The Islamic University of Gaza
Deanery of Higher Studies
Faculty of Science
Department of Mathematics

QUASI-IDEALS AND BI-IDEALS ON SEMIGROUPS AND SEMIRINGS

Presented By
Neveen R. El-Madhoun

Supervised By
Dr. Ahmed El-Mabhouh

Submitted in Partial Fulfillment of
the requirement For the Degree of Master of Science
At The Islamic University of Gaza
Gaza. Palestine

1428-2007

Contents

Acknowledgement

My special thanks after thanking Allah almighty, to the Department of Mathematics in the Islamic University of Gaza, specially Dr. Ahmed El-Mabhouh for his great support and right ideas. I thank my husband Mohammed for typing this thesis with me .Also, I thank my parents , and every one from my family for providing a comfortable environment to me.
Finally , I would like to thank everyone who contributed in achieving this thesis.

Abstract

In this thesis, we introduce some concepts of semigroups and semirings, as quasi-ideals, bi-ideals, regular elements and regular semigroups and semirings Also, we shall introduce some semigroups whose sets of bi-ideals and quasi-ideals coincide .

It has been shown that each quasi-ideal on a semigroup or a semiring is a bi-ideal , but the converse is not true.

We close this thesis by finding semigroups contain bi-ideals which are not quasi-ideals.

Introduction

In this thesis, firstly we introduce the definition of semigroups and semirings, and we give some basic results.
The notion of quasi-ideals was first introduced by O. Steinfeld [18] and [19] for rings and semigroups, respectively.
We mean by a quasi-ideal on a semigroup S is a subsemigroup Q of $(S,+)$ satisfying $SQ \cap QS \subseteq Q$.
Kiyoshi Iseki [8] introduced this concept for semirings without zero and proved some results.
By a quasi-ideal of a semiring S we mean a subsemigroup Q of $(S,+)$ satisfying $\langle SQ \rangle \cap \langle QS \rangle \subseteq Q$.
R. A. Good and D. R. Hughes [4] introduced the notion of bi-ideals.
We mean by a bi-ideal is a subsemigroup B of a semigroup S such that $BSB \subseteq B$. In a semiring we have $\langle BSB \rangle \subseteq B$.
Then bi-ideals are generalization of quasi-ideals.

In this thesis, we will give the notion **BQ** to denote the class of all semigroups whose bi-ideals are quasi-ideals.
In fact, Calais [1] has characterized the semigroups in **BQ**.

The important of this study is that to introduce some semigroups whose bi-ideals and quasi-ideals coincide
Also, we introduce semigroups contain bi-ideals which are not quasi-ideals.
This thesis consists of four chapters.

Chapter one includes Preliminaries that will be used in the reminder of the thesis.

Section one contains some definitions of sets which will be applied throughout the thesis, see [17].

In the second section, we introduce basic definition of semigroups and semirings ,and we give some examples about them, see [2], [10].

In the third section ,we talk about quasi-ideals , bi-ideals and the relation between them, that is , we show that each quasi-ideal is a bi-ideal, but the converse is not true.
Also, we define the quasi-ideal and bi-ideal generated by a nonempty set X of a semigroup or a semiring S, and we examine some of the elementary properties of quasi-ideals for semirings, see [3], [11], [10], [16].

In Chapter two,
we characterize regular semirings and regular elements of semirings using quasi-ideals, see [3]

Also, we define **BQ** and we prove that simple and semigroups are in **BQ**, see [1] and [9].

In Chapter three ,
we will study in the first section multiplication and additive interval semigroups of $\mathbb{R}$ which contain bi-ideals and quasi-ideals that coincide, see [2], [1], [14]

In the second section , we have transformation semigroups $T_1(X), T_2(X), T_3(X), T_4(X)$ and $T_5(X)$, all contains sets of bi-ideals and quasi-ideals that coincide, see [2], [6], [17], and [12].

Chapter four talk about semigroups that contain bi-ideals which are not quasi-ideals.

In section 4.1, we show that there are bi-ideals in the semigroup of continuous mapping which are not quasi-ideals.
Also, we show that $C(I)$ is not regular , see [11].

In section 4.2, we show that there are bi-ideals in the semigroup of differentiable mappings which are not quasi-ideals.
Also, we show that $D(I)$ is not regular, see [11] .

Chapter 1

Preliminaries

In this chapter we introduce definitions related to semigroups , semirings , left ideals , right ideals , quasi-ideals and bi-ideals .
We will study in section 1.3 the relation between these ideals , and we prove some basic theorems and results about quasi-ideals.
More information can be found in [2], [3], [17], [11], [10] and [16].

1.1 General Concepts on Sets

Definition 1.1.1. [2]
A binary operation $(*)$ on a set S is a mapping of $S{\times}S$ into S , where $S{\times}S$ is the set of all ordered pairs of elements of S,
$i.e$ $S{\times}S$ $=\{(a,b) : a,b \in S\}$ where $(a,b)\rightarrow a * b$.

Definition 1.1.2. [17]
Let $f : A \to B$, and let $D \subseteq A$. The restriction of f to D, denoted $f|_D$, is $\{(x, y) : (x, y) \in f$ and $x \in D\}$.

Definition 1.1.3. [17]
For any sets A and B,
$|A| = |B|$ iff there exists a bijection mapping from A onto B
where $|A|$ is the cardinal number of the set A.

Theorem 1.1.4. [17]

(1) every infinite set X contains a countable infinite subset.

(2) If $h : A \to C$ is one-to-one, $g : B \to D$ is one-to-one, $A \cap B = \emptyset$, and $C \cap D = \emptyset$ then $h \cup g : A \cup B \to C \cup D$ is one-to-one.

(3) Let $f : A \to B$. If $X \subseteq A$, and f is one-to-one , then $f(A \backslash X) = f(A) \backslash f(X)$

Proof. See [17] □

1.2 Semigroups and Semirings

In this section , we introduce the definitions of semigroups and semirings, and we study some properties about them .
Also , we give some examples on semigroups and semirings.

Definition 1.2.1. [10]
Let S be a nonempty set . S is said to be a semigroup if on S is defined a binary operation $(*)$ such that
for all $a, b, c \in S$, we have $(a * b)* c = a * (b * c)$, (associative law).
We denote the semigroup by $(S, *)$.

If $a * b = b * a$ for all a, b in S ,then we say S is a commutative semigroup.

Definition 1.2.2. [10]
Let $(S, *)$ be a semigroup

(a) If the number of elements in S is finite , we say S is a finite semigroup or a semigroup of finite order. Otherwise, S is called an infinite semigroup.

(b) If S contains an element e such that $e * a = a * e = a$ for all $a \in S$, we say S is a semigroup with identity e or S is a monoid .

(c) An element $x \in S$, where S is a monoid is said to have an inverse in S if there exists $y \in S$ such that $x * y = y * x = e$.

Examples 1.2.3. [10]

(a) $\mathbb{Z}_9 = \{0, 1, 2, ..., 8\}$ is a commutative semigroup of order nine under multiplication modulo 9 with identity 1 .

(b) The set of integers Z under usual multiplication is an infinite semigroup with identity 1 .

(c) Let $M_{2\times2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$. Then

$M_{2\times2}$ is a non-commutative semigroup of infinite order under matrix

multiplication with identity $I_{2\times2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Definition 1.2.4. [2]
A semigroup S with a zero element 0 is called a zero or a null semigroup if $ab = 0$ for all $a, b \in S$.

Definition 1.2.5. [2]
An element a of a semigroup S is said to be left [right] cancellable if , for any x and y in S , $ax = ay$ $[xa = ya]$ implies $x = y$.
A semigroup S is called left [right] cancellative if every element of S is left [right] cancellable.
We say that S is cancellative if it is both left and right cancellative.

Definition 1.2.6. [10]
let $(S, *)$ be a semigroup .
A nonempty subset H of S is said to be a subsemigroup of S if for all $a, b \in H$ we have $a * b \in H$.

Example 1.2.7. The set $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ is a subsemigroup of $M_{2\times 2}(\mathbb{Z})$

Definition 1.2.8. [10]
Let S be a non-empty set together with two binary operations addition $(+)$ and multiplication $(.)$. Then S is called a semiring with respect to these operations if the following properties hold.

1. $(S, +)$ is a commutative semigroup,
2. $(S, .)$ is a semigroup,
3. For all $a, b, c \in S$, we have $a.(b+c) = a.b + a.c$ and $(a+b).c = a.c + b.c$.

We denote the semiring by $(S, +, .)$.
If in addition, $(S, .)$ is commutative, then S is called a commutative semiring.

Definition 1.2.9. [3]
Let $(S, +, .)$ be semiring .
An element $0 \in S$ is called an absorbing element if $0 + a = a + 0 = a$ and $0.a = a.0 = 0$ for all $a \in S$.

Definition 1.2.10. [2]
In $(S, +, .)$,if $(S, .)$ is a monoid , that is there exists $1 \in S$ such that $a.1 = 1.a = a$ for all $a \in S$,
we say the semiring is a semiring with identity .

Definition 1.2.11. [10]
Let $(S, +, .)$ be a semiring .
We say S is finite if the number of elements in S is finite
If the number of elements in S is not finite we say S is of infinite order

Note : Denote $\mathbb{Z}^0 = \mathbb{Z}^+ \cup \{0\}$ by the set of positive integers with zero , or non negative integers .

Example 1.2.12. [10]
$(\mathbb{Z}^0, +, .)$ is a semiring of infinite order which is commutative with identity 1.

Example 1.2.13. [10]
Let $M_{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}^0 \right\}$.

Then $(M_{2\times 2}, +, .)$ is a semiring under matrix addition and matrix multiplication.
$M_{2\times 2}$ is a non-commutative semiring with identity element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and is of infinite order .

Definition 1.2.14. [10]
Let $(S, +, .)$ be a semiring .
A nonempty subset B of S is a subsemiring of S if for all $a, b \in B$ we have $a + b \in B$ and $a.b \in B$.

Example 1.2.15. $n\mathbb{Z}^0 = \{ \ 0 \ , n \ , 2n \ , ... \}$ is a subsemiring of $\mathbb{Z}^0$

Proof.
let kn , mn be two elements in $n\mathbb{Z}^0$, then $kn + mn = (k + m)n \in n\mathbb{Z}^0$.
$(kn)(mn) = (kmn)n \in n\mathbb{Z}^0$
Hence, $n\mathbb{Z}^0$ is a subsemiring of $\mathbb{Z}^0$ without identity.

□

Definition 1.2.16. [10]
Let S be a semiring .
An element $x \in S$ is called an idempotent if $x \ . \ x = x^2 = x$.

Example 1.2.17.
Let $\mathbb{Z}_6 = \{ \ 0 \ , 1 \ , 2 \ , 3 \ , 4 \ , 5 \ \}$ be the commutative semiring under addition and multiplication module 6 .
$3 \in \mathbb{Z}_6$ is an idempotent , since $3 \ . \ 3 = 3(mod6)$.

Definition 1.2.18. [10]
Let S be a semiring .
We say S is a strict semiring if $a + b = 0$ implies $a = 0$ and $b = 0$.

Example 1.2.19. $\mathbb{Z}^0$ is a strict semiring .

Definition 1.2.20. [10]
Let S be a semiring with identity 1 .
We say an element x in S has an inverse if there exists $y \in S$ such that $x \ . \ y = y \ . \ x = 1$.

1.3 Ideals , Quasi-ideals and Bi-ideals

We begin this section by defining the left , right and two sided ideals in semigroups and semirings, and so we give some examples about them .
Also, we define Quasi-ideals , Bi-ideals in semigroups and semirings and we discuss some basic results about Quasi-ideals.

Definition 1.3.1.
Let $(S, .)$ be a semigroup .
If $\emptyset \neq A, B \subseteq S$ then $AB = \{ab : a \in A, b \in B\}$

Definition 1.3.2. [3]
Let $(S, .)$ be a semigroup.
A non-empty subset A of S is called a left (right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$).
If A is both left ideal and right ideal, then it is called an ideal (or a two sided ideal) of S .

Definition 1.3.3. [3]
Let $(S, +, .)$ be a semiring.

Definition 1.3.4. [3]
Let $(S, +, .)$ be a semiring.
A nonempty subset A of S is called a left (right) ideal of S if A is a subsemigroup of $(S, +)$ and satisfying $SA \subseteq A$ $(AS \subseteq A)$.
If A is both left ideal and right ideal, then it is called an ideal (or a two sided ideal) of S .

Example 1.3.5.
Let $\mathbb{Z}_{14} = \{ 0 , 1 , 2 , ... , 13 \}$ be the semigroup under multiplication modulo 14 .
Then $I = \{ 0 , 7 \}$ and $J = \{ 0 , 2 , 4 , 6 , 8 , 10 , 12 \}$ are two sided ideals of $\mathbb{Z}_{14}$.

Example 1.3.6. [10]
In the semiring $\mathbb{Z}^0$, $n\mathbb{Z}^0$ is a two sided ideal for any $n \in \mathbb{N}$.

Proof.
In Example 1.2.15 we proved that $n\mathbb{Z}^0$ is a subsemiring of Z^0 .
To show that $n\mathbb{Z}^0$ is a two sided ideal of $\mathbb{Z}^0$, let $r \in \mathbb{Z}^0$ and $kn \in n\mathbb{Z}^0$.
Now $(kn)r \in (n\mathbb{Z}^0)Z^0$, so $(kn)r = (kr)n \in n\mathbb{Z}^0$ and so $(n\mathbb{Z}^0)\mathbb{Z}^0 \subseteq n\mathbb{Z}^0$.
Then $n\mathbb{Z}^0$ is a right ideal of $\mathbb{Z}^0$.
Similarly, $r(kn) \in n\mathbb{Z}^0$, and so $\mathbb{Z}^0(n\mathbb{Z}^0) \subseteq n\mathbb{Z}^0$
So, $n\mathbb{Z}^0$ is a left ideal of $\mathbb{Z}^0$.
Hence, $n\mathbb{Z}^0$ is a two sided ideal of $\mathbb{Z}^0$.

□

Definition 1.3.7. [7]
A semigroup S with a zero element is called left [right] simple if S contains no proper left [right] ideal , and S is called simple if it has no proper ideals. An ideal I of S is called a proper ideal if $I \neq S$.

Theorem 1.3.8. [7]
A semigroup S is left [right] simple if and only if $Sa = S[aS = S]$ for every $a \in S$

Proof.
We prove the case of left simple semigroup and the right is similarly.
Assume that S is a left simple semigroup, and $Sa \neq S$, for some $a \in S$.

But, Sa is a left ideal of S, then Sa is a proper left ideal of S, which contradicts the definition.
Conversly, assume that L is a proper left ideal of S and $a \in L$, then $SL \subseteq L \subset S$ and so $Sa \subseteq L \subset S$.
Hence, $Sa \neq S$.
Therefore, if $Sa = S$ then S is a left simple semigroup. $\square$

Definition 1.3.9. [3]
Let S be a semiring.
Let $\mathbb{N}$ be the set of positive integers and $\emptyset \neq X \subseteq S$.
Then $< X >= \{\sum_{i=1}^{n} x_i \mid x_i \in X, n \in \mathbb{N}\}$.

Remark 1.3.10. $< X >$ is a subsemigroup of $(S, +)$.

Proof.
Let x , $y \in < X >$, then $x = \sum_{i=1}^{n} x_i$ and $y = \sum_{i=1}^{m} y_i$ where x_i , $y_i \in X$, $n, m \in \mathbb{N}$.
Then, $x + y = \sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i = x_1 + ... + x_n + y_1 + ... + y_m \in < X >$, because x_i , $y_i \in X$, for all i .

Hence,$< X >$ is a subsemigroup of $(S$, $+)$. $\square$

Definition 1.3.11. [3]
Let S be a semiring, and $\emptyset \neq X$, $Y \subseteq S$.

Then $\langle XY \rangle = \{\sum_{i=1}^{n} x_i y_i : x_i \in X, y_i \in Y, n \in \mathbb{N}\}$.

We will write xY and $\langle xY\rangle$ instead of $\{x\}Y$ and $\langle\{x\}Y\rangle$.

Proposition 1.3.12. [3]
Let S be a semiring, and $\emptyset \neq X$, Y , $Z \subseteq S$ .Then

(1) $X \subseteq \langle X\rangle$

(2) $XY \subseteq \langle XY\rangle$

(3) $X \subseteq Y \Rightarrow \langle X\rangle \subseteq \langle Y\rangle$

(4) $\langle X \cup Y\rangle = \langle\langle X\rangle \cup \langle Y\rangle\rangle$

(5) $\langle XY\rangle = \langle\langle X\rangle Y\rangle = \langle X\langle Y\rangle\rangle = \langle\langle X\rangle\langle Y\rangle\rangle$

(6) $\langle XYZ\rangle = \langle\langle XY\rangle Z\rangle = \langle X\langle YZ\rangle\rangle$

(7) $\langle X \cap Y\rangle = \langle X\rangle \cap \langle Y\rangle$.

Proof.

(1) Let $x \in X$ then $x = \sum_{i=1}^{n} x_i$ where $n = 1$, $x_i = x$, and so $x_i \in X$
Then, $x \in \langle X\rangle$
Hence, $X \subseteq \langle X\rangle$.

(2) Follows from (1).

(3) Assume that $X \subseteq Y$.
let $z \in \langle X\rangle$ then $z = \sum_{i=1}^{n} x_i$, $x_i \in X$
But $X \subseteq Y$ then $x_i \in Y$ and so , $z \in \langle Y\rangle$.
Hence, $\langle X\rangle \subseteq \langle Y\rangle$.

(4) By (1) , $X \subseteq \langle X \rangle$ and $Y \subseteq \langle Y \rangle$.
Therefore, $X \cup Y \subseteq \langle X \rangle \cup \langle Y \rangle$.
So, by (3) $\langle X \cup Y \rangle \subseteq \langle \langle X \rangle \cup \langle Y \rangle \rangle$.
Conversely, let $z \in \langle \langle X \rangle \cup \langle Y \rangle \rangle$, then $z = \sum_{i=1}^{n} x_i$ where $x_i \in \langle X \rangle \cup \langle Y \rangle$.
Then, $x_i \in \langle X \rangle$ or $x_i \in \langle Y \rangle$, for each i .
Therefore, for each i , $x_i = \sum_{j=1}^{m} t_j$ where $t_j \in X$ or $x_i = \sum_{j=1}^{k} s_j$ where $s_j \in Y$.
Hence, z is a finite sum of elements from X or Y.
i.e. $z = \sum_{i=1}^{n} x_i^{'}$ where $x_i^{'} \in X \cup Y$.
Therefore, $z \in \langle X \cup Y \rangle$.
Then, $\langle \langle X \rangle \cup \langle Y \rangle \rangle \subseteq \langle X \cup Y \rangle$.
Hence, $\langle X \cup Y \rangle = \langle \langle X \rangle \cup \langle Y \rangle \rangle$.

