications:

Theorem 3.3.9. *[25] Let (X, τ) be an fts. Then the following implications holds for X:*

a. $FT_4 \Rightarrow FT_3 \Rightarrow FT_{2\frac{1}{2}} \Rightarrow FT_2 \Rightarrow FT_1 \Rightarrow FT_0$.

b. $FT_{s1} \Rightarrow FT_1 \Rightarrow FT_{w1}$.

In ordinary topology we have that every regular T_0 topological space is also T_3. However, this no longer holds in fuzzy topology. We have only another deviation from ordinary topology, that every FR-fts which is also T_0 is $FT_{2\frac{1}{2}}$.

Good Extensions

Theorem 3.3.10. *[2] For a topological space (X, δ),*
(X, δ) is T_i-space, if and only if $(X, \omega(\delta))$ is FT_i-space. for $i \in \{0, 1, 2, 3\}$.

Now, recall that if A is any fuzzy set, then supp$A = \{x : A(x) > 0\}$ the support of A, and if (X, δ) any topological space. Then $\tau_\delta = \{A \in I^X : suppA \in \delta\}$ is a fuzzy topological space.

Theorem 3.3.11. *Let (X, δ) be a topological space.*

a. (X, δ) is T_0-space if and only if (X, τ_δ) is FT_0 .

b. (X, δ) is T_1-space if and only if (X, τ_δ) is FT_1.

c. If (X, δ) is T_2-space, then (X, τ_δ) is FT_2.

Productive and Hereditary

Theorem 3.3.12. *[25]*

a. For every $i \in \{0, 1, s, 2, 2\frac{1}{2}, 3\}$ the corresponding FT_i property is hereditary.

b. Fuzzy normality is hereditary with respect to closed subspaces.

c. For every $i \in \{0, 1, s, 2, 2\frac{1}{2}\}$ the corresponding FT_i property is productive.

3.4 Compact Fuzzy Spaces

In the first paper of Chang, he gave a strong basement for the development of fuzzy topology in the [0,1], after the initial work of straight description of ordinary compactness, many authors tried to establish various notions of compactness to cover some of the fuzzy open sets instead of covers the whole space, by using different tools to obtained many important results. Each of them has its own advantages and shortcomings.

We will focus in our study on two definitions of compactness which defined by Chang and Lowen, then in short we study the other definitions and its prosperities.

Definition of Chang fuzzy compact topology

First we will define the cover of a fuzzy topology, and using it to define Chang compactness which is a natural extension of ordinary topology.

Definition 3.4.1. [7] A family $\mathcal{A}$ of fuzzy sets is *a cover* of a fuzzy set B iff $B \subseteq \bigvee\{A : A \in \mathcal{A}\}$. It is called *open cover* iff each member of $\mathcal{A}$ is an open fuzzy set. A *subcover* of $\mathcal{A}$ is a subfamily of $\mathcal{A}$ which is also a cover.

Definition 3.4.2. [7](Change definition of compactness)

A C-fts (X, τ) is *Chang compact* iff each open cover has a finite subcover.

That is, if $A_i \in \tau$ for every $i \in \Delta$ and $\bigvee_{i\in\Delta} A_i = 1$, then there are finitely many indices $i_1, i_2, ..., i_n \in \Delta$ such that $\bigvee_{j=1}^{n} A_{i_j} = 1$.

Definition 3.4.3. [8] A C-fts is *Chang countably compact* iff every countable open cover of the space has a finite subcover.

Also, C-fts is *Lindelöf* iff every open cover has a countable subcover.

In general if a C-fts is compact, then it is countably compact but the converse need not be true. If the C-fts satisfy the second axiom of countability which define in the following definition then the compactness and countably compactness will be equivalent as we shows in the next theorem.

Definition 3.4.4. [8] A fts (X, τ) is said to satisfy the *second axiom of countability* (briefly, C_{II}), if there exists a countable base of τ.

Theorem 3.4.5. *[8] If a C-fts (X, τ) is C_{II}, then X is Lindelöf.*

Proof. Since (X, τ) is C_{II} there exist $\mathfrak{B} = \{B_n : B_n \in \tau, n = 1, 2, ...\}$ a countable base for τ, and let $\mathcal{C} = \{A_\alpha : \alpha \in \Delta\}$ be any open cover of X. So, for each $\alpha \in \Delta, \exists \mathcal{N}_\alpha \subseteq \mathbb{N}$ such that $A_\alpha = \bigvee_{n \in \mathcal{N}_\alpha} B_n$. Set $\mathcal{N} = \bigvee_{\alpha \in \Delta} \mathcal{N}_\alpha$ and let $\mathcal{D} = \{B_n : n \in \mathcal{N}\}$.
Then $X = \bigvee_{\alpha \in \Delta} A_\alpha = \bigvee_{\alpha \in \Delta} (\bigvee_{n \in \mathcal{N}_\alpha} B_n) = \bigvee_{n \in \mathcal{N} B_n}$. That is $\mathcal{D}$ is countable open subcover of X, so X is Lindelöf. □

Theorem 3.4.6. *[8] If a C-fts (X, τ) is C_{II}, then X is compact iff it is countably compact.*

Proof. [8] The proof is complete if we prove that countably compactness implies compactness of X. So, let a C-fts (X, τ) be C_{II} and countable compact space. Let $\mathfrak{B} = \{B_n : B_n \in \tau, n = 1, 2, ...\}$ be a countable base for τ, and let $\mathcal{C} = \{A_\alpha : \alpha \in \Delta\}$, be any open cover of X. So for each $\alpha \in \Delta$, $\exists N_\alpha \subseteq \mathbb{N}$ such that $A_\alpha = \bigvee_{n \in N_\alpha} B_n$.
Set $\mathcal{N} = \bigcup_{\alpha \in \Delta} N_\alpha$ and let $\mathcal{D} = \{B_n : n \in \mathcal{N}\}$.
Then $X = \bigvee_{\alpha \in \Delta} A_\alpha = \bigvee_{\alpha \in \Delta} (\bigvee_{n \in \mathbb{N}_\alpha} B_n) = \bigvee_{\mathcal{D}} B_n$. That is $\mathcal{D}$ is countable open cover of X, so $\exists$ finite subcollection, say
$\grave{\mathcal{D}} = \{B_{\alpha_1}, B_{\alpha_2}, ..., B_{\alpha_k}\}$ such that $X = \bigvee_{i=1}^{k} B_{\alpha_i}$. But each B_{α_i} include in some A_α, so we get a finite subfamily $\{A_{\alpha_1}, A_{\alpha_2}, ..., A_{\alpha_k}\}$ of $\{A_\alpha : \alpha \in \Delta\}$ such that $X = \bigvee_{i=1}^{k} B_{\alpha_i} \subseteq \bigvee_{i=1}^{k} A_{\alpha_i}$. Hence X is compact.

□

Now, we put another proof for this theorem using Lindelöf property.

Proof. The proof is complete if we prove that countably compactness implies compactness of X. So, let a C-fts (X, τ) be C_{II} and countable compact space. Take any open cover of X say $\mathcal{C}$, since X is C_{II}, then X is Lindelöf, so there exist a countable subcover $\grave{\mathcal{C}} \subseteq \mathcal{C}$. But X is countably compact, so there exist a finit subcover $\grave{\grave{\mathcal{C}}} \subseteq \grave{\mathcal{C}} \subseteq \mathcal{C}$, so any open cover have a finite cover, that is X is compact. □

As in ordinary topology, Chang compactness and Lindelöf can be carried by F-continuity.

