The Islamic University of Gaza
Deanery of Higher Studies
Fuculty of Science
Departemnt of Mathematics

On Nonstandard Topology

M.Sc. Thesis

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Submitted in Partial Fulfillment of
the Requirements for M.Sc. Degree

October, 2011

Abstract

Nonstandard topology is a kind of topology constructed by means of nonstandard analysis. In the literature, most books and research articles talk about nonstandard analysis. They rarely talk about nonstandard topology. Indeed, it is hard to find a single book or article that gives a comprehensive study of nonstandard topology.

In this thesis, we did our utmost efforts to survey and collect most of the information that have been scattered in the literature and that deal with nonstandard topology. We present to the reader all what he wants to know about the subject in a simple and correlative way that mimics the presentation of standard topology by any elementary textbook. Thus this thesis can be considered as an introductory textbook on nonstandard topology that will be very helpful and useful for the researchers who will be interested in developing topology in the sense of the nonstandard methods.

Acknowledgements

*After Almighty Allah, I am grateful to **my parents** for their subsidization, patience, and love. Without them this work would never have come into existence. I am heartily thankful to my supervisor, Prof. **Eissa D. Habil**, whose encouragement, guidance and support from the initial to the final level enabled me to develop an understanding of the subject.*

*It is pleasure to express my great thanks to my best teacher Dr. **As'ad Y. As'ad** for his invaluable guidance for two years.*

*I would like to give my special thanks to **my wife** whose patience enabled me to complete this work.*

Lastly, I offer my regards and blessings to all of those who supported me in any respect during the completion of this thesis.

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Introduction

For over three hundred years, a basic question about calculus remained unanswered. Do the infinitesimals as conceptional understood by Leibniz and Newton, exist as formal mathematical objects? [When Leibniz and Newton invented calculus, they made use of infinitesimals. They argued that the theory of infinitesimals implies the introduction of ideal numbers which might be infinitely small or infinitely large compared with the real numbers but which were to possess the same properties as the latter.] The question was answered affirmatively by Abraham Robinson [22] and the subject termed "Nonstandard Analysis" was introduced to the scientific world.

According to Machover and Herschfeld [19], the aim of Robinson's theory can perhaps be explained best by discussing an example from topology: A mapping f of a topological space X into a topological space Y is continuous at p if $f(x)$ is near $f(p)$ provided x is near p, and this characterization can easily be turned into a precise definition by using the notion of open set; the familiar rigorous definition is then "for every open set V containing $f(p)$ there is an open set U containing p such that $f(U) \subseteq V$". In this rigorous version the notion of a point x being "near" a given point p has disappeared. In rigorous discussions one does not refer to a point x as being "near" p, but only as lying in a given neighbourhood of p. On the other hand, I think it is safe to use the notion "near" (e.g., when trying to find a proof or when we want to understand the meaning of topological concepts); we can say that p is surrounded by a tiny ball or cluster points which are "near" it.

Now a natural question to ask is: How this notion of "nearness" can be made precise? In particular, for the real line (with ordinary topology) this would mean giving a precise meaning to a number being "near" zero; i.e., "infinitesimal". This particular case pinpoints the difficulty of the problem. On the other hand, one cannot treat any of the already existing non-zero points of the real line as infinitesimals without getting an immediate contradiction. Now if one try to add the infinitesimals as new “ideal” elements of the real line, one would spoil its nice algebro-topological properties. (As is well known, the real line cannot be imbedded in a larger continuously ordered field.)

Robinson’s theory solves this problem completely, by showing, e.g., how a topological space X can be imbedded in a “topological space” ${}^*X$ such that:

(i) For any p in X the set $\{x : x \in X$ and x is “near” $p\}$ can be defined rigorously and has all desirable properties.

(ii) For any mathematical property of X, ${}^*X$ has the “same” property.

The reason for surrounding “topological space” and “same” by quotes is that ${}^*X$ does not really have the same properties as X, but only formally so. More precisely, given any mathematical property of X, one writes down a sentence (in a language specified in advance) expressing the fact that this property holds for X. Then one re-interprets this sentence (also in a way specified in advance) and under this new interpretation the sentence claims ${}^*X$ has a certain property; moreover, ${}^*X$ actually does have that property. Thus to every property of X there corresponds a property of ${}^*X$ which is expressed by the same formula. It follows that formal reasoning and calculation can be performed for ${}^*X$ in exactly the same way as for X. It turns out that one can prove theorems about X by first “going cut” to ${}^*X$ and later “coming back” to X (in much the same way as one can prove a theorem about a field by first considering an extension field). This is the essence of nonstandard analysis.

Haddad [5] wrote: Nonstandard methods can give a special insight in topological matters as they are mainly a new way to look at old things. But, as already noted by Fenstad [3] and Luxemburg [16], in the case of enlargements (or nonstandard extensions), ${}^*E$ bears much resemblance to the Stone space $\gamma(E)$ of ultrafilters on E. The following relation holds in nonstandard models: ${}^*(E \times F) = {}^*E \times {}^*F$, whereas, if E and F are infinite, the space $\gamma(E \times F)$ is not even homeomorphic to the product $\gamma(E) \times \gamma(F)$. From there stems, in Haddad’s opinion, the main technical superiority of nonstandard methods over standard methods in topology.

A main tool in nonstandard topology is the notion of enlargement (or nonstandard extension). An enlargement is a certain kind of nonstandard model satisfying a sort of saturation property. Now, it is known (see [16]) that “saturation” is a property akin to compactness. The fact that compactness is an essential feature of nonstandard methods in topology should be emphasized. It will be seen that it pervades the whole subject.

The notion of an enlargement has been defined in several ways differing by technical details, the main idea being essentially the same, of course. The notion was introduced by Robinson [22] using a type-theoretical version of higher-order logic. Machover and Herschfeld [19] dispense with the theory of types and use an instance of a first-order language with equality. Robinson and Zakon [23] describe a purely set-theoretical approach to the subject. Luxemburg [16] gives a “simplified” version of the theory developed by Robinson using again higher-order structures and higher-order languages. Edward Nelson [20] provides internal set theory (IST) which is an axiomatic basis for a portion of the nonstandard analysis introduced by Abraham Robinson. Instead of adding new elements to the real numbers, the axioms introduce a new term, “standard”, which can be used to make discriminations not possible under the conventional axioms for sets. Most recently, Todorov and Salbany [24] describe an axiomatic approach to the subject using the so-called superstructure. We will use this last version for our purposes.

In this thesis, we will concentrate on the main definitions and applications of topology using the nonstandard methods. Among other things, we shall concentrate on presenting nonstandard proofs of some well-known theorems and we shall compare these proofs with the standard (conventional) ones. The thesis is organized in four chapters.

In chapter 1, we give a brief summary of the main ideas of nonstandard analysis to help the reader understand the nonstandard methods in topology.

In chapter 2, we present the most important definition in nonstandard topology; namely monad, and we talk about its main properties. Later on, we make declaration of the main topological definitions, subspaces, continuity and product topology in the nonstandard context.

In chapter 3, we study compactness and separation axioms of topological spaces in terms of monads. Also we present the most famous theorems concerning compactness and separation axioms where their proofs utilize the nonstandard theory.

Finally, chapter 4 can be considered as a specialization of our general study in the previous chapters, where we give a specific example of constructing a nonstandard model ${}^*\mathbb{R}$ of the reals $\mathbb{R}$, to help the reader understand the idea of the whole subject of this thesis.

Chapter 1

Nonstandard Model

In this chapter, we give a brief summary of the main ideas of nonstandard analysis to help the reader understand the nonstandard methods in topology. The axiomatic approach, presented here, is, in our view, the best way to apply the nonstandard methods in other fields of mathematics (specially in topology) and science. Section one contains preliminaries. Section two includes preparation of the standard theory. Section three talks about axioms of nonstandard analysis. Section four proves the existence of the nonstandard model. In section five we give some basic properties of the nonstandard model.

1.1 Preliminaries

We give here some basic information which will be used in the remainder of the thesis.

Definition 1.1.1 (Filter [6]).

A nonempty $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a (proper) *filter* on X if and only if:

(i) for each A, $B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$ [Intersection Property],

(ii) if $A \subseteq B \subseteq X$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$ [Superset Property], and

(iii) $\emptyset \notin \mathcal{F}$.

Definition 1.1.2 (Principal Filter [6]).

Let $\emptyset \neq A \subseteq X$, and let $[A]\uparrow$ be the set of all subsets of X that contain A, or, more formally, $[A]\uparrow = \{x : (x \subseteq X) \wedge (A \subseteq x)\}$. Then $[A]\uparrow$ is a filter on X called the *principal filter* (generated by A).

Definition 1.1.3 (Ultrafilter [6]).

A filter $\mathcal{U}$ on X is called an *ultrafilter* iff whenever there's a filter $\mathcal{F}$ on X such that $\mathcal{U} \subseteq \mathcal{F}$, then $\mathcal{U} = \mathcal{F}$.

Theorem 1.1.4 (Ultrafilter Theorems [6]).

(i) Let $\mathcal{U}$ be an ultrafilter on X. If $A \cup B \in \mathcal{U}$, then $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

(ii) Let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is an ultrafilter iff for each $A \subseteq X$, either $A \in \mathcal{F}$ or $(X - A) \in \mathcal{F}$, not both.

(iii) Let $\mathcal{F}$ be a filter on X. Then there exists an ultrafilter $\mathcal{U}$ on X such that $\mathcal{F} \subseteq \mathcal{U}$.

Definition 1.1.5 (Free Ultrafilter [6]).

Nonprincipal ultrafilter is called *free ultrafilter*.

Example 1.1.6 (Free Ultrafilter on $\mathbb{N}$ [6]).

A free ultrafilter on $\mathbb{N}$ is a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, which is a collection of subsets of $\mathbb{N}$ closed under intersection and supersets, and for any set $A \subseteq \mathbb{N}$ we have either $A \in \mathcal{U}$ or $\mathbb{N} - A \in \mathcal{U}$, and contains no finite sets.

1.2 Preparation of The Standard Theory

In order to proceed to analysis, which is more general than calculus, we will need to consider mathematical systems which contain entities corresponding to sets of sets, sets of functions, and so on. Beginning with a basic set S, we can construct a superstructure $V(S)$ which contains all of the entities normally encountered in the mathematics of S by successively taking subsets.

Definition 1.2.1 (Superstructure).

Let S be an infinite set. The *superstructure* $V(S)$ on S is the union

$$V(S) = \bigcup_{k=0}^{\infty} V_k(S),$$

where the $V_k(S)$ are defined inductively by:

$$V_0(S) = S \text{ and } V_{k+1}(S) = V_k(S) \cup \mathcal{P}(V_k(S)).$$

We shall refer to the elements of $V(S)$ as *entities*, they are either *individuals* if belong to S, or *sets* if belong to $V(S) - S$.
If $A \in V(S)$, then we define the *type* $t(A)$ of A by

$$t(A) := \min\{k \in \mathbb{N}_0 : A \in V_k(S)\},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Notice that

$$S = V_0(S) \subseteq V_1(S) \subseteq V_2(S) \subseteq \cdots, \text{ and } S = V_0(S) \in V_1(S) \in V_2(S) \in \cdots.$$

Hence, it follows that $V_k(S) \subseteq V(S)$ and $V_k(S) \in V(S)\ \forall\, k \in \mathbb{N}_0$.

We now introduce the most distinguished property of the superstructure.

Lemma 1.2.2 (Transitivity [24])**.**

(i) Each $V_k(S)$ is transitive in $V(S)$ in the sense that $A \in V_k(S)$ implies either $A \in S$ or $A \subseteq V_k(S)$; i.e., $(\forall\, A \in V_k(S))\ [A \in S \vee (A \subseteq V_k(S))]$.

(ii) The superstructure $V(S)$ is transitive (in itself) in the sense that $A \in V(S)$ implies either $A \in S$ or $A \subseteq V(S)$; i.e., $(\forall\, A \in V(S))\ [A \in S \vee (A \subseteq V(S))]$.

Proof. (i) We want to show by induction that the statement

$$(\forall\, A \in V_k(S))[A \in S \vee (A \subseteq V_k(S))]$$

is true for all k.
For $k = 0$, since $V_0(S) = S$ we have

$$(\forall\, A \in V_0(S) = S)[A \in S \vee (A \subseteq V_0(S))].$$

Now assume that the statement is true for k; that is, $V_k(S)$ is transitive in $V(S)$, and so

$$(\forall\, A \in V_k(S))[A \in S \vee (A \subseteq V_k(S))].$$

Then we want to show that the statement is true for $k+1$. By definition of $V_{k+1}(S)$ we have $A \in V_{k+1}(S)$ implies either $A \in V_k(S)$ or $A \subseteq V_k(S)$. On the other hand, $A \in V_k(S)$ implies either $A \in S$ or $A \subseteq V_k(S)$, by assumption. So that $A \in V_{k+1}(S)$ implies $A \in S$ or $A \subseteq V_k(S) \subseteq V_{k+1}(S)$, therefore $V_{k+1}(S)$ is transitive. Hence the above statement is true for all k.
(ii) To show that $V(S)$ is transitive, let $A \in V(S)$. Then $A \in V_k(S)$ for some k, thus by transitivity of $V_k(S)$, we have either $A \in S$ or $A \subseteq V_k(S) \subseteq V(S)$. Hence $V(S)$ is transitive. □

Now we introduce a suitable language for superstructures and later on we show how to interpret sentences in this language.

Definition 1.2.3 (The language $\mathcal{L}$(V(S))).
We can make this language using:

(i) The set $\mathcal{L}$ of the bounded quantifier formulas (b.q.f) consists of all formulas of the type $\Phi(x_1, \ldots, x_n)$ that can be made by:

   (a) The symbols: $=$, $\in$, $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$, $\Rightarrow$, $\Leftrightarrow$, (), [ ].
   and / or

   (b) Countably many variables: x, y, x_i, A_i, $\ldots$ etc.
   and / or

   (c) Bounded quantifiers of the type $(\forall\, x \in x_i)$ or $(\exists\, y \in x_j)$, $i,\ j = 1, 2, \ldots, n$. The variables x and y are called bounded and those which are not bounded are called free. The variables $x_1, \ldots, x_n$ in $\Phi(x_1, \ldots, x_n)$ are exactly the free variables in it.

(ii) Let S be an infinite set and let $V(S)$ be a superstructure on S. The language $\mathcal{L}(V(S))$ consists of all statements of the form $\Phi(A_1, \ldots, A_n)$ for some (b.q.f) $\Phi(x_1, \ldots, x_n) \in \mathcal{L}$ and some $A_1, \ldots, A_n \in V(S)$. The "points" $A_1, \ldots, A_n \in \Phi(A_1, \ldots, A_n)$ are called constants of $\Phi(A_1, \ldots, A_n)$.

The statements in $\mathcal{L}(V(S))$ can be true or false.

Example 1.2.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function and let $x_0 \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^+$. For the set of individuals we choose $S = \mathbb{R}$. Then:

$$\Phi(\epsilon, x_0, f(x_0), \mathbb{R}^+, \mathbb{R}, f, <, |.|, -) =$$

$$= (\exists\, \delta \in \mathbb{R}^+)(\forall\, x \in \mathbb{R})(|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$$

is a bounded quantifier formula in $\mathcal{L}(V(\mathbb{R}))$, with constants: ϵ, x_0, $f(x_0)$, $\mathbb{R}^+$, $\mathbb{R}$, f, $<$, $|.|$, " $-$ " perceived as elements of $V(\mathbb{R})$ (where $<$, $|.|$ and " $-$ " are the order relation, absolute value and subtraction in $\mathbb{R}$, respectively). The above statement might be true or false depending on the choice of ϵ, x_0 and f.

1.3 Axioms of Nonstandard Model

Definition 1.3.1 ($^*-$Function)**.** Let S, $^*S$ be two infinite sets and $V(S)$, $V(^*S)$ be their superstructures. The map $^* : V(S) \to V(^*S)$ is called $^*-$*function*. Moreover, $B \in V(^*S)$ is called *internal* if $B \in {}^*A$ for some $A \in V(S)$, and $B \in V(^*S)$ is called *standard* if $B = {}^*A$ for some $A \in V(S)$.

We present nonstandard model by three axioms: the extension, transfer and saturation principles in the framework of the superstructure of a given infinite set.

Definition 1.3.2 (Nonstandard Model)**.**
Let S be an infinite set and let $V(S)$ be its superstructure. The superstructure $V(^*S)$ of a given set $^*S$ together with the $^*-$function is called a *nonstandard model* of S if they satisfy the following three axioms:

Axiom 1 (Extension Principle): $^*s = s \ \forall\, s \in S$ or, equivalently, $S \subseteq {}^*S$.

Axiom 2 (Transfer Principle): A bounded quantifier formula (b.q.f) $\Phi(A_1, \ldots, A_n)$ is true in $\mathcal{L}(V(S))$ iff its nonstandard counterpart $\Phi(^*A_1, \ldots, {}^*A_n)$ is true in $\mathcal{L}(V(^*S))$, where $\Phi(^*A_1, \ldots, {}^*A_n)$ is obtained from $\Phi(A_1, \ldots, A_n)$ by replacing all constants $A_1, \ldots,$ A_n by their $^*-$images $^*A_1, \ldots, {}^*A_n$, respectively.

Axiom 3 (Saturation Principle): Let κ be an infinite cardinal number. Then $V(^*S)$ is $\kappa-$saturated in the sense that

$$\bigcap_{\gamma \in \Gamma} A_\gamma \neq \emptyset,$$

for any family of internal sets $\{A_\gamma\}_{\gamma \in \Gamma}$ in $V(^*S)$ with the finite intersection property and index set Γ with card $\Gamma \leq \kappa$.

Example 1.3.3. Let $S = \mathbb{R}$ and Φ be the formula in $V(\mathbb{R})$ given in Example (1.2.4). Then its nonstandard counterpart in $\mathcal{L}(V(^*\mathbb{R}))$ is given by:

$$\Phi(\epsilon, x_0, f(x_0), {}^*\mathbb{R}^+, {}^*\mathbb{R}, {}^*f, <, |.|, -) =$$

$$= (\exists\, \delta \in {}^*\mathbb{R}^+)(\forall\, x \in {}^*\mathbb{R})(|x - x_0| < \delta) \Rightarrow |{}^*f(x) - f(x_0)| < \epsilon)$$

where the $^*-$images $^*\mathbb{R}$ and $^*\mathbb{R}^+$ (of $\mathbb{R}$ and $\mathbb{R}^+$, respectively) are the sets of the nonstandard real numbers and positive nonstandard real numbers, respectively, the $^*-$image $^*f$ of f is called the nonstandard extension of f, the asterisks in front of the standard reals are skipped since $\epsilon = {}^*\epsilon$, $x_0 = {}^*x_0$ and $f(x_0) = {}^*f(x_0)$, by the extension principle and, in addition, the asterisks in front of $^*<$, $^*|.|$, $^*-$, are also skipped, although these symbols now mean the order relation, absolute value and substraction in $^*\mathbb{R}$, respectively.

Definition 1.3.4 (Classification)**.**
Let S be an infinite set and let $V(S)$ be its superstructure. Then,

(1) If $A \subseteq V(S)$, then the set

$$^\sigma A = \{^*a : a \in A\}$$

is called the *standard copy* of A. In particular

$$^\sigma V(S) = \{^*A : A \in V(S)\}$$

is the set of all *standard entities* in $V(^*S)$.

(2) The set of all *internal entities* is denoted by $V_{int}(^*S)$; i.e.,

$$V_{int}(^*S) = \{A \in V(^*S) : A \in {}^*B \text{ for some } B \in V(S)\}.$$

(3) The entities in $V(^*S) - V_{int}(^*S)$ are called *external*.

Note: The nonstandard individuals in $^*S$ are internal entities. Moreover, if $s \in {}^*S$, then s is standard iff $s \in S$.

1.4 Existence of Nonstandard Model

The content of this section can be viewed either as a proof of the consistency of Axioms 1−3 of Nonstandard Model, presented in Section (1.3), or, alternatively, as an independent constructive approach to nonstandard analysis.

Theorem 1.4.1 (Consistency of nonstandard model [24])**.**
For any infinite set S and any infinite cardinality κ there exists a $\kappa-$saturated nonstdandard model $V(^*S)$ of S.

Proof. A sketch of the proof is presented in (A) and (B) bellow.
(A) Existence of $\aleph_0-$Saturated nonstandard extensions:
This part of our exposition can be viewed either as a proof of the consistency theorem above in the particular case $\kappa = \aleph_0$, where $\aleph_0 = \text{card } \mathbb{N}$, or as an independent "sequential

approach" to nonstandard analysis.

(i) Let $p : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ be a finitely additive measure such that $p(A) = 0$ for all finite $A \subseteq \mathbb{N}$ and $p(\mathbb{N}) = 1$. To see that there exist measures with these properties, take a free ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ on $\mathbb{N}$ (here the Axiom of Choice is involved) and define $p(A) = 0$ for $A \notin \mathcal{U}$ and $p(A) = 1$ for $A \in \mathcal{U}$.

We shall keep p fixed in what follows.

(ii) Let $S^{\mathbb{N}}$ be the set of all sequences in S. Define an equivalence relation $\sim$ in $S^{\mathbb{N}}$ by: $\{a_n\} = \{b_n\}$ if $a_n = b_n$ a.e., where $\{a_n\}$, $\{b_n\} \in S^{\mathbb{N}}$ and "a.e." stands for "almost everywhere"; i.e., if $p(\{n : \ a_n \neq b_n\}) = 0$. It can be easily checked that $\sim$ is an equivalence relation. Then the factor space ${}^*S = S^{\mathbb{N}}/\sim$ defines a set of nonstandard individuals. (Notice that ${}^*S$ depends on the choice of the measure p.) We shall denote by $\langle a_n \rangle$ the equivalence class determined by the sequence $\{a_n\}$. The inclusion $S \subseteq {}^*S$ is defined by $s \to \langle s, s, \dots \rangle$. We can now form the superstructure $V({}^*S)$ by using Definition (1.2.1) where S is replaced by ${}^*S$, and the latter is treated as a set of individuals (although it is, actually, a set of sets of sequences).

(iii) Let $V(S)^{\mathbb{N}}$ be the set of all sequences in $V(S)$; i.e., sequences of points in S, sequences of subsets of S, sequences of functions, sequences of mixture of points and functions, $\cdots$. A sequence $\{A_n\}$ in $V(S)^{\mathbb{N}}$ is called *tame* if there exists $m \in \mathbb{N}_0$ such that $A_n \in V_m(S)$ $\forall n \in \mathbb{N}$ (or, equivalently, for almost all $n \in \mathbb{N}$). If $\{A_n\}$ is a tame sequence in $V(S)^{\mathbb{N}}$, then its type $t(\{A_n\})$ is defined as the (unique) $k \in \mathbb{N}_0$ such that $t(A_n) = k$ a.e., where $t(A_n)$ is the type of A_n in $V(S)$ defined in Definition (1.2.1). To any tame sequence $\{A_n\}$ in $V(S)^{\mathbb{N}}$ we associate an element $\langle A_n \rangle$ in $V({}^*S)$ by induction on the type of $\{A_n\}$:

If $t(\{A_n\}) = 0$, then $\langle A_n \rangle$ is the element in ${}^*S$, defined in (ii). If $\langle B_n \rangle$ is already defined for all tame sequences $\{B_n\}$ in $V(S)^{\mathbb{N}}$ with $t(\{B_n\}) < k$ and $t(\{A_n\}) = k$, then:

$$\langle A_n \rangle = \{\langle B_n \rangle : \{B_n\} \in V(S)^{\mathbb{N}}\,;\ t(\{B_n\}) < k\,;\ B_n \in A_n \text{ a.e.}\}.$$

The element $A \in V({}^*S)$ is "internal" if it is of the form $A = \langle A_n \rangle$ for some tame sequence $\{A_n\}$ in $V(S)^{\mathbb{N}}$. The element of $V({}^*S)$ of the type ${}^*A = \langle A, A, \dots \rangle$, for some $A \in V(S)$, is "standard". Now we define the ${}^*-$mapping $A \to {}^*A$ from $V(S)$ into $V({}^*S)$ and the construction of the nonstandard model is complete. It can be checked (see [13]) that this model satisfies Axiom 1, Axiom 2 and Axiom 3 for $\kappa = \aleph_0$ treated now as theorems.