(5) By (1) , $X \subseteq \langle X \rangle$ and $Y \subseteq Y$. Then $XY \subseteq \langle X \rangle Y$.
So, by (3) $\langle XY \rangle \subseteq \langle \langle X \rangle Y \rangle$.
conversly, let $z \in \langle \langle X \rangle Y \rangle$, then $z = \sum_{i=1}^{n} x_i$ where $x_i \in \langle X \rangle Y$.
Hence, for each i , $x_i = \left(\sum_{j=1}^{m} t_j \right) y = (t_1 + t_2 + ... + t_m) y = t_1 y + t_2 y + ... + t_m y = \sum_{j=1}^{m} t_j y$ where $t_j \in X$, $y \in Y$
Hence, z is a finite sum of elements from XY.
Therefore, $z \in \langle XY \rangle$
Then, $\langle \langle X \rangle Y \rangle \subseteq \langle XY \rangle$.
Hence, $\langle XY \rangle = \langle \langle X \rangle Y \rangle$.
Similary, $\langle XY \rangle = \langle X \langle Y \rangle \rangle$.
Now , we show $\langle \langle X \rangle Y \rangle = \langle X \langle Y \rangle \rangle$.
By (1) , $Y \subseteq \langle Y \rangle$, then $\langle X \rangle Y \subseteq \langle X \rangle \langle Y \rangle$.
So, by (3) $\langle \langle X \rangle Y \rangle \subseteq \langle \langle X \rangle \langle Y \rangle \rangle$.
conversly, let $z \in \langle \langle X \rangle \langle Y \rangle \rangle$, then $z = \sum_{i=1}^{n} z_i$ where $z_i \in \langle X \rangle \langle Y \rangle$.

Hence, for each i , $z_i = \left(\sum_{j=1}^{m} x_j\right)\left(\sum_{k=1}^{l} y_k\right) = \left(\sum_{j=1}^{m} x_j\right)(y_1 + y_2 + ... + y_l) = \left(\sum_{j=1}^{m} x_j\right) y_1 + \left(\sum_{j=1}^{m} x_j\right) y_2 + ... + \left(\sum_{j=1}^{m} x_j\right) y_l$ where $x_j \in X$, $y_k \in Y$

Hence, z is a finite sum of elements from $\langle X \rangle Y$.

Therefore, $z \in \langle \langle X \rangle Y \rangle$

Then, $\langle \langle X \rangle \langle Y \rangle \rangle \subseteq \langle \langle X \rangle Y \rangle$.

Hence, $\langle \langle X \rangle Y \rangle = \langle \langle X \rangle \langle Y \rangle \rangle$.

So, $\langle XY \rangle = \langle \langle X \rangle Y \rangle = \langle \langle X \rangle \langle Y \rangle \rangle$.

(6) Follows from (5) , if we let $X = XY$ and $Y = Z$.

we have $\langle XYZ \rangle = \langle \langle XY \rangle Z \rangle$.

also, if $X = X$ and $Y = YZ$ we have $\langle XYZ \rangle = \langle X \langle YZ \rangle \rangle$

Therefore, $\langle XYZ \rangle = \langle \langle XY \rangle Z \rangle = \langle X \langle YZ \rangle \rangle$.

(7) Let $z \in \langle X \cap Y \rangle$, then $z = \sum_{i=1}^{n} z_i$ where $z_i \in X \cap Y$, $n \in \mathbb{N}$.

Then, $z_i \in X$ and $z_i \in Y$.

Then, $z \in \langle X \rangle$ and $z \in \langle Y \rangle$.

Therefore, $z \in \langle X \rangle \cap \langle Y \rangle$.

Hence, $\langle X \cap Y \rangle \subseteq \langle X \rangle \cap \langle Y \rangle$.

Conversly, let $z \in \langle X \rangle \cap \langle Y \rangle$, then $z \in \langle X \rangle$ and $z \in \langle Y \rangle$.

So, $z = \sum_{i=1}^{n} x_i = \sum_{i=1}^{m} y_i$ where $x_i \in X$, $y_i \in Y$.

Hence, z is a finite sum of elements from X and Y .

i.e. $z = \sum_{i=1}^{k} z_i$ where $z_i \in X \cap Y$.

Then, $z \in \langle X \cap Y \rangle$

Hence, $\langle X \rangle \cap \langle Y \rangle \subseteq \langle X \cap Y \rangle$.

Therefore, $\langle X \cap Y \rangle = \langle X \rangle \cap \langle Y \rangle$. $\square$

Remark 1.3.13. [3]
Let S be a semiring , and L, R be subsemigroups of $(S, +)$. Then
L is a left ideal of S iff $\langle SL \rangle \subseteq L$, and
R is a right ideal of S iff $\langle RS \rangle \subseteq R$

Proof.
We show the case of left ideal and we have similarly the case of right ideal.
Assume that L is a left ideal of S, then $SL \subseteq L$
let $x \in \langle SL \rangle$ then $x = \sum_{i=1}^{n} s_i l_i$ where $s_i \in S$, $l_i \in L$, $n \in \mathbb{N}$.
Then $s_i l_i \in SL$ and so , $s_i l_i \in L$
But L is subsemigroup of $(S, +)$ then $\sum_{i=1}^{n} s_i l_i \in L$.
Then , $x \in L$
Hence, $\langle SL \rangle \subseteq L$
Assume that $\langle SL \rangle \subseteq L$.
Since, $SL \subseteq \langle SL \rangle$ and $\langle SL \rangle \subseteq L$
then $SL \subseteq L$.

□

Definition 1.3.14. [11]
Let $(S, .)$ be a semigroup.
A subset $\emptyset \neq Q \subseteq S$ is called a quasi-ideal of S iff Q is a subsemigroup of $(S, .)$ satisfying $SQ \cap QS \subseteq Q$.

Definition 1.3.15. [3]
Let $(S, +, .)$ be a semiring.
A subset $Q \neq \emptyset$ of S is called a quasi-ideal of S iff Q is a subsemigroup of $(S, +)$ satisfying $\langle SQ \rangle \cap \langle QS \rangle \subseteq Q$.

Proposition 1.3.16. [13]
Let S be a semigroup or a semiring.
If S is commutative , then every quasi-ideal of S is a two sided ideal of S.

Proof.
Assume that S is commutative .
Let Q be a quasi-ideal of S . Then $\langle SQ \rangle \cap \langle QS \rangle \subseteq Q$.

But S is commutative and $Q \subseteq S$ then $SQ = QS$.
Then $\langle SQ\rangle \cap \langle QS\rangle = \langle SQ\rangle \cap \langle SQ\rangle = \langle SQ\rangle \subseteq Q$.
Hence, Q is a left ideal of S.
Similarly, Q is a right ideal of S.
Therefore, every quasi-ideal of S is a two sided ideal of S. $\square$

Example 1.3.17. [16]
let $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}^0 \right\}$.

Then, we see in Example 1.2.13, S is a semigroup under usual addition and multiplication of matrices .

Let $Q = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in \mathbb{Z}^0 \right\}$.

Then, Q is a quasi-ideal of S.

Proof. First, we show Q is a subsemigroup of $(S, +)$.

Let $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \in Q$, then $x, y \in \mathbb{Z}^0$
Now, $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x+y & 0 \\ 0 & 0 \end{pmatrix} \in Q$ since $x + y \in \mathbb{Z}^0$

Hence , Q is a subsemigroup of S.
We show Q is a quasi-ideal of S .

Let $A \in SQ \cap QS$. Then $A \in SQ$ and $A \in QS$.
So,

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ &= \begin{pmatrix} ax & 0 \\ cx & 0 \end{pmatrix} = \begin{pmatrix} ye & yf \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Then, $ax = ye$, $yf = 0$ and $cx = 0$.

Hence, $A = \begin{pmatrix} ax & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ye & 0 \\ 0 & 0 \end{pmatrix} \in Q$

Therefore, Q is a quasi-ideal of S. □

Lemma 1.3.18. [3]
Let S be a semiring , then

(a) Each one or two sided ideal of S is a quasi-ideal of S .

(b) The intersection of any system of quasi-ideals of S is either empty or a quasi-ideal of S .

(c) If L is a left and R a right ideal of S, then $RL \subseteq \langle RL \rangle \subseteq L \cap R$ holds and the intersection $Q = L \cap R$ is a quasi-ideal of S .

(d) For each $\emptyset \neq X \subseteq S$, $\langle SX \rangle$ is a left ideal , $\langle XS \rangle$ is a right ideal , $\langle SXS \rangle$ is an ideal and $\langle SX \rangle \cap \langle XS \rangle$ is a quasi-ideal of S .

(e) each quasi-ideal of a semiring S is a subsemiring of S.

Proof.

(a) First, we show if L is a left ideal of S then L is a quasi ideal of S. From Remark 1.3.12 , since L is a left ideal of S then $\langle SL \rangle \subseteq L$, but $\langle SL \rangle \cap \langle LS \rangle \subseteq \langle SL \rangle \subseteq L$.
Hence, L is a quasi-ideal of S .
Similarly, R is a quasi-ideal of S .
Since any two sided ideal is left and right ideal of S and any left or right ideal is a quasi-ideal of S then any two sided ideal is a quasi-ideal of S.

(b) let $\{Q_i : i \in I\}$ be any system of quasi-ideals of S.
We show $Q = \cap_{i\in I} Q_i$ is either empty or a quasi-ideal of S .
Assume that Q is non-empty .
Since Q_i is a quasi-ideal of S $\forall$ $i \in I$, then $\langle SQ_i\rangle \cap \langle Q_iS\rangle \subseteq Q_i$ $\forall$ i $\in I$.
Now, for each $i \in I$,
$\langle SQ\rangle = \langle S(\cap Q_i)\rangle = \langle \cap(SQ_i)\rangle \subseteq \langle SQ_i\rangle$, and
$\langle QS\rangle = \langle (\cap Q_i)S\rangle = \langle \cap(Q_iS)\rangle \subseteq \langle Q_iS\rangle$
Then,

$$\begin{aligned} \langle SQ\rangle \cap \langle QS\rangle &\subseteq \langle SQ_i\rangle \cap \langle Q_iS\rangle \\ &\subseteq Q_i \forall i \in I \end{aligned}$$

Therefore , $\langle SQ\rangle \cap \langle QS\rangle \subseteq \cap_{i\in I} Q_i = Q$.
Hence, Q is a quasi-ideal of S.

(c) Let L be a left ideal and R be a right ideal of S .
From Proposition 1.3.11(2) , we have
$RL \subseteq < RL >$.
Now , since $R \subseteq S$ and L is a left ideal of S , we have $RL \subseteq SL \subseteq L$.
Similarlly , $RL \subseteq RS \subseteq R$.
Hence , $RL \subseteq L \cap R$.
Next , we show $\langle RL\rangle \subseteq L \cap R$.
To see this , let $x \in \langle RL\rangle$ then $x = \displaystyle\sum_{i=1}^{n} r_il_i$ where $r_i \in R$, $l_i \in L$, $n \in \mathbb{N}$.
Then , $r_il_i \in RL \subseteq L \cap R$.
So , $r_il_i \in L$ and $r_il_i \in R$.
But L, R are subsemigroups of $(S, +)$, then $\displaystyle\sum_{i=1}^{n} r_il_i \in L$, and $\displaystyle\sum_{i=1}^{n} r_il_i \in R$
Then , $x \in L$, and $x \in R$.
So, $x \in L \cap R$.
Therefore, $\langle RL\rangle \subseteq L \cap R$.
Hence , $RL \subseteq \langle RL\rangle \subseteq L \cap R$.
Now, we show $Q = L \cap R$ is a quasi-ideal of S , i.e $\langle SQ\rangle \cap \langle QS\rangle \subseteq Q$.

$\langle SQ\rangle = \langle S(L\cap R)\rangle = \langle SL\cap SR\rangle \subseteq \langle SL\rangle \subseteq L$,
(since L is a left ideal of S). Also ,
$\langle QS\rangle = \langle (L\cap R)S\rangle \subseteq \langle RS\rangle \subseteq R$ (since R is a right ideal of S)
Then, $\langle SQ\rangle \cap \langle QS\rangle \subseteq L\cap R = Q$.
Hence, Q is a quasi-ideal of S.

(d) First , we show $\langle S\langle SX\rangle\rangle \subseteq \langle SX\rangle$, (by Remark 1.3.12) .
$\langle S\langle SX\rangle\rangle = \langle\langle SS\rangle X\rangle$, (by Proposition 1.3.11 (5)) .
But S is a left ideal of S then $\langle SS\rangle \subseteq S$,and so $\langle\langle SS\rangle X\rangle \subseteq \langle SX\rangle$.
Hence , $\langle SX\rangle$ is a left ideal of S .
Similarlly , $\langle\langle XS\rangle S\rangle \subseteq \langle XS\rangle$.
Now, since $\langle SX\rangle$ is a left ideal and $\langle XS\rangle$ is a right ideal of S then by part (c) $\langle SX\rangle \cap \langle XS\rangle$ is a quasi-ideal of S.
Finally, we show that $\langle SXS\rangle$ is a left and a right ideal of S .
Now, $\langle S\langle SXS\rangle\rangle = \langle SS\langle XS\rangle\rangle \subseteq \langle S\langle XS\rangle\rangle = \langle SXS\rangle$.
Then $\langle SXS\rangle$ is a left ideal of S .
Also, $\langle\langle SXS\rangle S\rangle = \langle\langle SX\rangle SS\rangle \subseteq \langle\langle SX\rangle S\rangle = \langle SXS\rangle$.
So, $\langle SXS\rangle$ is a right ideal of S .
Therefore, $\langle SXS\rangle$ is an ideal of S .

(e) let Q be a quasi ideal of a semiring S , we show Q is a subsemiring of S .
From Definition 1.3.14 we have Q is a subsemigroup of $(S, +)$
Let $a, b \in Q \subseteq S$. Then
$ab \in SQ \subseteq \langle SQ\rangle$, and $ab \in QS \subseteq \langle QS\rangle$.
Then, $ab \in SQ\cap QS \subseteq \langle SQ\rangle \cap \langle QS\rangle \subseteq Q$ since Q is quasi ideal of S .
Hence, $ab \in Q$.
Therefore , Q is a subsemiring of S. □

Remark 1.3.19.
The converse of Lemma 1.3.17 part (a) is not true , in general , that is , a quasi-ideal may not be a left , a right , or a two sided ideal of matrices.

Proof.
Let $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z^0 \right\}$ be a semiring under usual addition

and multiplication of matrices.

We showed that in Example 1.3.16.

$Q = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in Z^0 \right\}$ is a quasi-ideal of S.

Now, we prove Q is not a left ideal,

the element $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in SQ$, but $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \notin Q$.

Then, Q is not a left ideal of S.
Similararly, we can prove Q is not a right or a two sided ideal of S . □

Next, we observe that there exists a semiring S such that the sum and the product of two quasi-ideals of S is not a quasi-ideal of S .

Example 1.3.20. [16]

Let $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z^0 \right\}$, $Q_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in Z^0 \right\}$

, and $Q_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} : y \in Z^0 \right\}$.

Then Q_1 and Q_2 are quasi-ideals of S but $Q_1 + Q_2$ is not a quasi-ideal of S

Proof.
From Example 1.3.16 , we have Q_1 is a quasi-ideal of S , similarly Q_2 is a quasi-ideal of S .
Now, to prove $Q_1 + Q_2$ is not a quasi-ideal of S , let $Q = Q_1 + Q_2$.

Then, $Q = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in Z^0 \right\}$.

The element

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in SQ \cap QS$$

but it is not an element of Q .
Hence, the sum of two quasi-ideals is not a quasi-ideal of S. □

Example 1.3.21. [16]

Let $S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : a, b \in \mathbb{R}^+ \right\}$.

Then S is a semigroup under the usual matrix multiplication .

Let $A = S^0 = S \bigcup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Then, A is a semigroup with zero .

Define (+) on A as X + Y = 0 if X , Y $\in S$
X + 0 = 0 + X = X for all X $\in S$

Then, $(A, +, .)$ is a semiring.

Let $R = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : a, b \in \mathbb{R}^+, 0 < a < b \right\} \bigcup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *and*

$L = \left\{ \begin{pmatrix} p & 0 \\ q & 1 \end{pmatrix} : p, q \in \mathbb{R}^+, 0 < \text{p and } 5 < q \right\} \bigcup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Then R, L are quasi-ideals of A , but RL is not a quasi-ideal of A.

Proof.
First , we show $(A, +, .)$ is a semiring . It is clear that A is a semigroup under addition and multiplication.
To show the distributive law , let $X, Y, Z \in A$.
If $X, Y, Z \in S$, then
$X(Y + Z) = X.0 = 0$
$XY + XZ = 0$ since $XY, XZ \in S$, because S is a semigroup under multiplication.
Similarly, $(X + Y)Z = XZ + YZ$.
If $X, Y, Z \in A\backslash S$, then $X = Y = Z = 0$ and so
$X(Y + Z) = 0 = XY + XZ$
$X(Y + Z) = 0 = XZ + YZ$.
Hence , $(A, +, .)$ is a semiring.
Second, we show R is a right ideal of A .
Let

$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix} \in RA$ where $a, b, c, d \in \mathbb{R}^+$ and $0 < a < b$.

Then $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix} = \begin{pmatrix} ac & 0 \\ bc + d & 1 \end{pmatrix} \in R$, since $ac, bc + d \in \mathbb{R}^+$

$0 < ac < bc$ and $0 < ac < bc + d$

Then, $RA \subseteq R$
Hence, R is a right ideal of A and so by lemma 1.3.17(a) , R is a quasi-ideal of A .

Third, we show L is a left ideal of A .

Let $\begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ q & 1 \end{pmatrix} \in AL$ where c , d , p , $q \in \mathbb{R}^+$ and $0 < p$,$5 < q$

Then, $\begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} cp & 0 \\ dp + q & 1 \end{pmatrix} \in L$ since cp , $dp + q \in \mathbb{R}^+$,

$0 < p$ then $0 < cp$.

since, $5 < q$ and $0 < dp$ then $5 < dp + q$
Hence, $AL \subseteq L$.
Therefore L is a left ideal of A , and so L is a quasi-ideal of A.

Finally, we show RL is not a quasi-ideal of A.
To see this , note that

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} \in RL$$

$$\begin{pmatrix} 5 & 0 \\ 10 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 7 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 0 \\ 6 & 1 \end{pmatrix} \left[\begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} \right] \in A(RL)$$

$$= \left[\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 6 & 1 \end{pmatrix} \right] \begin{pmatrix} \frac{1}{4} & 0 \\ 1 & 1 \end{pmatrix} \in (RL)A$$

Then, $\begin{pmatrix} 5 & 0 \\ 10 & 1 \end{pmatrix} \in A(RL) \cap (RL)A$, but it is not an element of RL,

because $\begin{pmatrix} 10 & 0 \\ 6 & 1 \end{pmatrix} \notin R$.

Hence, the product of two quasi-ideals is not a quasi-ideal of A. □

Lemma 1.3.22. [3]
let S be a semigroup or a semiring , L a left and R a right ideal of S and consider elements $e = e^2$ and $f = f^2$ of S .
Then we have $eL = L \cap eS$, $Re = Se \cap R$ and $Sf \cap eS = eSf$, and all three subsets are quasi ideals of S .

Proof.