Theorem 3.4.7. *[8] Let f be an F-continuous function from the C-fts (X, τ) onto the C-fts (Y, γ). We have that if X is Chang compact (countably compact, Lindelöf), then Y is Chang compact (countably compact, Lindelöf respectively).*

Proof. [38] We will proof the first part and the proofs of other parts using the same argument. Let $B_i \in \gamma$ for each $i \in \Delta$ and assume that $\bigvee_{i \in \Delta} B_i = 1$. For each $x \in X$, $\bigvee_{i \in \Delta} f^{-1}[B_i](x) = \bigvee_{i \in \Delta} B_i[f(x)] = 1$. So, by the F-continuity of f the fuzzy sets $f^{-1}[B_i]$, $i \in \Delta$ are open and cover X. Thus for finitely many indices $i_1, i_2, ..., i_n \in \Delta$, $\bigvee_{j=1}^{n} f^{-1}[B_{i_j}] = 1$. Now, since f is a surjection, for any fuzzy set B in Y and for any $y \in Y$, we have that

$$\begin{aligned} f(f^{-1}[B])(y) &= sup\{f^{-1}[B](z) : \ z \in f^{-1}(y)\} \\ &= sup\{B(f(z)) : \ z \in f^{-1}(y)\} \\ &= sup\{B(y) : \ z \in f^{-1}(y)\} \\ &= B(y) \end{aligned}$$

so that $f(f^{-1}[B]) = B$, and since $f[\boldsymbol{X}](y) = sup\{\boldsymbol{X}(z) : z \in f^{-1}(y)\} = 1$. Thus,

$$1 \ = f[\boldsymbol{X}] \ = \ f(\bigvee_{j=1}^{n} f^{-1}[B_{i_j}]) \ = \ \bigvee_{j=1}^{n} f(f^{-1}[B_{i_j}]) \ = \ \bigvee_{j=1}^{n} B_{i_j}.$$

Therefore, (Y, γ) is compact. □

But this natural extension of the ordinary compactness is not consistent with natural fuzzy topology: If the topological space (X, δ) is ordinary compactness, then the fuzzy topological space $(X, \omega(\delta))$ need not be compact, as the following example shows.

Example 3.4.8. [30] Let (I, δ) be the unit interval with the usual topology which is compact space. $\forall x \in I$, $x \neq 0$ and $x \neq 1$, let A_x be the fuzzy set defined by:

$$A_x(y) = \begin{cases} 0, & y \in [0, \frac{x}{2}] \cup [\frac{1+x}{2}, 1]; \\ 1, & y = x; \\ \text{continuous linear}, & \text{on } [\frac{x}{2}, x] \text{ and on } [x, \frac{1+x}{2}]. \end{cases}$$

For

$$x = 0 \qquad A_0(y) = 1 - y \ \ \forall y \in I,$$

$$x = 1 \qquad A_1(y) = y \ \ \forall y \in I.$$

Clearly, we have that $\forall x \in I$, $\ A_x \in \omega(\delta)$ since $\forall \alpha \in I \ \ \alpha(A_1) = \{y : A_x(y) > \alpha\} = (\alpha, 1]$ which is an open set, and $(\bigvee\limits_{x \in I} A_x)(y) = sup\{A_x(y) : x \in I\} = 1, \forall y \in I$ since $\forall x \in (0, 1)$ $A_x(x) = 1$, $\ A_0(0) = 1$, and $A_1(1) = 1$. Then $\{A_x : x \in I\}$ is an open cover in $\omega(\delta)$.

Now, let $x_1, x_2, ..., x_n$ be a finite collection of I. Choose $y \in I - \{x_1, x_2, ..., x_n, 0, 1\}$. Then $A_{x_i}(y) \neq 1$. So, $\bigvee\limits_{i=1}^{n} A_x \neq 1$ and hence no finite subcover exist.

Now, we will give the definition of compactness and weakly compactness in the sense of Lowen by using the concept of Lowen compactness for the fuzzy sets as follows.

Definition of Lowen fuzzy compact topology

Definitions 3.4.9. 1. (Definition of Lowen fuzzy compact set [30]) Let (X, δ) be L-fts. (or C-fts). A fuzzy set $A \in I^X$ is *Lowen fuzzy compact* (briefly, Lf-compact set) iff whenever $A \subseteq \bigvee_{i \in \Delta} B_i$ for $B_i \in \tau$, $\forall i \in \Delta$, then for all $\epsilon > 0$, there exists a finitely many B_i's; say $B_{i_1}, ..., B_{i_n}$, such that $A - \epsilon \subseteq \bigvee_{j=1}^{n} B_{i_j}$

2. (Definition of Lowen fuzzy compact topology [30]) The Lowen fuzzy (or Chang fuzzy) topological space (X, τ) is *Lf-compact topology* iff each constant fuzzy set in (X, τ) is fuzzy compact.

3-[39] A L-fts (or C-fts) (X, τ) is *weakly fuzzy compact* if and only if the constant fuzzy set $\mathbf{X}$ in X is fuzzy compact.

Remark 3.4.10. Now the only relation which can be found is that if (X, τ) is a Chang fuzzy compact space then (X, τ) is weakly fuzzy compact.

Let us compare:

Chang's definition of compactness only makes sense in the class of Chang fuzzy topological spaces. Indeed, no Lowen fuzzy topological space can be Chang fuzzy compact.

Therefore the concept of Chang fuzzy compactness can be used on the class of Chang fuzzy topological spaces. But the concept of Lowen fuzzy compactness can be used on the class of Chang fuzzy topological spaces beside the class of Lowen fuzzy topological spaces.

Theorem 3.4.11. *[30] The fuzzy topological space $(X, \omega(\delta))$ is Lf-compact iff the space (X, δ) is compact.*

If $(X, \iota(\tau))$ is compact, then (X, τ) is Lowen fuzzy compact. But the converse is not necessarily true, see the counterexample in [30].

For the length of the proof, See [30].

Proposition 3.4.12. *[30]*

1. *If $f : (X, \tau) \to (Y, \gamma)$ is F-continuous and A is a Lowen fuzzy compact fuzzy set in (X, τ), then $f(A)$ is Lowen fuzzy compact in (Y, γ).*

2. *If* $f : (X, \tau) \to (Y, \gamma)$ *is* F*-continuous and onto,* X *is Lowen compact L-fts, Then* Y *is Lowen compact L-fts. But if* Y *is Lowen compact L-fts, then* X *needn't be Lowen fuzzy compact, see the counterexample in [30].*

3. *If* $f : (X, \tau) \to (Y, \gamma)$ *is* F*-continuous and onto,* X *is weakly compact L-fts, then* Y *is weakly compact L-fts.*

Proof. 1. Let A be a Lowen fuzzy compact set in (X, τ) and let $\beta \subseteq \gamma$ be such that

$$\bigvee_{B \in \beta} B \;>\; f[A].$$

Then by Lemma 1.2.23 and Theorem 1.2.25. part (c, f), we have that:

$$\bigvee_{B \in \beta} f^{-1}(B) = f^{-1}(\bigvee_{B \in \beta} B) \;>\; f^{-1}(f[A]) \geq A$$

Since f is continuous $(f^{-1}(B))_{B \in \beta} \subset \tau$ and A is fuzzy compact this implies that for any $\varepsilon > 0$ there exists a finite subfamily $\beta_0 \subset \beta$ such that

$$\bigvee_{B \in \beta_0} f^{-1}(B) \;>\; A - \varepsilon$$

Now it is immediately clear that:

$$\bigvee_{B \in \beta_0} B \;>\; f[A] - \varepsilon.$$

2. From the fact that, if f is onto, each constant fuzzy set on F is the image of the function f of constant fuzzy set on E with the same value.

3. The proof is a special case of proof 2, with $A = \boldsymbol{X}$.

□

We need the following lemma to prove the Tychonöf Theorem, which is an extension of the Alexander Subbase Lemma in the ordinary topology with similar argument of the proof.