(B) Existence of $\kappa-$Saturated nonstandard extensions:

In the case of a general cardinal κ, a similar construction and proofs to those presented in (A) can be carried out by replacing $\mathbb{N}$ with an index set I of cardinality κ, and a $\{0,1\}-$ valued measure on $\mathcal{P}(I)$ which is $\kappa-$*good* in the sense explained in [13], where κ^+ is the successor of κ $[\kappa^+ = |\inf\{\lambda \in ON : \kappa < \lambda\}|$ where ON is the class of ordinals.] Notice that every measure on $\mathcal{P}(\mathbb{N})$ given by a nonprinciple ultrafilter on $\mathbb{N}$ is $\aleph_1-$good, so this condition to be $\kappa-$good is not needed explicitly in the case $\kappa = \aleph_0$. □

1.5 Basic Properties of The Nonstandard Model

Let S be an infinite set and $V(^*S)$ be a nonstandard model of S. We shall study some very basic properties of $V(^*S)$ with focus on the standard and internal entities.

Theorem 1.5.1. [25] Let $^* : V(S) - S \to V(^*S) - {}^*S$. If $A, B \in V(S) - S$, then

$$^*\mathcal{P}(A) = \{x \subseteq {}^*A : x \in V_{int}(^*S)\},$$

$$^*(B^A) = \{f : {}^*A \to {}^*B, f \in V_{int}(^*S)\},$$

where $\mathcal{P}(A)$ denote the power set of A, and B^A denote the set of all functions $f : A \to B$.

Proof. Since $C := \mathcal{P}(A)$ and $D := B^A$ belong to $V(S)$, we find some $k \in \mathbb{N}_0$ such that $C, D \subseteq V_k(S)$ (recall that $V_k(S)$ is transitive), and so $^*C$, $^*D \subseteq {}^*V_k(S)$. Now

$$\forall\, x \in V_k(S)(x \in C \Leftrightarrow x \subseteq A)$$

and

$$\forall\, x \in V_k(S)(x \in D \Leftrightarrow (x : A \to B))$$

are true in $\mathcal{L}(V(S))$. Then, by transfer principle, we have

$$\forall\, x \in {}^*V_k(S)(x \in {}^*C \Leftrightarrow x \subseteq {}^*A)$$

and

$$\forall\, x \in {}^*V_k(S)(x \in {}^*D \Leftrightarrow (x : {}^*A \to {}^*B))$$

are true in $\mathcal{L}(V(^*S))$. In view of $^*C$, $^*D \in {}^*V_k(S)$, this implies the statements. □

Theorem 1.5.2 (Internal Entities and Transitivity [24, 25])**.**

(i) $V_{int}(^*S) = \bigcup_{k\in\mathbb{N}_0} {}^*V_k(S)$.

(ii) $^*V_k(S) = \{x \in V_k(^*S) : x \in V_{int}(^*S)\}$.

(iii) Each ${}^*V_k(S)$ is transitive in $V({}^*S)$ in the sense that $A \in {}^*V_k(S)$ implies either $A \in {}^*S$ or $A \subseteq {}^*V_k(S)$.

(iv) The whole set $V_{int}({}^*S)$ is transitive in $V({}^*S)$ in the sense that $A \in V_{int}({}^*S)$ implies either $A \in {}^*S$ or $A \subseteq V_{int}({}^*S)$.

Proof. (i) First, assume that $A \in V_{int}({}^*S)$. Then $A \in {}^*B$ for some $B \in V(S)$. That is, $B \in V_k(S)$ for some $k \in \mathbb{N}_0$, which implies that either $B \in S$ or $B \subseteq V_k(S)$, by transitivity of $V_k(S)$. It follows that either ${}^*B \in {}^*S = {}^*V_0(S)$ or ${}^*B \subseteq {}^*V_k(S)$, by transfer principle, therefore $A \in {}^*B \in {}^*V_0(S)$ or $A \in {}^*B \subseteq {}^*V_k(S)$ and hence $A \in {}^*V_k(S)$.

On the other hand assume that $A \in {}^*V_k(S)$ for some $k \in \mathbb{N}_0$ which implies that $A \in V({}^*S)$ since ${}^*V_k(S) \subseteq V({}^*S)$. Now since $V_k(S) \in V(S)$ and $A \in V({}^*S)$, by definition of $V_{int}({}^*S)$, we have $A \in V_{int}({}^*S)$.

(ii) We want to show by induction that the statement

$${}^*V_k(S) = \{x \in V_k({}^*S) : x \in V_{int}({}^*S)\}$$

is true for all k. For $k = 0$, since ${}^*V_0(S) = {}^*S = V_0({}^*S)$, the statement is true for $k = 0$. Now assume that the statement is true for k; that is

$${}^*V_k(S) = \{x \in V_k({}^*S) : x \in V_{int}({}^*S)\}.$$

Then we want to show that the statement is true for $k+1$; i.e.,

$${}^*V_{k+1}(S) = \{x \in V_{k+1}({}^*S) : x \in V_{int}({}^*S)\}.$$

By definition of $V_{k+1}(S)$, we have $V_{k+1}(S) = V_k(S) \cup \mathcal{P}(V_k(S))$, then, by transfer principle, we have

$${}^*V_{k+1}(S) = {}^*V_k(S) \cup {}^*\mathcal{P}(V_k(S)).$$

So that

$$\begin{aligned}
{}^*V_{k+1}(S) &= {}^*V_k(S) \cup {}^*\mathcal{P}(V_k(S)) \\
&= \{x \in V_k({}^*S) : x \in V_{int}({}^*S)\} \cup \{x \in \mathcal{P}(V_k({}^*S)) : x \in V_{int}({}^*S)\} \\
&= \{x \in V_k({}^*S) \cup \mathcal{P}(V_k({}^*S)) : x \in V_{int}({}^*S)\} \\
&= \{x \in V_{k+1}({}^*S) : x \in V_{int}({}^*S)\},
\end{aligned}$$

where the second equality follows from assumption and Theorem (1.5.1). Hence the above statement is true for all k.

(iii) To show the transitivity of ${}^*V_k(S)$, observe that

$$(\forall\, A \in V_k(S))[A \in S \vee (A \subseteq V_k(S))] \text{ is true in } \mathcal{L}(V(S)), \text{ by transitivity of } V_k(S).$$

Then, by transfer principle, we have

$$(\forall\, A \in\, {}^*V_k(S))[A \in\, {}^*S \vee (A \subseteq\, {}^*V_k(S))] \text{ is true in } \mathcal{L}(V({}^*S)).$$

Hence ${}^*V_k(S)$ is transitive in $V({}^*S)$.

(iv) To show the transitivity of $V_{int}({}^*S)$, let $A \in V_{int}({}^*S)$. Then, by (i), $A \in {}^*V_k(S)$ for some k, and by transitivity of ${}^*V_k(S)$ in (ii) we have $A \in {}^*S$ or $A \subseteq {}^*V_k(S) \subseteq V_{int}({}^*S)$. Hence $V_{int}({}^*S)$ is transitive. □

Theorem 1.5.3 (Boolean Properties [24])**.**

The extension mapping $A \to {}^*A$ from $V(S)$ into $V({}^*S)$ is injective and its restriction ${}^* : V(S) - S \to V({}^*S) - {}^*S$ preserves the Boolean operations; i.e., if A, $B \in V(S) - S$, then

(i) ${}^*(A \cup B) = {}^*A \cup {}^*B$.

(ii) ${}^*(A \cap B) = {}^*A \cap {}^*B$.

(iii) ${}^*(A - B) = {}^*A - {}^*B$.

Proof. First we want to show that the extension mapping is injective. Assume that ${}^*A = {}^*B$ for some A, $B \in V(S)$. We want to show that $A = B$. Now the formula

$$\Phi({}^*A, {}^*B) = [{}^*A = {}^*B]$$

is true in $\mathcal{L}(V({}^*S))$. So, by transfer principle, we have

$$\Phi(A, B) = [A = B]$$

is true in $\mathcal{L}(V(S))$, which means that $A = B$.

(i) We want to show that ${}^*(A \cup B) = {}^*A \cup {}^*B$. Suppose that $A \cup B = C$ for some A, B, $C \in V(S) - S$. We have to show that ${}^*A \cup {}^*B = {}^*C$. Now by definition of $V(S)$ we have A, B, $C \in V_k(S)$ for some $k \in \mathbb{N}$. And by transitivity of $V_k(S)$ we have A, B, $C \subseteq V_k(S)$. Now the equality $A \cup B = C$ can be formalized by the formula:

$$\Phi(A, B, C) = [(\forall\, x \in V_k(S))((x \in A) \vee (x \in B) \Rightarrow (x \in C))]$$
$$\wedge\, [(\forall\, z \in V_k(S))((z \in C) \Rightarrow (z \in A) \vee (z \in B))].$$

which is true in $\mathcal{L}(V(S))$. It follows from transfer principle that

$$\Phi(^*A, {}^*B, {}^*C) = [(\forall\, x \in {}^*V_k(S))((x \in {}^*A) \vee (x \in {}^*B) \Rightarrow (x \in {}^*C))]$$
$$\wedge\, [(\forall\, z \in {}^*V_k(S))((z \in {}^*C) \Rightarrow (z \in {}^*A) \vee (z \in {}^*B))].$$

is true in $\mathcal{L}(V(^*S))$. Hence $^*A \cup {}^*B = {}^*C$.

The remaining Boolean properties are checked similarly. □

Lemma 1.5.4 (Definable Sets [24])**.**

Let $\Phi(x, x_1, x_2, x_3, \ldots, x_n) \in \mathcal{L}$ be a (b.q.f) and $B, A_1, \ldots, A_n \in V(S)$. Then $^*\{x \in B : \Phi(x, A_1, \ldots, A_n)$ is true in $\mathcal{L}(V(S))\} = \{x \in {}^*B : \Phi(x, {}^*A_1, \ldots, {}^*A_n)$ is true in $\mathcal{L}(V(^*S))\}$.

Proof. Denote $A = \{x \in B : \Phi(x, A_1, \ldots, A_n)$ is true in $\mathcal{L}(V(S))\}$ and let $^*A$ be the nonstandard extension of A. We have to show that

$$\{x \in {}^*B : \Phi(x, {}^*A_1, \ldots, {}^*A_n) \text{ is true in } \mathcal{L}(V(^*S))\} = {}^*A.$$

Suppose, on the contrary, that $(\exists\, x \in {}^*A)(\Phi(x, {}^*A_1, \ldots, {}^*A_n))$ is false in $\mathcal{L}(V(^*S))$ $\vee\, (\exists\, x \in {}^*B - {}^*A)(\Phi(x, {}^*A_1, \ldots, {}^*A_n)$ is true in $\mathcal{L}(V_{int}(^*S))$. We have $^*B - {}^*A = {}^*(B - A)$, by the Boolean properties. As a result, the above formula becomes:

$(\exists\, x \in {}^*A)(\Phi(x, {}^*A_1, \ldots, {}^*A_n))$ is false in $\mathcal{L}(V(^*S))$ $\vee$ $(\exists\, x \in {}^*(B - A))(\Phi(x, {}^*A_1, \ldots, {}^*A_n)$ is true in $\mathcal{L}(V_{int}(^*S))$.

Using transfer principle we have

$(\exists\, x \in A)(\Phi(x, A_1, \ldots, A_n))$ is false in $\mathcal{L}(V(S))$ $\vee$ $(\exists\, x \in B - A)(\Phi(x, A_1, \ldots, A_n))$ is true in $\mathcal{L}(V(S))$. The latter contradicts the choice of A. □

Example 1.5.5 (Standard Intervals in $^*\mathbb{R}$)**.**

Let a, $b \in \mathbb{R}$, $a < b$. Let $S = \mathbb{R}$ and $V(^*\mathbb{R})$ be a nonstandard model of $\mathbb{R}$. We have

$$^*(a, b) = \{x \in {}^*\mathbb{R} : a < x < b\},$$
$$^*[a, b] = \{x \in {}^*\mathbb{R} : a \le x \le b\},$$
$$^*[a, b) = \{x \in {}^*\mathbb{R} : a \le x < b\},$$

by Lemma (1.5.4) (applied to $\Phi(x,a,b) = \{x \in \mathbb{R} : a < x < b\}$ for the first case and similarly for the others). Notice that the above subsets of ${}^*\mathbb{R}$ are intervals - open, closed and semi-open, respectively- in the order relation in ${}^*\mathbb{R}$.

Theorem 1.5.6 (Finite Sets [24])**.**

(i) If $A \in V(S) - S$ is a finite set, then ${}^*A = {}^\sigma A$. In particular, ${}^*\{a\} = \{{}^*a\}$ for any $a \in V(S)$.

(ii) If $A \subseteq S$ is a finite set, then ${}^*A = A$.

Proof. (i) We start with the case of a singleton. Let $a \in V(S)$. Then there exists $k \in \mathbb{N}$ such that $a \in V_k(S)$, and so, by transfer principle, we have ${}^*a \in {}^*V_k(S)$ for the same k. We observe that $\{a\}$ can be described as a definable set:

$$\{a\} = \{x \in V_k(S) : x = a \text{ in } \mathcal{L}(V(S))\}.$$

Applying Lemma (1.5.4) to $\Phi(x,\, a) = [x = a]$ implies that

$${}^*\{a\} = \{x \in {}^*V_k(S) : x = {}^*a \text{ in } \mathcal{L}(V({}^*S))\}.$$

The right hand side of the above formula is $\{{}^*a\}$, hence $\{{}^*a\} = {}^*\{a\}$.

In the case of arbitrary finite set A, the result follows from the Boolean properties of the extension mapping:

$${}^*A = {}^*(\bigcup_{a\in A}\{a\}) = \bigcup_{a\in A} {}^*\{a\} = \bigcup_{a\in A}\{{}^*a\} = {}^\sigma A.$$

(ii) Let $A \subseteq S$ be finite. By extension principle, we have

$${}^\sigma A = \bigcup_{a\in A}\{{}^*a\} = \bigcup_{a\in A}\{a\} = A.$$

Hence, by (i),

$${}^*A = {}^\sigma A = A$$

□

Theorem 1.5.7 (Cartesian Products [24])**.**

(i) The extension mapping * preserves the cartesian product; i.e., if $A,\, B \in V(S) - S$, then ${}^*(A \times B) = {}^*A \times {}^*B$. Consequently, the set of standard sets ${}^\sigma V(S) - S$ is closed under the cartesian product of finitly many sets.

(ii) The extnension mapping * preserves the ordered pairing of entities (individuals or sets); i.e., if $a,\, b \in V(S)$, then $^*(a,\, b) = (^*a,\, ^*b)$. Consequently, the set of standard sets $^\sigma V(S) - S$ is closed under the building of ordered $n-$tuples for $n \in \mathbb{N}$.

Proof. (i) Assume that $A \times B = C$ which can be formalized in $\mathcal{L}(V(S))$ by:

$$\begin{aligned}\Phi(A,\, B,\, C) = {} & [(\forall\, a \in A)(\forall\, b \in B)((a,b) \in C))] \\ & \wedge [(\forall\, c \in C)(\exists\, a \in A)(\exists\, b \in B)((a,b) = c)].\end{aligned}$$

By transfer principle, it follows that

$$\begin{aligned}\Phi(^*A,\, ^*B,\, ^*C) = {} & [(\forall\, a \in\, ^*A)(\forall\, b \in\, ^*B)((a,b) \in\, ^*C)] \\ & \wedge [(\forall\, c \in\, ^*C)(\exists\, a \in\, ^*A)(\exists\, b \in\, ^*B)((a,b) = c)]\end{aligned}$$

is true in $\mathcal{L}(V(^*S))$. Hence $^*A \times\, ^*B =\, ^*C$. The generalization for n many sets follows by induction.

(ii) Using Theorem (1.5.6), we have

$$\begin{aligned}^*(a,b) = {} & ^*\{\{a\},\{a,b\}\} \\ & = \{^*\{a\},\, ^*\{a,b\}\} \\ & = \{\{^*a\},\{^*a,^*b\}\} \\ & = (^*a,^*b).\end{aligned}$$

□

Theorem 1.5.8 (Nonstandard Extension [24])**.**
Let $A \in V(S) - S$ be a set in the superstructure, $^\sigma A$ be its standard image and $^*A$ be its nonstandard extension. Then:

(i) $^*A \cap\, ^\sigma V(S) =\, ^\sigma A$.

(ii) $^\sigma A \subseteq\, ^*A$.

(iii) $^\sigma A =\, ^*A$ iff A is a finite set.

Proof. (i) Suppose that $\alpha \in [^*A \cap\, ^\sigma V(S)]$. Then $\alpha \in\, ^\sigma V(S)$ means that $\alpha =\, ^*a$ for some $a \in V(S)$, at the same time $\alpha \in\, ^*A$ and so $^*a \in\, ^*A$; which is equivalent to $a \in A$, by transfer principle. So that $\alpha =\, ^*a$ for some $a \in A$, hence $\alpha \in\, ^\sigma A$.

On the other hand suppose that $\alpha \in {}^{\sigma}A$; i.e., $\alpha = {}^{*}a$ for some $a \in A$. Now $\alpha \in {}^{\sigma}A$ implies that $\alpha \in {}^{\sigma}V(S)$, since ${}^{\sigma}A \subseteq {}^{\sigma}V(S)$. And since, by transfer principle, $a \in A$ is equivalent to ${}^{*}a \in {}^{*}A$, we have $\alpha \in {}^{*}A$, thus $\alpha \in [{}^{*}A \cap {}^{\sigma}V(S)]$. Therefore ${}^{*}A \cap {}^{\sigma}V(S) = {}^{\sigma}A$.

(ii) From (i), since ${}^{\sigma}A = {}^{*}A \cap {}^{\sigma}V(S)$, we have ${}^{\sigma}A \subseteq {}^{*}A$.

(iii) By Theorem (1.5.6), if A is a finite set, then ${}^{*}A = A = {}^{\sigma}A$. Conversely, assume (by contraposition) that A is an infinite set. By (i) and (ii),we have ${}^{*}A \cap {}^{\sigma}V(S) = {}^{\sigma}A$ and ${}^{\sigma}A \subseteq {}^{*}A$. Consider first the case $A = \mathbb{N}$ which implies ${}^{*}\mathbb{N} \cap {}^{\sigma}V(S) = {}^{\sigma}\mathbb{N}$ and ${}^{\sigma}\mathbb{N} \subseteq {}^{*}\mathbb{N}$. We want to show that ${}^{*}\mathbb{N} - {}^{\sigma}\mathbb{N} \neq \emptyset$. Observe that if $n \in \mathbb{N}$, then the set ${}^{*}\mathbb{N} - \{{}^{*}n\}$ is internal (actually, standard), since

$${}^{*}\mathbb{N} - \{{}^{*}n\} = {}^{*}\mathbb{N} - {}^{*}\{n\} = {}^{*}(\mathbb{N} - \{n\}) \in {}^{\sigma}V(S) \subseteq V_{int}({}^{*}S).$$

The family of internal sets $\{{}^{*}\mathbb{N} - \{{}^{*}n\}\}_{n\in\mathbb{N}}$ has the finite intersection property, since ${}^{*}\mathbb{N}$ is an infinite set. It follows, by saturation principle, that its intersection is not empty,

$$\bigcap_{n\in\mathbb{N}}\{{}^{*}\mathbb{N} - \{{}^{*}n\}\} = {}^{*}\mathbb{N} -{}^{\sigma}\mathbb{N} \neq \emptyset.$$

We return to the general case of an infinite set A. Without loss of generality we may assume that $\mathbb{N} \subseteq A$, $\mathbb{N} \neq A$. The latter implies both ${}^{\sigma}\mathbb{N} \subseteq {}^{\sigma}A$, ${}^{\sigma}\mathbb{N} \neq {}^{\sigma}A$ and ${}^{*}\mathbb{N} \subseteq {}^{*}A$, ${}^{*}\mathbb{N} \neq {}^{*}A$. Suppose (for contradiction) that ${}^{\sigma}A = {}^{*}A$. By intersecting both sides by ${}^{*}\mathbb{N}$, we get ${}^{\sigma}A \cap {}^{*}\mathbb{N} = {}^{*}\mathbb{N}$. By (i), we have ${}^{*}\mathbb{N} \cap {}^{\sigma}A = {}^{\sigma}\mathbb{N}$. Hence ${}^{\sigma}\mathbb{N} = {}^{*}\mathbb{N}$, a contradiction. This contradiction shows that ${}^{\sigma}A \neq {}^{*}A$. □

Corollary 1.5.9 (Standard vs. Nonstandard Individuals [24]).
Let $A \subseteq S$. Then:

(i) ${}^{*}A \cap S = A$.

(ii) $A \subseteq {}^{*}A$.

(iii) $A = {}^{*}A$ iff A is a finite set.

Proof. Notice that $A = {}^{\sigma}A$ since $a = {}^{*}a \; \forall\, a \in A$, by the extension principle.

(i) First, let $\alpha \in {}^{*}A \cap S$. By Definition (1.3.4), we have that $S \in {}^{\sigma}V(S)$. So that $\alpha \in {}^{*}A \cap {}^{\sigma}V(S)$, and so, by Theorem (1.5.8) (i), we have $\alpha \in {}^{\sigma}A = A$.

On the other hand, let $\alpha \in A$. Then $\alpha \in S$ since $A \subseteq S$, and $\alpha \in {}^*A$ since $A \subseteq {}^*A$. Hence $\alpha \in {}^*A \cap S$. Therefore ${}^*A \cap S = A$.
(ii) and (iii) follow directly from Theorem (1.5.8) (ii) and (iii), respectively. □

In the above Theorem, if we take $A = S$, we have $S \subseteq {}^*S$, $S \neq {}^*S$, since S is an infinite set. The latter implies

$$V(S) \subseteq V({}^*S), V(S) \neq V({}^*S).$$

Example 1.5.10. Let us consider the important particular case $S = \mathbb{R}$. The nonstandard individuals are the nonstandard real numbers ${}^*\mathbb{R}$. It follows that ${}^*\mathbb{R}$ is a proper extension of $\mathbb{R}$, $\mathbb{R} \subseteq {}^*\mathbb{R}$, $\mathbb{R} \neq {}^*\mathbb{R}$, Since $\mathbb{R}$ is an infinite set.
Similarly, ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, ${}^*\mathbb{Q}$, etc., are proper extensions of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, respectively.

Theorem 1.5.11 (Nonstandard Functions [24])**.**
Let $f : A \to B$ be a function in $V(S)$; i.e., A, $B \in V(S)$. Let ${}^*f$ be the nonstandard extension of f. Then:

(i) ${}^*f$ is a function of the type ${}^*f : {}^*A \to {}^*B$.

(ii) ${}^*f$ is an extension of f in the sense that ${}^*f|_{\sigma A} = f$; i.e., ${}^*f({}^*a) = {}^*(f(a))$, $\forall\, a \in A$.