Let L be a left ideal and R a right ideal of S .
Let $e = e^2$ and $f = f^2$ of S .
First, we show that $eL = L \cap eS$.
Let $x \in eL$, then $x = ey$ for some $y \in L$. Since $y \in L$ and $e \in S$ then $ey \in L$ since L is a left ideal of S .
So that , $x \in L$.
But $y \in L \subseteq S$ then $y \in S$ and so $x = ey \in eS$.
Hence, $x \in L \cap eS$.
So that, $eL \subseteq L \cap eS$.
For the converse inclusion , let $x \in L \cap eS$, then $x \in L$ and $x \in eS$
Then $x = es$ for some $s \in S$.
Also , $x = es = e(es) = ex \in eL$.
Then, $L \cap eS \subseteq eL$.
Therefore, $eL = L \cap eS$.

Similarly , $Re = Se \cap R$.

Now , we prove that $Sf \cap eS = eSf$.
If $a \in Sf \cap eS$, then $a = sf$, $a = es^{'}$ for some s , $s^{'} \in S$.
Then , $af = sff = sf = a = es^{'} = ees^{'} = ea$.
Hence , $af = ea = a$, and so $eaf = ea = a$.
Therefore , $a \in eSf$.
Hence , $Sf \cap eS \subseteq eSf$.
Conversly , if $a \in eSf$, then $a = esf$ for some $s \in S$.
Hence , $a = esf = eesff = e(esf)f = eaf$.
Now , $a = eaf = eaff = af$ and $a = eaf = eeaf = ea$ that is , $a \in Sf \cap eS$.
Hence , $eSf \subseteq Sf \cap eS$.
Therefore , $Sf \cap eS = eSf$.

Finally , Since L is a left ideal and eS is a right ideal from Lemma (1.3.17)(d)
,
then the intersection $L \cap eS = eL$ is a quasi ideal of S by Lemma 1.3.17(c).
Similarly,since R is a right ideal and Se is a left ideal of S then we have $R \cap Se = Re$ is a quasi ideal of S.
Also, we have eS is a right ideal ,and Sf is a left ideal of S , then $Sf \cap eS = eSf$ is a quasi-ideal of S . □

Definition 1.3.23. [3]
let S be a semiring or a semigroup.
let $\phi \neq X \subseteq S$.
We define $(X)_l, (X)_r$ and $(X)_q$ to be the left , right , and the quasi-ideal of S generated by X , respectively .
Note that $(X)_l$ is the intersection of all left-ideals of S containing X , and all others are defined similarly.
The left ideal $(X)_l$ is finitely generated by X if there exists finitely many elements $x_1, ..., x_n \in X$ such that $(X)_l = s_1x_1 + s_2x_2 + ... + s_nx_n$, where $s_i \in S$, the right ideal is defined similarly.
We call them principle if X contains only one element x , we write $(x)_q$ instead of $(\{x\})_q$, for example.

Lemma 1.3.24. [3]
let S be a semigroup or a semiring .

(a) For each $\Phi \neq X \subseteq S$ we have $(X)_l = \langle X \cup SX \rangle$, $(X)_r = \langle X \cup XS \rangle$, and $(X)_q = \langle X \cup (\langle SX \rangle \cap \langle XS \rangle) \rangle$.

(b) For the principle left , right and two sided ideals generated by $s \in S$ we have $\langle S(s)_l \rangle = Ss, \langle (s)_r S \rangle = sS$.

Proof.

(a) Let $a \in (X)_l$ then $a = \sum_{i=1}^{n} s_i x_i$ where $s_i \in S, x_i \in X$ and $n \in \mathbb{N}$.
Then , $s_i x_i \in SX$, and so $s_i x_i \in X \cup SX$.
Hence , $a = \sum_{i=1}^{n} s_i x_i \in \langle X \cup SX \rangle$.
So, $(X)_l \subseteq \langle X \cup SX \rangle$.
Conversly , let $a \in \langle X \cup SX \rangle$, then $a = \sum_{i=1}^{n} a_i$ where $a_i \in X \cup SX$.
Now , $a_i \in X$ or $a_i \in SX$.
If $a_i \in SX$ then $a_i = s_i x_i$, where $x_i \in X$ and $s_i \in S$.
So, $a = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} s_i x_i \in (X)_l$.
If $a_i \in X$, then $a_i \in (X)_l$ (since $X \subseteq (X)_l$) and so $a \in (X)_l$.

Hence, $a \in (X)_l$, and so $\langle X \cup SX \rangle \subseteq (X)_l$.
Therefore, $(X)_l = \langle X \cup SX \rangle$.
Similarly,we have $(X)_r = \langle X \cup XS \rangle$.
Now, from lemma 1.3.17(c) we have.

$$\begin{aligned}
(X)_q &= (X)_l \cap (X)_r \\
&= \langle X \cup SX \rangle \cap \langle X \cup XS \rangle. \\
&= \langle \langle X \rangle \cup \langle SX \rangle \rangle \cap \langle \langle X \rangle \cup \langle XS \rangle \rangle, \text{ from Proposition 1.3.11 (4)} \\
&= \langle (\langle X \rangle \cup \langle SX \rangle) \cap (\langle X \rangle \cup \langle XS \rangle) \rangle, \text{ from Proposition 1.3.11 (7)} \\
&= \langle \langle X \rangle \cup (\langle SX \rangle \cap \langle XS \rangle) \rangle
\end{aligned}$$

(b) Let $x \in \langle S(s)_l \rangle$ then $x = \sum_{i=1}^{n} s_i' y_i$ where $s_i' \in S$ and $y_i \in (s)_l$,$n \in \mathbb{N}$

Then $y_i = y_i' s$ for some $y_i' \in S$, $\forall\ i \in \{1, 2, ..., n\}$.

Now,$x = \sum_{i=1}^{n} s_i'(y_i' s) = \sum_{i=1}^{n} (s_i' y_i') s$.

Since, $s_i' y_i' \in S$ then $(s_i' y_i') s \in Ss$, $\forall\ i \in \{1, 2, ..., n\}$.

But Ss is left ideal by 1.3.17(d) , then (Ss) is a subsemigroup of $(S, +)$, and so $\sum_{i=1}^{n} (s_i' y_i') s \in Ss$

Hence, $x \in Ss$.
Therefore, $\langle S(s)_l \rangle \subseteq Ss$.
Conversly, let $x \in Ss$ then $x = ys$ for some $y \in S$.
Since, $s \in (s)_l$ and $y \in S$ then $ys \in S(s)_l$

But, $x = ys = \sum_{i=1}^{n} y_i s_i$ where $n = 1$.
So $x \in \langle S(s)_l \rangle$

Then, $Ss \subseteq \langle S(s)_l \rangle$.
Therefore,$Ss = \langle S(s)_l \rangle$.
Similarly, $\langle (s)_l S \rangle = sS$.

□

Definition 1.3.25. [3]
let S be a semiring or a semigroup.
A quasi-ideal Q of S is said to have the intersection property , iff $Q = L \cap R$ holds for some left and right ideals L and R of S .

Definition 1.3.26. [11]
let S be a semigroup.
A subset $\Phi \neq B \subseteq S$ of a semigroup $(S, .)$ is called a bi-ideal of S iff B is a subsemigroup of $(S, .)$ satisfying $BSB \subseteq B$.

Definition 1.3.27. [3]
let S be a semiring .
A subset $\Phi \neq B \subseteq S$ of a semiring $(S, +, .)$ is called a bi-ideal of S iff B is a subsemiring of $(S, +, .)$ satisfying $\langle BSB \rangle \subseteq B$.

Lemma 1.3.28. [3]
Let S be a semiring and T a two sided ideal of S .Then each quasi-ideal Q of T is a bi-ideal of S.
Especially, each quasi-ideal Q of S is also a bi-ideal of S , hence Q satisfies $\langle QSQ \rangle \subseteq Q$.

Proof.
Let T be a two sided ideal of S , then $ST \subseteq T$ and $TS \subseteq T$.
let Q be a quasi-ideal of T , then $\langle TQ \rangle \cap \langle QT \rangle \subseteq Q$.
Now, we show that Q is a bi-ideal of S , i.e. $\langle QSQ \rangle \subseteq Q$.
From 1.3.11(6) $\langle QSQ \rangle = \langle Q\langle SQ \rangle\rangle = \langle\langle QS \rangle Q\rangle$..................................(∗)
Then

$$\begin{aligned}
\langle QSQ \rangle &= \langle Q\langle SQ \rangle\rangle \cap \langle\langle QS \rangle Q\rangle \\
&\subseteq \langle T\langle SQ \rangle\rangle \cap \langle\langle QS \rangle T\rangle \text{ since } Q \subseteq T \subseteq S \\
&= \langle\langle TS \rangle Q\rangle \cap \langle Q\langle ST \rangle\rangle \\
&\subseteq \langle TQ \rangle \cap \langle QT \rangle \\
&\subseteq Q
\end{aligned}$$

Next, we show that if Q is a quasi-ideal of S then Q is a bi-ideal of S.

Now, from $(*)$ we have

$$\begin{aligned}\langle QSQ\rangle &= \langle\langle QS\rangle Q\rangle \cap \langle Q\langle SQ\rangle\rangle \\ &\subseteq \langle\langle QS\rangle S\rangle \cap \langle S\langle SQ\rangle\rangle \text{ since } Q\subseteq S \\ &\subseteq \langle QS\rangle \cap \langle SQ\rangle \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(**)\end{aligned}$$

because from 1.3.17(d) ,$\langle QS\rangle$ is a right ideal of S and so $\langle\langle QS\rangle S\rangle \subseteq \langle QS\rangle$. also, $\langle SQ\rangle$ is a left ideal of S then $\langle S\langle SQ\rangle\rangle \subseteq \langle SQ\rangle$.
Since, Q is a quasi-ideal of S , then $\langle QS\rangle \cap \langle SQ\rangle \subseteq Q$
Hence, from $(**)$ we have $\langle QSQ\rangle \subseteq Q$.
Therefore, Q is a bi-ideal of S. □

Note that the converse of Lemma 1.3.27 does not hold in general , that is a bi-ideal of S may not be a quasi-ideal of S

Remark 1.3.29.
Since every left , right , and two sided ideal of a semiring S is a quasi-ideal of S , it follows that every left ,right , and two sided ideal of S is a bi-ideal of S , but the converse is not true , in general , [see Remark 1.3.18 since every quasi-ideal is a bi-ideal of S].

Definition 1.3.30.
let S be a semiring or a semigroup.
let $\phi \neq X \subseteq S$.
We define $(X)_b$ to be the bi-ideal of S generated by X .
The bi-ideal ideal $(X)_b$ is finitely generated by X if there exists finitely many elements $x_1, \ldots\ldots., x_n \in X$ such that $(X)_b = x_1s_1x_1 + x_2s_2x_2 + \ldots\ldots + x_ns_nx_n$, where $s_i \in S$.

Proposition 1.3.31. [11]
Let X be a nonempty subset of a semigroup or a semiring S. Then $(X)_b = \langle XSX \cup X \cup X^2\rangle$.

Proof.
Let $a \in (X)_b$, then $a = \sum_{i=1}^{n} x_i s_i x_i$ where $x_i \in X, s_i \in S$ and $n \in \mathbb{N}$.
Then , $x_i s_i x_i \in XSX$, and so $x_i s_i x_i \in XSX \cup X \cup X^2$.
Hence , $a = \sum_{i=1}^{n} x_i s_i x_i \in \langle XSX \cup X \cup X^2 \rangle$.
So, $(X)_b \subseteq \langle XSX \cup X \cup X^2 \rangle$.
Conversly , let $a \in \langle XSX \cup X \cup X^2 \rangle$, then $a = \sum_{i=1}^{n} a_i$ where $a_i \in XSX \cup X \cup X^2, \forall\ i \in \{1, 2, ..., n\}$.
Now , $a_i \in XSX$ or $a_i \in X$ or $a_i \in X^2, \forall\ i \in \{1, 2, ..., n\}$.
If $a_i \in XSX$ then $a_i = x_i s_i x_i$, where $x_i \in X$ and $s_i \in S$.
Thus, $a_i \in (X)_b$.
If $a_i \in X$, then $a_i \in (X)_b$ (since $X \subseteq (X)_b$).
If $a_i \in X^2$ then $a_i \in (X)_b$ (since $X^2 = X.X \subseteq ((X)_b)^2 \subseteq (X)_b$).
$\forall\ i \in \{1, 2, ..., n\}$, $a_i \in (X)_b$.
Thus, $a = \sum_{i=1}^{n} a_i \in (X)_b$.
Hence, in all cases $\langle XSX \cup X \cup X^2 \rangle \subseteq (X)_b$.
Therefore, $(X)_b = \langle XSX \cup X \cup X^2 \rangle$. □

Proposition 1.3.32.
Let X be a nonempty subset of a semigroup S. Then
$(X)_b \subseteq (X)_q$.

Proof.
Let $x \in (X)_b = XSX \cup X \cup X^2$, from Proposition 1.3.30.
Then, $x \in XSX$, $x \in X$ or $x \in X^2$.
If $x \in XSX$, then $x = asb$ for some $a, b \in X$, $s \in S$.
Now, $x = asb = (as)b \in SX$ since $X \subseteq S$ and $as \in S$.
Also, $x = asb = a(sb) \in XS$ since $sb \in S$.
Then, $x \in SX \cap XS$, and so $x \in (SX \cap XS) \cup X = (X)_q$ (From Lemma 1.3.23(a)).
If $x \in X$ then $x \in (SX \cap XS) \cup X = (X)_q$.
If $x \in X^2$ then $X^2 = X.X \subseteq SX$, since $X \subseteq S$.

also, $X^2 = X.X \subseteq XS$.
So that, $x \in X^2 \subseteq SX \cap XS$.
Hence, $x \in SX \cap XS$ and so $x \in (SX \cap XS) \cup X = (X)_q$.
Therefore,$(X)_b \subseteq (X)_q$. $\square$

Chapter 2

Quasi-ideals and Regularity

Throughout this chapter, S will denote a semigroup or a semiring.

2.1 Regular Semigroups and Semirings

Definition 2.1.1. [3]

An element $s \in S$ is called regular in S iff $s \in sSs$.
S is called regular iff each element of S is regular in S .

Proposition 2.1.2. [3]
An element s of S is regular iff one of the following statements holds:

(1) There is an element $x \in S$ satisfying $xs = e = e^2$, $s = se$.

(2) There is an element $y \in S$ satisfying $sy = f = f^2$, $s = fs$.

(3) There is an element $e = e^2 \in S$ satisfying $(s)_l = Se = Ss$.

(4) There is an element $f = f^2 \in S$ satisfying $(s)_r = fS = sS$.

Proof.
We prove (1),(3) and (2),(4) are similar to (1),(3) respectively.

(1) Assume that s is regular in S .
Then $s \in sSs$, that is $s = sxs$ for some element $x \in S$.
Now, let $xs = e$ then $s = se$.
Since $s = sxs$ then $xs = xsxs = (xs)^2 = e^2$.
Hence, $xs = e = e^2$.
Conversly, assume that $\exists\ x \in S$ such that $xs = e = e^2$, $s = se$.
Then, $s = se = sxs \in sSs$.
Hence, s is regular.

(3) Assume that $s = sxs$ for some $x \in S$
let $e = xs$ then $e = xs = xsxs = (xs)^2 = e^2$.
Hence, $e = e^2$.
First, we show $Se = Ss$.
$Se = Sxs = (Sx)s \subseteq Ss$
$Ss = Ssxs = (Ss)xs \subseteq Sxs = Se$
Then $Se = Ss$.
Now, we show $(s)_l = Se$.
Let $x^{'} \in (s)_l$ then $x^{'} = ys$ for some $y \in S$.
Now, $x^{'} = ys = y(sxs) = ys(xs) = yse \in Se$, since, $ys \in S$.
Hence $(s)_l \subseteq Se$.
For the converse inclusion, let $x^{'} \in Se$ then

$$\begin{aligned} x^{'} &= s^{'}e \text{ for some } s^{'} \in S \\ &= s^{'}xs \\ &= ys \in (s)_l \text{ where } y = s^{'}x \in S. \end{aligned}$$

Then , $Se \subseteq (s)_l$.
Therefore, $(s)_l = Se$.
Conversly, assume that $(s)_l = Se = Ss$ with $e = e^2$.
Now, $s \in (s)_l = Se$, then $s = xe$ for some $x \in S$.
So, $se = xee = xe^2 = xe = s$ and,
$e \in Se = Ss$ then $e = ys$ for some $y \in S$.
So that $s = se = sys \in sSs$.
Therefore, s is regular. □

Theorem 2.1.3. [3]
The following are equivalent for S:

(1) S is regular .

(2) Each left ideal L and each right ideal R of S satisfy $RL = \langle RL \rangle = L \cap R$.

(3) Each left ideal L and each right ideal R of S satisfy.

  (a) $\langle L^2 \rangle = L$

  (b) $\langle R^2 \rangle = R$ and

  (c) $\langle RL \rangle$ is a quasi-ideal of S .

(4) The set $\mathcal{Q}$ of all quasi-ideals of S is regular semigroup with respect to the product $\langle Q_1 Q_2 \rangle$.

(5) Each quasi-ideal Q of S satisfies $Q = \langle QSQ \rangle$.
Moreover, the statements $3a$) and $3b$) imply that each quasi-ideal Q of S has the intersection property since it satisfies $Q = \langle SQ \rangle \cap \langle QS \rangle$.

Proof. At first we prove the last statement .
We apply ($3a$) to the left ideal $(Q)_l = \langle Q \cup SQ \rangle$ (by Lemma 1.3.23(a))of S generated by a quasi-ideal Q of S and obtain

$$\begin{aligned} Q \subseteq \langle Q \cup SQ \rangle &= \langle \langle Q \cup SQ \rangle^2 \rangle = \langle \langle Q \cup SQ \rangle \langle Q \cup SQ \rangle \rangle \\ &= \langle \langle \langle Q \rangle \cup \langle SQ \rangle \rangle \langle Q \cup SQ \rangle \rangle \\ &\subseteq \langle \langle \langle S \rangle \cup \langle S \rangle \rangle \langle Q \cup SQ \rangle \rangle \text{ since } Q \subseteq S \end{aligned}$$

then $\langle Q \rangle \subseteq \langle S \rangle$ and $SQ \subseteq S^2 = S$, so that $\langle SQ \rangle \subseteq \langle S \rangle$

$$\begin{aligned} Q \subseteq \langle Q \cup SQ \rangle &= \langle \langle S \rangle \langle Q \cup SQ \rangle \rangle = \langle S \langle Q \cup SQ \rangle \rangle \\ &\subseteq \langle SQ \rangle \cup \langle SQ \rangle = \langle SQ \rangle \end{aligned}$$

Similarly, we get $Q \subseteq \langle Q \cup QS \rangle = \langle \langle Q \cup QS \rangle^2 \rangle \subseteq \langle QS \rangle$.
Therefore,$Q \subseteq \langle SQ \rangle \cap \langle QS \rangle \subseteq Q$.
Hence, $\langle SQ \rangle \cap \langle QS \rangle = Q$.