Lemma 3.4.13. *Let S be a subbase for a fuzzy topological space (X, τ). Then (X, τ) is Lowen fuzzy compact if and only if whenever α is a constant fuzzy set in X, $\bigvee_{i \in \Delta} A_i \geq \alpha$ where $A_i \in S$ for $i \in \Delta$, and $\varepsilon > 0$ then there are finitely many indices $i_1, ..., i_n \in \Delta$ such that $\bigvee_{j=1}^{n} A_{i_j} \geq \alpha - \varepsilon$.*

Recall that: a subbase for the product fuzzy topology on $(X, \tau) = (\Pi_{i=1}^{n} X_i,\ \Pi_{i=1}^{n} \tau_i)$ is given by $S = \{\pi_i^{-1} A_i : A_i \in \tau_i, i \in I\}$. In [39], the author give a long proofs for the following two Tychonöf theorems, we give a short way to prove them. So, for the first theorem we give our proof, then for the second theorem we give the proof in [39].

Theorem 3.4.14. *[39]* ***Finite Tychonöf Theorem for Chang Compactness*** *Let n be a positive integer, and for each $i = 1, ..., n$, let (X_i, τ_i) be a Chang compact C-fts. Then $(X, \tau) = (\Pi_{i=1}^{n} X_i,\ \Pi_{i=1}^{n} \tau_i)$ is fuzzy Chang compact.*

Proof. Let $\mathcal{S} = \{\pi_i^{-1}(A_i) :\ A_i \in \tau_i,\ i = 1, 2, .., n\}$ be a subbase for (X, τ). Assume $\mathcal{C}$ be a subcollection of S that covers X, let $\mathcal{C}_i = \{A_i \in \tau_i : \pi_i^{-1}(A_i) \in \mathcal{C}\}$. Now, for every $x = (x_i)_{i \in \Delta}$,

$$\bigvee_{i=1}^{n} \mathcal{C}_i(x_i) = \bigvee_{i=1}^{n} (\bigvee_{A_i \in \mathcal{C}_i} A_i(x_i)) = \bigvee_{i=1}^{n} (\bigvee_{A_i \in \mathcal{C}_i} A_i(\pi_i(x))) = \bigvee_{i=1}^{n} (\bigvee_{\pi_i^{-1}(A_i) \in \mathcal{C}} \pi_i^{-1}(A_i(x))) = \bigvee \mathcal{C}(x) = 1.$$

So $\forall i, \mathcal{C}_i$ is a cover of (X_i, τ_i) which is Chang compact, then $\exists A_{i_1}, A_{i_2}, ..., A_{i_k}$, $\bigvee_{j=1}^{k} A_{i_j}(x_i) = 1$. We have $\forall x = (x_i)_{i \in \Delta}$, $\bigvee_{j=1}^{k_i} \pi_i^{-1}(A_{i_j})(x) = \bigvee_{j=1}^{k_i} A_{i_j} \pi_i(x) = \bigvee_{j=1}^{k_i} A_{i_j}(x_i) = 1$.

Hence $[(\bigvee_{j=1}^{k_1} \pi_1^{-1}(A_{1_j})) \vee (\bigvee_{j=1}^{k_2} \pi_2^{-1}(A_{2_j})) \vee ... \vee (\bigvee_{j=1}^{k_n} \pi_n^{-1}(A_{n_j}))] = \sup\ \{1, 1, ..., 1\} = 1$.
So we get finite subfamily of $\mathcal{C}$ which covers τ, then (X, τ) is Chang compact.

□

Definition 3.4.15. If $0 < \varepsilon < \alpha$, then we will say that a collection of open fuzzy sets of a fuzzy topological space has ε-FUP for α if none of its finite subcollections covers $\alpha - \varepsilon$ (That is, if none of its finite subcollections has spermium greater than or equal to $\alpha - \varepsilon$).

Theorem 3.4.16. *[39]* ***Tychonöf Theorem for Lowen Compactness*** *(Lowen, 1977).*
For each $i \in \Delta$, let (X_i, τ_i) be a Lowen compact L-fts. Then $(X, \tau) = (\Pi_{i\in\Delta} X_i,\ \Pi_{i\in\Delta}\tau_i)$ is Lowen compact.

Proof. [39]

Let $\alpha > 0$ be a constant fuzzy set in X. We wish to show that α is Lowen fuzzy compact in (X, τ), so let $\mathcal{S} = \{\pi_i^{-1}(A_i) :\ A_i \in \tau_i,\ i \in \Delta\}$ be a subbase for (X, τ). Let $0 < \varepsilon < \alpha$ and let $\mathcal{C}$ be a subcollection of $\mathcal{S}$ with ε-FUP for α.
Now, we want to show that $\mathcal{C}$ does not cover α.
Let $i \in \Delta$ and define $\mathcal{C}_i = \{A \in \tau_i :\ \ \Pi_i^{-1}(A) \in \mathcal{C}\}$.
We claim that this collection $\mathcal{C}_i$ of open fuzzy sets in (X_i, τ_i) has $\frac{\varepsilon}{2}$-FUP for $\alpha - \frac{\varepsilon}{2}$.
To see this, let $A_{i_1}, ..., A_{i_k} \in \mathcal{C}_i$. Then $\{\pi_i^{-1}\ (A_{i_j}) :\ \ j = 1, ..., k\}$ is a finite subcollection of $\mathcal{C}$, and hence this subcollection dosent has spermium greater than or equal to $\alpha - \varepsilon$), so there exists some point $x = (x_i)_{i\in\Delta} \in X$ such that $\bigvee_{j=1}^{k} \pi_i^{-1}\ (A_{i_j})\ (x) < \alpha - \varepsilon$. It then follows that

$$\bigvee_{j=1}^{k} A_{i_j}\ (x_i) = \bigvee_{j=1}^{k} A_{i_j}\ (\pi_i(x)) = \bigvee_{j=1}^{k} \pi_i^{-1}\ (A_{i_j})\ (x) < \alpha - \varepsilon = (\alpha - \frac{\varepsilon}{2}) - \frac{\varepsilon}{2},$$

so $\mathcal{C}_i$ of open fuzzy sets in (X_i, τ_i) has $\frac{\varepsilon}{2}$-FUP for $\alpha - \frac{\varepsilon}{2}$.
Since (X_i, τ_i) is Lowen fuzzy compact, the constant fuzzy set $\alpha - \frac{\varepsilon}{2}$ is fuzzy compact in (X_i, τ_i). So, $\mathcal{C}_i$ cannot cover $\alpha - \frac{\varepsilon}{2}$
(since if $\mathcal{C}_i$ covers it, τ_i compact, then there exist finite subcollection cover $\alpha - \frac{\varepsilon}{2}$ which contradicts the $\frac{\varepsilon}{2}$-FUP, so $\bigvee \mathcal{C}_i < \alpha - \frac{\varepsilon}{2}$)

and we can find some point $y_i \in X_i$ such that $(\bigvee \mathcal{C}_i)\ (y_i)\ <\ \alpha\ -\ \frac{\varepsilon}{2}$. Having done this for all $i \in \Delta$, set $y\ =\ (y_i)_{i\in\Delta}$. If we set $\mathcal{C}'_i = \{\Pi_i^{-1}\ (A) :\ A \in \tau_i\} \bigcap \mathcal{C}$, then it follows that

$(\bigvee \mathcal{C}'_i)(x) = \bigvee\{\pi_i^{-1}(A)(x) : A \in \tau_i, \text{and } \pi_i^{-1}(A) \in \mathcal{C}\} = \bigvee\{(A)(x_i) : A \in \tau_i, \pi_i^{-1}(A) \in \mathcal{C}\} = (\bigvee \mathcal{C}_i)(x_i) = a_i$. So $(\bigvee \mathcal{C}'_i)(y)\ =\ (\bigvee \mathcal{C}_i)(y_i)$. Further, noting that $\mathcal{C}\ =\ \bigcup_{i\in\Delta} \mathcal{C}'_i$, then we obtain that

$$(\bigvee \mathcal{C})(y)\ =\ \bigvee_{i\in\Delta}\ (\bigvee \mathcal{C}'_i)(y)\ =\ \bigvee_{i\in\Delta}\ (\bigvee \mathcal{C}_i)(y_i)\ \leq\ \alpha\ -\ \frac{\varepsilon}{2}\ <\ \alpha.$$

Hence, $\mathcal{C}$ dose not cover α so we get that any collection has ε-FUP for α can not cover α.