(iii) Let dom(f) and ran(f) be the domain and the range of f, respectively, and let dom$({}^*f)$ and ran$({}^*f)$ be the domain and the range of ${}^*f$, respectively. Then

$${}^*(\text{dom}(f)) = \text{dom}({}^*f) \text{ and } {}^*(\text{ran}(f)) = \text{ran}({}^*f).$$

Proof. (i) We can formalize the function $f : A \to B$ by:

$$(\forall\, z \in f)(\exists\, x \in A)(\exists\, y \in B)[z = (x,\, y)] \wedge$$
$$(\forall\, x \in A)(\exists\, y \in B)[(x,\, y) \in f] \wedge$$
$$(\forall\, x \in A)(\forall\, y \in B)[((x,\, y) \in f) \Leftrightarrow (y = f(x))]$$

which is true in $\mathcal{L}(V(S))$. The first line of the above formula simply says that f is a relation between A and B, the second line says that A is the domain of f, and the third line expresses the uniqueness of the value $y = f(x)$ for any x in A. By transfer principle,

$$(\forall\, z \in {}^*f)(\exists\, x \in {}^*A)(\exists\, y \in {}^*B)[z = (x,\, y)] \wedge$$
$$(\forall\, x \in {}^*A)(\exists\, y \in {}^*B)[(x,\, y) \in {}^*f] \wedge$$
$$(\forall\, x \in {}^*A)(\forall\, y \in {}^*B)[((x,\, y) \in {}^*f) \Leftrightarrow (y = {}^*f(x))]$$

is true in $\mathcal{L}(V(^*S))$. The above formula means that $^*f$ is a function of the type $^*f : {}^*A \to {}^*B$.

(ii) Suppose that $a \in A$ and $b \in B$. Using transfer principle, we have

$$\begin{aligned}[f(a) = b] &\Leftrightarrow [(a \in A) \wedge ((a,\, b) \in f)] \\ &\Leftrightarrow [(^*a \in {}^*A) \wedge ((^*a,\, ^*b) \in {}^*f)] \\ &\Leftrightarrow [^*f(^*a) = {}^*b],\end{aligned}$$

hence $^*(f(a)) = {}^*b = {}^*f(^*a)$.

(iii) Using (i) we have: $^*(\mathrm{dom}(f)) = {}^*A = \mathrm{dom}(^*f)$. Observe that $\mathrm{ran}(f)$ is described by

$$\mathrm{ran}(f) = \{y \in B : (\exists\, x \in \mathrm{dom}(f))[(x,\, y) \in f]\}.$$

Hence, it follows from Lemma (1.5.4) that

$$^*(\mathrm{ran}(f)) = \{y \in {}^*B : (\exists\, x \in {}^*\mathrm{dom}(f))[(x,\, y) \in {}^*f]\}.$$

Replacing $^*(\mathrm{dom}(f))$ by $\mathrm{dom}(^*f)$, we get that

$$^*(\mathrm{ran}(f)) = \{y \in {}^*B : (\exists\, x \in \mathrm{dom}(^*f))[(x,\, y) \in {}^*f]\} = \mathrm{ran}(^*f).$$

□

Corollary 1.5.12 (Functions on S [24])**.**

Let $f : A \to B$ be a function in the set of individuals S; i.e., $A,\, B \subseteq S$. Then $^*f$ is an extension of f in the usual sense; i.e., $^*f|_A = f$, or $^*f(a) = f(a),\, \forall\, a \in A$.

Proof. Since $A, B \subseteq S$, the extension principle implies that

$$^*a = a \text{ and } ^*(f(a)) = f(a)\, \forall\, a \in A.$$

Hence, by Theorem (1.5.11), we have

$$^*f(a) = {}^*f(^*a) = {}^*(f(a)) = f(a)\, \forall\, a \in A,$$

as desired. □

Theorem 1.5.13. [25] Let $A \in V(S) - S$, and, for $n \in \mathbb{N}$, let $A_n = \{x \in A : t(A) = n\}$; i.e., $x \in V_n(S) - V_{n-1}(S)$. Then

$$^*A_n = \{x \in {}^*A : x \in V_n(^*S) - V_{n-1}(^*S)\}.$$

Proof. Since $A_n = \{x \in A : t(A) = n\}$, by definition of $t(A)$, we have $A_n = \{x \in A : x \in V_n(S) - V_{n-1}(S)\}$, and so, by transfer principle, we have

$$^*A_n = \{x \in {^*A} : x \in {^*V_n(S)} - {^*V_{n-1}(S)}\}.$$

Since Theorem (1.5.2 part (ii)) implies that $^*V_n(S) - {^*V_{n-1}(S)}$ contains all internal elements of $V_n(^*S) - V_{n-1}(^*S)$, the statement follows. □

Theorem 1.5.14. [25] Let $A \in V(S) - S$, and let $A_0 = \{B \in V(S) - S : B \in A\}$. Then

(i) $^*(\bigcup A_0) = \bigcup\{B \in V(S) - S : B \in {^*A}\}$.

(ii) $^*(\bigcap A_0) = \bigcap\{B \in V(S) - S : B \in {^*A}\}$.

Proof. Since entities are the elements of at least type 1, Theorem (1.5.13) implies that $^*A_0 = \{B \in V(S) - S : B \in {^*A}\}$. Hence, without loss of generality we may assume that $A = A_0$.

(i) Put $U = \bigcup A$. Then

$$\forall x \in U : \exists y \in A : x \in y$$

is true in $\mathcal{L}(V(S))$, and so, by transfer principle, we have

$$\forall x \in {^*U} : \exists y \in {^*A} : x \in y$$

is true in $\mathcal{L}(V(^*S))$; i.e., $^*U \subseteq \bigcup {^*A}$.

On the other hand, since $\bigcup A = U$, we have

$$\forall x \in A : \forall y \in x : y \in U$$

is true in $\mathcal{L}(V(S))$, and so, by transfer principle, we have

$$\forall x \in {^*A} : \forall y \in x : y \in {^*U}$$

is true in $\mathcal{L}(V(^*S))$; i.e., $\bigcup {^*A} \subseteq {^*U}$. Hence we have the equality.

(ii) Put $D = \bigcap A$, and observe that

$$D = \{x \in U : \forall y \in A : x \in y\}$$

is true in $\mathcal{L}(V(S))$, and so, by transfer principle, we have

$$^*D = \{x \in {^*U} : \forall y \in {^*A} : x \in y\}$$

is true in $\mathcal{L}(V(^*S))$. Since $^*U = \bigcup {^*A}$, we find $^*D = \bigcap {^*A}$. □

Notation: If $\{X_i : i \in I\}$ is a family of sets, one would like to describe the $^*-$value of $\bigcup_{i\in I} X_i$. A natural conjecture is that this value is $\bigcup_{i\in {}^*I} {}^*X_i$. To make this more precise, we have to define what we mean by ${}^*X_i$ when $i \in {}^*I$:
Let $X_i, I, \mathcal{H} = \{X_i : i \in I\} \in V(S) - S$. Then we may define a bijection $f : I \to \mathcal{H}$ by $f(i) = X_i$. Then ${}^*f : {}^*I \to {}^*\mathcal{H}$. In a slight misuse of notation, we define ${}^*X_i = {}^*f(i)(i \in {}^*I)$. However, there is no real danger of such confusion, since for $i \in I$ we have ${}^*X_{{}^*i} = {}^*f({}^*i) = {}^*(f(i)) = {}^*(X_i)$.

Corollary 1.5.15. [25] With the above notation, we have

$${}^*(\bigcup_{i\in I} X_i) = \bigcup_{i\in {}^*I} {}^*X_i.$$

Proof. The functions $f : I \to \mathcal{H}$ and ${}^*f : {}^*I \to {}^*\mathcal{H}$ are both onto. Hence $\bigcup_{i\in I} X_i = \bigcup \mathcal{H}$ and $\bigcup_{i\in {}^*I} {}^*X_i = \bigcup {}^*\mathcal{H}$. Thus, the statement follows from Theorem (1.5.14). □

Definition 1.5.16. With the above notation, we define the cartesian product

$$\prod_{i\in I} X_i = \{f : I \to \bigcup_{i\in I} X_i : f(i) \in X_i \text{ for each } i \in I\}.$$

Theorem 1.5.17 (Cartesian Product [25])**.** We have

$${}^*(\prod_{i\in I} X_i) = \{x \in \prod_{i\in {}^*I} {}^*X_i : x \in V_{int}({}^*S)\}$$

Proof. Putting $U = \bigcup_{i\in I} X_i$ and $X = \prod_{i\in I} X_i$, we have

$$X = \{y \in U^I : \forall\, x \in I : y(x) \in f(x)\},$$

where f denotes the mapping $i \to X_i : I \to \bigcup_{i\in I} X_i = U$. Then, by transfer principle, we have

$${}^*X = \{y \in {}^*(U^I) : \forall\, x \in {}^*I : y(x) \in {}^*f(x)\}.$$

By Theorem (1.5.1), we find

$${}^*X = \{y : {}^*I \to {}^*U : \forall\, x \in {}^*I : y(x) \in {}^*f(x) \wedge y \in V_{int}({}^*S)\}.$$

By Corollary (1.5.15), we have ${}^*U = \bigcup_{i\in {}^*I} {}^*X_i$. Since ${}^*f(i) = {}^*X_i(i \in {}^*I)$, we have

$${}^*X = \{y : {}^*I \to \bigcup_{i\in {}^*I} {}^*X_i : y(i) \in {}^*X_i \;\forall\, i \in {}^*I \wedge y \in V_{int}({}^*S)\}.$$

This implies the statement. □

Chapter 2

Nonstandard Topology

In this chapter we present the most important definition in nonstandard topology; namely monad, of course we talk about its main properties that we will use in the remainder of the thesis. We will present topological main definitions, subspaces, continuity and product topology in the nonstandard context. Section one talks about monad and it's main properties. Section two gives a brief definition of nonstandard compactification. Section three present the main topological nonstandard definitions. Section four talks about subspaces. Section five talks about continuity. In section six we talk about product topology.

2.1 Monads

Definition 2.1.1 (Monad of a Point)**.**

Let (X, T) be a topological space and let ${}^*X$ be the nonstandard extension of X. Then:

(i) For any $x \in X$ define the *monad* $\mu(x)$ of x by

$$\mu(x) := \bigcap\{{}^*G : G \in T_x\},$$

where T_x is the system of all open neighborhoods (abbreviated: nhoods) of x in X.

(ii) For any $x \in {}^*X$ define the *monad* $\mu(x)$ of x by

$$\mu(x) := \bigcap\{{}^*G : x \in {}^*G,\, G \in T\}.$$

It should be noted that (ii) supersedes (i) above.

Definition 2.1.2 (Monad of a Set)**.**

Let (X, T) be a topological space and let ${}^*X$ be the nonstandard extension of X. Then:

(i) For any $A \subseteq X$ define the *monad* $\mu(A)$ of A by

$$\mu(A) := \bigcap\{{}^*G : A \subseteq G,\, G \in T\}.$$

(ii) For any $A \subseteq {}^*X$ define the *monad* $\mu(A)$ of A by

$$\mu(A) := \bigcap\{{}^*G : A \subseteq {}^*G,\, G \in T\}.$$

It should be noted that for any $A \subseteq X$ we have $\mu(A) = \mu({}^*A)$. To see this, using transfer principle, we have

$$\mu(A) = \bigcap\{{}^*G : A \subseteq G,\, G \in T\} = \bigcap\{{}^*G : {}^*A \subseteq {}^*G,\, G \in T\} = \mu({}^*A).$$

A similar note is valid for any $x \in X$.

Example 2.1.3. As an example of how a monad looks like, let $X = \{0, 1, 2\}$, $T = \{\emptyset, \{0\}, \{0, 1\}, \{0, 2\}, X\}$. Since X is finite, we have ${}^*X = X$ and ${}^*T = T$. Now $\mu(0) = \{0\}$, $\mu(1) = \{0, 1\}$, $\mu(2) = \{0, 2\}$.

Theorem 2.1.4. [8] If $\mathcal{N}_x$ is the set of all nhoods of x in X, then

$$\mu(x) = \bigcap\{{}^*G : G \in \mathcal{N}_x\}.$$

Proof. Since $T_x \subseteq \mathcal{N}_x$, we have

$$\bigcap\{{}^*G : G \in T_x\} \supseteq \bigcap\{{}^*G : G \in \mathcal{N}_x\}.$$

On the other hand, for each $U \in \mathcal{N}_x$ there is $V \in T_x$ such that $V \subseteq U$, and so, by transfer principle, we have ${}^*V \subseteq {}^*U$, therefore

$$\bigcap\{{}^*V : V \in T_x\} \subseteq \bigcap\{{}^*U : U \in \mathcal{N}_x\}.$$

Hence $\mu(x) = \bigcap\{{}^*G : G \in \mathcal{N}_x\}$. □

Theorem 2.1.5. [8] If β_x is a nhood base at x in T, then

$$\mu(x) = \bigcap\{{}^*G : G \in \beta_x\}.$$

Proof. Since $\beta_x \subseteq \mathcal{N}_x$, we have

$$\bigcap\{^*G : G \in \beta_x\} \supseteq \bigcap\{^*G : G \in \mathcal{N}_x\}.$$

On the other hand, for each $U \in \mathcal{N}_x$ there exist $V_i \in \beta_x$ $(1 \leq i \leq n)$ with $\bigcap_{i=1}^n V_i \subseteq U$, and so, by transfer principle, we have $\bigcap_{i=1}^n {}^*V_i \subseteq {}^*U$. Therefore

$$\bigcap\{^*V : V \in \beta_x\} \subseteq \bigcap\{^*G : G \in \mathcal{N}_x\}.$$

Hence $\mu(x) = \bigcap\{^*G : G \in \beta_x\}$. □

Example 2.1.6 (Various types of topology [8])**.**

(1) Discrete Topology: A space (X, T) is discrete if $\{x\}$ is open for each $x \in X$. In this case $\mu(x) = \{x\}$ for each $x \in X$.

(2) Trivial Topology: A space (X, T) is trivial if $T = \{\emptyset, X\}$. In this case $\mu(x) = {}^*X$ for each $x \in X$.

(3) Finite Complement Topology: For simplicity let $X = \mathbb{N}$ (any infinite set would do), and let T be the collection consisting of the empty set and those subsets of $\mathbb{N}$ whose complements are finite; i.e., $T = \{A \subseteq \mathbb{N} : A = \emptyset \text{ or } \mathbb{N} - A \text{ is finite}\}$. In this case $\mu(x) = \{x\} \cup {}^*\mathbb{N}_\infty$, where ${}^*\mathbb{N}_\infty = {}^*\mathbb{N} - \mathbb{N}$.

Lemma 2.1.7. [24] If A, $B \subseteq {}^*X$, then:

(i) $A \subseteq \mu(A)$.

(ii) $A \subseteq B$ implies $\mu(A) \subseteq \mu(B)$.

(iii) $\mu(\mu(A)) = \mu(A)$.

Proof. (i) By definition of $\mu(A)$ we have $A \subseteq \mu(A)$.
(ii) Since $A \subseteq B$, we have

$$\mu(A) = \bigcap\{^*G : A \subseteq {}^*G, G \in T\} \subseteq \bigcap\{^*G : B \subseteq {}^*G, G \in T\} = \mu(B).$$

Hence $\mu(A) \subseteq \mu(B)$.
(iii) Since $A \subseteq \mu(A)$ by (i), we have $\mu(A) \subseteq \mu(\mu(A))$ by (ii). On the other hand, by definition, $\mu(A)$ is the largest subset such that every ${}^*G \supseteq A$ must contain $\mu(A)$; i.e., ${}^*G \supseteq \mu(A)$. Then $\mu(\mu(A))$ is the largest subset such that every ${}^*G \supseteq \mu(A)$ must contain $\mu(\mu(A))$; i.e., ${}^*G \supseteq \mu(\mu(A))$. Therefore, every ${}^*G \supseteq A$ must contain $\mu(\mu(A))$, and since $\mu(A)$ is the largest. Hence $\mu(\mu(A)) \subseteq \mu(A)$. □

Corollary 2.1.8. [24] For any $A \subseteq {}^*X$ and any α, $\beta \in {}^*X$ we have:

(i) $\alpha \in A$ implies $\mu(\alpha) \subseteq \mu(A)$.

(ii) $\alpha \in \mu(\beta)$ iff $\mu(\alpha) \subseteq \mu(\beta)$.

(iii) $\alpha \in \mu(\beta)$ and $\beta \in \mu(\alpha)$ iff $\mu(\alpha) = \mu(\beta)$.

Proof. (i) Since $\alpha \in A$ implies $\{\alpha\} \subseteq A$, by Lemma (2.1.7) part (ii), we have $\mu(\alpha) \subseteq \mu(A)$.

(ii) Let $\alpha \in \mu(\beta)$. Then using Lemma (2.1.7) part (iii), we have

$$\mu(\alpha) \subseteq \mu(\mu(\beta)) = \mu(\beta).$$

Conversely, assume $\mu(\alpha) \subseteq \mu(\beta)$. Then $\alpha \in \mu(\alpha) \subseteq \mu(\beta)$, so that $\alpha \in \mu(\beta)$.

(iii) Let $\alpha \in \mu(\beta)$ and $\beta \in \mu(\alpha)$. Then by part (ii), we have $\mu(\alpha) \subseteq \mu(\beta)$ and $\mu(\beta) \subseteq \mu(\alpha)$. Hence $\mu(\alpha) = \mu(\beta)$.

Conversely, let $\mu(\alpha) = \mu(\beta)$. Then $\alpha \in \mu(\alpha) = \mu(\beta)$ and $\beta \in \mu(\beta) = \mu(\alpha)$, so that $\alpha \in \mu(\beta)$ and $\beta \in \mu(\alpha)$. □

Theorem 2.1.9 (Balloon and Nuclei Principles [24])**.**
Let (X, T) be a topological space and let $x \in X$. Then:

(i) **Balloon Principle:** If $\mu(x) \subseteq B$ for some internal set $B \subseteq {}^*X$, then there exists $G \in T_x$ such that $\mu(x) \subseteq {}^*G \subseteq B$ (*ballooning* of $\mu(x)$ into ${}^*G$).

(ii) **Nuclei Principle:** There exists an internal set $A \subseteq {}^*X$ such that $x \in A \subseteq \mu(x)$. (The set A is called a *nuclei* of $\mu(x)$).

Proof. (i) Suppose there is no $G \in T_x$ such that $\mu(x) \subseteq {}^*G \subseteq B$; i.e.,

$$({}^*G - B) \neq \emptyset \; \forall \, G \in T_x.$$

Then the family of sets $\{{}^*G - B\}_{G \in T_x}$ has the finite intersection property, since

$$({}^*G_1 - B) \cap ({}^*G_2 - B) = ({}^*G_1 \cap {}^*G_2) - B = {}^*(G_1 \cap G_2) - B \neq \emptyset \, \forall \, G_1, \, G_2 \in T_x.$$

It follows that

$$\mu(x) - B = \bigcap_{G \in T_x} {}^*G - B = \bigcap_{G \in T_x} ({}^*G - B) \neq \emptyset$$

by saturation principle, since card$\{G : G \in T_x\} \leq$ card $T \leq \kappa$, by the choice of the nonstandard model. But $(\mu(x) - B) \neq \emptyset$ contradicts our assumption $[\mu(x) \subseteq B]$.
(ii) Define the family $\{S_G\}_{G \in T_x}$, where $S_G = \{H \in T : x \in H \subseteq G\}$, and observe that it has the finite intersection property, since $G \in S_G$ implies $S_G \neq \emptyset$ and on the other hand $S_{G_1} \cap S_{G_2} = S_{G_1 \cap G_2} \neq \emptyset \ \forall G_1, G_2 \in T_x$. It follows from transfer principle that the family $\{S_G\}_{G \in T_x}$ has the finite intersection property, and therefore there exists A in the intersection $\bigcap_{G \in T_x} {}^*S_G$, by saturation principle. On the other hand, observe that

$$^*S_G = \{H \in {}^*T : x \in H \subseteq {}^*G\}.$$

Thus, A is internal (as an element of $^*T$) and $x \in A \subseteq \mu(x)$. □

Lemma 2.1.10. [24] Let $A, B \subseteq {}^*X$. Then $\mu(A) \cap \mu(B) = \emptyset$ iff there exist two disjoint open sets G and H such that $A \subseteq {}^*G$ and $B \subseteq {}^*H$.

Proof. Let $\mu(A) \cap \mu(B) = \emptyset$ and suppose, on the contrary, that $G \cap H \neq \emptyset$ for all open G and H such that $A \subseteq {}^*G$ and $B \subseteq {}^*H$. By saturation principle, we have

$$\begin{aligned}\mu(A) \cap \mu(B) &= \bigcap\{{}^*G : G \in T,\, A \subseteq {}^*G\} \cap \bigcap\{{}^*H : H \in T,\, B \subseteq {}^*H\} \\ &= \bigcap\{{}^*(G \cap H) : G,\, H \in T,\, A \subseteq {}^*G \text{ and } B \subseteq {}^*H\} \neq \emptyset.\end{aligned}$$

So that $\mu(A) \cap \mu(B) \neq \emptyset$, a contradiction.

Conversely, suppose that there exists two disjoint open sets G and H such that $A \subseteq {}^*G$ and $B \subseteq {}^*H$. Now by the definition of monad, we have

$$A \subseteq \mu(A) \subseteq {}^*G \text{ and } B \subseteq \mu(B) \subseteq {}^*H.$$

Hence $\mu(A) \cap \mu(B) = \emptyset$. □

2.2 Compactification

Notations:

Let (X, T) be a topological space. Then, a simple observation shows that the collection of sets $^\sigma T = \{{}^*G : G \in T\}$ forms a base for a topology on $^*X$ [24]. Denote this topology by ^sT and the corresponding topological space by $({}^*X, {}^sT)$. Notice that the collection of sets $\mathcal{F} = \{{}^*F : (X - F) \in T\}$ forms a base for the closed sets of $^*X$ in $({}^*X, {}^sT)$.

Recall the following definition.

Definition 2.2.1 (Standard Compactness [26]).

A set $A \subseteq X$ is compact iff each open cover has a finite subcover, or equivalently each family ζ of closed subsets of A with the finite intersection property has nonempty intersection.

Definition 2.2.2 (Nonstandard Compactification).

Let (X, T) be a topological space and $({}^*X,\, {}^sT)$ be the corresponding topological space defined as above. Then

(i) ${}^sT$ will be called the *standard topology* on ${}^*X$.

(ii) The topological space $({}^*X,\, {}^sT)$ will be called the *nonstandard compactification* of (X, T).

The designation standard topology for ${}^sT$ arises from the fact that, in the literature on nonstandard analysis, all sets of the type ${}^*G$, where $G \subseteq X$, are called "standard sets" (even though ${}^*G$ is, in fact, a subset of ${}^*X$).

The terminology nonstandard compactification is justified by the following result.

Theorem 2.2.3. Let (X, T) be a topological space and $({}^*X,\, {}^sT)$ be its nonstandard compactification (in the sense of Definition 2.2.2). Then

(i) Every internal subset A of ${}^*X$ is compact in $({}^*X,\, {}^sT)$.

(ii) $({}^*X,\, {}^sT)$ is a compact topological space and (X, T) is a dense subspace of $({}^*X,\, {}^sT)$.

Proof. (i) Let $\{{}^*F_i \in \mathcal{F} : i \in I\}$ be a family of basic closed sets in ${}^*X$ such that the family $\{{}^*F_i \cap A\colon\, i \in I\}$ has the finite intersection property. Then, by saturation principle,

$$\bigcap_{i \in I} ({}^*F_i \cap A) \neq \emptyset,$$

using, Definition (2.2.1), A is compact.