(1) $\rightarrow$ (2) Assume that S is regular .
let L be a left ideal and R be a right ideal of S .
Then by Lemma 1.3.17(c) we have $RL \subseteq \langle RL \rangle \subseteq L \cap R$.
Now, we show that $L \cap R \subseteq RL$.
let $d \in L \cap R$ then $d \in L$ and $d \in R$, but $L, R \subseteq S$ then $d \in S$ and so,$(\exists\ x \in S) \ni d = dxd$ since S is regular.
Since $d \in L$ and L is a left ideal then $xd \in SL \subseteq L$, but $d \in R$ then $dxd \in RL$.
So, $d = dxd \in RL$.
Hence, $L \cap R \subseteq RL$.
To show $L \cap R \subseteq \langle RL \rangle$, note that $L \cap R \subseteq RL$ and $RL \subseteq \langle RL \rangle$ (from Proposition 1.3.11(1)).
Then , $L \cap R \subseteq RL \subseteq \langle RL \rangle$.
Hence, $L \cap R \subseteq \langle RL \rangle$.
Therefore, $RL = \langle RL \rangle = L \cap R$.

(2) $\rightarrow$ (3) let L be a left ideal and R a right ideal of S satisfy $RL = \langle RL \rangle = L \cap R$.
To prove (a) let $(L)_r$ the right ideal of S generated by the left ideal L of S .
From Lemma 1.3.23(a) we have $(L)_r = \langle L \cup LS \rangle$ and so $L \subseteq (L)_r$.
Then $L = L \cap (L)_r = \langle (L)_r L \rangle = \langle \langle L \cup LS \rangle L \rangle \subseteq \langle LL \cup LSL \rangle$...................$(*)$
But $SL \subseteq L$ then $LSL \subseteq LL$ and so $LSL \cup LL = LL$
Now, from $(*)$ we have.
$L \subseteq \langle LL \cup LSL \rangle = \langle LL \rangle \subseteq \langle SL \rangle \subseteq L$ (since L is left ideal).
Then,$L \subseteq \langle L^2 \rangle \subseteq L$.
Hence, $\langle L^2 \rangle = L$.
Similarly, $\langle R^2 \rangle = R$.
Now,from (2) we have $\langle RL \rangle = L \cap R$ then, $\langle RL \rangle$ is a quasi-ideal by Lemma (1.3.17)(c) .

(3) $\rightarrow$ (4) let Q_1 and Q_2 be quasi-ideals of S .
Then $L = \langle SQ_1Q_2 \rangle$ and $R = \langle Q_1Q_2S \rangle$ are left and right ideals of S .

Hence, $3a)$ and $3b)$ imply $S = \langle S^2 \rangle$ and

$$\begin{aligned}\langle SQ_1Q_2\rangle &= \langle\langle SQ_1Q_2\rangle\langle SQ_1Q_2\rangle\rangle \\ &= \langle\langle SQ_1Q_2\rangle\langle SSQ_1Q_2\rangle\rangle \\ &= \langle S\langle Q_1Q_2S\rangle\langle SQ_1Q_2\rangle\rangle\end{aligned}$$

$$\begin{aligned}\langle Q_1Q_2S\rangle &= \langle\langle Q_1Q_2S\rangle\langle Q_1Q_2S\rangle\rangle \\ &= \langle\langle Q_1Q_2SS\rangle\langle Q_1Q_2S\rangle\rangle \\ &= \langle\langle Q_1Q_2S\rangle\langle SQ_1Q_2\rangle S\rangle\end{aligned}$$

Since $\langle RL\rangle$ is a quasi-ideal of S by $(3c)$ and satisfy
$\langle SRL\rangle \cap \langle RLS\rangle = \langle RL\rangle$ from the last statement of the theorem .
Then,

$$\begin{aligned}\langle SQ_1Q_2\rangle \cap \langle Q_1Q_2S\rangle &= \langle S\langle Q_1Q_2S\rangle\langle SQ_1Q_2\rangle\rangle \cap \langle\langle Q_1Q_2S\rangle\langle SQ_1Q_2\rangle S\rangle \\ &= \langle SRL\rangle \cap \langle RLS\rangle = \langle RL\rangle \\ &= \langle\langle Q_1Q_2S\rangle\langle SQ_1Q_2\rangle\rangle \\ &= \langle Q_1Q_2S^2Q_1Q_2\rangle = \langle Q_1Q_2SQ_1Q_2\rangle \\ &\subseteq \langle\langle Q_1Q_2S\rangle SQ_2\rangle \text{ since } Q_1 \subseteq S \\ &\subseteq \langle\langle Q_1Q_2S\rangle Q_2\rangle \text{ since } \langle Q_1Q_2S\rangle \text{ is a right ideal of } S \\ &= \langle Q_1\langle Q_2SQ_2\rangle\rangle \\ &\subseteq \langle Q_1Q_2\rangle \text{ from Lemma 1.3.27}\end{aligned}$$

Hence,$\langle Q_1Q_2\rangle$ is a quasi-ideal of S , i.e $\mathcal{Q}$ is a semigroup and it remains to show that it is regular.
let Q be a quasi-ideal of S then $Q = \langle SQ\rangle \cap \langle QS\rangle$ by the last statement of the theorem.
Now, in a similar way as above we can conclude that

$$\begin{aligned} Q = \langle SQ\rangle \cap \langle QS\rangle &= \langle SQ\rangle^2 \cap \langle QS\rangle^2 \\ &= \langle\langle SQ\rangle\langle SQ\rangle\rangle \cap \langle\langle QS\rangle\langle QS\rangle\rangle \\ &= \langle\langle SQ\rangle\langle SSQ\rangle\rangle \cap \langle\langle QSS\rangle\langle QS\rangle\rangle \\ &= \langle S\langle QS\rangle\langle SQ\rangle\rangle \cap \langle\langle QS\rangle\langle SQ\rangle S\rangle \\ &= \langle\langle QS\rangle\langle SQ\rangle\rangle \\ &= \langle QSQ\rangle \end{aligned}$$

Hence, Q is regular in $\mathcal{Q}$.

(4) $\rightarrow$ (5) Assume that the set $\mathcal{Q}$ of all quasi-ideals of S is a regular semigroup with respect to the product $\langle Q_1Q_2\rangle$.
For each quasi-ideal Q of S , $\exists$ a quasi-ideal X of S such that
$Q = \langle QXQ\rangle \subseteq \langle QSQ\rangle \subseteq \langle SQ\rangle \cap \langle QS\rangle \subseteq Q$.
Hence, $Q = \langle QSQ\rangle$.

(5) $\rightarrow$ (1) Assume that each quasi-ideal Q of S satisfies $Q = \langle QSQ\rangle$.
For each element $s \in S$ the intersection $(s)_l \cap (s)_r$ is a quasi-ideal of S by Lemma 1.3.17(c).
Using (5) and Lemma (1.3.23)(b) we have

$$\begin{aligned} s \in (s)_l \cap (s)_r &= \langle((s)_l \cap (s)_r)S((s)_l \cap (s)_r)\rangle \\ &\subseteq \langle(s)_r S(s)_l\rangle = \langle sS(s)_l\rangle = \langle sSs\rangle = sSs \end{aligned}$$

Hence, $s \in sSs$.
Therefore, each $s \in S$ is regular in S . □

Lemma 2.1.4. [3]
Each two-sided ideal T of a regular semigroup or semiring S is a regular subsemigroup or subsemiring of S.

Proof. Since by Lemma 1.3.17(a) each two-sided ideal T of S is a quasi-ideal of S , then by Lemma (1.3.17)(e) we have each quasi-ideal of a semigroup or

a semiring S is a subsemigroup or a subsemiring of S.
Now, it remains to show T is regular
Each element $s \in T \subseteq S$ is regular in S , so $\exists\ x \in S$ such that $s = sxs = sx(sxs) = s(xsx)s$
but xsx is an element in T .
Then s is regular in T. □

Theorem 2.1.5. [3]
The following statement about an element s of a semigroup or a semiring S are equivalent :

(1) The element s is regular in S

(2) The principle left ideal $(s)_l$ and the principle right ideal $(s)_r$ of S satisfy $\langle (s)_r(s)_l \rangle = (s)_l \cap (s)_r$.

(3) The principle left ideal $(s)_l$ and the principle right ideal $(s)_r$ of S satisfy

  (a) $\langle (s)_l^2 \rangle = (s)_l$,

  (b) $\langle (s)_r^2 \rangle = (s)_r$ and

  (c) $\langle (s)_r(s)_l \rangle$ is a quasi-ideal of S .

(4) The principle quasi-ideal $(s)_q$ of S satisfies $(s)_q = \langle (s)_q S (s)_q \rangle$.

Proof. (1) $\Rightarrow$ (3) for *a*) let $s \in S$ be regular in S .Then by Proposition 2.1.2(3) and (4) we have $(s)_l = Se$ for some $e = e^2 \in S$ and $(s)_r = fS$ for some $f = f^2 \in S$.
Then $(s)_l = Se = See = Seee \subseteq SeSe \subseteq \langle SeSe \rangle = \langle (s)_l^2 \rangle$, so that $(s)_l \subseteq \langle (s)_l^2 \rangle$.
Conversly, since $(s)_l \subseteq S$ and $(s)_l$ is the principle left ideal generated by s , then $\langle S(s)_l \rangle \subseteq (s)_l$.
Now, $(s)_l^2 \subseteq S(s)_l$ and so , $\langle (s)_l^2 \rangle \subseteq \langle S(s)_l \rangle \subseteq (s)_l$.

Hence , $\langle (s_l)^2 \rangle = (s)_l$.
Similarly, we have $\langle (s)_r^2 \rangle = (s)_r$.
For (c), $\langle (s)_r(s)_l \rangle = \langle fSSe \rangle = \langle fS^2e \rangle$, but S is a left ideal then from (a) , $\langle S^2 \rangle = S$.
Now,$\langle (s)_r(s)_l \rangle = \langle fS^2e \rangle = \langle fSe \rangle = fS \cap Se$ by Lemma 1.3.21.
Hence,$\langle (s)_r(s)_l \rangle$ is a quasi-ideal of S.

(3) $\Rightarrow$ (2) Assume that (3) is true .
For $(3a)$ we have $(s)_l = \langle (s)_l^2 \rangle \subseteq \langle S(s)_l \rangle = Ss$ by Lemma 1.3.23 (b), and $Ss = \langle S(s)_l \rangle \subseteq (s)_l$ since $(s)_l$ is the principle left ideal generated by s.
Hence,$(s)_l = Ss$.
Similary $3b)$ yields $(s)_r = sS$.
We use again $3a)$ and $3b)$ to obtain

$$\begin{aligned} (s)_l \cap (s)_r &= Ss \cap sS \\ &= \langle (Ss)^2 \rangle \cap \langle (sS)^2 \rangle \\ &= \langle (Ss)(Ss) \rangle \cap \langle (sS)(sS) \rangle \\ &= \langle (Ss)(Ss)^2 \rangle \cap \langle sS(sS)^2 \rangle \\ &= \langle (Ss)^3 \rangle \cap \langle (sS)^3 \rangle \end{aligned}$$

Now, $\langle (s)_r(s)_l \rangle = \langle sSSs \rangle$ is by $3c)$ a quasi-ideal of S . From this it follows $\langle SsSSs \rangle \cap \langle sSSsS \rangle \subseteq \langle sSSs \rangle = \langle (s)_r(s)_l \rangle$.
Therefore,

$$\begin{aligned} (s)_l \cap (s)_r &= \langle (Ss)^3 \rangle \cap \langle (sS)^3 \rangle \\ &= \langle SsSsSs \rangle \cap \langle sSsSsS \rangle \\ &\subseteq \langle SsSSs \rangle \cap \langle sSSsS \rangle \\ &\subseteq \langle (s)_r(s)_l \rangle \end{aligned}$$

Then, $(s)_l \cap (s)_r \subseteq \langle (s)_r(s)_l \rangle$.
For the converse inclusion ,$\langle (s)_r(s)_l \rangle = \langle sSSs \rangle = \langle sS^2s \rangle = \langle sSs \rangle$, since S is a left ideal .
But from Lemma 1.3.21 we have $\langle sSs \rangle = Ss \cap sS = (s)_l \cap (s)_r$.

(2) $\Rightarrow$ (4) Assume that (2) is true , that is $\langle (s)_r(s)_l \rangle = (s)_l \cap (s)_r$.
Now, $\langle (s)_q S(s)_q \rangle \subseteq \langle S(s)_q \rangle \cap \langle (s)_q S \rangle \subseteq (s)_q$, since $(s)_q$ is the principle

quasi-ideal generated by s.
From (2) and Lemma 1.3.23(b) we have $s \in (s)_l \cap (s)_r = \langle (s)_r (s)_l \rangle \subseteq \langle S(s)_l \rangle = Ss$.
But $(s)_l \cap (s)_r \subseteq (s)_l$, $s \in (s)_l$ and $s \in Ss$ then $(s)_l \subseteq Ss$.
Also, $Ss = \langle S(s)_l \rangle \subseteq (s)_l$
Hence, $(s)_l = Ss$ and similarly , we have $(s)_r = sS$.
So, we obtain $(s)_q \subseteq (s)_l \cap (s)_r = \langle (s)_r (s)_l \rangle = \langle sSSs \rangle \subseteq \langle (s)_q S (s)_q \rangle$.
Therefore, $\langle (s)_q S (s)_q \rangle = (s)_q$.

(4) $\Rightarrow$ (1) Assume that the principle quasi-ideal $(s)_q$ of S satisfies $(s)_q = \langle (s)_q S (s)_q \rangle$.
Now, $s \in (s)_q = \langle (s)_q S (s)_q \rangle \subseteq \langle (s)_r S (s)_l \rangle = \langle sS(s)_l \rangle = sSs$
by Lemma 1.3.23(b).
Hence, $s \in sSs$.
So, s is regular in S . □

Theorem 2.1.6. [3]
The following statements are true for S:

(a) Each quasi-ideal Q of S satisfies
$Q = L \cap R = \langle RL \rangle$ with $L = (Q)_l = \langle SQ \rangle$ and $R = (Q)_r = \langle QS \rangle$.

(b) Each quasi-ideal Q of S satisfies $\langle Q^2 \rangle = \langle Q^3 \rangle$.

(c) Each bi-ideal B of S is a quasi-ideal of S .

(d) Each bi-ideal B of a two sided ideal T of S is a quasi-ideal of S .

Proof. (a) From Theorem 2.1.3 we have each quasi-ideal Q of S has the intersection property $Q = \langle SQ \rangle \cap \langle QS \rangle = (Q)_l \cap (Q)_r = L \cap R$, and condition (2) of Theorem 2.1.3 implies $Q = L \cap R = \langle RL \rangle$.

(b) For each quasi-ideal Q of S it follows by Theorem 2.1.3(4) that $\langle Q^2 \rangle$ is also a quasi-ideal of S and that there is a quasi-ideal X of S such that $\langle Q^2 \rangle = \langle Q^2 X Q^2 \rangle \subseteq \langle Q^2 S Q^2 \rangle = \langle Q(QSQ)Q \rangle \subseteq$

$\langle QQQ \rangle = \langle Q^3 \rangle$, since $\langle QSQ \rangle \subseteq Q$
Hence,$\langle Q^2 \rangle \subseteq \langle Q^3 \rangle$.
But,
$\langle Q^3 \rangle = \langle QQ^2 \rangle \subseteq \langle SQ^2 \rangle$
$\langle Q^3 \rangle = \langle Q^2 Q \rangle \subseteq \langle Q^2 S \rangle$
$\langle Q^3 \rangle \subseteq \langle SQ^2 \rangle \cap \langle Q^2 S \rangle \subseteq \langle Q^2 \rangle$ since Q^2 is a quasi-ideal.
Therefore,$\langle Q^3 \rangle = \langle Q^2 \rangle$.

(c) let B be a bi-ideal of S , then $\langle BSB \rangle \subseteq B$.
Since $\langle SB \rangle$ is a left ideal and $\langle BS \rangle$ is a right ideal of S then by Theorem 2.1.3(2) we obtain $\langle SB \rangle \cap \langle BS \rangle = \langle BSSB \rangle \subseteq \langle BSB \rangle \subseteq B$
Hence, B is a quasi-ideal of S .

(d) By Lemma (2.1.4) the two-sided ideal T of S is a regular subsemiring or subsemigroup of S .
So, by part (c) the bi-ideal B of T is a quasi-ideal of T ,But by Lemma (1.3.27) , each quasi-ideal of T is a bi-ideal of S , then by part (c) B is a quasi-ideal of S.

□

Theorem 2.1.7. [3]
The following conditions on a semiring or a semigroup are equivalent:

(1) Each left ideal L and each right ideal R of S satisfy
$\langle RL \rangle = L \cap R \subseteq \langle LR \rangle$.

(2) The set $\mathcal{Q}$ of all quasi-ideals of S is an idempotent semigroup with respect to the "product" $\langle Q_1 Q_2 \rangle$

(3) Each quasi-deal Q of S satisfies $Q = \langle Q^2 \rangle$.

Proof. To prove (1) $\Rightarrow$ (2) , we state at first that the equality of condition (1) is just the condition (2) of Theorem (2.1.3) which yields that S is regular and that the set $\mathcal{Q}$ of all quasi-ideals of S is regular semigroup with respect

to the product $\langle Q_1 Q_2 \rangle$.
So,we only have to show that it is in fact idempotent.
let Q be any element of the set $\mathcal{Q}$ of all quasi-ideals of S, then by condition (5) of Theorem (2.1.3) we have $Q = \langle QSQ \rangle$ for each quasi-ideal Q of S .
But by condition (3) of Theorem (2.1.3) $\langle S^2 \rangle = S$ for the left ideal S of S .
Combining this we can conclude ,

$$\begin{aligned} Q &= \langle QSQ \rangle \\ &= \langle (QSQ)S(QSQ) \rangle \\ &= \langle (QSQ)(SS)(QSQ) \rangle \\ &= \langle QS(QSSQ)SQ \rangle \\ &\subseteq \langle QS\langle SQQS \rangle SQ \rangle \\ &= \langle (QSSQ)(QSSQ) \rangle \\ &= \langle (QSQ)(QSQ) \rangle \\ &= \langle Q^2 \rangle \end{aligned}$$

where the inclusion follows from(1) since $\langle SQ \rangle$ and $\langle SQ \rangle$ are left and right ideals of S , respectively.
The other inclusion,

$$\begin{aligned} \langle Q^2 \rangle &= \langle QQ \rangle \subseteq \langle SQ \rangle \subseteq \langle QS \rangle \\ &\subseteq \langle SQ \rangle \cap \langle QS \rangle \\ &\subseteq Q \end{aligned}$$

Hence,$\langle Q^2 \rangle = Q$
(2) $\Rightarrow$ (3) is only a restriction.
(3) $\Rightarrow$ (1) Assume that each quasi-ideal Q of S satisfies $Q = \langle Q^2 \rangle$.
Now, by Lemma (1.3.17)(c) for each left ideal L and each right ideal R of S the inclusion $\langle RL \rangle \subseteq L \cap R$ holds and the intersection $L \cap R$ is a quasi-ideal of S.
So, by (3)

$$\begin{aligned} L \cap R &= \langle (L \cap R)^2 \rangle \\ &= \langle (L \cap R)(L \cap R) \rangle \\ &\subseteq \langle RL \rangle \\ &\subseteq \langle LR \rangle \end{aligned}$$

Hence, we obtain $(L \cap R) = \langle RL \rangle$ and $(L \cap R) \subseteq \langle LR \rangle$. □

2.2 Regularity in BQ

Throughout this section let **BQ** denotes the class of all semigroups whose bi-ideals and quasi-ideals coincide.
Also, Calais, J. in 1968 gives a characterization to all semigroups in **BQ** as in Proposition 2.2.4 .