So, if we have any open cover of X, then it can not has ε-FUP for α and hence there exist finite subcollection of it which covers X, so X is compact in the sense of Lowen. □

Theorem 3.4.17. {*Finite Tychonöf Theorem for Weak Fuzzy Compactness (Lowen, 1977)*}.

Let n be a positive integer, and for each $i = 1, ..., n$, let (X_i, τ_i) be a weakly fuzzy compact L-fts. Then $(X, \tau) = (\Pi_{i=1}^{n} X_i,\ \ \Pi_{i=1}^{n} \tau_i)$ is weakly fuzzy compact.

Proof. The proof is a spacial case of the last proof with the constant fuzzy set $\alpha = \boldsymbol{X}$. □

We will give one counterexample for the product of countably many Change compact C-fts.

Example 3.4.18. [39] A product of countably many Chang compact C-fts need not be Chang fuzzy compact.

For each positive integer i let $X_i\ =\ \mathbb{N}$, the set of positive integers, let β_i be the constant fuzzy set in $\mathbb{N}$ given by $\beta_i(x) = \frac{i-1}{i}$ for $x \in \mathbb{N}$, and let

$\tau_i\ =\ \{\Phi, \mathbb{N}, \beta_i\} \cup \{\beta_i \mathcal{X}_{\{1,2,...,n\}}\ :\ n \in N\}$. Where $\beta_i \mathcal{X}_{\{1,2,...,n\}}(x) = \frac{i-1}{i}$ if $x \leq n$ or 0 elsewhere.

τ_i is a fuzzy topology on X_i, (since if $A_\alpha \in X_i, \alpha \in \Delta$, then

$$\bigvee_{\alpha\in\Delta} A_\alpha = \begin{cases} \mathrm{N}, & \mathbb{N} \in \{A_\alpha, \alpha \in \Delta\}; \\ \beta_i, & \beta_i \in \{A_\alpha, \alpha \in \Delta\}, \mathbb{N} \notin \{A_\alpha, \alpha \in \Delta\}; \\ \beta_i \mathcal{X}_{\{1,2,\ldots,sup\{n:\beta_i \mathcal{X}_{\{1,2,\ldots,n\}} \in A_\alpha\}\}}, & \text{otherwise.} \end{cases}$$

belongs to τ_i). Moreover, if $\bigvee_{\alpha\in\Delta} B_\alpha = 1$, where $B_\alpha \in \tau_i$, then $B_\alpha = \mathbb{N}$ for some α (since if $\forall \alpha, B_\alpha \neq \mathbb{N}$, then $\bigvee_{\alpha\in\Delta} B_\alpha = \frac{i-1}{i} \neq 1$). So (X_i, τ_i) is compact.

Now let $(X, \tau) = (\Pi_{i\in\mathbb{N}} X_i, \quad \Pi_{i=1}^{n} \tau_i)$. For $(i, n) \in \mathbb{N} \times \mathbb{N}$, the member of the fuzzy topology τ of a fixed $x = (x_i) \in X$,

$$A_{(i,n)}(x) = \pi_i^{-1}(\beta_i \mathcal{X}_{\{1,2,\ldots,n\}})(x) = \beta_i \mathcal{X}_{\{1,2,\ldots,n\}} \circ \pi_i(x) = \beta_i \mathcal{X}_{\{1,2,\ldots,n\}}(x_i)$$

$$= \begin{cases} \frac{i-1}{i} & x_i \leq n, \\ 0 & x_i > n. \end{cases}$$

Given $\varepsilon > 0$, find i with $1 - \varepsilon < \frac{i-1}{i}$. Then for all $n \geq x_i, A_{i,n}(x) = \frac{i-1}{i} > 1 - \varepsilon$. So $\bigvee_{(i,n)\in\mathbb{N}\times\mathbb{N}} A_{i,n} = 1$. But if S is a finite subset $\mathbb{N} \times \mathbb{N}$, then we can find $N \in \mathbb{N}$ such that if $(i, n) \in S$ then $n < N$. It follows that for $x = (N, N, N, ...)$ we have $A_{i,n}(x) = 0$ for all $(i, n) \in S$, and certainly $\bigvee_{(i,n)\in S} A_{i,n} < 1$. Thus, we conclude that (X, τ) is not compact.

Example 3.4.19. A product of countably many weakly compact L-fts need not be weakly fuzzy compact.

The counterexample is similar to the above one with slightly changes in [39].

Another Definitions of Compactness and Some Prosperities

The concept of compactness in fuzzy set theory was first introduced by Chang[7] in terms of open cover. Goguen [18] was the first to point out a deficiency in Chang's compactness theory by showing that the Tychonoff Theorem is false .

Since Chang's compactness has some limitations, T.E.Gantner [42] introduced α-compactness, Lowen [30] introduced fuzzy compactness, strong fuzzy compactness and ultra-fuzzy compactness.

Definition 3.4.20. [32] A family of fuzzy subsets $\beta \subset I^X$ is called α- (resp. α^*-) shading iff for all $x \in X$ there exists $B \in \beta$ such that $B(x) > \alpha$ (resp. $B(x) \geq \alpha$).

T.E.Gantner [42], and others used the concept of shading families to study compactness. The shading families are a very natural generalization of coverings.

Definition 3.4.21. [32] Let (X, τ) be a Lowen or Chang fuzzy topological space,

Definition I. Let $0 \leq \alpha < 1$, and (X, τ) is a fts then (X, τ) is α-*compact* iff each α-shading family in τ has a finite α-shading subfamily.

Definition II. Let $0 < \alpha \leq 1$, and (X, τ) is a fts then (X, τ) is α^*-*compact* iff each α^*-shading family in τ has a finite α^*-shading subfamily.

Definition III. A fts (X, τ) is *strong fuzzy compact* iff it is α-compact for each $\alpha \in [0, 1)$.

Definition IV. A fts (X, τ) is *ultra-fuzzy compact* iff $(X, \iota(\tau))$ is compact.

Now we will mention some properties of those definitions (without proof since we gave similar proofs for Chang compactness, Lowen compactness and weakly compactness):

Theorem 3.4.22. *Definitions (I), (III), (IV), weakly fuzzy compact, and Lowen compact are good extensions of compactness.*

Theorem 3.4.23. *[33] A continuous image of an α-compact, weak compact, Chang-compact, Lowen-compact space is α-compact, weak compact, Chang-compact, Lowen-compact space, respectively.*

Theorem 3.4.24. *[33] If $(X_j, \tau_j)_{j \in J}$ is a family of a Lowen or Chang fuzzy topological spaces, then*

$(\prod_{j\in J} X_j, \prod_{j\in J} \tau_j)$ *is* α*-compact strong fuzzy compact, ultra-fuzzy compact, Lowen-compact iff* $\forall j \in J$, (X_j, τ_j) *is* α*-compact strong fuzzy compact, ultra-fuzzy compact, Lowen-compact respectively. Moreover, Chang fuzzy compactness and weak fuzzy compactness are preserved when taking finite products.*

Theorem 3.4.25. *A fuzzy space* X *is* α*-compact if and only if the ordinary topological space* ι_α *is compact. (For* ι_α*, see Definition 2.3.4)*

conclusion

If the condition of Lowen-compact satisfies for at least the constant $\alpha = 1$*, then we have weakly compact space.*

For (I), (III), (IV), and Lowen compact which are good extensions and have the Tychonoff product property, we have the following implications:

$$(IV) \Rightarrow (III) \ \Rightarrow (I)$$

$$\Downarrow$$

(Lowen compact)

In this chapter we state Example 3.2.3, Example 3.2.4, Remark 3.2.5, Example 3.2.6. Also we give the proof the following: Theorem 3.1.1, Proposition 3.2.7, Theorem 3.2.9, Proposition 3.2.10, Proposition 3.2.12, Theorem 3.3.3, Theorem 3.4.6, Theorem 3.4.15, Theorem3.4.18.