(ii) The compactness of $({}^*X,\, {}^sT)$ follows from (i) as a particular case for $A = {}^*X$. The original space (X, T) is a subspace of $({}^*X,\, {}^sT)$ since ${}^*G \cap X = G \neq \emptyset$ for any $G \subseteq X$, hence $T = \{{}^*G \cap X : G \in T\}$. To show the denseness of (X, T), notice that ${}^*G \cap X = G \neq \emptyset$ for any basic open set ${}^*G \neq \emptyset$, $G \in T$. $\square$

Lemma 2.2.4. For any $H \subseteq X$ we have ${}^*(\mathrm{cl}_X H) = \mathrm{cl}_{{}^*X}({}^*H)$, where cl_X and $\mathrm{cl}_{{}^*X}$ are the closure operators in (X, T) and $({}^*X,\ {}^sT)$, respectively.

Proof. Note that $\mathrm{cl}_X H = \bigcap\{F : H \subseteq F, X - F \in T\}$, so that

$$\begin{aligned}
{}^*(\mathrm{cl}_X H) &= {}^*\big(\bigcap\{F : H \subseteq F,\ X - F\ \in T\}\big) \text{ (by transfer principle)} \\
&= \bigcap\{{}^*F : H \subseteq F,\ X - F\ \in T\} \text{ (by Theorem (1.5.14))} \\
&= \bigcap\{{}^*F : {}^*H \subseteq {}^*F,\ X - F\ \in T\} \\
&= \mathrm{cl}_{{}^*X}({}^*H).
\end{aligned}$$

Hence ${}^*(\mathrm{cl}_X H) = \mathrm{cl}_{{}^*X}({}^*H)$. □

2.3 Topological Main Definitions

Recall the following definitions.

Definition 2.3.1 (Standard Topological Main Definitions [14]).
Let (X, T) be a topological space and let $A \subseteq X$. Then

(1) A point $x \in A$ is an *interior point* of A iff there exists an open set U containing x such that $U \subseteq A$. The set of all interior points of A is denoted by A^o. Consequently, A is an *open* set iff each $x \in A$ is an interior point.

(2) The *closure* of A is the intersection of all closed sets in X which contains A and is denoted by $\mathrm{Cl}_X A$.

(3) A point $x \in X$ is an *exterior point* of A iff there exists an open set U containing x such that $U \cap A = \emptyset$. The set of all exterior points of A is denoted by $\mathrm{Ext}_X A$.

(4) A point $x \in X$ is a *boundary point* of A iff every open set containing x contains at least one point of A. The set of all boundary points of A is called the *frontier* of A and is denoted by $\mathrm{Fr}_X A$.

(5) A point $x \in X$ is an *accumulation point* of A iff every open set containing x contains at least one point of A different from x. The set of all accumulation points of A is denoted by A'.

(6) The set A is *dense* in X iff $\mathrm{Cl}_X A = X$.

(7) A point $x \in A$ is an *isolated point* of A iff there exists an open set U containing x such that $U \cap A = \{x\}$.

Proposition 2.3.2 (Interior point & Open Set)**.**
Let $A \subseteq X$ and A^o be the interior of A in (X, T). Then $x \in A^o$ iff $\mu(x) \subseteq {}^*A$. So that

$$A^o = \{x \in X : \mu(x) \subseteq {}^*A\}.$$

Consequently, A is *open* in (X, T) iff $\mu(x) \subseteq {}^*A \; \forall x \in A$.

Proof. Suppose that x is an interior point of A, then $\mu(x) \subseteq {}^*A$, by the definition of $\mu(x)$.

Conversely, assume that $\mu(x) \subseteq {}^*A$. To show that $x \in A^o$, suppose, on the contrary, that x is not an interior point of A; i.e., $G - A \neq \emptyset \; \forall G \in T_x$. Observe that the family of sets $\{G - A\}_{G \in T_x}$ has the finite intersection property. It follows, by transfer principle, that the family of internal (actually, standard) sets $\{{}^*G - {}^*A\}_{G \in T_x}$ has the finite intersection property, since ${}^*(G - A) = {}^*G - {}^*A$, by the Boolean properties. Hence, by the saturation principle,

$$\bigcap\{{}^*G - {}^*A : G \in T_x\} = \bigcap\{{}^*G : G \in T_x\} - {}^*A = \mu(x) - {}^*A \neq \emptyset,$$

a contradiction. □

Proposition 2.3.3 (Closed Set)**.**
A set $A \subseteq X$ is *closed* in (X, T) iff $\mu(x) \cap {}^*A \neq \emptyset$ implies $x \in A$ for each $x \in X$.

Proof. Suppose, by contraposition, that $x \notin A$. Then $x \in X - A$. Now since $X - A$ is open, then, using Proposition (2.3.2), we have $\mu(x) \subseteq {}^*(X - A) = {}^*X - {}^*A$. Hence $\mu(x) \cap {}^*A = \emptyset$.

Conversely, suppose that $\mu(x) \cap {}^*A \neq \emptyset$ implies $x \in A$ for each $x \in X$. Then $\mu(x) \cap {}^*A = \emptyset \; \forall x \in (X - A)$. So that $\mu(x) \subseteq {}^*(X - A) \; \forall \; x \in (X - A)$. Hence, by Proposition (2.3.2), $(X - A)$ is open. Therefore A is closed. □

Proposition 2.3.4 (Closure)**.**
Let $A \subseteq X$ and $\mathrm{cl}_X(A)$ be the closure of A in (X, T). Then $x \in \mathrm{cl}_X(A)$ iff $\mu(x) \cap {}^*A \neq \emptyset$. So that

$$\mathrm{cl}_X(A) = \{x \in X : \mu(x) \cap {}^*A \neq \emptyset\}.$$

Proof. Let $x \in \mathrm{cl}_X A$; i.e., $x \in F$ for all F such that $A \subseteq F \subseteq X$, $X - F \in T$. Suppose, on the contrary, that ${}^*A \cap \mu(x) = \emptyset$. Then, by Balloon principle (applied to the internal set $B = {}^*X - {}^*A$), there exists $G \in T_x$, such that ${}^*G \subseteq {}^*X - {}^*A$. Thus, we have ${}^*A \subseteq {}^*X - {}^*G = {}^*(X - G)$, implying $A \subseteq X - G$ [by transfer principle]. Since $X - G$ is closed, our assumption yields that $x \in X - G$, a contradiction.

Conversely let $x \in X$ and ${}^*A \cap \mu(x) \neq \emptyset$. We have to show that $x \in F$ for all F such that $A \subseteq F \subseteq X$ and $X - F \in T$. Suppose, on the contrary, that $x \notin F$ for some F such that $A \subseteq F \subseteq X$ and $X - F \in T$. Then $\mu(x) \subseteq {}^*(X - F) = {}^*X - {}^*F$ [by Boolean property], and $A \subseteq F$ implies ${}^*A \subseteq {}^*F$, by transfer principle. Hence, ${}^*A \cap ({}^*X - {}^*F) = \emptyset$, which implies that ${}^*A \cap \mu(x) \subseteq {}^*A \cap ({}^*X - {}^*F) = \emptyset$; i.e., ${}^*A \cap \mu(x) = \emptyset$, a contradiction. □

Proposition 2.3.5 (Exterior)**.**
Let $A \subseteq X$ and $\mathrm{Ext}_X A$ be the exterior of A in (X, T). Then $x \in \mathrm{Ext}_X A$ iff $\mu(x) \subseteq {}^*(X - A)$. So that

$$\mathrm{Ext}_X A = \{x \in X : \mu(x) \subseteq {}^*(X - A)\}.$$

Proof.

$$\begin{aligned} x \in \mathrm{Ext}_X A &\Leftrightarrow (\exists U \in T,\, x \in U,\, U \subseteq (X - A)) \\ &\Leftrightarrow (\exists {}^*U \in {}^*T,\, x \in {}^*U,\, {}^*U \subseteq {}^*(X - A)) \text{ (by transfer principle)} \\ &\Leftrightarrow (\mu(x) \subseteq {}^*(X - A)). \end{aligned}$$

□

Proposition 2.3.6 (Frontier)**.**
Let $A \subseteq X$ and $\mathrm{Fr}_X(A)$ be the frontier of A in (X, T). Then $x \in \mathrm{Fr}_X(A)$ iff $\mu(x) \cap {}^*A \neq \emptyset$ and $\mu(x) \cap {}^*(X - A) \neq \emptyset$. So that

$$\mathrm{Fr}_X(A) = \{x \in X : \mu(x) \cap {}^*A \neq \emptyset \text{ and } \mu(x) \cap {}^*(X - A) \neq \emptyset\}.$$

Proof. Using Proposition (2.3.4), we have

$$\begin{aligned} x \in \mathrm{Fr}_X(A) &\Leftrightarrow x \in \mathrm{cl}_X A \cap \mathrm{cl}_X (X - A) \\ &\Leftrightarrow {}^*A \cap \mu(x) \neq \emptyset \text{ and } {}^*(X - A) \cap \mu(x) \neq \emptyset. \end{aligned}$$

□

Proposition 2.3.7 (Derived Set)**.**

Let $A \subseteq X$ and A' be the derived set of A [the set of all *accumulation points* of A] in (X, T). Then $x \in A'$ iff $\mu(x)$ contains a point $y \in {}^*A$ different from x; i.e., $y \in \mu(x) \cap {}^*A$ where $y \neq x$.

Proof.

$$\begin{aligned} x \in A' &\Leftrightarrow (\forall U \in T,\, x \in U)\,(\exists\, y \in U \cap A)\,[y \neq x] \\ &\Leftrightarrow (\forall\, {}^*U \in \ {}^*T,\ x \in {}^*U)\,(\exists\, y \in \ {}^*U \cap {}^*A)\,[y \neq x] \\ &\Leftrightarrow (\exists\, y \in (\mu(x) \cap {}^*A))\,[y \neq x]. \end{aligned}$$

□

Proposition 2.3.8 (Dense)**.**

A set $A \subseteq X$ is dense in (X, T) iff $(\forall\ x \in X)\ (\mu(x) \cap {}^*A \neq \emptyset)$.

Proof.

$$\begin{aligned} \text{A set } A \subseteq X \text{ is dense } &\Leftrightarrow \ \mathrm{cl}_X A = X \\ &\Leftrightarrow \mu(x) \cap {}^*A \neq \emptyset\ (\forall\, x \in X)\ [\text{By definition of closure}]. \end{aligned}$$

□

Theorem 2.3.9. If A, $B \subseteq X$ are open dense, then $A \cap B$ is open dense.

Proof. $A \cap B$ is clearly open. Suppose, on the contrary, that $A \cap B$ is not dense, so that $\exists\, c \in X$ such that $\mu(c) \cap {}^*(A \cap B) = \emptyset$. Then there exists an open set U such that $c \in U$ and $({}^*U \cap {}^*A \cap {}^*B) = {}^*U \cap {}^*(A \cap B) = \emptyset$. Hence, by transfer principle, $U \cap A \cap B = \emptyset$. Now since A is open dense, we have $U \cap A \neq \emptyset$. And since B is open dense and $U \cap A$ is open, we have $U \cap A \cap B \neq \emptyset$, a contradiction. Hence $A \cap B$ is open dense. □

Proposition 2.3.10 (Isolated Point)**.**

A point x is an isolated point of a set $A \subseteq X$ iff $\mu(x) \cap {}^*A = \{x\}$.

Proof.

$$\begin{aligned} x \text{ is an isolated point of } A &\Leftrightarrow (\exists\, U \in T,\, x \in U \text{ such that } U \cap A = \{x\}) \\ &\Leftrightarrow (\exists\, {}^*U \in \ {}^*T,\, x \in \ {}^*U,\, {}^*U \cap {}^*A = \{x\}) \\ &\Leftrightarrow (\mu(x) \cap {}^*A = \{x\}).[\text{ since } x \in \mu(x) \subseteq \ {}^*U \text{ for any } U \in T_x]. \end{aligned}$$

□

2.4 Subspaces

Theorem 2.4.1 (Monad of Relative Topology)**.**
If Y is a subset of a topological space (X, T), then the monads in (Y, T_Y) are given by

$$\hat{\mu}(y) = \mu(y) \cap {}^*Y,$$

where $\mu(y)$ is the monad of y in (X, T) and (Y, T_Y) is the relative topology induced on Y by T.

Proof. By definition of monad, the monad on Y is given by

$$\hat{\mu}(y) = \bigcap \{{}^*U : U \in T_Y,\ y \in U\}.$$

Now since $U \in T_Y$ iff $U = V \cap Y$ for some $V \in T$, by transfer principle, we have ${}^*U = {}^*V \cap {}^*Y$ for some ${}^*V \in {}^*T$. So that

$$\begin{aligned}
\hat{\mu}(y) &= \bigcap \{({}^*V \cap {}^*Y) : V \in T,\ y \in V \cap Y\} \\
&= \bigcap \{{}^*V : V \in T,\ y \in V\} \cap {}^*Y \\
&= \mu(y) \cap {}^*Y.
\end{aligned}$$

□

Theorem 2.4.2. If A is a subspace of a topological space X, then:

(i) $H \subseteq A$ is open in A iff $\hat{\mu}(x) \subseteq {}^*H$ for each $x \in H$.

(ii) $F \subseteq A$ is closed in A iff $\hat{\mu}(x) \cap {}^*F \neq \emptyset$ implies $x \in F$ for each $x \in A$.

(iii) If $E \subseteq A$, then $\text{Cl}_A E = \{x \in A : \hat{\mu}(x) \cap {}^*E \neq \emptyset\}$.

Proof. The proof is the obvious modifications of those we have just proved in Section (2.3) with $\hat{\mu}$ replacing μ. □

2.5 Continuity

Recall the following definitions.

Definition 2.5.1 (Standard Continuous and Open Functions [14])**.**
Let (X, T_1), (Y, T_2) be two topological spaces, and let $f : X \to Y$ be a function. Then

(1) f is *continuous at the point* $x \in X$ iff given any open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(U) \subseteq V$. If f is continuous at every point of X, then f is said to be a *continuous* function.

(2) f is *open* iff for each open set $U \subseteq X$, $f(U)$ is open in Y.

Theorem 2.5.2 (Continuity at a point)**.**
Let (X, T_1), (Y, T_2) be two topological spaces with monads $\mu(x)(x \in X)$ and $\bar{\mu}(y)(y \in Y)$, respectively, and let $f : X \to Y$ be a function. Then f is continuous at $x \in X$ iff ${}^*f(\mu(x))$ $\subseteq \bar{\mu}(f(x))$.

Proof. Suppose that f is continuous at $x \in X$. Now since

$$\begin{aligned} {}^*f(\mu(x)) &= {}^*f(\bigcap_{x \in U \in T_1} {}^*U) \\ &\subseteq \bigcap_{x \in U \in T_1} {}^*f({}^*U) \\ &\subseteq \bigcap_{f(x) \in V \in T_2} {}^*V \\ &= \bar{\mu}(f(x)), \end{aligned}$$

where the second inclusion follows from the definition of continuity at x and the transfer principle. So that ${}^*f(\mu(x)) \subseteq \bar{\mu}(f(x))$.

Conversely, suppose that ${}^*f(\mu(x)) \subseteq \bar{\mu}(f(x))$. Let $V \in T_2$ be an open set containing $f(x)$, so that $\bar{\mu}(f(x)) \subseteq {}^*V$. Let ${}^*U \in {}^*T_1$ be such that $x \in {}^*U \subseteq \mu(x)$. Then

$${}^*f({}^*U) \subseteq {}^*f(\mu(x)) \subseteq \bar{\mu}(f(x)) \subseteq {}^*V.$$

So that, by transfer principle, we have $f(U) \subseteq V$ for some $U \in T_1$ such that $x \in U$. Hence f is continuous at $x \in X$. □

Definition 2.5.3. We say that f is continuous on X iff $\forall x \in X$ $[{}^*f(\mu(x)) \subseteq \bar{\mu}(f(x))]$.

Theorem 2.5.4 (Continuity)**.**
Let (X, T_1), (Y, T_2) be two topological spaces and $f : X \to Y$ be a function. Then f is continuous iff $\forall\ V \in T_2$ $(f^{-1}(V) \in T_1)$.

Proof. Suppose that f is continuous, and let V $\in T_2$. Choose $x \in X$ such that $f(x) \in V$. Then, by continuity of f at x, we have ${}^*f(\mu(x)) \subseteq \bar{\mu}(f(x)) \subseteq {}^*V$. It follows that $\mu(x) \subseteq {}^*f^{-1}({}^*V) = {}^*(f^{-1}(V))$, and so $f^{-1}(V)$ is open in $(X,\, T_1)$.

Conversely, suppose that $f^{-1}(V) \in T_1$ for every $V \in T_2$. Fix $x \in X$ and choose $V \in T_2$ such that $x \in f^{-1}(V)$. Then, by openness of $f^{-1}(V)$, we have $\mu(x) \subseteq {}^*(f^{-1}(V)) = {}^*f^{-1}({}^*V)$. Then, by definition of $\mu(x)$, we have ${}^*U \subseteq {}^*f^{-1}({}^*V)$ for some $U \in T_1$, $x \in U$, and therefore

$$ {}^*f({}^*U) \subseteq {}^*f({}^*f^{-1}({}^*V)) \subseteq {}^*V. $$

So that

$$ \bigcap_{x\in U\in T_1} {}^*f({}^*U) \subseteq \bigcap_{f(x)\in V\in T_2} {}^*V. $$

Hence ${}^*f(\mu(x)) \subseteq \bar{\mu}(f(x))$. □

Theorem 2.5.5 (Open Mapping)**.**
Let (X, T_1), (Y, T_2) be two topological spaces with monads $\mu(x)(x \in X)$ and $\bar{\mu}(y)(y \in Y)$, respectively, and let $f : X \to Y$ be a function. Then f is open iff

$$ \forall\, x \in X\, [{}^*f(\mu(x)) \supseteq \bar{\mu}(f(x))]. \tag{2.1} $$

Proof. Assume that f is open. Let $x \in X$, and consider $z \in \bigcap_{x\in U\in T_1} {}^*f({}^*U)$. Since the set $\{{}^*U : x \in U,\, U \in T_1\}$ is closed under finite intersections, by saturation principle, for some $y \in \bigcap_{x\in U\in T_1} {}^*U$, we have $z = {}^*f(y)$; in other words,

$$ \begin{aligned} \bigcap_{x\in U\in T_1} {}^*f({}^*U) &\subseteq {}^*f(\bigcap_{x\in U\in T_1} {}^*U) \\ &= {}^*f(\mu(x)). \end{aligned} $$

And since ${}^*f(\mu(x)) \subseteq \bigcap_{x\in U\in T_1} {}^*f({}^*U)$, we have ${}^*f(\mu(x)) = \bigcap_{x\in U\in T_1} {}^*f({}^*U)$. Since f is open, we have $f(U) \in T_2$ for every $U \in T_1$. It follows that

$$ \begin{aligned} {}^*f(\mu(x)) &= \bigcap_{x\in U\in T_1} {}^*f({}^*U) \\ &\supseteq \bigcap_{f(x)\in V\in T_2} {}^*V \\ &= \bar{\mu}(f(x)). \end{aligned} $$

Thus (2.1) holds.

Conversely, assume (2.1) holds. To show f is open, let $U \in T_1$ and $y \in f(U)$. Then $y = f(x)$ for some $x \in U$, and

$$ \bar{\mu}(y) = \bar{\mu}(f(x)) \subseteq {}^*f(\mu(x)) \subseteq {}^*f({}^*U). $$

Hence $f(U) \in T_2$, and therefore f is open. □

Combining Theorems (2.5.4) and (2.5.5), we obtain the following result.

Theorem 2.5.6 (Homeomorphism)**.**
Let (X, T_1), (Y, T_2) be two topological spaces with monads $\mu(x)(x \in X)$ and $\bar{\mu}(y)(y \in Y)$, respectively, and let $f : X \to Y$ be a bijective function. Then f is *homeomorphism* iff

$$\forall\, x \in X\, [{}^*f(\mu(x)) = \bar{\mu}(f(x))],$$

or, equivalently,

$$\forall\, A \subseteq X\, [{}^*f(\mu(A)) = \bar{\mu}(f(A))].$$

2.6 Product Topology

Definition 2.6.1 (Product Topology)**.** Let $X = \prod_{i \in I} X_i$ where $\{(X_i, T_i) : i \in I\}$ is a family of topological spaces. Let $\mathscr{B}$ denote the system of all sets of the form $\prod_{i \in I} O_i$ where $O_i = X_i$ for all except finitely many $i \in I$, and $O_i \subseteq X_i$ is open in X_i for the remaining $i \in I$. Then $\mathscr{B}$ is a base for a topology on X, which is called the *product topology*. Thus a set $O \subseteq X$ is open in the product topology iff it is a union of sets from $\mathscr{B}$. This topology on X is the smallest topology such that each of the projections $\pi_i : X \to X_i$ is continuous.

Theorem 2.6.2. [25] In the product topology, if $x \in X$, then $y \in \mu(x)$ iff $y_i \in \mu_i(x_i)$ for all standard $i \in {}^*I$, where $\mu_i(x_i)$ is the monad of x_i in X_i.

Proof. Suppose that $y_i \in \mu_i(x_i)$ for any standard $i \in {}^*I$. Let $O \subseteq X$ be open with ${}^*x \in {}^*O$. We have to prove that $y \in {}^*O$. But ${}^*x \in {}^*O$ implies $x \in O$ [by transfer principle] and so we have $x \in B \subseteq O$ for some $B \in \mathscr{B}$ where $\mathscr{B}$ is as in Definition (2.6.1). By definition of $\mathscr{B}$, we find finitely many $i_1, i_2, \ldots, i_n \in I$ and open sets $O_{i_k} \subseteq X_{i_k}$, $k = 1, 2, \ldots, n$ such that

$$B = O_{i_1} \times O_{i_2} \times \cdots \times O_{i_n} \times \prod_{i \in I} \{X_i : i \neq i_1, \ldots, i_n\}.$$

Then, by transfer principle, we have

$${}^*B = {}^*O_{i_1} \times {}^*O_{i_2} \times \cdots \times {}^*O_{i_n} \times {}^*(\prod_{i \in I} \{X_i : i \neq i_1, \ldots, i_n\}).$$

By Theorem (1.5.17), we have

$${}^*B = {}^*O_{i_1} \times {}^*O_{i_2} \times \cdots \times {}^*O_{i_n} \times \prod_{i \in {}^*I} \{{}^*X_i : i \neq i_1, \ldots, i_n\}.$$

Now $y_{i_k} \in \mu_i(x_{i_k}) \subseteq {}^*O_{i_k}$, so we have $y_{i_k} \in {}^*O_{i_k}$ for some standard $i_k \in {}^*I$. Thus y satisfies $y_i \in {}^*O_i$ for all standard $i \in {}^*I$ (for $i \neq i_1, \ldots, i_k$ we have ${}^*O_i = {}^*X_i$, as we have shown). This proves that $y \in {}^*B \subseteq {}^*O$. Hence $y \in \mu(x)$.

Conversely, let $y \in \mu(x)$. If $i_0 \in {}^*I$ is standard and $O_{i_0} \subseteq X_{i_0}$ is open with $x_{i_0} \in O_{i_0}$ are given, put

$$O = O_{i_0} \times \prod_{i \in I} \{X_i : i \neq i_0\}.$$

Then O is open with $x \in O$. Hence ${}^*x \in {}^*O$, and so our assumption implies that $y \in {}^*O$. Since, by transfer principle, we have

$${}^*O = {}^*O_{i_0} \times {}^*(\prod_{i \in I} \{X_i : i \neq i_0\}),$$

and, by Theorem(1.5.17), we have

$${}^*O = {}^*O_{i_0} \times \prod_{i \in {}^*I} \{{}^*X_i : i \neq i_0\},$$

we must have $y_{i_0} \in {}^*O_{i_0}$. Hence $y_{i_0} \in \mu_i(x_{i_0})$. $\square$

Chapter 3

Compactness and Separation Axioms

In this chapter we study compactness and separation axioms of topological spaces in terms of monads. We present the most famous theorems concerning compactness and separation axioms where their proofs utilize the nonstandard theory. Section one presents A. Robinson's compactness theorem and two characterizations of the compactness. In section two we present separation axioms.