Example 2.2.1. [11]
All zero semigroups are in **BQ**.

Proof.
Assume that S is zero semigroup.
It is known that every quasi-ideal is a bi-ideal of S , from Lemma 1.3.27 .
Now, we show that every bi-ideal is a quasi-ideal of S
Let B be a bi-ideal of S, then $BSB \subseteq B$.
We show $BS \cap SB \subseteq B$
Since B is a nonempty subsemigroup of S , and $0 \in S$ then $0 \in B$.
Now, let $x \in BS \cap SB$ then $x = bs = s^{'}b^{'}$
But $B \subseteq S$ then $bs = s^{'}b^{'} \in S$.
Since S is zero semigroup, that $x = bs = s^{'}b^{'} = 0$.
So, $x = 0$
Hence, $BS \cap SB = \{0\} \subseteq B$.
Therfore, B is a quasi-ideal of S.
Hence, $S \in$ **BQ**. □

Proposition 2.2.2. [15]
Every regular semigroup belongs to **BQ**.

Proof.
From Lemma 1.3.27 and Theorem 2.1.6(c) we have the statement is true. □

Proposition 2.2.3. [11]
If a semigroup S is a left [right] simple then $S \in$ **BQ**.

Proof.
Assume that S is a left simple semigroup, then $Sa = S$for all $a \in S$, from Theorem 1.3.7.
To show $S \in \mathbf{BQ}$, by Proposition 2.2.2, it is enough to show that S is regular
By 2.1.2(3) , $(s)_l = Ss = S$ since S is left simple.
Also, $(s)_l = Se = S$ since S is left simple.
Hence, $s = se$ for some $e \in S$.
Since $Ss = S$ and $e \in S$ then $e = xs$ for some $x \in S$.
Now, we show $e = e^2 \in S$.
Since, $s = se$ then $xs = xse = xsxs$.
Hence, $e = e^2$.
Therefore, S is regular.
Similarly, If S is right simple , then $S \in \mathbf{BQ}$. □

Proposition 2.2.4. [1]
A semigroup S is in $\mathbf{BQ}$ if and only if $(x, y)_q = (x, y)_b$ for all $x, y \in S$.

Proof.
See [1] □

Chapter 3

Semigroups whose Sets of Bi-ideals and Quasi-ideals Coincide

In this chapter , we will give some examples of semigroups contain bi-ideals which are quasi-ideals , interval semigroups on $\mathbb{R}$ and some transformation semigroups , as $T_1(X), T_2(X), T_3(X), T_4(X)$ and $T_5(X)$,
More information can be found in, [2],[17],[12] and [14].

3.1 Interval Semigroups on $\mathbb{R}$

In this section, we show that for both multiplicative interval semigroups and additive interval semigroups on $\mathbb{R}$, regularity is necessary for them to belong to **BQ** where $\mathbb{R}$ denotes the set of real numbers. We prove the following statements:
For a multiplicative interval semigroup S on $\mathbb{R}$, $S \in \mathbf{BQ}$ if and only if S is $\mathbb{R}$, $\{0\}$, $\{1\}$, $(0, \infty)$ or $[0, \infty]$. For an additive interval semigroup S on $\mathbb{R}$, $S \in \mathbf{BQ}$ if and only if S is $\mathbb{R}$ or $\{0\}$.

There are exactly 15 types of multiplicative interval semigroups on $\mathbb{R}$ which are

1) $\mathbb{R}$

2) $\{0\}$

3) $\{1\}$

4) $(0, \infty)$

5) $[0, \infty)$

6) (a, ∞) where $a \geq 1$

7) $[a, \infty)$ where $a \geq 1$

8) $(0, b)$ where $0 < b \leq 1$

9) $(0, b]$ where $0 < b \leq 1$

10) $[0, b)$ where $0 < b \leq 1$

11) $[0, b]$ where $0 < b \leq 1$

12) (a, b) where $-1 \leq a < 0 < a^2 \leq b \leq 1$

13) $(a, b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$

14) $[a, b)$ where $-1 < a < 0 < a^2 \leq b < 1$

15) $[a, b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$

The above 15 multiplicative interval semigroups on $\mathbb{R}$ are the only ones, since for example if we let any interval with $a < 1$ in (6) or (7) , as $[\frac{1}{2}, \infty)$, then $\frac{1}{2}.\frac{1}{2} = \frac{1}{4} \notin [\frac{1}{2}, \infty)$.
So, it is not a semigroup on $\mathbb{R}$

Also, there are exactly 6 types of additive interval semigroups on $\mathbb{R}$ which are

1) $\mathbb{R}$

2) $\{0\}$

3) (a, ∞) where $a \geq 0$

4) $[a, \infty)$ where $a \geq 0$

5) $(-\infty, b)$ where $b \leq 0$

6) $(-\infty, b]$ where $b \leq 0$

3.1.1 Multiplicative Interval Semigroups on $\mathbb{R}$

In this subsection, we show that a multiplicative interval semigroup S on $\mathbb{R}$ belongs to **BQ** if and only if S is one of types $(1)-(5)$ and each of the multiplicative interval semigroups of types $(6)-(15)$ has uncountably many bi-ideals which are not quasi-ideals.

Theorem 3.1.1. [14]
Let S be a multiplicative interval semigroup on $\mathbb{R}$. Then
S is regular iff S is $\mathbb{R}, \{0\}, \{1\}, (0, \infty)$ or $[0, \infty)$

Proof.
If $S = \mathbb{R}$, we show S is regular.
Let $0 \neq a \in \mathbb{R}$ be arbitrary , then $\frac{1}{a} \in \mathbb{R}$. and so, $a = a.\frac{1}{a}.a \in a\mathbb{R}a$.
If $a = 0$, then $a = axa$ for some $x \in \mathbb{R}$.
So, in both cases , $a \in a\mathbb{R}a$.
Then, a is regular.
Since, a is arbitrary element in S, then S is regular.
If $S = \{0\}$, then clearly S is regular.
If $S = \{1\}$, then $1 = 1.1.1$ and so S is regular.
If $S = (0, \infty)$, let $a \in S$ be arbitrary, then $\frac{1}{a} \in S$ and so , $a = a.\frac{1}{a}.a \in aSa$
So, S is regular.
If $S = [0, \infty)$,
Let $a \in S$ be arbitrary , if $a = 0$ then it is a trivial case.
If $a \in (0, \infty)$.
Then, S is regular as above.
Conversly, we show S is regular if $S = \mathbb{R}$, $\{0\}, \{1\}, (0, \infty)$ or $[0, \infty)$.
To prove this , we must show the multiplicative interval semigroups on $\mathbb{R}$ $(6)-(15)$ are not regular.
If S is of type (6), let $a = 2$ then the interval $(2, \infty)$ is not regular, since if we

let for example $3 \in (2, \infty)$ then 3 is not regular element , because $3 = 3.\frac{1}{3}.3$ but $\frac{1}{3} \notin (2, \infty)$.
Similarly, if S is of type (7).
If S is of type (8) , let $b = \frac{1}{2}$ then the interval $(0, \frac{1}{2})$ is not regular since $\frac{1}{3} \in (0, \frac{1}{2})$ but $\frac{1}{3}$ is not regular element because $\frac{1}{3} = \frac{1}{3}.3.\frac{1}{3}$ and $3 \notin (0, \frac{1}{2})$.
Similarly, if S is of type $(9), (10)$ and (11).
If S is of type (12), let $a = \frac{-1}{2}$, $b = \frac{1}{2}$ then the interval $(\frac{-1}{2}, \frac{1}{2})$ is not regular, since $\frac{1}{4} \in (\frac{-1}{2}, \frac{1}{2})$ but $\frac{1}{4}$ is not regular element because $\frac{1}{4} = \frac{1}{4}.4.\frac{1}{4}$ and $4 \notin (\frac{-1}{2}, \frac{1}{2})$.
Similarly, if S is of type $(13), (14)$ and (15).
Now, since the multiplicative interval semigroups on $\mathbb{R}$ from (1) to (5) are regular, and (6) - (15) are not regular, then the theorem is proved. □

Theorem 3.1.2. [14]
let S be a multiplicative interval semigroup on $\mathbb{R}$ .Then

(i) $S \in \mathbf{BQ}$ if and only if S is $\mathbb{R}$, $\{0\}$, $\{1\}$, $(0, \infty)$ or $[0, \infty]$.

(ii) If $S \notin \mathbf{BQ}$, then S has an uncountable number of bi-ideals which are not quasi-ideals.

Proof.
(i) From Theorem (3.1.1) and Proposition (2.2.2) we have that for a multiplicative interval semigroup S on $\mathbb{R}$, $S \in \mathbf{BQ}$ if and only if S is regular .
To prove part (ii) , it suffices to show that there are uncountably many bi-ideals of S which are not quasi-ideals of S if S is one of type $(6)-(15)$.

Case (1) S is of type (6) or (7).
Then, $S = (a, \infty)$ or $[a, \infty)$ for some $a \geq 1$.
let $r \in (0, \infty)$ and set $B_r = (a + r, a + 2r) \cup ((a + r)^2, \infty)$.
Then, $B_r \subseteq S$, and $B_r^2 = ((a + r)^2, (a + 2r)^2) \cup ((a + r)^4, \infty) \subseteq (a + r)^2, \infty) \subseteq B_r$,
so we have that $SB_r^2 = (a, \infty)B_r^2 \subseteq (a, \infty)B_r \subseteq (a(a + r)^2, \infty) \subseteq ((a + r)^2, \infty) \subseteq B_r$.
But S is commutative and $B_r \subseteq S$ then $B_rSB_r \subseteq B_r$.

Hence, B_r is a bi-ideal of S.
Since $a+\frac{r}{2}\in S$ and $a+\frac{3r}{2}\in B_r$, $(a+\frac{r}{2})(a+\frac{3r}{2})\in SB_r$.
But $(a+\frac{r}{2})(a+\frac{3r}{2})=a^2+2ra+\frac{3r^2}{4}<a^2+2ra+r^2=(a+r)^2$,
and $a+2r\leq a(a+2r)<a^2+2ra+\frac{3r^2}{4}=(a+\frac{r}{2})(a+\frac{3r}{2})$.
Then, $a+2r<(a+\frac{r}{2})(a+\frac{3r}{2})<(a+r)^2$.
So, $(a+\frac{r}{2})(a+\frac{3r}{2})\notin B_r$.
Hence, B_r is not a quasi-ideal of S.

Case (2) S is of type (8) or (9).
Then, $S=(0,b)$ or $(0,b]$ for some $0<b\leq 1$.
Let $r\in(0,1]$ and put $B_r=(0,\frac{r^2b^2}{4})\cup(\frac{rb}{4},\frac{rb}{2})$.
Then $B_r\subseteq S$, and $B_r^2\subseteq(0,\frac{r^2b^2}{4})\subseteq B_r$ and thus $SB_r^2\subseteq SB_r=(0,b)B_r\subseteq(0,\frac{r^2b^3}{4})\subseteq(0,\frac{r^2b^2}{4})\subseteq B_r$.
Then, $SB_rB_r\subseteq B_r$, but S is commutative and $B_r\subseteq S$ then
$B_rSB_r\subseteq B_r$.
Therefore, B_r is a bi-ideal of S .
Since, $\frac{3rb}{8}\in(\frac{rb}{4},\frac{rb}{2})\subseteq B_r$ and $\frac{2rb}{3}\in S$, $\frac{r^2b^2}{4}=(\frac{2rb}{3})(\frac{3rb}{8})\in SB_r$ but $\frac{r^2b^2}{4}\notin B_r$
Thus, B_r is not a quasi-ideal of S .

Case (3) S is of type (10) or (11) .
Let S be $[0,b)$ or $[0,b]$, $r\in(0,1]$ and $B_r=[0,\frac{r^2b^2}{4})\cup(\frac{rb}{4},\frac{rb}{2})$.
Then, it is similar to case (2) we have B_r is a bi-ideal but not a quasi-ideal of S.

Case (4) S is of type $(12),(13),(14)$ or (15).
Then there exist $a,b\in\mathbb{R}$ such that

1) $S=(a,b),(a,b]$ or $[a,b]$ and $-1\leq a<0<a^2\leq b\leq 1$ or
2) $S=[a,b)$ and $-1<a<0<a^2<b\leq 1$

let $r\in(0,1]$ and set $B_r=(\frac{ra}{2},\frac{ra^2}{4})\bigcup(\frac{ra^2}{3},\frac{ra^2}{2})$.
Then, $B_r\subseteq S$.

Since, $0 < r \leq 1$ and $-1 \leq a < 0 < a^2 \leq b \leq 1$ we have that

$$\begin{aligned} B_r^2 &\subseteq (\frac{r^2a^3}{4}, \frac{r^2a^2}{4}) \\ &\subseteq (\frac{ra}{4}, \frac{r^2a^2}{4}) \\ &\subseteq B_r \end{aligned}$$

and $[a,b](\frac{r^2a^3}{4}, \frac{r^2a^2}{4}) = (\frac{r^2a^3}{4}, \frac{r^2a^2b}{4}) \subseteq (\frac{r^2a^3}{4}, \frac{r^2a^2}{4}) \subseteq (\frac{ra}{2}, \frac{ra^2}{4}) \subseteq B_r$
Then $SB_r^2 \subseteq B_r$.
Hence , B_r is a bi-ideal of S.
Since $\frac{ra}{3} \in B_r$ and $\frac{3a}{4} \in S$, $\frac{ra^2}{4} = (\frac{3a}{4})(\frac{ra}{3}) \in SB_r$
But, $(\frac{ra^2}{4}) \notin B_r$.
So, B_r is not a quasi-ideal of S .
In each case , r is arbitrary in an uncountable set and $B_r \neq B_{r'}$ if $r \neq r'$ in that set. □

3.1.2 Additive Interval Semigroups on $\mathbb{R}$

We show in this subsection that an additive interval semigroup S on $\mathbb{R}$ belongs to **BQ** if and only if S is $\mathbb{R}$ or $\{0\}$, and if $S \notin$ **BQ** ,S has uncountably many bi-ideals which are not quasi-ideals.

Theorem 3.1.3. [14]
Let S be an additive interval semigroup on $\mathbb{R}$. Then
S is regular iff S is $\mathbb{R}$ or $\{0\}$.

Proof.
If $S = \mathbb{R}$, we show S is regular.
Let $a \in \mathbb{R}$ be arbitrary , then $(-a) \in \mathbb{R}$, and so, $a = a+(-a)+a \in a+\mathbb{R}+a$.
Then, a is regular.
Since, a is arbitrary element in S, then S is regular.
If $S = \{0\}$, then $0 = 0 + 0 + 0$ and so S is regular
Conversly, we show S is regular if S is $\mathbb{R}$ or $\{0\}$.

To prove this, we must show an additive interval semigroups on $\mathbb{R}$ (3) − (6) are not regular.
If S is of type (3) , let $a = 1$ then the interval $(1, \infty\ )$ is not regular , since $2 \in (1, \infty)$ but 2 is not regular element , because $2 = 2 + (-2) + 2$ and $-2 \notin (1, \infty)$.
Similary, if S is of type (4)
If S is of type (5) , let $b = -1$ then the interval $(-\infty, -1)$ is not regular, since $-3 \in (-\infty, -1)$, but (-3) is not regular element, because $-3 = (-3) + 3 + (-3)$ and $3 \notin (-\infty, -1)$
Similary, if S is of type (6) □

Theorem 3.1.4. [14]
Let S be an additive interval semigroup on $\mathbb{R}$ .Then

(i) $S \in \mathbf{BQ}$ if and only if S is $\mathbb{R}$ or $\{0\}$

(ii) If $S \notin \mathbf{BQ}$, then S has an uncountable number of bi-ideals which are not quasi-ideals.

Proof.
(i) From Theorem (3.1.3) and Proposition (2.2.2) we have that for an additive interval semigroup S on $\mathbb{R}$, $S \in \mathbf{BQ}$ if and only if S is regular .
To prove (ii), it suffices to show that if S is one of type (3) − (6), S has uncountably many bi-ideals which are not quasi-ideals .

Case (1) S is of type (3) or (4).
Then $S = (a, \infty)$or $[a, \infty)$ for some $a \in R$ with $a \geq 0$.
Let $r \in (0, \infty)$ and set $B_r = (a + 2r, a + 3r] \cup (2a + 4r, \infty)$.
Then $B_r \subseteq S$,
and $B_r + B_r = (2a + 4r, 2a + 6r] \cup (4a + 8r, \infty) \subseteq (2a + 4r, \infty) \subseteq B_r$
So,

$$\begin{aligned} S + B_r + B_r &\subseteq S + (2a + 4r, \infty) \\ &= (a, \infty) + (2a + 4r, \infty) \\ &\subseteq (3a + 4r, \infty) \\ &\subseteq B_r \end{aligned}$$

Hence, B_r is a bi-ideals of S.
Since , $a+r \in S$ and $a+3r \in B_r$, $2a+4r = (a+r)+(a+3r) \in (S+B_r)\backslash B_r$.
Hence, $S + B_r \nsubseteq B_r$, and so $(S + B_r) \cap (B_r + S) \nsubseteq B_r$.
Therefore, B_r is not a quasi-ideal of S.
Case (2) S is of type (5) or (6) .
Then $S = (-\infty, b)$ or $(-\infty, b]$ for some $b \in \mathbb{R}$ with $b \leq 0$.
Let $r \in (0, \infty)$ and put $B_r = (-\infty, 2b - 4r) \cup [b - 3r, b - 2r)$.
Then $B_r \subseteq S$ and,

$$\begin{aligned} B_r + B_r &= (-\infty, 4b - 8r) \cup [2b - 6r, 2b - 4r) \\ &\subseteq (-\infty, 2b - 4r) \\ &\subseteq B_r \end{aligned}$$

and so,

$$\begin{aligned} S + B_r + B_r &\subseteq S + (-\infty, 2b - 4r) \\ &= (-\infty, b) + (-\infty, 2b - 4r) \\ &\subseteq (-\infty, 3b - 4r) \\ &\subseteq (-\infty, 2b - 4r) \\ &\subseteq B_r \end{aligned}$$

Thus, B_r is a bi-ideal of S .
Since, $b - r \in S$ and $b - 3r \in B_r$, we have $2b - 4r = (b - r) + (b - 3r) \in S + B_r\backslash B_r$.
Therefore, B_r is not a quasi-ideal of S.
Since $(0, \infty)$ is an uncountable set and $B_r \neq B_{r'}$ if $r \neq r'$ in both cases, the theorem is proved. □

3.2 Some Transformation Semigroups

Definition 3.2.1. [2]

(a) Let X be a nonempty set.
A transformation of a set X is a single-valued mapping α of X into itself.

(b) The composition of two transformations α and β of X is the transformation $\alpha\beta$ defined by
$x(\alpha\beta) = (x\alpha)\beta$ for all x in X.

The associative law $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds for the transformations α, β and γ of X, because for every x in X,
$x((\alpha\beta)\gamma) = (x(\alpha\beta))\gamma = ((x\alpha)\beta)\gamma = (x\alpha)(\beta\gamma) = x(\alpha(\beta\gamma))$.