Chapter 4

Sostak Fuzzy Topological Spaces

4.1 Sostak's Definition

The concept of fuzzy topology was first defined in 1968 by Chang. But a Changs fuzzy topology is a crisp subfamily of some family of fuzzy sets and fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of the topological space. Therefore, in 1985 Sostak, introduced the fundamental concept of a fuzzy topological structure as an extension of both ordinary topology and Changs fuzzy topology, in the sense that not only the object were fuzzifed, but also the axiomatic. Sostak gave some rules and showed how such an extension can be realized. The fuzzy topology in the sense of Chang was referred to be a topology on a fuzzy set.

A Changs fuzzy topology on X can be regarded as a map $\tau : I^X \to \{0, 1\}$ which satisfies the following three conditions:

a. $\tau(\Phi) = \tau(X) = 1$,

b. if $\tau(A) = \tau(B) = 1$, then $\tau(A \wedge B) = 1$,

c. if $\tau(A_\alpha) = 1$ for each $\alpha \in \Delta$, then $\tau(\bigvee A_\alpha) = 1$.

But fuzziness in the concept of openness of a fuzzy subset is absent in the above Changs definition of fuzzy topology. So for fuzzifying the openness of a fuzzy subset, some authors gave other definitions of fuzzy topology.

Definition 4.1.1. [4] Let X be a non-empty set and $\tau : I^X \to I$ be a mapping satisfies the following conditions:

O1. $\tau(\Phi) = \tau(\mathbf{X}) = 1$.

O2. If $A, B \in I^X$, then $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$.

O3. For any $\{A_i : i \in \Delta\} \subseteq I^X$, we have that $\tau(\bigvee\limits_{i\in\Delta} A_i) \geq \bigwedge\limits_{i\in\Delta} \tau(A_i)$.

Then τ is called a *sostak fuzzy topology* on X (or *gradation of openness on X*, or *smooth topology*), or S-ft for short.

Example 4.1.2. Let $X = \{a, b\}$. Define the fuzzy sets A, B in X as:

$$A(x) = \begin{cases} 1 & x = a, \\ 0 & x = b. \end{cases}$$

$$B(x) = \begin{cases} 0 & x = a, \\ 1 & x = b. \end{cases}$$

We define a mapping $\tau : I^X \to [0, 1]$ by:

$\tau(\mathbf{X}) = \tau(\Phi) = 1$, $\tau(A) = 0.15, \tau(B) = 0.80$, and $\tau(C) = 0.10$, if $C \notin \{\Phi,\ \mathbf{X}, A, B\}$. Note, for example, that $A \vee B = \boldsymbol{X}, \tau(A \vee B) = 1 \geq 0.15 = \tau(A) \wedge \tau(B)$. It is clearly that (X, τ) is a S-ft on X, which is not C-ft.

Definition 4.1.3. [21] Let X be a non-empty set and $\mathfrak{F} : I^X \to I$ be a mapping satisfies the following conditions:

C1. $\mathfrak{F}(\Phi) = \mathfrak{F}(\boldsymbol{X}) = 1$.

C2. For $A, B \in I^X$, $\mathfrak{F}(A \vee B) \geq \mathfrak{F}(A) \wedge \mathfrak{F}(B)$.

C3. For $\{A_i : i \in \Delta\} \subseteq I^X$, $\mathfrak{F}(\bigwedge_{i\in\Delta} A_i) \geq \bigwedge_{i\in\Delta} \mathfrak{F}(A_i)$.

Then the mapping τ is called a *gradation of closedness* on X or *smooth cotopological space.*

Proposition 4.1.4. *Let (X, τ) be a S-tfs and $\mathfrak{F}_\tau : I^X \to I$ be a mapping defined by $\mathfrak{F}_\tau(A) = \tau(A^c)$, where A^c is the complement of A. Then $\mathfrak{F}$ is a gradation of closedness.*

Proof. C1. $\mathfrak{F}(\Phi) = \tau(\Phi^c) = \tau(\boldsymbol{X}) = 1$.

C2. Flows from the De Morgan laws that is: $\mathfrak{F}(A \vee B) = \tau(A^c \wedge B^c) \geq \tau(A^c) \wedge \tau(B^c) = \mathfrak{F}(A) \wedge \mathfrak{F}(B)$.

C3. Follows from the De Morgan's Low $\mathfrak{F}(\bigwedge_{i\in\Delta} A_i) = \tau(\bigwedge_{i\in\Delta} A_i)^c = \tau(\bigvee_{i\in\Delta} (A_i)^c) \geq \bigwedge_{i\in\Delta} \tau(A_i)^c = \bigwedge_{i\in\Delta} \mathfrak{F}(A_i)$.

□

Definition 4.1.5. If (X, τ) is an S-fts and $A \in I^X$, then:

i. $\tau(A)$ is referred to as *grade of openness* of A.

ii. $\mathfrak{F}_\tau(A)$ (simply, $\mathfrak{F}(A)$) is referred to as *grade of closedness* of A.

Example 4.1.6. Let X be a non-empty set. Define $\tau_o, \tau_1 : I^X \to I$ by the rules: $\tau_0(\Phi) = \tau_0(\boldsymbol{X}) = 1, \tau_0(A) = 0, \forall A \in I^X - \{\Phi, \boldsymbol{X}\}$, and $\tau_1(A) = 1, \forall A \in I^X$. Then τ_o and τ_1 are two Sostak fuzzy topologies (S-fts) on X satisfy that for any S-ft τ on X, we have that $\tau_o(A) \leq \tau(A) \leq \tau_1(A),\ \forall A \in I^X$. In short, $\tau_o \leq \tau \leq \tau_1$.

The following theorem is parallel with the usual topology, C-ft and L-ft.

Proposition 4.1.7. *[21] An arbitrary intersection of S-ft is a S-ft.*

Proof. Let X be a non-empty set and let $\{\tau_k : k \in \Delta\}$ be an arbitrary family of S-fts on X. Take $\tau = \bigwedge_{k\in\Delta} \tau_k$, Clearly $\tau(\Phi) = \tau(\mathbf{X}) = 1$. Moreover,

$$\tau(A \wedge B) = \bigwedge_{k\in\Delta} \tau_k(A \wedge B) \geq \bigwedge_{k\in\Delta} \{\tau_k(A) \wedge \tau_k(B)\} = \inf_{k\in\Delta} \{ \inf(\tau_k(A) , \tau_k(B)) \}$$

$$\geq \inf\{ (\inf_{k\in\Delta} \tau_k(A)) , (\inf_{k\in\Delta} \tau_k(A)) \} = (\bigwedge_{k\in\Delta} \tau_k(A)) \wedge (\bigwedge_{k\in\Delta} \tau_k(B)) = \tau(A) \wedge \tau(B).$$

Finally,$\tau(\bigvee_i A_i) = \bigwedge_{k\in\Delta} \tau_k(\bigvee_i A_i) \geq \bigwedge_{k\in\Delta} (\bigwedge_i \tau(A_i)) \geq \bigwedge_i \bigwedge_{k\in\Delta} \tau(A_i) = \bigwedge_i \tau(A_i).$ □

If we have a S-fts, we can convert it to a C-fts as follows:

Theorem 4.1.8. *[4] Let τ be a S-ft, and for each $\alpha \in (0, 1]$, let $\tau_\alpha = \{A : A \in I^X, \tau(A) \geq \alpha\}$. Then τ_α is a Chang fuzzy topology on X. Moreover $\alpha \leq \beta \Rightarrow \tau_\alpha \geq \tau_\beta$. In addition, $\tau_1(A) = sup\{\alpha : A \in \tau_\alpha\}$ is a Sostak fuzzy topology on X such that $\tau_1 = \tau$.*

Proof. Let $\tau_\alpha = \{A : A \in I^X, \tau(A) \geq \alpha\}$, $\alpha \in (0, 1]$
$\tau(\Phi) = \tau(\boldsymbol{X}) = 1 \geq \alpha$ so Φ , $\boldsymbol{X} \in \tau_\alpha$. For every $A,\ B \in I^X$, and $A,\ B \in \tau_\alpha$, we have that $\tau(A) \geq \alpha$ and $\tau(B) \geq \alpha$ so $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B) \geq \alpha$. For $i \in \Delta$, if every $A_i \in \tau_\alpha$, then $\tau(A_i) \geq \alpha, \forall i$. So, $\tau(\bigvee_i A_i) \geq \bigwedge_i \tau(A_i) \geq \bigwedge_i \alpha = \alpha$. Hence $\bigvee A_i \in \tau_\alpha$ and τ_α is a S-ft on X.

Now, let $\alpha \leq \beta$, and let $A \in \tau_\beta$. Then $\tau(A) \geq \beta \geq \alpha$, so $A \in \tau_\alpha$.
Finally, let $\tau_1(A) = sup\{\alpha : A \in \tau_\alpha\}$. Then we have that :

i. $\Phi, \boldsymbol{X} \in \tau_\alpha\ \forall \alpha \in (0, 1]$, So $\tau_1(\Phi) = \tau_1(\boldsymbol{X}) = 1$.

ii. For every $A, B \in I^X$, and $\alpha > 0$, if $A, B \in \tau_\alpha$ then $A \wedge B \in \tau_\alpha$ (since τ_α is a C-ft). So, $\{\alpha : A \wedge B \in \tau_\alpha\} \supseteq \{\alpha : A \in \tau_\alpha \text{ and } B \in \tau_\alpha\}$.

Hence $\sup\{\alpha : A \wedge B \in \tau_\alpha\} \geq \sup\ \{\alpha : A \in \tau_\alpha \text{ and } B \in \tau_\alpha\}$. That is, $\tau_1(A \wedge B) \geq \tau_1(A) \wedge \tau_1(B)$.

iii. For every $A_i \in \tau_\alpha, i \in \Delta$, we have that $\bigvee A_i \in \tau_\alpha$ (τ_α is a C-ft).

So, $\{\alpha : \bigvee A_i \in \tau_\alpha\} \supseteq \{\alpha : \forall i \in \Delta, A_i \in \tau_\alpha\}$. This implies that

$\sup\ \{\alpha : \bigvee A_i \in \tau_\alpha\} \geq \inf\ \{\alpha : \forall i \in \Delta, A_i \in \tau_\alpha\}$.

That is $\tau_1(\bigvee_{i\in\Delta} A_i) \geq \bigwedge_{i\in\Delta} \tau_1(A_i)$.

iv. $\tau_1(A) = \sup\ \{\alpha : A \in \tau_\alpha\} = \sup\ \{\alpha : \tau(A) \geq \alpha\} = \tau(A)$.

□

Conversely, to convert a C-fts to a S-fts we use the following mapping:

Theorem 4.1.9. *[21] Let (X, T) be a C-fts. Define for each $\beta \in (0, 1]$, a mapping $T^\beta : I^X \to I$ defined by:*

$$T^\beta(A) = \begin{cases} 1 & A \in \{\Phi, \boldsymbol{X}\}, \\ \beta & A \in T - \{\Phi, \boldsymbol{X}\}, \\ 0 & \textit{otherwise}. \end{cases}$$

Then T^β is a Sostak fuzzy topology on X such that $(T^\beta)_\beta = T$.

Proof. $T^\beta(\boldsymbol{X}) = T^\beta(\Phi) = 1$ Let $A, B \in I^X$, then if $T^\beta(A) = 0$ or $T^\beta(B) = 0$ we have $T^\beta(A) \wedge T^\beta(B) = 0 \leq T^\beta(A \wedge B)$. But if $T^\beta(A) \neq 0$ and $T^\beta(B) \neq 0$ (i.e $A, B \in T, A \wedge B \in T$) we have two cases

case 1. If $A \wedge B = \Phi$ or $A \wedge B = \boldsymbol{X}$,then $T^\beta(A \wedge B) = 1$ so $T^\beta(A) \wedge T^\beta(B) \leq 1 = T^\beta(A \wedge B)$.

case 2. At least A or $B \in T - \{\phi, \boldsymbol{X}\}$ then $T^\beta(A) \wedge T^\beta(B) = \beta \leq T^\beta(A \wedge B)$.

Now, let $A_i \in I^X$, there is three cases

case 1. $\bigwedge T^\beta(A_i) = 0 \leq \bigvee T^\beta(A_i)$.

case 2. $\bigwedge T^{\beta}(A_i) = 1$, so $\forall i A_i \in \{\Phi, \boldsymbol{X}\}$, then $\bigvee A_i \in \{\Phi, \boldsymbol{X}\}, T^{\beta}(\bigvee A_i) = 1 = \bigwedge T^{\beta}(A_i)$.

case 3. $\bigwedge T(A_i) = \beta$, so $\forall i T^{\beta}(A_i) \geq \beta$ i.e.$\forall i T^{\beta}(A_i) \in \{1, \beta\}$ that is $\forall i (A_i) \in T$, and since T is a C-fts $\bigvee A_i \in T$, and hence $T^{\beta}(\bigvee A_i) \in \{1, \beta\}$, then we have $T^{\beta}(\bigvee A_i) \geq \beta = \bigwedge T^{\beta}(A_i)$.

□

Also we use the following mapping to convert an ordinary topology to a S-ft.

Theorem 4.1.10. *[5] Let (X, δ) be an ordinary topological space. Then the mapping $\psi_{\delta} : I^X \longrightarrow I$, defined by: $\psi_{\delta}(A) = sup\{\alpha \in I : A^{-1}(\alpha, 1] \in \delta\}$ is a Sostak fuzzy topology on X, where $A \in I^X$.*

4.2 Topological Concept on Sostak Fuzzy Topology

CONTINUITY

Definition 4.2.1. [23] Let (X, τ_1), (X, τ_2) be two S-fts. A map $f : X \to Y$ is called *smooth continuous* with respect to τ_1 and τ_2, iff for every $A \in I^Y$ we have $\tau_1(f^{-1}[A]) \geq \tau_2(A)$, where $f^{-1}[A]$ is defined by $f^{-1}[A](x) = A(f(x)), \forall x \in X$.
A map $f : X \to Y$ is called *weakly smooth continuous* with respect to τ_1 and τ_2 iff for every $A \in I^Y$ such that $\tau_2(A) > 0$ we have that $\tau_1(f^{-1}[A]) > 0$.