3.1 Compactness

Definition 3.1.1 (Concurrent Relation [8])**.**
A binary relation P is *concurrent* on $A \subseteq \text{dom } P$ if for each finite set $x_1, \ldots, x_n$ in A there is a $y \in \text{ran } P$ so that $(x_i, y) \in P$, $1 \leq i \leq n$.

Theorem 3.1.2 ([8])**.** The superstructure $V({}^*X)$ is an enlargement of the superstructure $V(X)$ iff for each concurrent relation $P \in V(X)$ there is an element $b \in \text{ran } {}^*P$ so that $({}^*x, b) \in {}^*P$ for all $x \in \text{dom } P$.

Theorem 3.1.3. Let (X, T) be a topological space and $({}^*X,\, {}^sT)$ be its nonstandard compactification. If $A \subseteq {}^*X$ is compact in $({}^*X,\, {}^sT)$, then

$$\bigcup_{\alpha \in A} \mu(\alpha) = \mu(A).$$

Proof. Since $\mu(\alpha) \subseteq \mu(A)\ \forall\ \alpha \in A$, we have

$$\bigcup_{\alpha \in A} \mu(\alpha) \subseteq \mu(A).$$

On the other hand, let $\alpha \in \mu(A)$ and suppose that $\alpha \notin \mu(\beta)$ for all $\beta \in A$. Then, for any $\beta \in A$ there is $G_\beta \in T$ such that $\beta \in {}^*G_\beta$ and $\alpha \notin {}^*G_\beta$. Now, we obviously have the cover:

$$A \subseteq \bigcup \{{}^*G_\beta : \beta \in A\}.$$

By the compactness of A, there exist $G_{\beta_1}, \ldots, G_{\beta_n}$ such that

$$A \subseteq \bigcup_{i=1}^{n} {}^*G_{\beta_i} = {}^*(\bigcup_{i=1}^{n} G_{\beta_i}).$$

Hence, by the definition of $\mu(A)$, we obtain

$$\mu(A) \subseteq {}^*(\bigcup_{i=1}^{n} G_{\beta_i}),$$

so that $\alpha \in {}^*G_{\beta_i}$, for some i, which is a contradiction. Therefore

$$\mu(A) \subseteq \bigcup_{\alpha \in A} \mu(\alpha).$$

□

Corollary 3.1.4. Let (X, T) be a topological space and ${}^*X$ be the nonstandard extension of X. Then $\bigcup_{\alpha \in B} \mu(\alpha) = \mu(B)$ holds for any internal subset B of ${}^*X$.

Proof. The internal subsets of ${}^*X$ are compact in $({}^*X, {}^sT)$, by Theorem (2.2.3). Hence the result follows immediately from Theorem (3.1.3). □

Theorem 3.1.5 (Robinson's Compactness)**.**
Let (X, T) be a topological space. Then $A \subseteq X$ is compact iff for every $y \in {}^*A$ there is $x \in A$ such that $y \in \mu(x)$.

Proof. Suppose, to contrary, that A is compact but there is a point $y \in {}^*A$ such that $y \notin \mu(x)\ \forall\, x \in A$. Then each $x \in A$ possesses an open neighborhood U_x, with $y \notin {}^*U_x$. Now, since A is compact, the open cover $\{U_x : x \in A\}$ of A has a finite subcover, say $\{U_1, \ldots, U_n\}$; i.e.,

$$A \subseteq U_1 \cup \cdots \cup U_n.$$

Using transfer principle, we have

$${}^*A \subseteq {}^*U_1 \cup \cdots \cup {}^*U_n.$$

This contradicts the fact that $y \notin {}^*U_i$ for $1 \leq i \leq n$.

Conversely, suppose, by contraposition, that A is not compact. Then there is an open cover $B = \{U_i : i \in I\}$ of A which has no finite subcover. The binary relation P on $B \times A$ defined by $P(U, x)$ iff $x \notin U$ is concurrent. Indeed, since A has no finite subcover, there exists $y \in A$ such that $y \notin U_i$ for some U_i, $1 \leq i \leq n$, and so $(U_i, y) \in P$; hence P is a concurrent relation. Now by Theorem (3.1.2), there is a point $y \in {}^*A$ with $y \notin {}^*U$ for all $U \in B$. Hence if $x \in A$, then $x \in U$ for some $U \in B$ but $y \notin {}^*U$ so that $y \notin \mu(x)$. □

Example 3.1.6. Here are some examples of compact spaces.

1. In the discrete topology every finite subset A is compact. For, if $\emptyset \neq A \subseteq X$ is finite, then ${}^*A = A$. Hence, for any $y \in {}^*A$, $y \in A$, so that $y \in \mu(x)$, for some $x \in A$, by definition of $\mu(x)$.

2. All subsets in the trivial topology are compact. For, if $\emptyset \neq A \subseteq X$ and $y \in {}^*A$, then $y \in {}^*X$. And since $\mu(x) = {}^*X$ for any $x \in X$, choose $x \in A$ such that $y \in \mu(x)$.

3. In the finite complement topology for $\mathbb{N}$, every subset A is compact. For, if $A \neq \emptyset$ and $y \in {}^*A$ then either $y \in A$ or $y \in {}^*\mathbb{N}_\infty$. In the first case $y \in \mu(x)$ for some $x \in A$, since $A \subseteq \mu(A)$, and in the second case $y \in \mu(x)$ for any $x \in \mathbb{N}$ and, in particular, for some $x \in A$. Recall that a set must be finite to be closed in this topology, so there are compact subsets which are not closed in this non-Hausdorff topology.

Theorem 3.1.7 (Characterization of Compactness)**.**
Let $A \subseteq X$. Then the following conditions are equivalent:

(i) A is compact in (X, T).

(ii) ${}^*A \subseteq \bigcup_{x \in A} \mu(x)$.

(iii) $\bigcup_{x \in A} \mu(x) = \bigcup_{\alpha \in {}^*A} \mu(\alpha)$.

(iv) $\bigcup_{x \in A} \mu(x) = \mu(A)$.

Proof. (i) ⇔ (ii) Proved in Theorem (3.1.5).
(ii) ⇒ (iii): Let ${}^*A \subseteq \bigcup_{x \in A} \mu(x)$ so that $\alpha \in {}^*A$ implies $\alpha \in \mu(x)$ for some $x \in A$, which

implies that $\mu(\alpha) \subseteq \mu(x)$ by Corollary (2.1.8). So that

$$\mu(\alpha) \subseteq \bigcup_{x\in A} \mu(x) \, \forall \, \alpha \in {}^*A \text{ ; i.e., } \bigcup_{\alpha\in {}^*A} \mu(\alpha) \subseteq \bigcup_{x\in A} \mu(x).$$

On the other hand, since $A \subseteq {}^*A$, we have

$$\bigcup_{x\in A} \mu(x) \subseteq \bigcup_{\alpha\in {}^*A} \mu(\alpha).$$

Hence, $\bigcup_{x\in A} \mu(x) = \bigcup_{\alpha\in {}^*A} \mu(\alpha)$.

(iii) $\Rightarrow$ (iv): Suppose that $\bigcup_{x\in A} \mu(x) = \bigcup_{\alpha\in {}^*A} \mu(\alpha)$. Since $A \subseteq X$, ${}^*A$ is an internal subset of ${}^*X$. So that, using Corollary (3.1.4) (applied to $B = {}^*A$), we have

$$\bigcup_{\alpha\in {}^*A} \mu(\alpha) = \mu({}^*A).$$

Now using our assumption and the fact that $\mu(A) = \mu({}^*A)$ for any $A \subseteq X$, we have

$$\bigcup_{x\in A} \mu(x) = \bigcup_{\alpha\in {}^*A} \mu(\alpha) = \mu({}^*A) = \mu(A).$$

Hence $\bigcup_{x\in A} \mu(x) = \mu(A)$.

(iv) $\Rightarrow$ (ii): Suppose that $\bigcup_{x\in A} \mu(x) = \mu(A)$. Since ${}^*A \subseteq \mu({}^*A)$, using the fact that $\mu(A) = \mu({}^*A)$ for $A \subseteq X$, we have ${}^*A \subseteq \mu(A)$. Now using our assumption, we have

$${}^*A \subseteq \mu(A) = \bigcup_{x\in A} \mu(x).$$

Hence ${}^*A \subseteq \bigcup_{x\in A} \mu(x)$. □

Theorem 3.1.8. A closed subset of a compact set is compact.

Proof. Let (X, T) be a topological space, and let $A \subseteq K \subseteq X$, such that K compact and A closed. Then, by transfer principle, we have ${}^*A \subseteq {}^*K \subseteq {}^*X$. Hence if $y \in {}^*A$, then $y \in {}^*K$, so that by the compactness of K, $y \in \mu(x)$ for some $x \in K$. But since $y \in \mu(x)$ and $y \in {}^*A$, we have $x \in A$ [since A is closed], so that $y \in \mu(x)$ for some $x \in A$. Hence A is compact. □

Theorem 3.1.9 (Tychonoff)**.**

Let $(X_i, T_i)(i \in I)$ be a family of spaces and let $X = \prod_{i\in I} X_i$. Then X is compact in the product topology T iff each X_i is compact.

Proof. Suppose that X is compact and fix $i \in I$. To show that X_i is compact, we have to show that for each $b \in {}^*X_i$ there is some $a \in X_i$ such that $b \in \mu(a)$. By axiom of choice, choose some $y \in {}^*X$ with $y_i = b$ for some standard $i \in {}^*I$. Since X is compact, we find some $x \in X$ with $y \in \mu(x)$. Then, using Theorem (2.6.2), we have $y_i \in \mu_i(x_i)$. Taking $a = x_i \in X_i$, we have $b \in \mu(a)$.

Conversely, suppose that X_i is compact for all $i \in I$. Let $y \in {}^*X$. For each standard $i \in {}^*I$, we have $y_i \in {}^*X_i$. Since X_i is compact, we find some $x_i \in X_i$ such that $y_i \in \mu_i(x_i)$. Then $x \in X$ (using the axiom of choice), and Theorem (2.6.2) implies that $y \in \mu(x)$. Hence X is compact. $\square$

Theorem 3.1.10. Let (X, T_1), (Y, T_2) be two topological spaces with monads $\mu(x)(x \in X)$ and $\bar{\mu}(y)(y \in Y)$, respectively, let $f : X \to Y$ be a continuous function, and let K be a compact subset of X. Then $f(K)$ is a compact subset of Y.

Proof. Let $y \in {}^*f(K)$. Then $y = {}^*f(z)$ for some $z \in {}^*K$. Since K is compact, we have $z \in \mu(x)$ for some $x \in K$. Now, by continuity of f, we have

$$y = {}^*f(z) \in {}^*f(\mu(x)) \subseteq \bar{\mu}(f(x)),$$

so that $y \in \bar{\mu}(f(x))$ for some $f(x) \in f(K)$. Hence $f(K)$ is compact. $\square$

3.2 Separation Axioms

Recall the following definitions.

Definition 3.2.1 (Standard Separation Axioms [14])**.** A topological space (X, T) is:

(1) T_0 iff whenever x and y are distinct points in X, then there is an open set containing one but not the other.

(2) T_1 iff whenever x and y are distinct points in X, then there is an open set of each not containing the other. Equivalently, iff every singleton is closed.

(3) T_2 iff whenever x and y are distinct points in X, then there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

(4) *Regular* iff whenever F is closed in X and $x \notin F$, then there are disjoint open sets U and V in X with $x \in U$ and $F \subseteq V$. Equivalently, iff whenever U is open in X and $x \in U$, there is an open set V containing x such that $\overline{V} \subseteq U$.

(5) *Normal* iff whenever F_1 and F_2 are disjoint closed sets in X, there are disjoint open sets U and V in X with $F_1 \subseteq U$ and $F_2 \subseteq V$.

(6) *Completely normal* iff whenever A and B are disjoint sets in X, there are disjoint open sets U and V in X with $A \subseteq U$ and $B \subseteq V$.

Theorem 3.2.2. [T_0-Space] Let (X, T) be a topological space. Then the following statements are equivalent:

(i) (X, T) is a T_0-space.

(ii) $x,\, y \in X,\, x \neq y \Rightarrow \mu(x) \neq \mu(y)$.

(iii) $x,\, y \in X,\, x \neq y \Rightarrow x \notin \mu(y)$ or $y \notin \mu(x)$.

Proof. (i)⇔(ii): Suppose that (X, T) is a T_0-space and $x \neq y$ in X. Then $\exists\, G \in T$ such that $x \in G$ and $y \notin G$. That is, $x \in G \subseteq {}^*G$ and $y \in X - G \subseteq {}^*X - {}^*G$, so that $x \in {}^*G$ and $y \notin {}^*G$ which implies that $y \notin \mu(x)$. Since $y \in \mu(y)$, we have $\mu(x) \neq \mu(y)$. Thus (ii) holds.

Conversely, suppose that (ii) holds; i.e., $\mu(x) \neq \mu(y)$ for $x \neq y$ in X. Without loss of generality, we may assume that $\alpha \in \mu(y) - \mu(x)$; i.e., $\exists\, G \in T$ such that $x \in G$ but $\alpha \notin {}^*G$. Notice now that $y \notin G$. For, otherwise $y \in G$ implies $\alpha \in \mu(y) \subseteq {}^*G$ which is a contradiction. So that there exist an open set G such that $x \in G$ and $y \notin G$. Thus (X, T) is a T_0-space.
(i)⇔(iii): Suppose that (X, T) is a T_0-space and $x \neq y$ in X. Then, by (i)⇔(ii), $\mu(x) \neq \mu(y)$, so that, by Corollary (2.1.8), we have $x \notin \mu(y)$ or $y \notin \mu(x)$. Hence (iii) holds.

Conversely, suppose (iii) holds and let $x \neq y$ in X. Then, by (iii), $x \notin \mu(y)$ or $y \notin \mu(x)$; hence in both cases, $\mu(x) \neq \mu(y)$, and therefore by (i)⇔(ii), (X, T) is a T_0-space. □

Theorem 3.2.3.

(i) The property of being a T_0-space is a topological property.

(ii) Every subspace of a T_0-space is T_0.

(iii) Every nonempty product space is T_0 iff each factor space is T_0.

Proof. (i) Let (X, T_X) be a T_0-space and let (Y, T_Y) be any space homeomorphic to (X, T_X) with monads $\mu(x)(x \in X)$ and $\bar{\mu}(y)(y \in Y)$, respectively. It is required to show that (Y, T_Y) is also a T_0-space. Let $f : X \to Y$ be the homeomorphism, and let $f(x) \neq f(y)$ in Y. Since f is homeomorphism, by Theorem (2.5.6), ${}^*f(\mu(x)) = \bar{\mu}(f(x))$ and ${}^*f(\mu(y)) = \bar{\mu}(f(y))$ where $x \neq y$ in X. Now since X is T_0, Theorem (3.2.2) implies that $\mu(x) \neq \mu(y)$, so that ${}^*f(\mu(x)) \neq {}^*f(\mu(y))$. Therefore $\bar{\mu}(f(x)) \neq \bar{\mu}(f(y))$. Hence, by Theorem (3.2.2) again, (Y, T_Y) is a T_0-space.
(ii) Let A be a subspace of a T_0-space X, and let a, b be two distinct points in A. Since a, b are distinct points in X, by Theorem (3.2.2), we have $a \notin \mu(b)$ or $b \notin \mu(a)$. Since $a \in (\mu(a) \cap {}^*A)$ and $b \in (\mu(b) \cap {}^*A)$, it follows that $(\mu(a) \cap {}^*A) \neq (\mu(b) \cap {}^*A)$, and therefore $\hat{\mu}(b) \neq \hat{\mu}(a)$. Hence, by Theorem (3.2.2) again, A is a T_0-space.
(iii) Let $\{X_i : i \in I\}$ be a family of spaces, and suppose that X_{i_0} is not T_0 for some $i_0 \in I$. Then, by Theorem(3.2.2), there are two distinct points $a, b \in X_{i_0}$ such that $a \in \mu_{i_0}(b)$ and $b \in \mu_{i_0}(a)$. Choose some x, $y \in X = \prod_{i \in I} X_i$ with $x_{i_0} = a$, $y_{i_0} = b$ and $y_i = x_i$ for $i \in I - \{i_0\}$. Then $x \neq y$ in X and, $x_{i_0} \in \mu_{i_0}(y_{i_0})$ and $y_{i_0} \in \mu_{i_0}(x_{i_0})$. Now, using Theorem (2.6.2), we have

$$x \in \mu(y) \text{ and } y \in \mu(x).$$

Hence, by Theorem(3.2.2) again, X is not a T_0-space.

Conversely, suppose that X is not a T_0-space. Then we find elements $x \neq y$ in X such that $x \in \mu(y)$ and $y \in \mu(x)$. Choose some $i_0 \in I$ such that $x_{i_0} \neq y_{i_0}$. Then, by Theorem (2.6.2), we have

$$x_{i_0} \in \mu_{i_0}(y_{i_0}) \text{ and } y_{i_0} \in \mu_{i_0}(x_{i_0}).$$

Hence, by Theorem (3.2.2), X_{i_0} is not a T_0-space. □

Example 3.2.4. The trivial topology on a set X with two or more points is not T_0.

Theorem 3.2.5 (T_1-Space)**.** Let (X, T) be a topological space. Then the following statements are equivalent:

(i) (X, T) is a T_1-space.

(ii) x, $y \in X$, $x \neq y \Rightarrow \mu(x) \nsubseteq \mu(y)$ and $\mu(y) \nsubseteq \mu(x)$.

(iii) $x, y \in X$, $x \neq y \Rightarrow x \notin \mu(y)$ and $y \notin \mu(x)$.

Proof. (i)$\Leftrightarrow$(ii): Suppose that (X, T) is a T_1-space and let $x \neq y$ in X. To show (ii) holds, assume, on the contrary, that $\mu(x) \subseteq \mu(y)$ or $\mu(y) \subseteq \mu(x)$. Without loss of generality, we may assume that $\mu(x) \subseteq \mu(y)$. Since $x \neq y$, we have an open set $G = X - \{x\}$ with $x \notin G$ and $y \in G$. That is, $x \in X - G \subseteq {}^*X - {}^*G$ and $y \in G \subseteq {}^*G$ so that $x \notin \mu(y)$, which contradicts the assumption that $\mu(x) \subseteq \mu(y)$. Thus we have shown that $x \neq y$ in $X \Rightarrow \mu(x) \nsubseteq \mu(y)$ and $\mu(y) \nsubseteq \mu(x)$, and so (ii) holds.

Conversely, suppose that (ii) holds. If (X, T) is not a T_1-space, then there exist some x and y, $x \neq y$ such that $y \in \mathrm{cl}_X\{x\}$. Hence $x \in \mu(y)$, and so $\mu(x) \subseteq \mu(y)$, which implies that $x = y$, a contradiction. Thus, (X, T) is a T_1-space.

(i)$\Leftrightarrow$(iii): Suppose that (X, T) is a T_1-space and let $x \neq y$ in X. Then by (i)$\Leftrightarrow$(ii) we have $\mu(x) \nsubseteq \mu(y)$ and $\mu(y) \nsubseteq \mu(x)$. If $x \in \mu(y)$ or $y \in \mu(x)$, then $\mu(x) \subseteq \mu(y)$ or $\mu(y) \subseteq \mu(x)$, which is a contradiction. So that $x \notin \mu(y)$ and $y \notin \mu(x)$.

Conversely, suppose (iii) holds and let $x \neq y$ in X. Then $x \notin \mu(y)$ and $y \notin \mu(x)$. Hence, by Corollary (2.1.8), we have $\mu(x) \nsubseteq \mu(y)$ and $\mu(y) \nsubseteq \mu(x)$. So, by (i)$\Leftrightarrow$(ii), (X, T) is a T_1-space. □

Theorem 3.2.6.

(i) The property of being a T_1-space is a topological property.

(ii) Every subspace of a T_1-space is T_1.

(iii) Every nonempty product space is T_1 iff each factor space is T_1.

Proof. (i) Let (X, T_X) be a T_1-space and let (Y, T_Y) any space homeomorphic to (X, T_X) with monads $\mu(x)(x \in X)$ and $\bar{\mu}(y)(y \in Y)$, respectively. It is required to show that (Y, T_Y) is also a T_1-space. Let $f : X \to Y$ be the homeomorphism, and let $f(x) \neq f(y)$ in Y. Since f is a homeomorphism, by Theorem (2.5.6), we have ${}^*f(\mu(x)) = \bar{\mu}(f(x))$ and ${}^*f(\mu(y)) = \bar{\mu}(f(y))$ where $x \neq y$ in X. Now since X is T_1, by Theorem (3.2.5), we have $\mu(x) \nsubseteq \mu(y)$ and $\mu(y) \nsubseteq \mu(x)$, so that ${}^*f(\mu(x)) \nsubseteq {}^*f(\mu(y))$ and ${}^*f(\mu(y)) \nsubseteq {}^*f(\mu(x))$. Therefore $\bar{\mu}(f(x)) \nsubseteq \bar{\mu}(f(y))$ and $\bar{\mu}(f(y)) \nsubseteq \bar{\mu}(f(x))$. Hence, by Theorem (3.2.5) again, (Y, T_Y) is a T_1-space.

(ii) Let A be a subspace of a T_1-space X, and let a, b be two distinct points in A. Since a, b are distinct points in X, by Theorem(3.2.5), we have $a \notin \mu(b)$ and $b \notin \mu(a)$. Since

$a \in (\mu(a) \cap {}^*A)$ and $b \in (\mu(b) \cap {}^*A)$, it follows that $\mu(a) \cap {}^*A \nsubseteq \mu(b) \cap {}^*A$ and $\mu(b) \cap {}^*A \nsubseteq \mu(a) \cap {}^*A$. Hence $\hat{\mu}(a) \nsubseteq \hat{\mu}(b)$ and $\hat{\mu}(b) \nsubseteq \hat{\mu}(a)$. Therefore, by Theorem (3.2.5), A is a T_1-space.

(iii) Let $\{X_i : i \in I\}$ be a family of spaces, and suppose that X_{i_0} is not T_1 for some $i_0 \in I$. Then, by Theorem(3.2.5), there are two distinct points $a, b \in X_{i_0}$ with $a \in \mu_{i_0}(b)$ or $b \in \mu_{i_0}(a)$. Choose some x, $y \in X = \prod_{i \in I} X_i$ with $x_{i_0} = a$, $y_{i_0} = b$ and $y_i = x_i$ for $i \in I - \{i_0\}$. Then $x \neq y$ in X with $x_{i_0} \in \mu_{i_0}(y_{i_0})$ or $y_{i_0} \in \mu_{i_0}(x_{i_0})$. Now, using Theorem (2.6.2), we have

$$x \in \mu(y) \text{ or } y \in \mu(x).$$

Hence, by Theorem(3.2.5) again, X is not a T_1-space.

Conversely, suppose that X is not a T_1-space. Then we find elements $x \neq y$ in X such that $x \in \mu(y)$ or $y \in \mu(x)$. Choose some $i_0 \in I$ such that $x_{i_0} \neq y_{i_0}$. Then, by Theorem (2.6.2), we have

$$x_{i_0} \in \mu_{i_0}(y_{i_0}) \text{ or } y_{i_0} \in \mu_{i_0}(x_{i_0}).$$

Hence, by Theorem (3.2.5), X_{i_0} is not a T_1-space. □

Example 3.2.7 (T_0-space that is not T_1)**.**

Let $X = \{a, b\}$, with the topology $T = \{\emptyset, \{a\}, X\}$. Since X is finite, we have ${}^*X = X$ and ${}^*T = T$. Now $\mu(a) = \{a\}$, $\mu(b) = X$. Since $a \neq b$ and $b \notin \mu(a)$, Theorem(3.2.2) yields that (X, T) is a T_0-space. Since $a \neq b$ but $a \in \mu(b)$, Theorem(3.2.5) yields that (X, T) is not T_1.