Hence,we have the set T_X of all transformations of X is a semigroup under composition of mappings.

We call T_X *the full transformation semigroup on* X.

Let I_X denotes the identity map on X,that is $I_X : X \to X$ defined by $x(I_X) = x$
.
The range of an element $\alpha \in T_X$ will be denoted by ranα.
ran$\alpha = \{y \in X : x\alpha = y$ for some $x \in X\}$
For convenience, for $a \in X$, let $X_a \in T_X$ be such that its range is $\{a\}$.

Theorem 3.2.2. [12]
$T_X \in$ **BQ**.

Proof. To see this, we must show T_X is regular.
Let $\alpha \in T_X$ be arbitrary. We show that α is regular , i.e $\alpha\beta\alpha = \alpha$ for some $\beta \in T_X$.
Let $y \in X$ be fixed.

Define β as $x\beta = \begin{cases} x, & \text{if } x \notin \text{ran } \alpha; \\ \\ y \in x\alpha^{-1}, & \text{otherwise.} \end{cases}$

Now, for any $x \in X$,
Then $x\alpha\beta\alpha = (x\alpha)\beta\alpha = x\alpha$.

becase $x\alpha \in$ranα and so $(x\alpha)\beta = y \in (x\alpha)\alpha^{-1} = (x)\alpha\alpha^{-1} = x$.
So that , $y = x$.
Hence, $\alpha\beta\alpha = \alpha$ for some $\beta \in T_X$.
Therefore, T_X is regular , and so by Proposition (2.2.2) $T_X \in$ **BQ**. □

.

Remark 3.2.3. [17]
ran$\alpha\beta \subseteq$ ranβ , for all $\alpha, \beta \in T_X$.

Proof.
Let $y \in$ ran$\alpha\beta$, then $x\alpha\beta = y$ for some $x \in X$.
Now, $(x\alpha)\beta = x\alpha\beta = y$.
Put, $x\alpha = y^{'}$ for some $y^{'} \in X$, so that $y^{'}\beta = y$.
Hence, $y \in$ ranβ.
Therefore, ran$\alpha\beta \subseteq$ ranβ , for all $\alpha, \beta \in T_X$. □

Definition 3.2.4. $T_1(X) = \{\alpha \in T_X :$ ranα is finite $\}$.

Proposition 3.2.5. [12]
$T_1(X)$ is a subsemigroup of T_X.

Proof.
Let $\alpha, \beta \in T_1(X)$.
Then , $\alpha, \beta \in T_X$ and ranα , ranβ are finite.
But , ran$\alpha\beta \subseteq$ ranβ.
Then , ran$\alpha\beta$ is finite, and so $\alpha\beta \in T_1(X)$
Hence , $T_1(X)$ is a subsemigroup of T_X
. □

Proposition 3.2.6. [12]
$T_1(X) \in$ **BQ**.

Proof.
We must show , $T_1(X)$ is regular.
let $\alpha \in T_1(X)$ be arbitrary.
Then , $\alpha \in T_X$ and ranα is finite.
We show that $\alpha = \alpha\beta\alpha$ for some $\beta \in T_1(X)$.

Let ran$\alpha = \{a_1, a_2, ..., a_n\}$,
$i.e\ x\alpha = a_i$, for some $i = 1, 2, ..., n$, $\forall\ x \in X$.
Let $y \in X$ be fixed.

Define β as $x\beta = \begin{cases} y, & \text{if } x \notin ran\alpha \\ x\beta = x\alpha^{-1}, & x \in ran\alpha. \end{cases}$

Now, for any $x \in X$,
$x\alpha\beta\alpha = (x\alpha)\beta\alpha = (a_i)\beta\alpha = x\alpha$ for some $a_i \in$ ran α.
Since $(a_i)\beta\alpha \in (a_i)\alpha^{-1}\alpha = (a_i)\alpha\alpha^{-1} = a_i = x\alpha$.
Hence, $\alpha\beta\alpha = \alpha$ for some $\beta \in T_1(X)$.
Therefore, $T_1(X)$ is regular , and so by Proposition (2.2.2) $T_1(X) \in$ **BQ**. □

Definition 3.2.7. [12]
$T_2(X) = \{\alpha \in T_X : X\setminus$ ranα is infinite $\}$.

Proposition 3.2.8. [12]
$T_2(X)$ is a subsemigroup of T_X if X is infinite.

Proof.
Assume that X is infinite.
Let $\alpha, \beta \in T_2(X)$.
Then $\alpha, \beta \in T_X$ and $X\setminus$ ranα , $X\setminus$ ranβ are infinite.
Since , ran$\alpha\beta \subseteq$ ranβ, then $X\setminus$ ran$\beta \subseteq X\setminus$ ran$\alpha\beta$.
But , $X\setminus$ ranβ is infinite than $X\setminus$ ran$\alpha\beta$ is infinite, and so $\alpha\beta \in T_2(X)$.
Hence , $T_2(X)$ is a subsemigroup of T_X. □

Proposition 3.2.9. [12]
$T_2(X)$ is a left ideal of T_X, if X is infinite.

Proof.
Assume that X is infinite.
let $\alpha \in T_X$ and $\beta \in T_2(X)$, then $X\setminus$ ranβ is infinite.
As in the proof of Proposition 3.2.8 , $\alpha\beta \in T_2(X)$.
Therefore, $T_2(X)$ is a left ideal of T_X. □

Proposition 3.2.10. [12]
Every one-to-one map in $T_2(X)$ is not regular in $T_2(X)$.

Proof.
Let α be one-to-one map in $T_2(X)$.
Assume that α is regular in $T_2(X)$, then $\alpha = \alpha\beta\alpha$ for some $\beta \in T_2(X)$.
Since , α is one-to-one, then α^{-1} exists.
So , $\alpha\alpha^{-1} = \alpha\beta\alpha\alpha^{-1}$ then $\alpha\beta = I_X$.
Now , since α is one-to-one and $\alpha\beta = I_X$ then we claim that ran$\beta = X$
Let $y \in X$, we show there exist $x \in X$ such that $x\beta = y$.
Let $x = y\alpha$, then $x\beta = y\alpha\beta = y$ and so $y \in$ ranβ.
Hence, $X \subseteq$ ranβ ,and ran$\beta \subseteq X$ then ran$\beta = X$, and so $X\setminus$ ranβ finite.
This contradicts with $\beta \in T_2(X)$. □

Proposition 3.2.11. [12]
$T_2(X)$ is neither left simple nor right simple semigroup.

Proof.
By Theorem 1.2.9, we show that $T_2(X)\alpha \neq T_2(X)$ and $\alpha T_2(X) \neq T_2(X)$ for some $\alpha \in T_2(X)$.
let $a \in X$. Then $X_a \in T_2(X)$.
$T_2(X)X_a = \{X_a\} \neq T_2(X)$ and
$X_aT_2(X) = \{X_x : x \in X\} \neq T_2(X)$.
Hence, $T_2(X)$ is neither left simple nor right simple semigroup. □

Theorem 3.2.12. [12]
For any infinite set X ,$T_2(X) \in BQ$.

Proof.
Since for any nonempty subset A of $T_2(X)$, $(A)_b \subseteq (A)_q$, from Proposition 1.3.31 .
To prove the theorem, by Proposition 1.3.23(a) and Proposition 2.2.4, it suffices to show that
$T_2(X)A \cap AT_2(X) \subseteq AT_2(X)A$ for any nonempty subset A of $T_2(X)$.
let A be an arbitrary fixed nonempty subset of $T_2(X)$ and let $\alpha \in T_2(X)A \cap AT_2(X)$.

Then, $\alpha \in T_2(X)A$ and $\alpha \in AT_2(X)$
So, $\alpha = \beta\gamma = \lambda\eta$ for some $\beta, \eta \in T_2(X)$ and $\gamma, \lambda \in A$.
Since, $\lambda \in A$ and $A \subseteq T_X$, T_X is regular by the proof of Theorem 3.2.2, then $\lambda = \lambda\mu\lambda$ for some $\mu \in T_X$.
Then we have ,
$\alpha = \lambda\eta = \lambda\mu\lambda\eta = \lambda\mu\beta\gamma = \lambda(\mu\beta)\gamma$
Since $T_2(X)$ ia a left ideal of T_X by Proposition 3.2.9, then we have $\mu\beta \in T_2(X)$.
It follows that $\alpha \in \lambda T_2(X)\gamma \subseteq AT_2(X)A$.
Hence,$T_2(X)A \cap AT_2(X) \subseteq AT_2(X)A$.
Then, $T_2(X) \in BQ$
. □

Definition 3.2.13. [12]

$T_3(X) = \{\alpha \in T_X : \alpha$ is one-to-one and $X\setminus$ ranα is infinite $\}$.

Proposition 3.2.14. [12]
$T_3(X)$ is a subsemigroup of T_X, if X is infinite.

Proof.
Assume that X is infinite.
Let $\alpha, \beta \in T_3(X)$. Then α, β are one-to-one mappings and $X\setminus$ ranα , $X\setminus$ ranβ are infinite.
Now, we show that $\alpha\beta \in T_3(X)$.
Since, α and β are one-to-one mappings then $\alpha\beta$ is one-to-one.
Since , ran$\alpha\beta \subseteq$ ranβ then $X\setminus$ ran$\beta \subseteq X\setminus$ ran$\alpha\beta$.
But , $X\setminus$ ranβ is infinite then $X\setminus$ ran$\alpha\beta$ is infinite and so , $\alpha\beta \in T_3(X)$.
Therfore , $T_3(X)$ is a subsemigroup of T_X
. □

Proposition 3.2.15. [12]
$T_3(X)$ is right cancellative.

Proof.
let $\alpha \in T_3(X)$ be arbitrary.
We show that α is right cancellable.
let β and γ in $T_3(X)$ such that $\beta\alpha = \gamma\alpha$.
Since, $\alpha \in T_3(X)$, then α is one-to-one and so α^{-1} exists.

Then
$\beta\alpha\alpha^{-1} = \gamma\alpha\alpha^{-1}$ which implies that $\beta = \gamma$.
Hence , α is right cancellable.
Since , α is arbitrary element in $T_3(X)$, then , $T_3(X)$ is right cancellative. $\square$

Proposition 3.2.16. [12]
$T_3(X)$ contains no idempotents.

Proof.
Assume that $T_3(X)$ contains an idempotent .
Then $\alpha = \alpha^2$ for some $\alpha \in T_3(X)$.
Thus, α is one-to-one and $X\backslash$ ranα is infinite.
That is , $X\backslash X\alpha$ is infinite , and $X\alpha = X\alpha^2$.
Hence, $X\alpha\backslash X\alpha^2 = \phi$, and so $|X\alpha\backslash X\alpha^2| = |\phi| = 0$.
But $|X\alpha\backslash X\alpha^2| = |X\alpha\backslash X\alpha\alpha| = |(X\backslash X\alpha)\alpha|$ (since α is one-to-one).
Since $X\backslash X\alpha$ is infinite , then $(X\backslash X\alpha)\alpha$ is infinite and so , $|(X\backslash X\alpha)\alpha|$ is infinite (since α is one-to-one).
This is a contradiction.
Therefore, $T_3(X)$ contains no idempotents . $\square$

Proposition 3.2.17. [6]
If X is countably infinite, then $T_3(X)$ is right simple.

Proof. See [6] $\square$

Theorem 3.2.18. [12]
Assume that X is infinite .
Then $T_3(X) \in$ **BQ** if and only if X is countably infinite.

Proof.
If X is countable infinite, by Proposition 3.2.17 $T_3(X)$ is right simple.
So,by Proposition 2.2.3 $T_3(X) \in$ **BQ**
For the converse, assume that X is uncountable.
Since, X is infinite then $\exists$ a subset Z of X such that ,$|X \setminus Z| = |Z| = |X|$ and since Z is infinite then $\exists$ $Y \subseteq Z$ such that $|Z \setminus Y| = |Y| = |Z|$.

Now,by infiniteness of X , let W be a countably infinite subset of X.
Since X is uncountable, $|X \setminus W| = |X|$.
Then there are bijections $\alpha : X \to X \setminus Z$ and $\beta : X \to X \setminus W$.

Now, we claim that $\alpha, \beta \in T_3(X)$.
It is clear , α and β are one-to-one mappings.
Next, ran$\alpha = X \setminus Z$ since α is bijection, then $X \setminus (X \setminus Z) = X\setminus$ ranα and so $Z = X\setminus$ ranα but Z is infinite then $X\setminus$ ranα is infinite.
Similarly, $X\setminus$ ranβ is infinite.
Hence, α, β are in $T_3(X)$.
Moreover , Since α is one-to-one, then $\alpha^{-1} : X \setminus Z \to X$ exist and one-to-one.
Since $W \subseteq X$ then $X\backslash W \subseteq X$ and so the restriction of α to $X\backslash W$ is $\alpha|_{X\backslash W} : X\backslash W \to X\backslash Z$
Then the composition $\alpha^{-1}\beta\alpha : X \setminus Z \to X \setminus Z$ is one-to-one.
Since $|Z| = |Y|$, there is a bijection $\varphi : Z \to Y$.
Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} x\alpha^{-1}\beta\alpha, & \text{if } x \in X \setminus Z; \\ x\varphi, & \text{if } x \in Z. \end{cases}$$

Since, $Y \subseteq Z$ then $(X \setminus Z) \cap Y = \emptyset$.
Hence,by Theorem 1.1.4(2) , $\gamma : (X \setminus Z) \cup Z \to (X \setminus Z) \cup Y$ is one-to-one.
Also, we have ran$\gamma \subseteq (X \setminus Z) \cup Y$.
Which implies that

$$\begin{aligned} X \setminus ran\gamma &\supseteq X \setminus ((X \setminus Z) \cup Y) \\ &= Z \setminus Y \end{aligned}$$

since

$$\begin{aligned} X \setminus (X \setminus Z) \cup Y) &= X \cap ((X \setminus Z) \cup Y)' \\ &= X \cap ((X \setminus Z)' \cap Y') \\ &= (X \cap Z) \cap Y' \\ &= Z \cap Y' \\ &= Z \setminus Y. \end{aligned}$$

Hence, $|X\setminus$ ran$\gamma| \geq |Z \setminus Y| = |X|$

But, X is infinite , than $X\setminus$ ranγ is infinite.
Therefore,$\gamma \in T_3(X)$.
By the definition of γ , $\gamma|_{ran\alpha} = \alpha^{-1}\beta\alpha$.
Thus $\beta\alpha = \alpha\gamma \in T_3(X)\alpha \cap \alpha T_3(X) \subseteq (\alpha)_q$, that is $\beta\alpha \in (\alpha)_q$.
Now, we show $\beta\alpha \notin (\alpha)_b$.
Assume that $\beta\alpha \in (\alpha)_b$. By Proposition 1.3.30, $(\alpha)_b = \alpha T_3(X)\alpha \cup \{\alpha, \alpha^2\}$.

case (1) $\beta\alpha = \alpha$. Since α is one-to-one , then α^{-1} exist and so $\beta\alpha\alpha^{-1} = \alpha\alpha^{-1}$ which implies that $\beta = I_X$. This is a contradiction since $\beta : X \to X\setminus W$ and $I_X : X \to X$, that is $\beta \neq I_X$.

case (2) $\beta\alpha = \alpha^2$. Then $\beta\alpha\alpha^{-1} = \alpha\alpha\alpha^{-1}$ and so $\beta = \alpha$. This is a contradiction because $X\setminus$ ran$\beta = X \setminus (X \setminus W) = W$ which is countable , but $X\setminus$ ran $\alpha = X \setminus (X \setminus Z) = Z$ which is uncountable.

case (3) $\beta\alpha \in \alpha T_3(X)\alpha$. Then $\beta\alpha = \alpha\lambda\alpha$ for some $\lambda \in T_3(X)$.
Since α is one-to-one , we have
$\beta\alpha\alpha^{-1} = \alpha\lambda\alpha\alpha^{-1}$ and so, $\beta = \alpha\lambda$. Hence

$$\begin{aligned} W &= X \setminus ran\beta \\ &= X \setminus ran\alpha\lambda \\ &\supseteq X\lambda \setminus (ran\alpha)\lambda \\ &= (X \setminus ran\alpha)\lambda \quad \text{(since } \lambda \text{ is one to one)} \\ &= Z\lambda \end{aligned}$$

So, $|W| \geq |Z\lambda|$ which is impossible since W is countable Z is uncountable and λ is one-to-one.
Hence,$\beta\alpha \notin (\alpha)_b$.
Therefore $(\alpha)_q \neq (\alpha)_b$.
By Proposition 2.2.4 $T_3(X) \notin BQ$.
Therefore, $T_3(X) \in BQ$ if X is countably infinite

□

Definition 3.2.19. [12]
$T_4(X) = \{\alpha \in T_X : X\backslash \text{ ran}\alpha \text{ is finite } \}$.

Proposition 3.2.20. [12]
$T_4(X)$ is a subsemigroup of T_X if X is infinite.

Proof.
Assume that X is infinite.
Let $\alpha, \beta \in T_4(X)$. Then
$\alpha, \beta \in T_X$, and $X\backslash$ ranα,$X\backslash$ ranβ are finite.
We show that $\alpha\beta \in T_4(X)$.
Now,

$$\begin{aligned} X\backslash ran\alpha\beta &= (X\backslash ran\beta) \cup (ran\beta\backslash ran\alpha\beta) \\ &= (X\backslash ran\beta) \cup (X\beta\backslash X\alpha\beta) \\ &\subseteq (X\backslash ran\beta) \cup (X\backslash X\alpha)\beta \text{ from Theorem 1.1.4 (3)} \end{aligned}$$

But $X\backslash$ ranβ is finite and $X\backslash X\alpha = X\backslash$ ranα is finite and so, $(X\backslash X\alpha)\beta$ is finite.
Hence,$(X\backslash \text{ ran}\beta) \cup (X\backslash X\alpha)\beta$ is finite.
Then, $X\backslash$ ran$\alpha\beta$is finite and so $\alpha\beta \in T_4(X)$.
Therefore, $T_4(X)$ is a subsemigroup of T_X. □

Theorem 3.2.21. [12]
$T_4(X) \in$ **BQ** if and only if X is finite.

Proof.
If X is finite, then we claim that $T_4(X) = T_X$.
Note that $T_4(X) \subseteq T_X$, to show the converse let $\alpha \in T_X$.
Then, ran$\alpha \subseteq X$ and so $X\backslash$ ran$\alpha \subseteq X$.
But X is finite , then $X\backslash$ ranα is finite.
Hence, $\alpha \in T_4(X)$.
Therefore, $T_4(X) = T_X$.
Since, $T_X \in$ **BQ** from Theorem 3.2.2 , then $T_4(X) \in$ **BQ** .
For the converse , assume that by contrapositive X is infinite.