One may deduce that smooth continuous implies weakly continuous, but the converse may not be true in general, as the following example shows:

Example 4.2.2. Let $X = \{a, b, c, d\}$. Define the fuzzy sets A, B in X as:

$$A(x) = \begin{cases} 1 & x = b, d, \\ 0 & x = a, c. \end{cases}$$

$$B(x) = \begin{cases} 1 & x = a, c, \\ 0 & x = b, d. \end{cases}$$

For each $i = 1, 2$ we define a mapping $\tau_i : I^X \to \{0, 1\}$ by:

$$\tau_i(\boldsymbol{X}) = \tau_i(\Phi) = 1,$$

$$\tau_1(A) = \tau_1(B) = \frac{1}{2},$$

$$\tau_2(A) = \tau_2(B) = 1.$$

$$\tau_i(C) = 0,\ C \in I^X - \{\boldsymbol{X}, \Phi, A, B\}.$$

Since $\{\boldsymbol{X}, \Phi, A, B\}$ is Chang fts and by using Theorem 4.1.9. we have that (X, τ_1) and (X, τ_2) are two S-ft on X. The identity mapping $id : (X, \tau_1) \to (X, \tau_2)$ is weakly smooth continuous. To see this, $\tau_2(\Phi) = 1 > 0$ and $\tau_1(id^{-1}[\Phi]) = \tau_1(\Phi(id)) = \tau_1(\Phi) = 1 > 0$. Similarly for $\boldsymbol{X}$. Also $\tau_2(A) = 1 > 0$ and $\tau_1(id^{-1}[A]) = \tau_1(A(id)) = \tau_1(A) = \frac{1}{2} > 0$, we do the same for B, so id is weakly smooth continuous. But it is not smooth continous since $\tau_1(id^{-1}[A]) = \frac{1}{2} \leq 1 = \tau_2(A)$.

Proposition 4.2.3. *[4] Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping. Then:*
(1) f is smooth continuous iff $\forall A \in I^Y$ we have $\mathfrak{F}_{\tau_1}(f^{-1}[A]) \geq \mathfrak{F}_{\tau_2}(A)$.
(2) f is weakly smooth continuous iff $\forall A \in I^Y$, $\mathfrak{F}_{\tau_1}(A) > 0$ we have that $\mathfrak{F}_{\tau_2}(f^{-1}[A]) > 0$.

Proof. For part (1), $\mathfrak{F}_{\tau_1}(f^{-1}[A]) = \tau_1(f^{-1}[A])^c = \tau_1(A(f))^c = \tau_1(A^c(f)) = \tau_1(f^{-1}[A^c]) \geq \tau_2(A^c) = \mathfrak{F}_{\tau_2}(A)$. Similarly for part (2) □

Proposition 4.2.4. *[4] Let $(X, \tau_1), (Y, \tau_2)$ and (Z, τ_3) be S-fts. If $f : X \to Y$, and $g : Y \to Z$ are smooth continuous, then so is $g \circ f$.*

Proof. Using the smooth continuity of g and f , and theorem 1.2.25 it follows that, for $A \in I^Z$,

$$\tau_1((g \circ f)^{-1}(A)) = \tau_1(f^{-1}(g^{-1}(A))) \geq \tau_2(g^{-1}(A)) \geq \tau_3(A).$$ □

Definition 4.2.5. [23] Let (X, τ_1), (X, τ_2) be two S-fts. A map $f : X \to Y$ is called a *smooth open (resp., a smooth closed*) map if for each $A \in I^X$ we have $\tau_1(A) \leq \tau_2(f(A))$ (resp. $\mathfrak{F}_1(A) \leq \mathfrak{F}_2(f(A))$) where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are fuzzy families of closed sets in X and Y respectively.
f is called a *smooth homeomorphism* iff it is one-to-one, onto and f and f^{-1} are smooth continuous. Its clearly that f is smooth open equivalent to f^{-1} is smooth continuous.

As we showed above any Chang open function need not to be Chang closed function, But in Sostak topology both of them are equivalent as follows:

Proposition 4.2.6. *Let* (X, τ_1)*,* (X, τ_2) *be two S-fts. and* $f : X \to Y$ *be a bijective map. The following statements are equivalent:*

a. f^{-1} *is smooth continuous.*

b. f *is smooth open.*

c. f *is smooth closed.*

Proof. Follows from Definitions 4.2.1, 4.2.5 and Proposition 4.2.3. □

Proposition 4.2.7. *[23] Let* f *be a map from a S-fts* (X, τ) *to a S-fts* $(Y, \dot{\tau})$*. If for every* $\alpha \in (0, 1]$*,* $f : (X, \tau_\alpha) \to (Y, \dot{\tau}_\alpha)$ *is Chang fuzzy continuous (resp, fuzzy open) between the tow C-fts* τ_α *and* $\dot{\tau}_\alpha$*, then it is smooth continuous (resp. smooth open) with respect to the smooth topologies* τ *and* $\dot{\tau}$*.*

Proof. Remember that $\tau_\alpha = \{A : A \in I^X, \tau(A) \geq \alpha\}$.
f is Change fuzzy continuous, if $B \in \dot{\tau}_\alpha$ then $f^{-1}[B] \in \tau_\alpha$.
f is fuzzy open, if $A \in \tau_\alpha$ then $f(A) \in \dot{\tau}_\alpha$.
Now, let $B \in I^Y$ and suppose $\dot{\tau}(B) = \beta$. So by definition of $\dot{\tau}_\beta, B \in \dot{\tau}_\beta$, but f is Change continuous so $f^{-1}[B] \in \tau_\beta$. Hence $\tau(f^{-1}[B]) \geq \beta = \dot{\tau}(B)$. So, f is smooth continuous.
If $A \in I^X$ and suppose $\tau(A) = \alpha$. So by definition of $\tau_\alpha, A \in \tau_\alpha$, but f is fuzzy open so $f(A) \in \dot{\tau}_\alpha$. Hence $\dot{\tau}(f(A)) \geq \alpha = \tau(A)$. So, f is smooth open. □

SEPARATIONS AXIOMS

Definition 4.2.8. [24] Let (X, τ) be a S-fts. Then we say that X is:

a. ST_0 space iff for each $x, y \in X$ with $x \neq y$ there exists $A \in I^X$ such that $(A(x) > 0,\ A(y) = 0$ and $A(x) \leq \tau(A))$ or $(A(y) > 0,\ A(x) = 0$ and $A(y) \leq \tau(A))$.

b. ST_1 space iff for each $x, y \in X$ with $x \neq y$ there exists $A, B \in I^X$ such that $(B(x) = 0$ and $0 < A(x) \leq \tau(A))$, or $(A(y) = 0$ and $0 < B(y) \leq \tau(B))$.

c. ST_2 space iff for each $x, y \in X$ with $x \neq y$ there exists $A, B \in I^X$ such that $(0 < A(x) \leq \tau(A))$, $(0 < B(y) \leq \tau(B))$ and $A \wedge B = \Phi$.

d. S_1-regular (resp. S_2-regular) space iff for each $C \in I^X$, satisfying $\mathfrak{F}_\tau(C) > 0$, and each $x \in X$ satisfying $C(x) = 0$, there exist $A, B \in I^X$ such that $A(x) > 0$ (resp. $(A \backslash B)(x) > 0$), $A(x) \leq \tau(A)$, $C \subseteq B$, $\mathfrak{F}_\tau(C) \leq \tau(B)$ and $A \wedge B = \Phi$ (resp. $A \subseteq B^c$).

e. S_1-normal (resp. S_2-normal) space iff for each $C, D \in I^X$, satisfying $\mathfrak{F}_\tau(C) > 0$, $\mathfrak{F}_\tau(D) > 0$ and $C \subseteq (D^c)$ (resp. $C \wedge D = \Phi$), there exist $A, B \in I^X$ such that $C \subseteq A$, $\mathfrak{F}_\tau(C) \leq \tau(A)$, $D \subseteq B$, $\mathfrak{F}_\tau(D) \leq \tau(B)$ and $A \wedge B = \Phi$ (resp. $A \subseteq B^c$).

The definitions given above can also be modified in several ways. For example if the condition $A \wedge B = \Phi$ is changed into $A \subseteq B^c$, then we obtain another forms. The investigation of this and other forms will be a topic of further research.