Theorem 3.2.8 (T_2-Space)**.**

A topological space (X, T) is T_2 (Hausdorff) iff $\mu(x) \cap \mu(y) = \emptyset \; \forall \; x, \, y \in X, \, x \neq y$.

Proof. Suppose that (X, T) is a T_2-space and let $x \neq y$ in X. Then there are two open sets $U, V \in T$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Hence, by transfer principle, ${}^*U \cap {}^*V = \emptyset$, and since $\mu(x) \subseteq {}^*U$, $\mu(y) \subseteq {}^*V$, we have

$$\mu(x) \cap \mu(y) = \emptyset.$$

Conversely, suppose that $\mu(x) \cap \mu(y) = \emptyset$ for $x \neq y$ in X. Then, by nuclei principle, there exist two open internal sets ${}^*U$, ${}^*V \in {}^*T$ such that $x \in {}^*U \subseteq \mu(x)$, $y \in {}^*V \subseteq \mu(y)$, so that ${}^*U \cap {}^*V = \emptyset$. Hence, by transfer principle, there exist two open sets U, $V \in T$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Therefore (X, T) is a T_2-space. □

Theorem 3.2.9.

(i) The property of being a T_2-space is a topological property.

(ii) Every subspace of a T_2-space is T_2.

(iii) Every nonempty product space is T_2 iff each factor space is T_2.

Proof. (i) Let (X, T_X) be a T_2-space and let (Y, T_Y) be any space homeomorphic to (X, T_X) with monads $\mu(x)(x \in X)$ and $\bar{\mu}(y)(y \in Y)$ respectively. It is required to show that (Y, T_Y) is also a T_2-space. Let $f : X \to Y$ be the homeomorphism, and let $f(x) \neq f(y)$ in Y. Since f is homeomorphism, by Theorem (2.5.6), we have ${}^*f(\mu(x)) = \bar{\mu}(f(x))$ and ${}^*f(\mu(y)) = \bar{\mu}(f(y))$ where $x \neq y$ in X. Now since X is T_2, we have $\mu(x) \cap \mu(y) = \emptyset$, so that ${}^*f(\mu(x)) \cap {}^*f(\mu(y)) = \emptyset$. Hence $\bar{\mu}(f(x)) \cap \bar{\mu}(f(y)) = \emptyset$, and therefore (Y, T_Y) is a T_2-space.

(ii) Let A be a subspace of a T_2-space X, and let a, b be two distinct points in A. Since a, b are distinct points in X, by Theorem(3.2.8), we have $\mu(a) \cap \mu(b) = \emptyset$ and so $(\mu(a) \cap {}^*A) \cap (\mu(b) \cap {}^*A) = \emptyset$. Hence $\hat{\mu}(a) \cap \hat{\mu}(b) = \emptyset$, and therefore, by Theorem (3.2.8), A is a T_2-space.

(iii) Let $\{X_i : i \in I\}$ be a family of spaces, and suppose that X_{i_0} is not T_2 for some $i_0 \in I$. Then, by Theorem(3.2.8), there exist $a, b \in X_{i_0}$ and $c \in {}^*X_{i_0}$, such that $a \neq b$ and $c \in \mu_{i_0}(a) \cap \mu_{i_0}(b)$. Choose some x, $y \in X = \prod_{i \in I} X_i$ with $x_{i_0} = a$, $y_{i_0} = b$ and $y_i = x_i$ for $i \in I - \{i_0\}$. Then $x \neq y$ in X. Consider the function $z_i = {}^*y_i$ for $i \in I - \{i_0\}$ and $z_{i_0} = c$. Then z is an internal function and since $z_{i_0} = c \in \mu_{i_0}(a) \cap \mu_{i_0}(b) = \mu_{i_0}(x_{i_0})$ $\cap\ \mu_{i_0}(y_{i_0})$, by Theorem (2.6.2), we have $z \in \mu(x) \cap \mu(y)$. Hence, by Theorem (3.2.8), X is not a T_2-space.

Conversely, suppose that X is not a T_2-space. Then we find elements $x \neq y$ in X such that $\mu(x) \cap \mu(y)$ contains some element $z \in {}^*X$. Choose some $i_0 \in I$ such that $x_{i_0} \neq y_{i_0}$. Now since $z \in \mu(x) \cap \mu(y)$, by Theorem (2.6.2), we have $z_{i_0} \in \mu_{i_0}(x_{i_0}) \cap \mu_{i_0}(y_{i_0})$. Hence, by Theorem (3.2.8), X_{i_0} is not a T_2-space. □

Example 3.2.10.

The discrete topology is Hausdorff, and every subset is both open and closed.

Example 3.2.11 (T_1-space that is not T_2)**.**

The finite complement topology T on $\mathbb{N}$ with $\mu(x) = (\{x\} \cup {}^*\mathbb{N}_\infty)$, is a T_1-space, since for $x \neq y$ we have $x \notin (\{y\} \cup {}^*\mathbb{N}_\infty) = \mu(y)$ and $y \notin (\{x\} \cup {}^*\mathbb{N}_\infty) = \mu(x)$. On the other hand, for $x \neq y$ we have $\mu(x) \cap \mu(y) = (\{x\} \cup {}^*\mathbb{N}_\infty) \cap (\{y\} \cup {}^*\mathbb{N}_\infty) = {}^*\mathbb{N}_\infty \neq \emptyset$. Hence $(\mathbb{N}, T)$ is not a T_2-space.

Theorem 3.2.12 (Regular Space)**.** Let (X, T) be a topological space. Then the following statements are equivalent:

(i) (X, T) is a regular space.

(ii) $\alpha \notin \mu(x) \Rightarrow \mu(\alpha) \cap \mu(x) = \emptyset$ for any $\alpha \in {}^*X$ and $x \in X$.

(iii) $x \notin F \Rightarrow \mu(F) \cap \mu(x) = \emptyset$ for any $x \in X$ and any closed set $F \subseteq X$.

Proof. (i)⇔(ii): Suppose that (X, T) is a regular space, and let $\alpha \in {}^*X$ and $x \in X$ be such that $\alpha \notin \mu(x)$. Then, there is $G \in T$ such that $x \in G$ and $\alpha \notin {}^*G$. By regularity, there is $U \in T$ such that $x \in U$ and $\text{cl}_X U \subseteq G$. Hence we have $\mu(x) \subseteq {}^*U$ and also $\mu(\alpha) \subseteq {}^*(X - \text{cl}_X U)$, since $\alpha \in {}^*X - {}^*G = {}^*(X - G) \subseteq {}^*(X - \text{cl}_X U)$. Thus $\mu(\alpha) \cap \mu(x) = \emptyset$, and we have shown that (i)⇒(ii).

Conversely, suppose (by contraposition) that (X, T) is not regular. We must show that there exist $\alpha \in {}^*X$ and $x \in X$ such that $\alpha \notin \mu(x)$ and $\mu(\alpha) \cap \mu(x) \neq \emptyset$. Indeed, since X is not regular, there are $x \in X$ and $G \in T$ such that $x \in G$ and $\text{cl}_X H \cap (X - G)$ $\neq \emptyset \; \forall \, H \in T_x$. Observe that the family of sets $\{\text{cl}_X H \cap (X - G) : H \in T_x\}$ has the finite intersection property. It follows that the family of internal sets $\{{}^*(\text{cl}_X H) \cap {}^*(X - G) : H \in T_x\}$ has the finite intersection property. Then, by saturation principle there exists $\alpha \in {}^*X$ such that

$$\alpha \in \bigcap_{H \in T_x} {}^*(\text{cl}_X H) - {}^*G.$$

Since $\alpha \in {}^*(\text{cl}_X H) = \text{cl}_{{}^*X} \, {}^*H$, we have ${}^*O \cap {}^*H \neq \emptyset \, \forall \, O, \, H \in T$ such that $\alpha \in {}^*O$ and $x \in H$. Also we have $\alpha \notin \mu(x)$, since $\alpha \notin {}^*G$. Observe that the family of sets $\{{}^*O \cap {}^*H\}_{O, H \in T}$ has the finite intersection property. Then using saturation principle again, we obtain

$$\mu(\alpha) \cap \mu(x) = \bigcap \{{}^*O \cap {}^*H : O, H \in T, \alpha \in {}^*O, x \in H\} \neq \emptyset.$$

This complete the proof that (ii)⇒(i).

(i)⇔(iii): Suppose that (X, T) is a regular space and let $x \in X$, $F \subseteq X$ be a closed set such that $x \notin F$. Since F is closed, by definition of closed sets, we have ${}^*F \cap \mu(x) = \emptyset$; i.e., $\alpha \notin \mu(x)\ \forall\, \alpha \in {}^*F$. By (i)⇒(ii), $\mu(\alpha) \cap \mu(x) = \emptyset\ \forall\, \alpha \in {}^*F$, so that

$$\bigcup_{\alpha \in {}^*F} \mu(\alpha) \cap \mu(x) = \emptyset.$$

Now since ${}^*F$ is an internal set, by Corollary (3.1.4), we have $\mu({}^*F) \cap \mu(x) = \emptyset$, and since $\mu({}^*F) = \mu(F)$, we get $\mu(F) \cap \mu(x) = \emptyset$. Thus we have shown (i)⇒(iii).

Conversely, suppose (iii) holds, and let $\alpha \in {}^*X$, $x \in X$ be such that $\alpha \notin \mu(x)$. Then there exists an open set G such that $x \in G$ and $\alpha \notin {}^*G$. Let $F = X - G$. Then F is closed and $x \notin F$, so by (iii) we have $\mu(F) \cap \mu(x) = \emptyset$. Since $\mu({}^*F) = \mu(F)$, we have $\mu({}^*F) \cap \mu(x) = \emptyset$. Since $\alpha \in {}^*F$, we have $\mu(\alpha) \cap \mu(x) = \emptyset$; hence, by (i)⇔(ii), (X, T) is a regular space. □

Theorem 3.2.13.

(i) The property of being a regular space is a topological property.

(ii) Every subspace of a regular space is regular.

(iii) Every nonempty product space is regular iff each factor space is regular.

Proof. (i) Let (X, T_X) be a regular space and let (Y, T_Y) be any space homeomorphic to (X, T_X) with monads $\mu(x)(x \in X)$ and $\bar{\mu}(y)(y \in Y)$, respectively. It is required to show that (Y, T_Y) is also a regular space. Let $f : X \to Y$ be the homeomorphism, and let $f(x) \notin f(F)$ in Y for any $f(x) \in Y$ and any closed set $f(F) \subseteq Y$. Since f is a homeomorphism, by Theorem (2.5.6), we have ${}^*f(\mu(x)) = \bar{\mu}(f(x))$ and ${}^*f(\mu(F)) = \bar{\mu}(f(F))$. Since X is regular, $\mu(x) \cap \mu(F) = \emptyset$, so that ${}^*f(\mu(x)) \cap {}^*f(\mu(F)) = \emptyset$. Therefore $\bar{\mu}(f(x)) \cap \bar{\mu}(f(F)) = \emptyset$. Hence, by Theorem (3.2.12), (Y, T_Y) is a regular space.

(ii) Let A be a subspace of a regular space X, and let $F \subseteq A$ be a closed set in A and let $x \in A$ be such that $x \notin F$. Then $F = A \cap K$ for some closed set K in X. Now $x \in A$ and $x \notin F$ implies $x \notin K$. Hence, by regularity of X, we have $\mu(K) \cap \mu(x) = \emptyset$, and so $(\mu(K) \cap {}^*A) \cap (\mu(x) \cap {}^*A) = (\mu(K) \cap \mu(x)) \cap {}^*A = \emptyset$; i.e., $\hat{\mu}(F) \cap \hat{\mu}(x) = \emptyset$. Since $F \subseteq K$ implies that $\hat{\mu}(F) \subseteq \hat{\mu}(K)$, it follows that $\hat{\mu}(F) \cap \hat{\mu}(x) = \emptyset$, and therefore, by Theorem (3.2.12), A is regular.

(iii) Suppose that $X = \prod_{i \in I} X_i$ is regular and fix $i \in I$. To show X_i is regular, we have to show that for any $b \in {}^*X_i$ and $a \in X_i$ with $b \notin \mu(a)$ we have $\mu(a) \cap \mu(b) = \emptyset$. Choose some $y \in {}^*X$ with $y_i = b$ for some standard $i \in {}^*I$. Since X is regular, we find $x \in X$ with $y \notin \mu(x)$ and $\mu(x) \cap \mu(y) = \emptyset$. Then using Theorem (2.6.2), we have $y_i \notin \mu_i(x_i)$ and $\mu_i(x_i) \cap \mu_i(y_i) = \emptyset$. Taking $a = x_i \in X_i$, we have $b \notin \mu(a)$ and $\mu(a) \cap \mu(b) = \emptyset$. Hence X_i is regular.

Conversely, suppose that X_i is regular for all $i \in I$. Let $y \in {}^*X$. For each standard $i \in {}^*I$, we have $y_i \in {}^*X_i$. Since X_i is regular, we find some $x_i \in X_i$ such that $y_i \notin \mu(x_i)$ and $\mu_i(x_i) \cap \mu_i(y_i) = \emptyset$. Then $x \in X$ (using the axiom of choice), and Theorem (2.6.2) implies that $y \notin \mu(x)$ and $\mu(x) \cap \mu(y) = \emptyset$. Hence X is regular. □

Example 3.2.14 (A regular space that is not Hausdorff)**.**
Let $X = \{a, b, c\}$, with the topology $T = \{\emptyset, \{a\}, \{b, c\}, X\}$. Since X is finite, we have ${}^*X = X$ and ${}^*T = T$. Now $\mu(a) = \{a\}$, $\mu(b) = \{b, c\}$ and $\mu(c) = \{b, c\}$. Moreover, $a \notin \mu(b)$ and $\mu(a) \cap \mu(b) = \emptyset$, $b \notin \mu(a)$ and $\mu(b) \cap \mu(a) = \emptyset$, $a \notin \mu(c)$ and $\mu(a) \cap \mu(c) = \emptyset$, $c \notin \mu(a)$ and $\mu(c) \cap \mu(a) = \emptyset$. Hence, by Theorem (3.2.12), X is regular. Finally, since $\mu(b) \cap \mu(c) = \{b, c\} \neq \emptyset$, Theorem (3.2.8) implies that X is not Hausdorff.

Definition 3.2.15 (T_3-space)**.** A regular T_1-space is called a T_3-*space*.

Theorem 3.2.16 (Normal Space)**.**

(i) A topological space (X, T) is normal iff $F_1 \cap F_2 = \emptyset$ implies $\mu(F_1) \cap \mu(F_2) = \emptyset$ for any closed sets F_1, $F_2 \subseteq X$.

(ii) The property of being a normal space is a topological property.

(iii) A closed subspace of a normal space is normal.

Proof. (i) Suppose that (X, T) is a normal space and let F_1, $F_2 \subseteq X$ be two disjoint closed sets. Then by normality there are two disjoint open sets U_1, U_2 such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$. So that, by transfer principle, we have ${}^*F_1 \subseteq {}^*U_1$, ${}^*F_2 \subseteq {}^*U_2$ with ${}^*U_1 \cap {}^*U_2 = \emptyset$, hence $\mu(F_1) \subseteq {}^*U_1$ and $\mu(F_2) \subseteq {}^*U_2$, and therefore $\mu(F_1) \cap \mu(F_2) = \emptyset$.

Conversely, suppose (by contraposition) that (X, T) is not normal. Then, there exist two disjoint closed sets F_1 and F_2 such that $U_1 \cap U_2 \neq \emptyset \ \forall \ U_1, U_2 \in T$ such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$. Observe that the family of sets $\{U_1 \cap U_2 : U_1, U_2 \in T\}$ has the finite intersection property. It follows that the family of internal sets $\{^*U_1 \cap {}^*U_2 : U_1,\, U_2 \in T\}$ has the finite intersection property. Hence, by saturation principle, we have:

$$\mu(F_1) \cap \mu(F_2) = \bigcap \{^*U_1 \cap {}^*U_2 : F_1 \subseteq U_1,\ F_2 \subseteq U_2,\, U_1, U_2 \in T\} \neq \emptyset.$$

(ii) Let (X, T_X) be a normal space and let (Y, T_Y) be any space homeomorphic to (X, T_X) with monads $\mu(x)(x \in X)$ and $\bar{\mu}(y)(y \in Y)$, respectively. It is required to show that (Y, T_Y) is also a normal space. Let $f : X \to Y$ be the homeomorphism, and let $f(F_1)$ and $f(F_2)$ be closed sets in Y such that $f(F_1) \cap f(F_2) = \emptyset$. Since f is a homeomorphism, by Theorem (2.5.6), we have $^*f(\mu(F_1)) = \bar{\mu}(f(F_1))$ and $^*f(\mu(F_2)) = \bar{\mu}(f(F_2))$. Now since X is normal and $F_1 \cap F_2 = \emptyset$, we have $\mu(F_1) \cap \mu(F_2) = \emptyset$, so that $^*f(\mu(F_1)) \cap {}^*f(\mu(F_2)) = \emptyset$. Therefore $\bar{\mu}(f(F_1)) \cap \bar{\mu}(f(F_2)) = \emptyset$. Hence (Y, T_Y) is a normal space.
(iii) Let A be a closed subspace of a normal space X, and let $F_1, F_2 \subseteq A$ be two disjoint closed sets in A. Then $F_1 = A \cap K_1$ and $F_2 = A \cap K_2$ for some closed sets K_1, K_2 in X. Now, by normality of X, we have $\mu(K_1) \cap \mu(K_2) = \emptyset$, and so $(\mu(K_1) \cap {}^*A) \cap (\mu(K_2) \cap {}^*A) = (\mu(K_1) \cap \mu(K_2)) \cap {}^*A = \emptyset$; i.e., $\hat{\mu}(K_1) \cap \hat{\mu}(K_2) = \emptyset$. Since $F_1 \subseteq K_1$ and $F_2 \subseteq K_2$, $\hat{\mu}(F_1) \subseteq \hat{\mu}(K_1)$ and $\hat{\mu}(F_2) \subseteq \hat{\mu}(K_2)$, it follows that $\hat{\mu}(F_1) \cap \hat{\mu}(F_2) = \emptyset$, and therefore, by (i), A is normal. □

Example 3.2.17 (A normal space that is not regular)**.**
Let $X = \{a, b, c\}$, with the topology $T = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Observe that the set of closed sets $\mathcal{F} = \{\emptyset, \{b, c\}, \{a, c\}, \{c\}, X\}$. Since X is finite, we have $^*X = X$ and $^*T = T$ and $^*\mathcal{F} = \mathcal{F}$. Now $\mu(a) = \{a\}$, $\mu(b) = \{b\}$ and $\mu(c) = X$. The only disjoint closed sets are X and $\emptyset$. Now since $\mu(X) \cap \mu(\emptyset) = \emptyset$, by Theorem (3.2.16), X is normal. Finally, since $c \notin \mu(a)$ but $\mu(c) \cap \mu(a) = X \cap \{a\} = \{a\} \neq \emptyset$, Theorem (3.2.12) implies that X is not regular.

Definition 3.2.18 (T_4-space)**.** A normal T_1-space is called a T_4-*space.*

Theorem 3.2.19 (Completely Normal Space)**.**
A topological space (X, T) is *completely normal* iff $A_1 \cap A_2 = \emptyset$ implies $\mu(A_1) \cap \mu(A_2) = \emptyset$ for any two sets $A_1, A_2 \subseteq X$.

Proof. Suppose that (X, T) is a completely normal space and let $A_1,\ A_2 \subseteq X$ be two disjoint sets. Then by complete normality there are two disjoint sets $U_1,\ U_2$ such that

$A_1 \subseteq U_1$ and $A_2 \subseteq U_2$. So that, by transfer principle, we have ${}^*A_1 \subseteq {}^*U_1$, ${}^*A_2 \subseteq {}^*U_2$ with ${}^*U_1 \cap {}^*U_2 = \emptyset$, hence $\mu(A_1) \subseteq {}^*U_1$ and $\mu(A_2) \subseteq {}^*U_2$, and therefore $\mu(A_1) \cap \mu(A_2) = \emptyset$.

Conversely, suppose (by contraposition) that (X, T) is not completely normal. Then, there exist two disjoint sets A_1 and A_2 such that $U_1 \cap U_2 \neq \emptyset\ \forall\ U_1,\ U_2 \in T$ such that $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$. Observe that the family of sets $\{U_1 \cap U_2 : U_1,\ U_2 \in\ T\}$ has the finite intersection property. It follows that the family of internal sets $\{{}^*U_1 \cap {}^*U_2 : U_1,\ U_2 \in\ T\}$ has the finite intersection property. Hence, by saturation principle, we have

$$\mu(A_1) \cap \mu(A_2) = \bigcap \{{}^*U_1 \cap {}^*U_2 : A_1 \subseteq U_1,\ A_2 \subseteq U_2,\ U_1, U_2 \in\ T\} \neq \emptyset.$$

□

Theorem 3.2.20. The property of being a completely normal space is a topological property.

Proof. Let (X, T_X) be a completely normal space and let (Y, T_Y) be any space homeomorphic to (X, T_X) with monads $\mu(x)(x \in X)$ and $\bar{\mu}(y)(y \in Y)$, respectively. It is required to show that (Y, T_Y) is also a completely normal space. Let $f : X \to Y$ be the homeomorphism, and let $f(A_1) \cap f(A_2) = \emptyset$ for any sets $f(A_1),\ f(A_2) \in Y$. Since f is a homeomorphism, by Theorem (2.5.6), we have ${}^*f(\mu(A_1)) = \bar{\mu}(f(A_1))$ and ${}^*f(\mu(A_2)) = \bar{\mu}(f(A_2))$. Now since X is completely normal, and $A_1 \cap A_2 = \emptyset$, Theorem (3.2.19) implies that $\mu(A_1) \cap \mu(A_2) = \emptyset$, so that ${}^*f(\mu(A_1)) \cap {}^*f(\mu(A_2)) = \emptyset$. Therefore $\bar{\mu}(f(A_1)) \cap \bar{\mu}(f(A_2)) = \emptyset$. Hence, by Theorem (3.2.19) again, (Y, T_Y) is completely normal. □

Definition 3.2.21 (T_5-space)**.** A completely normal T_1-space is called a T_5-*space.*

Theorem 3.2.22. If (X, T) is T_0 and regular, then (X, T) is Hausdorff.

Proof. Let $x,\ y \in X$ be such that $x \neq y$. Since (X, T) is T_0, we have either $x \notin \mu(y)$ or $y \notin \mu(x)$, so that $\mu(x) \neq \mu(y)$. On the other hand, by regularity, we have $\mu(\alpha) \cap \mu(y) = \emptyset\ \forall\, \alpha \in {}^*X$ such that $\alpha \notin \mu(y)$, in particular, $\mu(x) \cap \mu(y) = \emptyset$. Hence (X, T) is Hausdorff. □

Theorem 3.2.23. Let f and g be continuous mappings of a topological space X into a Hausdorff space Y. Then the set $B = \{x : f(x) \neq g(x)\}$ is open in X.