Since every infinite set contains a countable infinite subset, let A be a countably infinite subset of X and let $A = \{a_n | n \in \mathbb{N}\}$, $a_i \neq a_j$ if $i \neq j$ where $\mathbb{N}$ is the set of positive integers.
Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a_{\frac{n}{2}}, & \text{if } x = a_n \text{ for some even } n \in \mathbb{N}; \\ a_1, & \text{if } x = a_n \text{ for some odd } n \in \mathbb{N}; \\ x, & \text{otherwise.} \end{cases}$$

It is clear ranα = X , but X is infinite , then ranα is infinite and so , $\phi = X\setminus$ ranα is finite.
So, $\alpha \in T_4(X)$.
To prove $T_4(X) \notin BQ$, we show $(\alpha)_q \neq (\alpha)_b$ in $T_4(X)$.
To see this, define β , $\gamma : X \to X$ by

$$x\beta = \begin{cases} a_{n-2}, & \text{if } x = a_n \text{ and } n \in \mathbb{N}\setminus\{1,2\}; \\ a_1, & \text{if } x = a_2; \\ x, & \text{otherwise.} \end{cases}$$

and

$$x\gamma = \begin{cases} a_{n-1}, & \text{if } x = a_n \text{ and } n \in \mathbb{N}\setminus\{1\}; \\ x, & \text{otherwise.} \end{cases}$$

Then ranβ = ranγ = X.
Similarlly as α , we have that $\beta, \gamma \in T_4(X)$.
Also , we have $x\beta\alpha = x\alpha\gamma$, for all $x \in X$,
because ,

(1) If $x \in X\setminus A$, then
$x\beta\alpha = (x\beta)\alpha = x\alpha = x$ and ,
$x\alpha\gamma = (x\alpha)\gamma = x\gamma = x$
Hence, $x\beta\alpha = x = x\alpha\gamma$

(2) If $x \in A$ and $n = 2$ or n is odd natural number, then
$x\beta\alpha = a_n\beta\alpha = (a_n\beta)\alpha = a_1$
$x\alpha\gamma = a_n\alpha\gamma = (a_n\alpha)\gamma = a_1\gamma = a_1$
hence , $x\beta\alpha = a_1 = x\alpha\gamma$.

(3) If $x \in A$, n is even and n is natural number > 2, then

$x\beta\alpha = a_n\beta\alpha = (a_n\beta)\alpha = (a_{n-2})\alpha = a_{\frac{n-2}{2}}$
$x\alpha\gamma = a_n\alpha\gamma = (a_n\alpha)\gamma = (a_{\frac{n}{2}})\gamma = a_{\frac{n}{2}-1} = a_{\frac{n-2}{2}}$
Hence, $x\beta\alpha = a_{\frac{n-2}{2}} = x\alpha\gamma$.

Therefore, $\alpha \neq \beta\alpha = \alpha\gamma \in T_4(X)\alpha \cap \alpha T_4(X) \subseteq (\alpha)_q$.
Hence, $\alpha\gamma \in (\alpha)_q$.
Now, we show $\alpha\gamma \notin (\alpha)_b$.
Suppose that $\alpha\gamma \in (\alpha)_b$. By Proposition 1.3.30 .
$(\alpha)_b = \alpha T_4(X)\alpha \cup \{\alpha, \alpha^2\}$.

case(1): If $\alpha\gamma = \alpha$, then $\gamma = I_X$ since $ran\alpha = X$.
This is a contradiction .

case(2): If $\alpha\gamma = \alpha^2$, then $\gamma = \alpha$ since ran$\alpha = X$
This is a contradiction .

case(3): If $\alpha\gamma \in \alpha T_4(X)\alpha$ then $\alpha\gamma = \alpha\lambda\alpha$ for some $\lambda \in T_4(X)$.
Since , ran$\alpha = X$ then $X\alpha = X$ and so,
$X(\alpha\gamma) = X(\alpha\lambda\alpha)$.
$(X\alpha)\gamma = (X\alpha)\lambda\alpha$, then $(X\gamma) = X(\lambda\alpha)$ and so, $\gamma = \lambda\alpha$.
By the definition of γ, we have
$x\gamma = x = (x\lambda)\alpha$ for all $x \in X\backslash A$.
and $(A\backslash\{a_1, a_2\})\gamma = A\backslash\{a_1\} = ((A\backslash\{a_1, a_2\})\lambda)\alpha$
From the definition of α , we get,
$x\lambda = x$ for all $x \in X\backslash A$.
and
$(A\backslash\{a_1, a_2\})\lambda = \{a_n | n \in \mathbb{N}$, $n > 2$ and n is even $\}$.
Thus we have,
$(X\backslash\{a_1, a_2\})\lambda = X\backslash(\{a_n | n \in \mathbb{N}$, and n is odd $\} \cup \{a_2\})$.
This implies that $X\backslash$ ranλ must be infinite.
This is a contradiction since $\lambda \in T_4(X)$.
Hence, $(\alpha)_q \neq (\alpha)_b$.
Therefore, $T_4(X) \notin$ **BQ** by Proposition 2.2.4.

□

Definition 3.2.22. [12]

$T_5(X) = \{\alpha \in T_X |\ \alpha$ is one-to-one , and $X\backslash$ ranα is finite $\}$.

Proposition 3.2.23. [12]
$T_5(X)$ is a subsemigroup of T_X.

Proof.
Let $\alpha, \beta \in T_5(X)$. Then
α, β are one-to-one mappings and $X\backslash$ ranα,$X\backslash$ ranβ are finite.
We show that $\alpha\beta \in T_5(X)$.
Since α and β are one-to-one , then $\alpha\beta$ is one-to-one mapping.
Now,

$$\begin{aligned} X\backslash ran\alpha\beta &= (X\backslash ran\beta) \cup (ran\beta\backslash ran\alpha\beta) \\ &= (X\backslash ran\beta) \cup (X\beta\backslash X\alpha\beta) \\ &= (X\backslash ran\beta) \cup (X\backslash X\alpha)\beta \text{ since } \beta \text{ is one-to-one} \end{aligned}$$

But $X\backslash$ ranβ is finite and $X\backslash X\alpha = X\backslash$ ranα is finite and so, $(X\backslash X\alpha)\beta$ is finite.
Hence,$(X\backslash$ ran$\beta) \cup (X\backslash X\alpha)\beta$ is finite.
Then, $X\backslash$ran$\alpha\beta$is finite and so $\alpha\beta \in T_5(X)$.
Therefore, $T_5(X)$ is a subsemigroup of T_X. □

Theorem 3.2.24. [12]
$T_5(X) \in$ **BQ**, if and only if X is finite.

Proof.
let $\alpha \in T_5(X)$.
Since X is finite , and $\alpha : X \to X$ is one-to-one then α is also onto.
So, $T_5(X)$ is the set of all permutation on X , i.e. $T_5(X)$ is the symmetric group on X .
Then , $T_5(X)$ is regular on X and so from Proposition 2.2.2 , $T_5(X) \in BQ$.
Conversly, assume that X is infinite .
Let $A = \{a_n | n \in \mathbb{N}\}$ be a subset of X where $a_i \neq a_j$ if $i \neq j$.
Define $\alpha, \beta, \gamma \in T_X$ by

$$x\alpha = \begin{cases} a_{n+3}, & \text{if } x = a_n \text{ for } n \in \mathbb{N}; \\ x, & \text{otherwise.} \end{cases}$$

and

$$x\beta = \begin{cases} a_{n+1}, & \text{if } x = a_n \text{ for } n \in \mathbb{N}\backslash\{1\}; \\ x, & \text{otherwise.} \end{cases}$$

and

$$x\gamma = \begin{cases} a_{n+1}, & \text{if } x = a_n \text{ for } n \in \mathbb{N}\backslash\{1,2,3,4\}; \\ x, & \text{otherwise.} \end{cases}$$

Then, we claim that α, β and γ are all one-to-one.
For α , let $x_1, x_2 \in X$ such that $x_1 \neq x_2$.
If x_1 , $x_2 \in A$ then $x_1 = a_{n_1}$, $x_2 = a_{n_2}$ for $n_1, n_2 \in \mathbb{N}$ and $n_1 \neq n_2$. then $a_{n_1} \neq a_{n_2}$ and so , $a_{n_1+3} \neq a_{n_2+3}$
Thus, we have $x_1\alpha \neq x_2\alpha$.
If x_1 , $x_2 \in X\backslash A$ then $x_1 \neq x_2$ and so $x_1\alpha \neq x_2\alpha$.
If $x_1 \in A$, $x_2 \notin A$, then $x_1\alpha \in A$ and $x_2\alpha = x_2 \notin A$.
Hence, $x_1\alpha \neq x_2\alpha$.
Therefore, α is one-to-one
Similarlly, β and γ are one-to-one.
ran$\alpha = \{a_4, a_5, a_6, ...\} \cup X\backslash A$ then , $X\backslash$ ran$\alpha = \{a_1, a_2, a_3\}$.
Also,ran$\beta = \{a_1, a_3, a_4, a_5.........\} \cup X\backslash A$ then , $X\backslash$ ran$\beta = \{a_2\}$.
ran$\gamma = \{a_1, a_2, a_3, a_4, a_6, a_7.........\} \cup X\backslash A$ then , $X\backslash$ ran$\gamma = \{a_5\}$.
So, we have $X\backslash$ ranα, $\backslash$ ranβ and $\backslash$ ranγ are finite sets.
Then $\alpha, \beta, \gamma \in T_5(X)$.
We show $(\alpha)_q \neq (\alpha)_b$ in $T_5(X)$.
Now, $x\beta\alpha = x\alpha\gamma$ for $x \in X$, because

(1) If $x \in X\backslash A$, then
$x\beta\alpha = x = x\alpha\gamma$.

(2) If $n \in \mathbb{N}\backslash\{1\}$, then
$x\beta\alpha = (x\beta)\alpha = (a_n\beta)\alpha = (a_{n+1})\alpha = a_{n+4}$.
$x\alpha\gamma = (x\alpha)\gamma = (a_{n+3})\gamma = a_{n+4}$

(3) If $n = 1$
$a_1\beta\alpha = (a_1\beta)\alpha = a_1\alpha = a_4$.
$a_1\alpha\gamma = (a_1\alpha)\gamma = a_4\gamma = a_4$.

Therfore, $\alpha \neq \beta\alpha = \alpha\gamma \in T_5(X)\alpha \cap \alpha T_5(X) \subseteq (\alpha)_q$.
Hence, $\beta\alpha \in (\alpha)_q$.
Now , we show $\beta\alpha \notin (\alpha)_b$.
Suppose that $\beta\alpha \in (\alpha)_b$. By Proposition 1.3.30, $(\alpha)_b = \alpha T_5(X)\alpha \cup \{\alpha, \alpha^2\}$

case(1) : If $\beta\alpha = \alpha$ then $\beta = I_X$, since α is one-to-one.
This is a contradiction , since $ran\beta \neq X$.

case(2) : If $\beta\alpha = \alpha^2$ then $\beta\alpha = \alpha\alpha$ and so $\beta\alpha\alpha^{-1} = \alpha\alpha\alpha^{-1}$, $\beta = \alpha$ since α is one-to-one.
This is a contradiction , ran$\beta \neq$ ranα.

case(3) : If $\beta\alpha \in \alpha T_5(X)\alpha$ then $\beta\alpha = \alpha\lambda\alpha$ for some $\lambda \in T_5(X)$.
Since α is one-to-one , then α^{-1} exist and so $\beta = \alpha\lambda$.
But $a_1, a_2, a_3 \notin$ ranα and λ is one-to-one , so we deduce that $a_1\lambda, a_2\lambda, a_3\lambda \notin$ ran$\alpha\lambda$.
So, we have $|X\backslash$ ran$\alpha\lambda| \geq 3$.
From that $\beta = \alpha\lambda$, we get $|X\backslash$ ran$\beta| \geq 3$ which is a contradiction since $|X\backslash$ ran$\beta| = 1$
Hence , $\beta\alpha \notin (\alpha)_b$ and so , $(\alpha)_q \neq (\alpha)_b$.
Therefore, by Proposition 2.2.4 $T_5(X) \notin$ **BQ**.

□

Chapter 4

Semigroups Contain Bi-ideals Which are not Quasi-ideals

In this chapter, we shall find bi-ideals in the semigroup of continuous mappings and the semigroup of differentiable mappings which are not quasi-ideals.
Throughout this chapter , let I be an interval of $\mathbb{R}$ with $|I| > 1$, where $\mathbb{R}$ is the set of all real numbers. More information can be found in [11].

4.1 Semigroups of Continuous Mappings

In this section , we show that there are bi-ideals in the semigroup of continuous mappings which are not quasi-ideals.
Let $C(I)$ be the set of all continuous mappings $\alpha : I \to I$.
Then $C(I)$ is a semigroup under composition of mappings.

Theorem 4.1.1. [11]
There exists a bi-ideal in the semigroup $C(I)$ which is not a quasi-ideal.
Hence, $C(I) \notin \mathbf{BQ}$.

Proof.
Let $a, b \in I$ be such that $a < b$.
Since I is an interval of $\mathbb{R}$, $[a, b] \subseteq I$.

Then $\frac{a+b}{2}, \frac{3a+b}{4} \in [a,b]$.
Define $\alpha : I \to I$ by

$$x\alpha = \begin{cases} x, & \text{if } x \leq \frac{3a+b}{4}; \\ \frac{3a+b}{4}, & \text{if } \frac{3a+b}{4} < x \leq \frac{a+b}{2}; \\ \frac{1}{2}x + \frac{a}{2}, & \text{if } \frac{a+b}{2} < x \leq b; \\ \frac{a+b}{2}, & \text{if } x > b. \end{cases} \tag{4.1.1}$$

Then , it is clear that α is continuous and so $\alpha \in C(I)$.
Also,α is increasing on I and the graph of α on $[a,b] \times [a,b] \subseteq \mathbb{R}^2$ can be given as follows:

Now, α is one-to-one.
Hence , α^{-1} exists.
Therefore ,

$$x\alpha^{-1} = x \text{ for all } x \in [a, \tfrac{3a+b}{4}) \tag{4.1.2}$$

Now, if $x \in (\frac{a+b}{2}, b]$, then
$x\alpha = \frac{1}{2}x + \frac{a}{2}$ i.e $y = \frac{1}{2}x + \frac{a}{2}$
So, $x = 2y - a$, then $y = 2x - a$
If $x = \frac{a+b}{2}$ then
$\frac{a+b}{2} = 2y - a$ and so, $y = \frac{a+b}{4} + \frac{a}{2} = \frac{3a+b}{4}$
If $x = b$ then $b = 2y - a$, so $y = \frac{a+b}{2}$.
Hence, $y = 2x - a$ for all $x \in (\frac{3a+b}{4}, \frac{a+b}{2})$.
i.e.,

$$x\alpha^{-1} = 2x - a \text{ for all } x \in (\tfrac{3a+b}{4}, \tfrac{a+b}{2}) \tag{4.1.3}$$

By Proposition 1.3.30 $(\alpha)_b = \alpha C(I)\alpha \cup \{\alpha\} \cup \{\alpha^2\}$.
We shall show that $(\alpha)_b$ is not a quasi-ideal of $C(I)$.
By Proposition 2.2.4 , it suffices to show that $(\alpha)_b \neq (\alpha)_q$.
Define $\beta : I \to I$ by

$$x\beta = \begin{cases} \frac{7a+b}{8}, & \text{if } x \leq \frac{7a+b}{8}; \\ x, & \text{if } x > \frac{7a+b}{8}. \end{cases} \tag{4.1.4}$$

Then β is continuous mapping , and so $\beta \in C(I)$.
We claim that $\alpha\beta = \beta\alpha$.

case(1) : $x \leq \frac{7a+b}{8}$. Then $x < \frac{3a+b}{4}$ (since $a < b$, then $\frac{7a+b}{8} = \frac{6a+a+b}{8} < \frac{6a+2b}{8} = \frac{3a+b}{4}$).
$x\alpha\beta = (x\alpha)\beta = x\beta = \frac{7a+b}{8}$ and $x\beta\alpha = (x\beta)\alpha = (\frac{7a+b}{8})\alpha = \frac{7a+b}{8}$.
So, $\alpha\beta = \beta\alpha$.

case(2) : $x > \frac{7a+b}{8}$. Then $x\alpha > (\frac{7a+b}{8})\alpha$ since α is increasing
From(4.1.4),
$x\alpha\beta = (x\alpha)\beta = x\alpha$ and
$x\beta\alpha = (x\beta)\alpha = x\alpha$.
Hence, $\alpha\beta = \beta\alpha$.

It follows from Proposition 1.3.23(a) that $\beta\alpha = \alpha\beta \in C(I)\alpha \cap \alpha C(I) \subseteq (\alpha)_q$.
To show $\alpha\beta \notin (\alpha)_b$, suppose on the contrary that $\alpha\beta \in (\alpha)_b = \alpha C(I)\alpha \cup \{\alpha\} \cup \{\alpha^2\}$.

case(1) : If $\alpha\beta = \alpha$, then $a\alpha\beta = a\beta = \frac{7a+b}{8} \neq a = a\alpha$.
Hence, $a\alpha\beta \neq a\alpha$. This is a contradiction .

case(2) : If $\alpha\beta = \alpha^2$ then $\beta = \alpha$, which is a contradiction.

case(3) : If $\alpha\beta \in \alpha C(I)\alpha$, then $\alpha\beta = \alpha\gamma\alpha$ for some $\gamma \in C(I)$.
By (4.1.1) and (4.1.4) , we have ,
for every $x \in (\frac{7a+b}{8}, \frac{3a+b}{4})$, $x = x\beta = (x\alpha)\beta = x\alpha\beta = x\alpha\gamma\alpha = (x\alpha)\gamma\alpha = (x\gamma)\alpha$.
So, $(x\gamma)\alpha = x$.
Now, from (4.1.2) we have ,

$$x\gamma = x\alpha^{-1} = x \text{ for all } x \in (\tfrac{7a+b}{8}, \tfrac{3a+b}{4}) \tag{4.1.5}$$

By (4.1.3) we have ,
for every $x \in (\frac{3a+b}{4}, \frac{a+b}{2})$,
$x = x\beta = ((2x-a)\alpha)\beta = (2x-a)\alpha\beta = (2x-a)\alpha\gamma\alpha = (x\gamma)\alpha$.
So, $(x\gamma)\alpha = x$
From (4.1.3) we have

$$x\gamma = x\alpha^{-1} = 2x - a \text{ for all } x \in (\tfrac{3a+b}{4}, \tfrac{a+b}{2}) \tag{4.1.6}$$

Since $\gamma \in C(I)$, then γ is continuous at $\frac{3a+b}{4}$.

Then, $\lim\limits_{x \to (\frac{3a+b}{4})^-} x\gamma = \lim\limits_{x \to (\frac{3a+b}{4})^+} x\gamma$.

Now , from (4.1.5) and (4.1.6) we have

$$\lim_{x \to (\frac{3a+b}{4})^-} x\gamma = \lim_{x \to (\frac{3a+b}{4})^-} x = \frac{3a+b}{4} \ ,$$
$$\lim_{x \to (\frac{3a+b}{4})^+} x\gamma = \lim_{x \to (\frac{3a+b}{4})^+} 2x - a = 2(\frac{3a+b}{4}) - a = \frac{a+b}{2} \ ,$$

which is a contradiction since $a < b$.
Hence, $(\alpha)_b \neq (\alpha)_q$.
therefore , $(\alpha)_b$ is a bi-ideal of $C(I)$ which is not a quasi-ideal.