Remark 4.2.9. Directly from the definition above, we have that:

a. S_1 regular $\Rightarrow$ S_2 regular and S_1 normal $\Rightarrow$ S_2 normal.

b. $ST_2 \Rightarrow ST_1 \Rightarrow ST_0$.

c. If (X, δ) is an ordinary topological space, and if (X, ψ_δ) is the generated S-fts, then these concepts coincide with the ordinary concepts.
Hence there will be no difference between S_1-regular and S_2-regular and between S_1-normal and S_2-normal.

d. If τ is a Chang fuzzy topology on X , then these concepts also coincide with the corresponding fuzzy concepts as introduced in *Definition 2.3.3.* which are studied befor

Proposition 4.2.10. *Let $f : X \to Y$ be a smooth homeomorphism from a S-fts (X, τ_1) onto a S-fts (Y, τ_2) then:*

(a) If X is $ST_i, i = 0, 1, 2$, then (Y, τ_2) is ST_i space.

(b) If X is S_i-regular (resp. normal), $i = 1, 2$, then (Y, τ_2) is S_i-regular (resp. normal) space.

Proof. We give the proof for ST_2.

Let $x, y \in Y$ with $x \neq y$. Since f is bijective, then $f^{-1}(x) \neq f^{-1}(y)$. Since (X, τ_1) is ST_2 , there exist $A, B \in I^X$ such that $0 < A(f^{-1}(x)) \leq \tau_1(A))$ and $0 < B(f^{-1}(y)) \leq \tau_1(B))$, and $A \wedge B = \Phi$.

Since f is smooth open it follows that $\tau_1(A) \leq \tau_2(f(A))$ and $\tau_1(B) \leq \tau_2(f(B))$.

As $A(f^{-1}(x)) = f[A](x)$, $B(f^{-1}(y)) = f[B](y)$, so we have

$0 < f[A](x) = A(f^{-1}(x)) \leq \tau_1(A) \leq \tau_2(f(A))$, and $f[B](y) \leq \tau_2(f[B])$

Moreover $f(A] \wedge f[B] = \Phi$. Therefore (Y, τ_2) is ST_2. □

Proposition 4.2.11. *For $i = 0, 1, 2$, let $f : X \to Y$ be an injective, smooth continuous map from a smooth space (X, τ_1) to ST_i space (Y, τ_2). Then (X, τ_1) is ST_i space.*

This will be true for The $S_i (i = 1, 2)$ regularity (resp. normality) property, if f is smooth closed in addition to above conditions.

Proof. We give the proof for ST_1.

Let $x, y \in Y$ with $x \neq y$. Since f is injective, then $f^{-1}(x) \neq f^{-1}(y)$. Since (Y, τ_2) is ST_1 , there exist $A, B \in I^Y$ such that $(B(f(x)) = 0$ and $0 < A(f(x)) \leq \tau_2(A))$, or $(A(f(y)) = 0$ and $0 < B(f(y)) \leq \tau_2(B))$.

Assume $(B(f(x)) = 0$ and $0 < A(f(x)) \leq \tau_2(A))$ hold , since f is smooth continuous it follows that $f^{-1}(B(x)) = 0$ and $0 < f^{-1}(A(x)) \leq \tau_2(A) \leq \tau_1(f^{-1}(A))$. Therefore (X, τ_1) is ST_1.

□

COMPACTNESS

Definition 4.2.12. [4] A smooth topological space (X, τ) is smooth compact if for every $\alpha > 0$, (X, τ_α) is Chang fuzzy compact.

Proposition 4.2.13. *[4] Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an onto smooth continuous mapping. If (X, τ_1) is compact then so is (Y, τ_2).*

Proof. Claim: If $f : (X, \tau) \to (Y, \gamma)$ is smooth continuous mapping, then
$\forall \alpha > 0$, $f : (X, \tau_\alpha) \to (Y, \gamma_\alpha)$ is fuzzy continuous mapping in sense of Change. To see this, Let $A \in \gamma_\alpha$, then $\gamma(A) \geq \alpha$, but f is smooth continuous, so $\tau(f^{-1}(A)) \geq \gamma(A) \geq \alpha$. Hence $f^{-1}(A) \in \tau_\alpha$ and so f is fuzzy continuous.
Now, we want to show that (Y, γ) is smooth continuous, that is for any $\alpha > 0$, (Y, γ_α) is Chang fuzzy compact. So, choose any open cover of (Y, γ_α) say $\{A_i : i \in \Delta\}$, where $\bigvee_{i \in \Delta} A_i = \boldsymbol{Y}$ and $A_i \in \gamma_\alpha$. By the claim above, f is fuzzy continuous so $\forall i \in \Delta, f^{-1}(A_i) \in \tau_\alpha$.
But (X, τ) is smooth compact, so (X, τ_α) is Chang compact. Since $\bigvee_{i \in \Delta} A_i = \boldsymbol{Y}$ and by Lemma 1.2.23 we get an open cover of (X, τ_α) as follows:

$$\boldsymbol{X} = f^{-1}(\boldsymbol{Y}) = f^{-1}(\bigvee_{i \in \Delta} A_i) = \bigvee_{i \in \Delta} f^{-1}(A_i).$$

Hence their exist a finite subcover of (X, τ_α) say $\{f^{-1}(A_i) : i = 1, 2, ..., n\}$ such that $\bigvee_{i=1}^{n} f^{-1}(A_i) = \boldsymbol{X}$. By using surjection of f, Lemma 1.2.23 and Theorem 1.2.25 we get:

$$\boldsymbol{Y} = f(\boldsymbol{X}) = f(\bigvee_{i=1}^{n} f^{-1}(A_i)) = f(f^{-1}(\bigvee_{i=1}^{n}(A_i))) \subseteq \bigvee_{i=1}^{n}(A_i) \subseteq \boldsymbol{Y}.$$

So, $\bigvee_{i=1}^{n}(A_i) = \boldsymbol{Y}$ hence we have a finite cover of (Y, γ_α), so (Y, γ_α) is Chang compact. But α is arbitrary, so (Y, γ) is smooth compact. □

In this chapter, we state the proof of Proposition 4.1.4, Proposition 4.1.9, Proposition 4.2.3, Proposition 4.2.7, Proposition 4.2.9, Proposition 4.2.10, Proposition 4.2.11.

Chapter 5

Conclusion

In this thesis, we study a new topic of topology, called fuzzy topology which was defined by many author as Chang, Lowen, Sostak and others. We note that Chang use similar concepts and definitions in ordinary topology with relapsing the ordinary sets by fuzzy sets. Lowen's definitions depend on the good extension property (i.e the fuzzy topological space has a property P if and only if the ordinary topology has the same property). Therefore, in 1985, Sostak fuzzy topology was an extension of both ordinary topology and Changs fuzzy topology, in the sense that not only the elements were fuzzifed, but also the topology. We give many mapping to convert one of them to another

We study the definitions of points and singletons and giving reasons to depend on some definition and ignore others. The same we do when we study continuity, compactness and separation axioms. We depend on Chang continuity since it has the good extension property, and even the research witch depend on Lowen fuzzy topology use the Chang continuity, also If any function is Chang continuous then it is Lowen continuous.

There are several definitions of separation axioms. For an example we give some definitions (not all) of FT_1-property, and comparing between them. Then we redefine them to get the definitions of FT_1, strong FT_1 and week FT_1. We choose the separation axioms depending on the property that if the fuzzy space is FT_i, then it is FT_{i-1} space, for $i = 1, 2, 3, 4$), beside the productive, hereditary and good extension property.

By the same way, we give seven different definition of compactness, we concentrate on Chang, Lowen and weekly compactness. giving reasons to ignore Chang definition. The Tychonoff theorem was proofed with complex and long way, we give simple method to proof it.

God well our next study will have the following steps:

1. generalizing some concepts of the ordinary compactness to fuzzy compactness, and using the several definitions of fuzzy compactness in our study.

2. applied of our study(of fuzzy compactness) in Image Processing.

3. Introducing Alexandrof fuzzy sets.

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