Proof. Let $x \in B$ and $y \in \mu(x)$. By Proposition (2.3.2), we have to show that $y \in {}^*B$. Since f and g are continuous at the point x, we have

$$^*f(\mu(x)) \subseteq \bar{\mu}(f(x)) \text{ and } {}^*g(\mu(x)) \subseteq \bar{\mu}(g(x)),$$

and so ${}^*f(y) \subseteq \bar{\mu}(f(x))$ and ${}^*g(y) \subseteq \bar{\mu}(g(x))$. Now since Y is Hausdorff and $f(x) \neq f(y)$, we have $\bar{\mu}(f(x)) \cap \bar{\mu}(g(x)) = \emptyset$ and so ${}^*f(y) \cap {}^*g(y) = \emptyset$. Hence ${}^*f(y) \neq {}^*g(y)$, and therefore $y \in {}^*B$. □

Theorem 3.2.24. If (X, T) is compact and Hausdorff, then (X, T) is regular.

Proof. Let $\alpha \in {}^*X$ and $x \in X$ be such that $\alpha \notin \mu(x)$. Since (X, T) is compact, we have $\alpha \in \mu(y)$ for some $y \in X$. By the choice of x and y we have $\mu(x) \neq \mu(y)$, and hence $x \neq y$. Now since (X, T) is Hausdorff, by Theorem (3.2.8), we have $\mu(x) \cap \mu(y) = \emptyset$. On the other hand, $\mu(\alpha) \subseteq \mu(y)$ by Corollary (2.1.8), so that $\mu(x) \cap \mu(\alpha) = \emptyset$. Hence, by Theorem (3.2.12), (X, T) is regular. □

Theorem 3.2.25. If (X, T) is compact and regular, then (X, T) is normal.

Proof. Let F_1 and F_2 be two disjoint closed sets in X. Since F_1 is closed and $F_2 \subseteq X - F_1$, for any $x \in F_2$ we have ${}^*F_1 \cap \mu(x) = \emptyset$. Hence, $\alpha \notin \mu(x)$ for any $\alpha \in {}^*F_1$ and any $x \in F_2$. Since (X, T) is regular, Theorem (3.2.12) implies that $\mu(\alpha) \cap \mu(x) = \emptyset$, and hence that

$$\mu(\alpha) \cap \bigcup_{x \in F_2} \mu(x) = \emptyset \text{ for any } \alpha \in {}^*F_1. \tag{3.1}$$

Now, being a closed subset of a compact space, F_2 is compact, so that (3.1) and Theorem (3.1.7) yield that

$$\mu(\alpha) \cap \mu(F_2) = \emptyset \text{ for any } \alpha \in {}^*F_1,$$

which implies that

$$(\bigcup_{\alpha \in {}^*F_1} \mu(\alpha)) \cap \mu(F_2) = \emptyset.$$

Since F_1 is also compact, Theorem (3.1.7) implies that $(\bigcup_{\alpha \in {}^*F_1} \mu(\alpha)) = (\bigcup_{\alpha \in F_1} \mu(\alpha))$, and hence

$$(\bigcup_{\alpha \in F_1} \mu(\alpha)) \cap \mu(F_2) = \emptyset.$$

Also Theorem (3.1.7) implies that $\mu(F_1) = (\bigcup_{\alpha \in F_1} \mu(\alpha))$; hence we have that $\mu(F_1) \cap \mu(F_2) = \emptyset$. Therefore, by Theorem (3.2.16), (X, T) is normal. □

Theorem 3.2.26. A compact set in a Hausdorff space is closed.

Proof. Let (X, T) be a Hausdorff space and let $K \subseteq X$ be compact. Suppose $x \in X$ is such that $y \in \mu(x)$ for some $y \in {}^*K$. By proposition (4.3.6), we have to show that $x \in K$. Now, by compactness of K, Theorem (3.1.7) implies that $y \in \mu(z)$ for some $z \in K$, and hence $y \in \mu(x) \cap \mu(z) \neq \emptyset$. Since (X, T) is Hausdorff, it follows that $x = z \in K$, and therefore K is closed. □

Chapter 4

Nonstandard Real Numbers ${}^*\mathbb{R}$

In this chapter we give a specific example of constructing a nonstandard model ${}^*\mathbb{R}$ of the reals $\mathbb{R}$, to help the reader understand the main idea of the whole subject of this thesis. Section one contains an altrapower construction of the nonstandard real numbers ${}^*\mathbb{R}$, using the sequential approach presented in Tom Lindstrom [13]. Section two includes properties of elements of ${}^*\mathbb{R}$. In section three we talk about the (usual) topology on $\mathbb{R}$ using the nonstandard definitions.

4.1 Altrapower Construction of ${}^*\mathbb{R}$

Depending on the sequential approach presented in Tom Lindstrom [13], we give here a construction of the nonstandard model ${}^*\mathbb{R}$.

Definition 4.1.1. Let $\mathbb{R}^{\mathbb{N}}$ represents the set of all sequences with domain $\mathbb{N}$ and range values (images) in $\mathbb{R}$. Of course, sequences are functions, (maps, mappings, etc.). We define *binary operations* $+$ and $\cdot$, for sequences by simply taking any two f, $g \in \mathbb{R}^{\mathbb{N}}$ and defining $f + g = h$ to be the sequence h where the values of h are $h(n) = f(n) + g(n)$ and $f \cdot g = fg = k$ to be the sequence k where the values of k are $k(n) = f(n)g(n)$ for each $n \in \mathbb{N}$. This forms what is called a ring with unity.

What we will do later is to show that there's a subset of $\mathbb{R}^{\mathbb{N}}$ that behaves like the real numbers, with respect to the defined operations, and we will use this subset as if it is the real numbers.

In all that follows, $\mathcal{U} = \mathcal{U}_{\mathbb{N}}$ will always be a free ultrafilter.

Definition 4.1.2 (Equality in $\mathcal{U}$)**.**
Let $A, B \in \mathbb{R}^{\mathbb{N}}$. Define $A =_{\mathcal{U}} B$ iff $\{n : A_n = B_n\} = U \in \mathcal{U}$. (The set of all $n \in \mathbb{N}$ such that the values of the sequences A and B are equal.)

Theorem 4.1.3. [8] The relation $=_{\mathcal{U}}$ is an equivalence relation on $\mathbb{R}^{\mathbb{N}}$.

Proof. Of course, properties of the = for members of $\mathbb{R}$ are used. First, notice that $\{n : A_n = A_n\} = \mathbb{N} \in \mathcal{U}$ for any $A \in \mathbb{R}^{\mathbb{N}}$. Thus, the relation is reflexive. Clearly, for any $A, B \in \mathbb{R}^{\mathbb{N}}$, if $\{n : A_n = B_n\} \in \mathcal{U}$, then $\{n : B_n = A_n\} \in \mathcal{U}$. Thus, the relation is symmetric. Finally, suppose that $A, B, C \in \mathbb{R}^{\mathbb{N}}$ and $A =_{\mathcal{U}} B$ and $B =_{\mathcal{U}} C$. Then, $\{n : A_n = B_n\} \in \mathcal{U}$ and $\{n : B_n = C_n\} \in \mathcal{U}$. Since $\mathcal{U}$ is a filter, the word "and" implies

$$\{n : A_n = B_n\} \cap \{n : B_n = C_n\} \in \mathcal{U}.$$

Of course, this "intersection" need not give all the values of $\mathbb{N}$ that these three sequences have in common, but that does not matter since the "superset" property for a filter implies from the result

$$\{n : A_n = B_n\} \cap \{n : B_n = C_n\} \subseteq \{n : A_n = C_n\},$$

that $\{n : A_n = C_n\} \in \mathcal{U}$. Thus, the relation is transitive. Hence $=_{\mathcal{U}}$ is an equivalence relation. □

Definition 4.1.4 (Equivalence Classes)**.**
We now use the relation $=_{\mathcal{U}}$ to define subsets of $\mathbb{R}^{\mathbb{N}}$. For each $A \in \mathbb{R}^{\mathbb{N}}$, let the set

$$[A] = \{x \in \mathbb{R}^{\mathbb{N}} : x =_{\mathcal{U}} A\}.$$

Note that for each $A, B \in \mathbb{R}^{\mathbb{N}}$, either $[A] = [B]$ or $[A] \cap [B] = \emptyset$ (The = here is the set-theoretic equality). Denote the set of all of these equivalence classes by ${}^*\mathbb{R}$; i.e., ${}^*\mathbb{R} := \mathbb{R}^{\mathbb{N}}/{=_{\mathcal{U}}}$ and call this set the set of all *hyperreal numbers*. (The * is often translated as "hyper"). Consequently,

$${}^*\mathbb{R} = \{[A] : A \in \mathbb{R}^{\mathbb{N}}\}.$$

After various relations are defined on ${}^*\mathbb{R}$, the resulting "structure" is generally called an *ultrapower*.

The reals $\mathbb{R}$ are identified with the equivalence classes of constant sequences [example: 1 in $\mathbb{R}$ is defined in ${}^*\mathbb{R}$ by $(1, 1, 1, \ldots)$], so that ${}^*\mathbb{R}$ is then an extension of $\mathbb{R}$.

Definition 4.1.5 (Addition and multiplication in ${}^*\mathbb{R}$)**.**

Consider any a, b, $c \in {}^*\mathbb{R}$. Define $a \ {}^*\!+ \ b := c$ iff $\{n : A_n + B_n = C_n\} \in \mathcal{U}$. [Note: such definition assumes that you have selected some sequences $A_n \in a$, $B_n \in b$, $C_n \in c$.] Now define $a \ {}^*\!\cdot \ b := c$ iff $\{n : A_n \cdot B_n = C_n\} \in \mathcal{U}$.

Theorem 4.1.6. The operations ${}^*+$ and ${}^*\cdot$ are well-defined.

Proof. Let a, $b \in {}^*\mathbb{R}$, and let $[A]$, $[D] \in a$, $[B]$, $[F] \in b$. Now, since $\{n : A_n = D_n\} \in \mathcal{U}$ and $\{n : B_n = F_n\} \in \mathcal{U}$, we have

$$\{n : A_n = D_n\} \cap \{n : B_n = F_n\} \in \mathcal{U},$$

so that

$$\{n : A_n = D_n\} \cap \{n : B_n = F_n\} \subseteq \{n : A_n + B_n = D_n + F_n\},$$

and by the superset property, we have

$$\{n : A_n + B_n = D_n + F_n\} \in \mathcal{U}.$$

Thus the ${}^*+$ is well-defined. In like manner for the ${}^*\cdot$. □

Theorem 4.1.7. [6] For the structure $\langle \ {}^*\mathbb{R}, {}^*+, {}^*\cdot \ \rangle$, the following holds:

(i) $[0]$ is the *additive identity.*

(ii) For each $a = [A] \in {}^*\mathbb{R}$, $-a = [-A]$ is the *additive inverse.*

(iii) $[1]$ is the *multiplicative identity.*

(iv) If $a \neq [0]$, then there exists $b = [B] \in {}^*\mathbb{R}$ such that $a \ {}^*\!\cdot \ b = [1]$.

(v) For each $n \in \mathbb{N}$, if $D_n = A_n + B_n$ and $E_n = A_n \cdot B_n$, then $[A] \ {}^*\!+ \ [B] = [D]$ and $[A] \ {}^*\!\cdot \ [B] = [E]$. That is, our definitions for addition and multiplication of sequences and the hyper operations ${}^*+$, ${}^*\cdot$ are compatible.

Proof. (i) Let $[A] \ {}^*\!+ \ [0] = [C]$. Considering that $\{n : A_n + 0_n = C_n\} \in \mathcal{U}$ and $\{n : A_n + 0_n = C_n\} \subseteq \{n : A_n = C_n\} \in \mathcal{U}$, then $[A] = [C]$.

(ii) Since $\{n : A_n + (-A_n) = 0 = 0_n\} = \mathbb{N} \in \mathcal{U}$, we have $[A] \ {}^*\!+ \ [-A] = [0]$.

(iii) Let $[A] \ {}^*\!\cdot \ [1] = [C]$. Considering that $\{n : A_n \cdot 1_n = C_n\} \in \mathcal{U}$ and $\{n : A_n \cdot 1_n = C_n\} \subseteq \{n : A_n = C_n\} \in \mathcal{U}$, then $[A] = [C]$.

(iv) Let $[A] \neq [0]$. Then $\{n : A_n = 0 = 0_n\} = U \notin \mathcal{U}$. Hence, $\mathbb{N} - U = \{n : A_n \neq 0\} \in \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter. Define

$$B_n = \begin{cases} A_n^{-1} & \text{if } A_n \neq 0 \\ 0 & \text{if } A_n = 0 \end{cases}.$$

Notice that $\{n : A_n \cdot B_n = 1 = 1_n\} = \{n : A_n \neq 0\} \in \mathcal{U}$. Hence $[A] \,{}^*\!\cdot\, [B] = [1]$.

(v) By definition, $[A] \,{}^*\!+\, [B] = [C]$ iff $\{n : A_n + B_n = C_n\} \in \mathcal{U}$. However, $\{n : A_n + B_n = D_n\} = \mathbb{N} \in \mathcal{U}$. Hence,

$$\{n : A_n + B_n = C_n\} \cap \{n : A_n + B_n = D_n\} = \{n : C_n = D_n\} \in \mathcal{U}.$$

Thus $[C] = [D]$. In like manner, the result holds for multiplication. □

Definition 4.1.8 (Order)**.**
For each $a = [A]$, $b = [B] \in {}^*\mathbb{R}$ define $a \,{}^*\!\leq b$ iff $\{n : A_n \leq B_n\} \in \mathcal{U}$.

Recall the following definition.

Definition 4.1.9 (Totally Ordered Field [11])**.**
A field $\langle F, +, \cdot \rangle$ with a total order $\leq$ is an ordered field if the order satisfies the following properties:

(i) If $a \leq b$ in F then $a + c \leq b + c$ for any $c \in F$.

(ii) If $0 \leq a$ and $0 \leq b$ in F then $0 \leq a \cdot b$.

Theorem 4.1.10. [6] The structure $\langle\, {}^*\mathbb{R}, {}^*+, {}^*\cdot, {}^*\leq \,\rangle$ is a totally ordered field.

Proof. First, notice that $\{n : A_n \leq A_n\} = \mathbb{N} \in \mathcal{U}$. Thus, ${}^*\leq$ is reflexive. Next, this relation needs to be anti-symmetric. So, assume that $[A] \,{}^*\!\leq [B]$, $[B] \,{}^*\!\leq [A]$. Then

$$\{n : A_n \leq B_n\} \cap \{n : B_n \leq A_n\} \subseteq \{n : A_n = B_n\} \in \mathcal{U}.$$

Hence, $[A] = [B]$. For transitivity, consider $[A] \,{}^*\!\leq [B]$, $[B] \,{}^*\!\leq [C]$. Then

$$\{n : A_n \leq B_n\} \cap \{n : B_n \leq C_n\} \subseteq \{n : A_n \leq C_n\} \in \mathcal{U}.$$

Thus, $[A] \,{}^*\!\leq [C]$. It follows that $\langle\, {}^*\mathbb{R}, {}^*\leq \,\rangle$ is a partially ordered set. (Notice that the same processes seem to be used each time. That is because $\mathcal{U}$ is closed under finite intersection and supersets.)

Next to show that $\langle {}^*\mathbb{R}, {}^*\leq \rangle$ is totally ordered, let $[A], [B] \in {}^*\mathbb{R}$. Then by trichotomy law for $\mathbb{R}$, we have $\{n : A_n < B_n\} \in \mathcal{U}$ or $\{n : A_n > B_n\} \in \mathcal{U}$ or $\{n : A_n = B_n\} \in \mathcal{U}$. Hence $[A] \ {}^*\!< [B]$ or $[A] \ {}^*\!> [B]$ or $[A] \ {}^*\!= [B]$. To show that $\langle {}^*\mathbb{R}, {}^*+, {}^*\cdot, {}^*\leq \rangle$ is a totally ordered field, all that's really needed is to show that it satisfies two properties related to this order and the ${}^*+$, ${}^*\cdot$ operations [11]. So, let $[A], [B], [C] \in {}^*\mathbb{R}$, and let $[A] \ {}^*\!\leq [B]$. Then

$$\{n : A_n \leq B_n\} \subseteq \{n : A_n + C_n \leq B_n + C_n\} \in \mathcal{U}.$$

Thus $[A] \ {}^*\!+ [C] \ {}^*\!\leq [B] \ {}^*\!+ [C]$. Now suppose that $[0] \ {}^*\!\leq [A], [B]$. Then

$$\{n : 0 \leq A_n\} \cap \{n : 0 \leq B_n\} \subseteq \{n : 0 \leq A_n \cdot B_n\} \in \mathcal{U}.$$

Hence $[0] \ {}^*\!\leq [A] \ {}^*\!\cdot [B] \ {}^*\!= [AB]$. $\square$

Definition 4.1.11 (Hyper * extensions of standard objects [6])**.**
For any $C \subseteq \mathbb{R}$ (a $1-$ary relation), let $b = [B] \in {}^*C$, iff $\{n : B_n \in C\} \in \mathcal{U}$. Let Φ be any $k-$ary $(k > 1)$ relation. Then

$$(a_1, \ldots, a_k) = ([A_1], \ldots, [A_k]) \in {}^*\Phi \Leftrightarrow \{n : (A_1(n), \ldots, A_k(n)) \in \Phi\} \in \mathcal{U}.$$

This extension process can be continued for other mathematical entities.

Theorem 4.1.12. [6] The hyper-extensions of standard objects are well-defined.

Proof. In general, for any $[B] \in {}^*\mathbb{R}$, let $[B] = [B']$. Then $\{n : B_n = B'_n\} \in \mathcal{U}$. That is, let $B' \in \mathbb{R}^{\mathbb{N}}$ be any other member of the equivalence class $[B]$. Let $C \subseteq \mathbb{R}$ be a 1-ary relation. Then

$$\{n : B_n = B'_n\} \subseteq \{n : (B_n \in C) \Leftrightarrow (B'_n \in C)\} \in \mathcal{U},$$
$$\{n : B_n \in C\} \cap \{n : (B_n \in C) \Leftrightarrow (B'_n \in C)\} \subseteq \{n : B'_n \in C\} \in \mathcal{U} \Rightarrow [B'] \in {}^*C,$$
$$\{n : B'_n \in C\} \cap \{n : (B_n \in C) \Leftrightarrow (B'_n \in C)\} \subseteq \{n : B_n \in C\} \in \mathcal{U} \Rightarrow [B] \in {}^*C.$$

Thus the 1-ary relation C is well defined. For the other $k-$ary relations, proceed as just done but alter the proof by starting with

$$\{n : B_1(n) = B'_1(n)\} \cap \cdots \cap \{n : B_k(n) = B'_k(n)\} \subseteq$$
$$\{n : (B_1(n), \ldots, B_k(n)) \in \Phi \Leftrightarrow (B'_1(n), \ldots B'_k(n)) \in \Phi\}.$$

Thus the $k-$ary relation Φ is well defined. $\square$

Definition 4.1.13 (Standard objects operator [6]).
For each $x \in \mathbb{R}$, let ${}^*x := [X] \in {}^*\mathbb{R}$, where $\{n : X_n = x\} = \mathbb{N}$ (the constant sequence). Then for $X \subseteq \mathbb{R}$, let ${}^\sigma X := \{{}^*x : x \in X\} \subseteq {}^*\mathbb{R}$. For $n > 1$ and each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let ${}^*x = ({}^*x_1, \ldots, {}^*x_n) \in {}^*(\mathbb{R}^n)$. For $X \subseteq \mathbb{R}^n$ let ${}^\sigma X := \{{}^*x : x \in X\} \subseteq {}^*(\mathbb{R}^n)$. Each such ${}^*x$ and ${}^\sigma X$ is called a *standard object*. Thus, ${}^\sigma\mathbb{R}$ is the set of *embedded real numbers*.

4.2 The Hyperreal Properties

Nonstandard analysis begins with the construction of a richer real line ${}^*\mathbb{R}$ called the hyperreals or nonstandard reals [2]. This is an ordered field that extends the (standard) reals $\mathbb{R}$ in two main ways:

(1) ${}^*\mathbb{R}$ contains non-zero infinitesimal numbers. (see Definition (4.2.3) to come).

(2) ${}^*\mathbb{R}$ contains positive and negative infinite numbers. (see Definition (4.2.3) to come).

Now using our definitions in Chapter 1, let $S = \mathbb{R}$, $V(\mathbb{R})$ be its superstructure and $\mathcal{L}(V(\mathbb{R}))$ be its language. We shall refer to $V(\mathbb{R})$ as standard analysis. Let $V({}^*\mathbb{R})$ be a nonstandard extension of $V(\mathbb{R})$ and $\mathcal{L}(V({}^*\mathbb{R}))$ be its language. We shall refer to $V({}^*\mathbb{R})$ as non-standard analysis and the elements of ${}^*\mathbb{R}$ as *nonstandard real numbers* or *hyperreal numbers*.

Let $A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $A(x, y) = x + y$, and $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $M(x, y) = x \cdot y$, be the addition and the multiplication in $\mathbb{R}$, respectively. Let $\mathbb{R}^+$ be the set of the positive real numbers. Let ${}^*A$, ${}^*M$ and ${}^*\mathbb{R}^+$ be the nonstandard extensions of A, M and $\mathbb{R}^+$, respectively. Observe that ${}^*A$ and ${}^*M$ are functions of the type ${}^*A : {}^*\mathbb{R} \times {}^*\mathbb{R} \to {}^*\mathbb{R}$ and ${}^*M : {}^*\mathbb{R} \times {}^*\mathbb{R} \to {}^*\mathbb{R}$, respectively.

Definition 4.2.1 (Field operations and order relation in ${}^*\mathbb{R}$).
We define the addition and multiplication in ${}^*\mathbb{R}$ by: $x + y = {}^*A(x, y)$ and $xy = {}^*M(x, y)$, respectively. The order relation in ${}^*\mathbb{R}$ is defined by $x > 0$ if $x \in {}^*\mathbb{R}^+$.

Theorem 4.2.2 (Properties of ${}^*\mathbb{R}$ [24]).
The set of nonstandard real numbers ${}^*\mathbb{R}$ is a totally ordered non-Archimedean field which is a proper extension of $\mathbb{R}$, in symbols, $\mathbb{R} \subseteq {}^*\mathbb{R}$, $\mathbb{R} \neq {}^*\mathbb{R}$.

Proof. Let 0 and 1 be the zero and the unit in $\mathbb{R}$, respectively. The fact that $\mathbb{R}$ is a totally ordered field can be formalized in $\mathcal{L}(V(\mathbb{R}))$ by the following statements:

$$(\forall\, x \in \mathbb{R})([(x + 0 = x) \wedge (x \cdot 0 = 0)]$$

$$(\forall\, x \in \mathbb{R})(\exists\, y \in \mathbb{R})[A(x,\, y) = 0)]$$

$$(\forall\, x \in \mathbb{R})[M(x, 1) = x]$$

$$(\forall\, x \in \mathbb{R})[(x \neq 0) \Rightarrow (\exists\, y \in \mathbb{R})[M(x,\, y) = 1]]$$

$$(\forall\, x \in \mathbb{R})(\forall\, y \in \mathbb{R})[A(x,\, y) = A(y,\, x)]$$

$$(\forall\, x \in \mathbb{R})(\forall\, y \in \mathbb{R})[A(A(x,\, y),\, z) = A(x,\, A(y,\, z))]$$

$$(\forall\, x \in \mathbb{R})(\forall\, y \in \mathbb{R})[M(x,\, y) = M(y,\, x)]$$

$$(\forall\, x \in \mathbb{R})(\forall\, y \in \mathbb{R})[M(M(x,\, y),\, z) = M(x,\, M(y,\, z))]$$

$$(\forall\, x \in \mathbb{R})(\forall\, y \in \mathbb{R})(\forall\, z \in \mathbb{R})[M(A(x,\, y),\, z) = A(M(x,\, z),\, M(y, z))]$$

$$0 \in \mathbb{R}^+$$

$$(\forall\, x \in \mathbb{R}^+)(\forall\, y \in \mathbb{R}^+)[(A(x,\, y) \in \mathbb{R}^+) \wedge (M(x,\, y) \in \mathbb{R}^+)]$$

$$(\forall\, y \in \mathbb{R})[(y = 0) \vee (y \in \mathbb{R}^+) \vee (-y \in \mathbb{R}^+)],$$

where $-y$ is the (unique) solution of the equation $A(x,\, y) = 0$ in $\mathbb{R}$. Then, by transfer principle, we have:

$$(\forall\, x \in\, {}^*\mathbb{R})([(x + 0 = x) \wedge (x \cdot 0 = 0)]$$

$$(\forall\, x \in\, {}^*\mathbb{R})(\exists\, y \in\, {}^*\mathbb{R})[{}^*A(x,\, y) = 0)]$$

$$(\forall\, x \in\, {}^*\mathbb{R})[{}^*M(x,\, 1) = x]$$

$$(\forall\, x \in\, {}^*\mathbb{R})[(x \neq 0) \Rightarrow (\exists\, y \in\, {}^*\mathbb{R})[{}^*M(x,\, y) = 1]]$$

$$(\forall\, x \in\, {}^*\mathbb{R})(\forall\, y \in\, {}^*\mathbb{R})[{}^*A(x,\, y) =\, {}^*A(y,\, x)]$$

$$(\forall\, x \in\, {}^*\mathbb{R})(\forall\, y \in\, {}^*\mathbb{R})[{}^*A({}^*A(x,\, y),\, z) =\, {}^*A(x,\, A(y,\, z))]$$

$$(\forall\, x \in\, {}^*\mathbb{R})(\forall\, y \in\, {}^*\mathbb{R})[{}^*M(x,\, y) =\, {}^*M(y,\, x)]$$

$$(\forall\, x \in\, {}^*\mathbb{R})(\forall\, y \in\, {}^*\mathbb{R})[{}^*M({}^*M(x,\, y),\, z) =\, {}^*M(x,\, {}^*M(y, z))]$$

$$(\forall\, x \in\, {}^*\mathbb{R})(\forall\, y \in\, {}^*\mathbb{R})(\forall\, z \in\, {}^*\mathbb{R})[{}^*M({}^*A(x,\, y),\, z) =\, {}^*A({}^*M(x,\, z),\, {}^*M(y,\, z))]$$

$$0 \in\, {}^*\mathbb{R}^+$$

$$(\forall\, x \in\, {}^*\mathbb{R}^+)(\forall\, y \in\, {}^*\mathbb{R}^+)[({}^*A(x,\, y) \in\, {}^*\mathbb{R}^+) \wedge ({}^*M(x,\, y) \in\, {}^*\mathbb{R}^+)]$$

$$(\forall\, y \in\, {}^*\mathbb{R})[(y = 0) \vee (y \in\, {}^*\mathbb{R}^+) \vee (-y \in\, {}^*\mathbb{R}^+)],$$

where $-y$ is the (unique) solution of the equation ${}^*A(x, y) = 0$ in ${}^*\mathbb{R}$. The interpretation of the above formulae means that ${}^*\mathbb{R}$ is a totally ordered field.

On the other hand, $\mathbb{R} \subseteq {}^*\mathbb{R}$, $\mathbb{R} \neq {}^*\mathbb{R}$ follows from Corollary (1.5.9) (applied to $A = S = \mathbb{R}$), since $\mathbb{R}$ is an infinite set. Thus, ${}^*\mathbb{R}$ turns out to be a proper totally ordered field extension of $\mathbb{R}$. It follows that ${}^*\mathbb{R}$ is a non-Archimedean field (ordered field that has infinitesimal and infinitely large elements [24]). □

Definition 4.2.3. [2] Let $x, y \in {}^*\mathbb{R}$. We say that:

(1) x is *infinitesimal* if $|x| < \epsilon$, for any positive real number ϵ; we write $x \approx 0$, where $|.|$ is the extension of the modolus function to ${}^*\mathbb{R}$. This takes its values in ${}^*\mathbb{R}$, and is defined just as in $\mathbb{R}$, so that $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

(2) x is *finite* if, for some positive real number ϵ, $|x| < \epsilon$.

(3) x is *infinite* (or *infinitely large*) if it is not finite; i.e., $|x| > \epsilon$ for any positive real number ϵ; we write $x \approx \infty$.

(4) x, y are *infinitely close* if $x - y$ is infinitesimal; we write $x \approx y$.

Let $\mathcal{I}({}^*\mathbb{R})$, $\mathcal{F}({}^*\mathbb{R})$ and $\mathcal{L}({}^*\mathbb{R})$ denote the sets of the infinitesimals, finite and infinitely large numbers in ${}^*\mathbb{R}$, respectively. It can be easily shown (as in any totally ordered field – see [6]) that

$$\begin{aligned}
&{}^*\mathbb{R} = \mathcal{F}({}^*\mathbb{R}) \cup \mathcal{L}({}^*\mathbb{R}),\ \mathcal{F}({}^*\mathbb{R}) \cap \mathcal{L}({}^*\mathbb{R}) = \emptyset,\\
&\mathcal{I}({}^*\mathbb{R}) \subseteq \mathcal{F}({}^*\mathbb{R}),\ \mathbb{R} \subseteq \mathcal{F}({}^*\mathbb{R}),\\
&\mathbb{R} \cap \mathcal{I}({}^*\mathbb{R}) = \{0\},\\
&\mathcal{L}({}^*\mathbb{R}) = \{1/x : x \in \mathcal{I}({}^*\mathbb{R}),\ x \neq 0\}.
\end{aligned}$$

Definition 4.2.4. For $x \in {}^*\mathbb{R}$, the *monad* of x is the subset of ${}^*\mathbb{R}$ given by:

$$\mu(x) := \{y \in {}^*\mathbb{R} : x \approx y\}.$$

Theorem 4.2.5 (Standard Part Theorem [8])**.**
If $x \in {}^*\mathbb{R}$ is finite, then there is a unique $r \in \mathbb{R}$ such that $x \approx r$; i.e., any finite hyperreal x is uniquely expressible as $x = r + \delta$ with r a standard real and δ infinitesimal.

Proof. For the existence, let $r = \sup\{b \in \mathbb{R} : b < x\}$. Since x is finite, r exists. We must show that $x - r$ is infinitesimal. Assume not, then there is a real number k such that $0 < k < |x - r|$. If $x - r > 0$, this implies that $r + k < x$, contradicting the choice of r. If $x - r < 0$, we get $x < r - k$, also contradicting the choice of r. The uniqueness is obvious since if $x = r_1 + \delta_1 = r_2 + \delta_2$, then $r_1 - r_2 = \delta_2 - \delta_1$ is both real and infinitesimal, so it must be zero. □

Definition 4.2.6 (Standard Part)**.**
If x is a finite hyperreal, then the unique real $r \approx x$ is called the *standard part* of x, and it is denoted by $\mathrm{st}(x)$.

Theorem 4.2.7. [6] The collection $\{\mu(x) : x \in {}^{\sigma}\mathbb{R}\}$ is a partition of $\mathcal{F}({}^*\mathbb{R})$.

Proof. Technically, to be a partition of $\mathcal{F}({}^*\mathbb{R})$, we have to show that $\mu(x) \cap \mu(y) \neq \emptyset$ implies $\mu(x) = \mu(y)$ and that $\bigcup\{\mu(x) : x \in {}^{\sigma}\mathbb{R}\} = \mathcal{F}({}^*\mathbb{R})$. For the first part, assume that there exists some $a \in \mu(x) \cap \mu(y)$. Then $a = \epsilon + x$, $a = \lambda + y$ where $\epsilon, \lambda \in \mathcal{I}({}^*\mathbb{R})$. But $\epsilon + x = \lambda + y$ implies that $\epsilon - \lambda = y - x$. This is only possible if $\epsilon - \lambda = 0$ since $y - x \in {}^{\sigma}\mathbb{R}$. Thus $x = y$ and so $\mu(x) = \mu(y)$.

For the second part, let $a \in \bigcup\{\mu(x) : x \in {}^{\sigma}\mathbb{R}\}$. Then $a = \epsilon + x$ for some $x \in {}^{\sigma}\mathbb{R}$. Then $|a| = |\epsilon + x| \leq |\epsilon| + |x| < |x| + 1$. Hence $a \in \mathcal{F}({}^*\mathbb{R})$. Consequently, $\bigcup\{\mu(x) : x \in {}^{\sigma}\mathbb{R}\} \subseteq \mathcal{F}({}^*\mathbb{R})$.
Now assume that $a \in \mathcal{F}({}^*\mathbb{R})$. Then there is some ${}^*x \in {}^{\sigma}\mathbb{R}^+$ such that $a < {}^*x$. So, consider the set $S = \{y : {}^*y < a\}$. This set is nonempty since $-x \in S$. Also since $a < {}^*x$, S is a set of real numbers that is bounded above, so it has a least upper bound z. Assume that $|z - a| \notin \mathcal{I}({}^*\mathbb{R})$. Then there is some $w \in \mathbb{R}$ such that $|{}^*z - a| > {}^*w$. Suppose that ${}^*z < a$, then $a - {}^*z > {}^*w$ implies that ${}^*z + {}^*w = {}^*(z + w) < a$, which in turn implies $z + w \in S$ and z is not a least upper bound of S. So, let $a < {}^*z$. This implies that $a < {}^*(z - w) < {}^*z$. But, $z - w$ is an upper bound for the set S. This contradicts the fact that z is the least upper bound of S. Hence, ${}^*z - a = \epsilon$ for some $\epsilon \in \mathcal{I}({}^*\mathbb{R})$, which implies that $a \in \mu(z)$. Therefore $\mathcal{F}({}^*\mathbb{R}) \subseteq \bigcup\{\mu(x) : x \in {}^{\sigma}\mathbb{R}\}$. □

4.3 Topology on $\mathbb{R}$

When we wish to examine the continuity, or otherwise, of a function f at a point a we find it necessary to consider that function's behavior at all points sufficiently near to a, if we are applying the standard criterion. In the case of the nonstandard criterion we would be concerned with all points infinitely close to a. What connects the two approaches is the fundamental idea of a neighborhood of a point.

Definition 4.3.1 (Standard neighborhood)**.**
If a is any point in $\mathbb{R}$ and if $r \in \mathbb{R}^+$, then we denote by $B(a, r)$ the set of all real points x whose distance from a is less than r:

$$B(a, r) = \{x \in \mathbb{R} : |a - x| < r\}.$$

Any set $M \subseteq \mathbb{R}$ will be called a *neighborhood* of a point $a \in \mathbb{R}$ iff there exists some $r > 0$ such that

$$a \in B(a, r) \subseteq M.$$

Theorem 4.3.2 (Nonstandard neighborhood [7])**.**
A set $M \subseteq \mathbb{R}$ is a neighborhood of a point $a \in \mathbb{R}$ iff every hyperreal x which is infinitely close to a necessarily belongs to the nonstandard extension ${}^*M$ of M; i.e., $M \subseteq \mathbb{R}$ is a neighborhood of $a \in \mathbb{R}$ iff:

$$\mu(a) \subseteq {}^*M.$$

Proof. First, let M be a neighborhood of a. Then there exists $r_0 \in \mathbb{R}^+$ such that $B(a, r_0) \subseteq M$. Let $x = [A] \in {}^*\mathbb{R}$ be such that $x \in \mu(a)$. Then for any real $r \in \mathbb{R}^+$ we have $|a - A_n| < r$ for almost all values of n; in particular this is true for r_0. It follows that $A_n \in B(a, r_0) \subseteq M$ for almost all values of n. So that $x \in {}^*M$.

On the other hand, let $\mu(a) \subseteq {}^*M$ and suppose, on the contrary, that M is not a neighborhood of a. Then for each $n \in \mathbb{N}$ we can find a point $A_n \in \mathbb{R}$ such that:

$$|a - A_n| < \frac{1}{n} \text{ and } A_n \in \mathbb{R} - M.$$

But this means that $x = [A]$ is a hyperreal which belongs to $\mu(a)$ but not to ${}^*M$, a contradiction. Hence M is a neighborhood of a. □

Using Theorem (4.3.2), we can define the open sets in $\mathbb{R}$ as follows.

Definition 4.3.3 (Nonstandard Open Set [22])**.**
A set $A \subseteq \mathbb{R}$ is *open* iff for every $x \in A$ we have $\mu(x) \subseteq {}^*A$. (In other words: for any $x \in A$ and $y \approx x$ then $y \in {}^*A$).

Theorem 4.3.4. [6, 25] Let $A \subseteq \mathbb{R}$, $p \in \mathbb{R}$. Then:

(i) p is an *accumulation point* of A iff $\mu(p) \cap {}^*A \neq \emptyset$. (In other words: $\exists\, y \in {}^*A$ such that $p \approx y$ but $p \neq y$).

(ii) p is an *isolated point* of A iff $\mu(p) \cap {}^*A = \{p\}$.

Proof. (i) Let $p \in \mathbb{R}$ be an accumulation point for $A \subseteq \mathbb{R}$. Then,

$$\forall\, x((x \in \mathbb{R}^+) \Rightarrow \exists\, y((y \in A) \wedge |y - p| < x)).$$

Then, by transfer principle, we have

$$\forall\, x((x \in {}^*\mathbb{R}^+) \Rightarrow \exists\, y((y \in {}^*A) \wedge |y - p| < x)),$$

so that $y \in \mu(p) \cap {}^*A$.

Conversely, assume that $\mu(p) \cap {}^*A \neq \emptyset$. By Theorem (4.3.2), $\mu(p) \subseteq {}^*(-w+p,\, p+w)$ $\forall\, w \in \mathbb{R}^+$. Hence, letting $y \in \mu(p) \cap {}^*A$ and $w \in \mathbb{R}^+$, we have

$$\exists\, y((y \in {}^*A) \wedge |y - p| < w),$$

so that, by transfer principle, we have

$$\forall\, x((x \in \mathbb{R}^+) \Rightarrow \exists\, y((y \in A) \wedge |y - p| < x)).$$

Hence p is an accumulation point for A.

(ii) Suppose that p is an isolated point for A. Then there exists some $w \in \mathbb{R}^+$ such that

$$(-w + p,\, p + w) \cap A = \{p\}.$$

Hence, by transfer principle, we have ${}^*(-w + p,\, p + w) \cap {}^*A = {}^*\{p\} = \{p\}$, and since $p \in \mu(p) \subseteq {}^*(-w + p,\, p + w)$, we have $\mu(p) \cap {}^*A = \{p\}$.

Conversely, suppose that $\mu(p) \cap {}^*A = \{p\}$. Then, by Theorem (4.3.2), there exists some $w \in \mathbb{R}^+$ such that $p \in \mu(p) \subseteq {}^*(-w + p,\, p + w)$, and hence

$${}^*(-w + p,\, p + w) \cap {}^*A = \{p\} = {}^*\{p\}.$$

By transfer principle, we have

$$(-w + p,\, p + w) \cap A = \{p\},$$

therefore p is an isolated point for A. $\square$

Definition 4.3.5 (Point of Closure [22])**.**

A point $x \in \mathbb{R}$ is said to be a *point of closure* of a set $F \subseteq \mathbb{R}$ iff there exists $y \in {}^*F$ such that $x \approx y$.

Definition 4.3.6 (Nonstandard Closed Set [22])**.**

A set $F \subseteq \mathbb{R}$ is *closed* iff $y \in {}^*F$ and $y \approx x \in \mathbb{R}$ always implies that $x \in F$. (In other words: $\mu(x) \cap {}^*F \neq \emptyset$ implies $x \in F$ for each $x \in \mathbb{R}$).

Theorem 4.3.7. $F \subseteq \mathbb{R}$ is closed iff its complement $G = \mathbb{R} - F$ is open.

Proof. Suppose that G is open and let y be any hyperreal in ${}^*F$. If $x := \text{st}(y) \in G$, then, since G is open, we must have

$$\mu(x) \subseteq {}^*G = {}^*(\mathbb{R} - F) = {}^*\mathbb{R} - {}^*F.$$

In particular $y \in {}^*\mathbb{R} - {}^*F$, which is a contradiction. Hence x must belong to $\mathbb{R} - G = F$ and so, by Definition (4.3.6), F is closed.

Conversely, suppose that F is closed and let x be any point of G. If there exists $y \approx x$ such that $y \in {}^*F = {}^*(\mathbb{R} - G)$, then, since F is closed, we must have $x \in F = \mathbb{R} - G$, which is a contradiction. Hence,

$$\mu(x) \subseteq {}^*\mathbb{R} - {}^*F = {}^*(\mathbb{R} - F) = {}^*G,$$

and so G is open. ☐

Theorem 4.3.8 (Boundedness [6])**.**

A nonempty set $A \subseteq \mathbb{R}$ is bounded iff ${}^*A \subseteq \mathcal{F}({}^*\mathbb{R})$.

Proof. Suppose that A is bounded. Then there is some positive real number x such that, for each $y \in A$, $|y| \leq x$. By transfer principle, for any $a \in {}^*A$ we have $|a| \leq {}^*x$. Consequently, ${}^*A \subseteq \mathcal{F}({}^*\mathbb{R})$.

Conversely, if A is not bounded, then for any $n \in \mathbb{N}$ there is some $x_n \in A$ such that $|x_n| > n$. Hence, by transfer principle, for any $\Lambda \in {}^*\mathbb{N}$ there is some $p_\Lambda \in {}^*A$ such that $|p_\Lambda| > \Lambda$. Choose $\Lambda \in \mathbb{N}_\infty = {}^*\mathbb{N} - \mathbb{N}$. Then $p_\Lambda \notin \mathcal{F}({}^*\mathbb{R})$. Hence ${}^*A \not\subseteq \mathcal{F}({}^*\mathbb{R})$. ☐

Theorem 4.3.9 (Continuity [22])**.**

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ iff ${}^*f(x) \approx {}^*f(a)$ whenever $x \in {}^*\mathbb{R}$ and $x \approx a$.

Proof. Suppose that f is continuous at $a \in \mathbb{R}$ and let x be a hyperreal such that $x \approx a$. We have to prove that $|{}^*f(x) - {}^*f(a)| < \epsilon$ for each $\epsilon \in \mathbb{R}^+$. For any such ϵ choose $\delta \in \mathbb{R}^+$ such that

$$|y - a| < \delta \Rightarrow |f(y) - f(a)| < \epsilon \, \forall \, y \in \mathbb{R}.$$

Then, by transfer principle, we have

$$|y - a| < \delta \Rightarrow |{}^*f(y) - {}^*f(a)| < \epsilon \, \forall \, y \in {}^*\mathbb{R}.$$

Taking $y = x$, since $x \approx a$, we have $|x - a| < \delta$ and so $|{}^*f(x) - {}^*f(a)| < \epsilon$.

Conversely, assume that ${}^*f(x) \approx {}^*f(a)$ whenever $x \approx a$, and let $\epsilon \in \mathbb{R}^+$ be given. Pick any positive infinitesimal $\delta \in \mathbb{R}^+$. Then $x \in {}^*\mathbb{R}$ and $|x - a| < \delta$ implies $x \approx a$; so, by assumption,

$$\exists \, \delta \in {}^*\mathbb{R}, \, \delta \in \mathbb{R}^+, \, (|x - a| < \delta \Rightarrow |{}^*f(x) - {}^*f(a)| < \epsilon).$$

Then, by transfer principle, we have

$$|x - a| < \delta \Rightarrow (|f(x) - f(a)| < \epsilon \, \forall \, x \in \mathbb{R}).$$

Hence f is continuous at a. □

Theorem 4.3.10. [7] A function $f : \mathbb{R} \to \mathbb{R}$ is continuous (on $\mathbb{R}$) iff the inverse image $f^{-1}(A) = \{x \in \mathbb{R} : f(x) \in A\}$ of any open set A is itself always an open set.

Proof. Suppose that f is continuous and let A be an open set in $\mathbb{R}$ and $x \in f^{-1}(A)$. Then $f(x) \in A$. If $y = [B] \in {}^*\mathbb{R}$ is such that $y \approx x$, then, by the continuity of f,

$${}^*f(x) \approx {}^*f(y).$$

But, since A is open, this implies that ${}^*f(y) \in {}^*A$. This means that $f(B_n) \in A$ for almost all n and therefore that $B_n \in f^{-1}(A)$ for almost all n. Thus $y \approx x$ always implies that $y \in {}^*(f^{-1}(A))$, and so $f^{-1}(A)$ is an open set.

Conversely, suppose that the inverse image under f of every open set A is always itself an open set. Let $x \in \mathbb{R}$ and $y \in {}^*\mathbb{R}$ be such that $y \approx x$. If it is false that ${}^*f(x) \approx {}^*f(y)$, then for some $r \in \mathbb{R}^+$ we must have $|{}^*f(x) - {}^*f(y)| > r$. Thus, ${}^*f(y) \notin {}^*A$ where

$$A = (f(x) - r, \, f(x) + r).$$

It follows that $y \notin {}^*(f^{-1}(A))$. This contradicts the hypothesis that $f^{-1}(A)$ is open and yet we have

$$y \approx x \in f^{-1}(A).$$

□

Definition 4.3.11. [22] A set $A \subseteq \mathbb{R}$ is *compact* iff for each $b \in {}^*A$ there is some $p \in A$ such that $b \in \mu(p)$ (i.e., $b \approx p$) iff ${}^*A \subseteq \bigcup\{\mu(p) : p \in A\}$.

Theorem 4.3.12 (Heine−Borel [7])**.**
A nonempty $A \subseteq \mathbb{R}$ is compact iff it is closed and bounded.

Proof. Assume that A is compact. Using Theorem (4.2.7), we have ${}^*A \subseteq \bigcup\{\mu(p) : p \in A\} \subseteq \mathcal{F}({}^*\mathbb{R})$. Then, by Theorem (4.3.8), A is bounded. Now let $\mu(q) \cap {}^*A \neq \emptyset$ for some $q \in \mathbb{R}$. Since ${}^*A \subseteq \bigcup\{\mu(p) : p \in A\}$, $\mu(q) \cap \mu(p) \neq \emptyset$ for some $p \in A$. Hence Theorem (4.2.7) implies that $q = p$. Thus $q \in A$. Hence, by Definition (4.3.6), A is closed.

Conversely, assume that A is closed and bounded. Since A is bounded, by Theorem (4.3.8), we have ${}^*A \subseteq \bigcup\{\mu(p) : p \in \mathbb{R}\} = \mathcal{F}({}^*\mathbb{R})$. Also, $A \neq \mathbb{R}$. Since A is closed, Definition (4.3.6) implies that $\mu(q) \cap {}^*A = \emptyset$ for any $q \in \mathbb{R} - A$. Thus ${}^*A \subseteq \bigcup\{\mu(p) : p \in A\}$. Hence, by Definition (4.3.11), A is compact. □

Theorem 4.3.13 (Bolzano−Weierstrass [7])**.**
If $A \subseteq \mathbb{R}$ is an infinite, compact subset of $\mathbb{R}$, then every infinite subset of A has a limit point in A.

Proof. If A has an infinite subset B, then we can choose a sequence C_n of distinct points of B which defines a hyperreal $y = [C]$. Then $y \in {}^*A$ (since $C_n \in B \subseteq A$ for all n), y is finite (since A is bounded) and $x = \mathrm{st}(y)$ exists and belongs to A (by compactness of A and Definition (4.3.11)). Finally, $x \approx y$ but $x \neq y$ (since the C_n are all distinct). It follows from Theorem (4.3.4) that $x \in A$ is a limit point of B. □

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