□

Corollary 4.1.2. [11]
The semigroup $C(I)$ is not regular

Proof.
By Theorem 4.1.1 we proved that $C(I) \notin \mathbf{BQ}$, and so by Proposition 2.2.2 we have $C(I)$ is not regular . □

Corollary 4.1.3. [11]
The semigroup $C(I)$ is neither left simple nor right simple

Proof.
From Theorem 4.1.1 and Proposition 2.2.3 we get this corollary. □

4.2 Semigroups of Differentiable Mappings

In this section , we will show that there are bi-ideals in semigroups of differentiable mappings which are not quasi-ideals.
Let $D(I)$ be the set of all differentiable mappings $\alpha : I \to I$.
Then $D(I)$ is a semigroup under composition of mappings.

Also, $D(I)$ is a subsemigroup of $C(I)$, since every differentiable mapping is continues.
For a mapping $\alpha \in D(I)$, $\alpha^{'}$ will denote the first derivative of α.

Theorem 4.2.1. [11]
There exists a bi-ideal in the semigroup $D(I)$ which is not a quasi-ideal.
Hence, $D(I) \notin \mathbf{BQ}$.

Proof.
All the possible types of I as follows :

1. $\mathbb{R}$,
2. $[c, \infty)$ or (c, ∞) for some $c \in \mathbb{R}$,
3. $(-\infty, d]$ or $(-\infty, d)$ for some $d \in \mathbb{R}$,
4. $[c, d], [c, d), (c, d]$ or (c, d) for some $c, d \in \mathbb{R}$, with $c < d$.

Choose $a, b \in \mathbb{R}$ such that $a < b$ and a, b have the following property.

If I is of type 1 , a, b can be any points of I .

If I is of type 2 , choose $a = c$.

If I is of type 3 , choose $b = d$.

If I is of type 4 , choose $a = c$ and $b = d$. Then $(a, b) \subseteq I$.

Define $\mu : \mathbb{R} \to \mathbb{R}$ by

$x\mu = \frac{1}{b-a}(x - \frac{a+b}{2})^2 + \frac{a+b}{2}$, for all $x \in \mathbb{R}$.

$x\mu^{'} = \frac{2}{b-a}(x - \frac{a+b}{2}) = \frac{2x}{b-a} - \frac{a+b}{b-a}$.

Then , μ is differentiable on $\mathbb{R}$, and so $\mu \in D(I)$
From the choice of a and b , we have
$I\mu \subseteq I$ for every type of I .
Let $\alpha = \mu|_I$. Thus $\alpha : I \to I$ where ,

$$x\alpha = \frac{1}{b-a}(x - \frac{a+b}{2})^2 + \frac{a+b}{2} \text{ for all } x \in I \tag{4.2.1}$$

and since $\mu \in D(I)$, then $\alpha \in D(I)$.
Now by Proposition 1.3.30, $(\alpha)_b = \alpha D(I)\alpha \cup \{\alpha\} \cup \{\alpha^2\}$.
Let $\mu_1, \mu_2 : \mathbb{R} \to \mathbb{R}$ be the straight lines defined by

$x\mu_1 = \frac{1}{2}(x - \frac{a+b}{2}) + \frac{a+b}{2}$ and

$x\mu_2 = \frac{1}{4}(x - \frac{a+b}{2}) + \frac{a+b}{2}$ for all $x \in \mathbb{R}$.

Then μ_1 and μ_2 are increasing on $\mathbb{R}$ since

$x\mu_1^{'} = \frac{1}{2} > 0$ and $x\mu_2^{'} = \frac{1}{4} > 0$

So, $\mu_1, \mu_2 \in D(I)$.

Now,

$$\begin{aligned} a\mu_1 &= \frac{1}{2}(a - \frac{a+b}{2}) + \frac{a+b}{2} \\ &= \frac{a}{2} - \frac{a+b}{4} + \frac{a+b}{2} \\ &= \frac{2a - a - b + 2a + 2b}{4} \\ &= \frac{3a+b}{4} \end{aligned}$$

$$
\begin{aligned}
b\mu_1 &= \frac{1}{2}(b - \frac{a+b}{2}) + \frac{a+b}{2} \\
&= \frac{b}{2} - \frac{a+b}{4} + \frac{a+b}{2} \\
&= \frac{2b - a - b + 2a + 2b}{4} \\
&= \frac{3b+a}{4}
\end{aligned}
$$

$$
\begin{aligned}
a\mu_2 &= \frac{1}{4}(a - \frac{a+b}{2}) + \frac{a+b}{2} \\
&= \frac{a}{4} - \frac{a+b}{8} + \frac{a+b}{2} \\
&= \frac{2a - a - b + 4a + 4b}{8} \\
&= \frac{5a+3b}{8}
\end{aligned}
$$

$$
\begin{aligned}
b\mu_2 &= \frac{1}{4}(b - \frac{a+b}{2}) + \frac{a+b}{2} \\
&= \frac{b}{4} - \frac{a+b}{8} + \frac{a+b}{2} \\
&= \frac{2b - a - b + 4a + 4b}{8} \\
&= \frac{5b+3a}{8}
\end{aligned}
$$

Since $a < \frac{3a+b}{4} < \frac{a+b}{2} < \frac{a+3b}{4} < b$ and

$\frac{3a+b}{4} < \frac{5a+3b}{8} < \frac{a+b}{2} < \frac{3a+5b}{8} < \frac{a+3b}{4}$,

that is , $a < a\mu_1 < \frac{a+b}{2} < b\mu_1 < b$ and

$a\mu_1 < a\mu_2 < \frac{a+b}{2} < b\mu_2 < b\mu_1$, from the choice of a and b, we have $I\mu_1 \subseteq I$ and $I\mu_2 \subseteq I$.

Let $\beta = \mu_1|_I$ and $\gamma = \mu_2|_I$. Then $\beta, \alpha \in D(I)$.

Since μ_1, μ_2 are in $D(I)$, and

$$x\beta = \tfrac{1}{2}(x - \tfrac{a+b}{2}) + \tfrac{a+b}{2} \text{ for all } x \in I \tag{4.2.2}$$

$$x\gamma = \tfrac{1}{4}(x - \tfrac{a+b}{2}) + \tfrac{a+b}{2} \text{ for all } x \in I \tag{4.2.3}$$

From (4.2.1). (4.2.2) and (4.2.3) , for $x \in I$,

$$\begin{aligned} x\alpha\gamma &= (x\alpha)\gamma \\ &= \left(\frac{1}{b-a}\left(x - \frac{a+b}{2}\right)^2 + \frac{a+b}{2}\right)\gamma \\ &= \frac{1}{4}\left(\frac{1}{b-a}\left(x - \frac{a+b}{2}\right)^2 + \frac{a+b}{2} - \frac{a+b}{2}\right) + \frac{a+b}{2} \\ &= \frac{1}{4(b-a)}\left(x - \frac{a+b}{2}\right)^2 + \frac{a+b}{2}, \end{aligned}$$

$$\begin{aligned}
x\beta\alpha &= (x\beta)\alpha \\
&= \left(\frac{1}{2}\left(x - \frac{a+b}{2}\right) + \frac{a+b}{2}\right)\alpha \\
&= \frac{1}{b-a}\left(\frac{1}{2}\left(x - \frac{a+b}{2}\right) + \frac{a+b}{2} - \frac{a+b}{2}\right)^2 + \frac{a+b}{2} \\
&= \frac{1}{4(b-a)}\left(x - \frac{a+b}{2}\right)^2 + \frac{a+b}{2}.
\end{aligned}$$

Hence,

$$x\alpha\gamma = x\beta\alpha = \tfrac{1}{4(b-a)}\left(x - \tfrac{a+b}{2}\right)^2 + \tfrac{a+b}{2} \text{ for all } x \in I \tag{4.2.4}$$

Thus, $\alpha\gamma = \beta\alpha \in \alpha D(I) \cap D(I)\alpha = (\alpha)_q$ by Proposition (1.3.23)(a)

To show $(\alpha)_b \neq (\alpha)_q$, we must show $\alpha\gamma \notin (\alpha)_b$.

To see this , suppose that $\alpha\gamma \in (\alpha)_b = \alpha D(I)\alpha \cup \{\alpha\} \cup \{\alpha^2\}$.

We have $\alpha\gamma \neq \alpha$ since by (4.2.1),

$$\begin{aligned}
\left(\frac{3a+b}{4}\right)\alpha &= \frac{1}{b-a}\left(\frac{3a+b}{4} - \frac{a+b}{2}\right)^2 + \frac{a+b}{2} \\
&= \frac{1}{b-a}\left(\frac{3a+b-2a-2b}{4}\right)^2 + \frac{a+b}{2} \\
&= \frac{1}{b-a}\left(\frac{a-b}{4}\right)^2 + \frac{a+b}{2} \\
&= \frac{-a+b+8a+8b}{16} \\
&= \frac{7a+9b}{16}
\end{aligned}$$

and by (4.2.3) ,

$$
\begin{aligned}
\left(\frac{3a+b}{4}\right)\alpha\gamma &= \left(\left(\frac{3a+b}{4}\right)\alpha\right)\gamma \\
&= \left(\frac{7a+9b}{16}\right)\gamma \\
&= \frac{1}{4}\left(\frac{7a+9b}{16}-\frac{a+b}{2}\right)+\frac{a+b}{2} \\
&= \frac{7a+9b}{64}-\frac{a+b}{8}+\frac{a+b}{2} \\
&= \frac{7a+9b-8a-8b+32a+32b}{64} \\
&= \frac{31a+33b}{64}
\end{aligned}
$$

and $\frac{7a+9b}{16} \neq \frac{31a+33b}{64}$ (because $a \neq b$)

Also, we have $\alpha\gamma \neq \alpha^2$ since by (4.2.1)

$$\begin{aligned}
\left(\frac{3a+b}{4}\right)\alpha^2 &= \left(\left(\frac{3a+b}{4}\right)\alpha\right)\alpha \\
&= \left(\frac{7a+9b}{16}\right)\alpha \\
&= \frac{1}{b-a}\left(\frac{7a+9b}{16}-\frac{a+b}{2}\right)^2+\frac{a+b}{2} \\
&= \frac{1}{b-a}\left(\frac{7a+9b-8a-8b}{16}\right)^2+\frac{a+b}{2} \\
&= \frac{1}{b-a}\left(\frac{b-a}{16}\right)^2+\frac{a+b}{2} \\
&= \frac{b-a}{256}+\frac{a+b}{2} \\
&= \frac{b-a+128a+128b}{256} \\
&= \frac{127a+129b}{256}
\end{aligned}$$

So, $\frac{127a+129b}{256} \neq \frac{31a+33b}{64} = \left(\frac{3a+b}{4}\right)\alpha\gamma$ (because $a \neq b$)

Next, if $\alpha\gamma \in \alpha D(I)\alpha$, then $\alpha\gamma = \alpha\eta\alpha$ for some $\eta \in D(I)$.
Then , it follows from (4.2.1) and (4.2.4) that , for every $x \in I$
$x(\alpha\eta\alpha) = x\alpha\gamma$
$(x\alpha)\eta\alpha = x\alpha\gamma$

$\left(\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)^2+\frac{a+b}{2}\right)\eta\alpha = \frac{1}{4(b-a)}\left(x-\frac{a+b}{2}\right)^2+\frac{a+b}{2}$

$\left(\left(\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)^2+\frac{a+b}{2}\right)\eta\right)\alpha = \frac{1}{4(b-a)}\left(x-\frac{a+b}{2}\right)^2+\frac{a+b}{2}$

which implies by (4.2.1) that

$\frac{1}{b-a}\left(\left(\left(\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)^2+\frac{a+b}{2}\right)\eta-\frac{a+b}{2}\right)^2+\frac{a+b}{2}=\frac{1}{4(b-a)}\left(x-\frac{a+b}{2}\right)^2+\frac{a+b}{2}\right.$,

$\left(\left(\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)^2+\frac{a+b}{2}\right)\eta-\frac{a+b}{2}\right)^2=\frac{1}{4}\left(x-\frac{a+b}{2}\right)^2$ for all $x \in I$.

Hence,
$\left(\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)^2+\frac{a+b}{2}\right)\eta=\mp\frac{1}{2}\left(x-\frac{a+b}{2}\right)+\frac{a+b}{2}$ for all $x \in I$.

If $x=\frac{a+b}{2}$ then , $\left(\frac{a+b}{2}\right)\eta=\frac{a+b}{2}$.

Let $t \in Im\alpha$. If $x \in I$ is such that

$\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)^2+\frac{a+b}{2}=t$, then

$\left(x-\frac{a+b}{2}\right)^2=(b-a)\left(t-\frac{a+b}{2}\right)$

$x=\pm\sqrt{(b-a)\left(t-\frac{a+b}{2}\right)}+\frac{a+b}{2}$

Hence,

$$\begin{aligned} t\eta &= \pm\frac{1}{2}\left(\sqrt{(b-a)\left(t-\frac{a+b}{2}\right)}+\frac{a+b}{2}-\frac{a+b}{2}\right)+\frac{a+b}{2} \\ &= \pm\tfrac{1}{2}\sqrt{(b-a)\left(t-\frac{a+b}{2}\right)}+\tfrac{a+b}{2} \quad \text{for all } t \in Im\alpha. \end{aligned}$$

But, $\left[\frac{a+b}{2},\frac{a+3b}{4}\right) \subseteq Im\alpha$, so

$t\eta=\pm\frac{1}{2}\sqrt{(b-a)\left(t-\frac{a+b}{2}\right)}+\frac{a+b}{2}$ for all $t \in \left[\frac{a+b}{2},\frac{a+3b}{4}\right)$

Now,

$$\text{for } t \in \left[\tfrac{a+b}{2},\tfrac{a+3b}{4}\right) \text{ , } t\eta=\tfrac{a+b}{2} \text{ if and only if } t=\tfrac{a+b}{2} \qquad (4.2.5)$$

Suppose that there exist $t_1, t_2 \in \left(\frac{a+b}{2}, \frac{a+3b}{4}\right)$ such that

$$\begin{aligned} t_1\eta &= \frac{1}{2}\sqrt{(b-a)\left(t_1 - \frac{a+b}{2}\right)} + \frac{a+b}{2} \quad \text{and} \\ t_2\eta &= -\frac{1}{2}\sqrt{(b-a)\left(t_2 - \frac{a+b}{2}\right)} + \frac{a+b}{2} \end{aligned}$$

Since , $\frac{a+b}{2}\eta = \frac{a+b}{2}$ then $\frac{a+b}{2} \in [t_1\eta, t_2\eta]$

Since η is differentiable on I , η is continuous on I .

Thus $\left(\frac{a+b}{2}, \frac{a+3b}{4}\right)\eta$ is an interval .

Since $t_1\eta, t_2\eta \in \left(\frac{a+b}{2}, \frac{a+3b}{4}\right)\eta$, it follows that

$[t_2\eta, t_1\eta] \subseteq \left(\frac{a+b}{2}, \frac{a+3b}{4}\right)\eta$, so $\frac{a+b}{2} \in \left(\frac{a+b}{2}, \frac{a+3b}{4}\right)\eta$.

This is a contradiction by (4.2.5).
This proves that

$t\eta = \frac{1}{2}\sqrt{(b-a)\left(t - \frac{a+b}{2}\right)} + \frac{a+b}{2}$ for all $t \in \left[\frac{a+b}{2}, \frac{a+3b}{4}\right)$

or

$t\eta = -\frac{1}{2}\sqrt{(b-a)\left(t - \frac{a+b}{2}\right)} + \frac{a+b}{2}$ for all $t \in \left[\frac{a+b}{2}, \frac{a+3b}{4}\right)$

Assume that $t\eta = \frac{1}{2}\sqrt{(b-a)\left(t - \frac{a+b}{2}\right)} + \frac{a+b}{2}$ for all $t \in \left[\frac{a+b}{2}, \frac{a+3b}{4}\right)$

Thus for $h > 0$ with $\frac{a+b}{2} + h \in \left(\frac{a+b}{2}, \frac{a+3b}{4}\right)$,

$$\begin{aligned}
\frac{1}{h}\left(\left(\frac{a+b}{2}+h\right)\eta - \left(\frac{a+b}{2}\right)\eta\right) &= \frac{1}{h}\left(\frac{1}{2}\sqrt{(b-a)\left(\frac{a+b}{2}+h-\frac{a+b}{2}\right)} + \frac{a+b}{2} - \frac{a+b}{2}\right) \\
&= \frac{1}{h}\left(\frac{1}{2}\sqrt{(b-a)h}\right) \\
&= \frac{1}{2}\sqrt{\frac{(b-a)h}{h^2}} \\
&= \frac{1}{2}\sqrt{\frac{b-a}{h}}
\end{aligned}$$

Which implies that

$$\lim_{h\to 0^+} \frac{1}{h}\left(\left(\frac{a+b}{2}+h\right)\eta - \left(\frac{a+b}{2}\right)\eta\right) =$$

$$\lim_{h\to 0^+} \frac{1}{2}\sqrt{\frac{a+b}{h}} = \infty$$, does not exist .

This is a contradiction since η is differentiable at $\frac{a+b}{2}$. We also get a contradiction similarly if

$$t\eta = -\tfrac{1}{2}\sqrt{(b-a)\left(t-\tfrac{a+b}{2}\right)} + \tfrac{a+b}{2} \quad \text{for all} \quad t \in \left[\tfrac{a+b}{2}, \tfrac{a+3b}{4}\right)$$

So, $\alpha\gamma \notin (\alpha)_b$
Hence , $(\alpha)_b \neq (\alpha)_q$
Therefore , by Proposition 2.2.4 , $(\alpha)_b$ is a bi-ideal of $D(I)$ which is not a quasi-ideal . □

Corollary 4.2.2. [11]
The semigroup $D(I)$ is not regular

Proof.

By Theorem 4.2.1 we proved that $D(I) \notin \mathbf{BQ}$, and so by Proposition 2.2.2 we have $D(I)$ is not regular . □

Corollary 4.2.3. [11]
The semigroup $D(I)$ is neither left simple nor right simple

Proof.
From Theorem 4.2.1 and Proposition 2.2.3 we get this corollary . □

Bibliography

[1] Calais, J. ,*Demi-groups Dans Lesquels Tout Bi-ideal est un Quasi-ideal.* Symp. semigroups, Smolenice, 1968.

[2] Clifford, A.H. and Preston, G.B. ,*The Algebraic Theory of Semigroups*, Number 7, Part I , American Mathematical Society (1961).

[3] Donges, C. , *On Quasi-ideals of Semirings*, Vol. 17 No. 1, DW-3392 Clausthal-Zellerfeld, Germany (1994) 47-58.

[4] Good, R.A. and Hughes, D.R. ,*Associated groups for a semigroup*, Bull.Amer.Math. Soc.**58** (1952), 624-625 (Abstract 575)

[5] Hebisch, U. and Weinert, H.J, *Semigroups and Semifields* , Elsevier Science Publishers B.V. , (1992).

[6] Higgins, P.M. Fundamentals. *In Techinques of Semigroup Theory*; Oxford University Press: Oxford-New York-Tokyo, (1992) 1-80.

[7] Howie, J.M. ,*Fundamentals of Semigroup Theory*, Oxford Science Publications(1995).

[8] Iseki, K.*Quasi-ideals in semirings without zero*,proc. Japan Acad **34**, 79-81(1958).

[9] Kapp, K.M. ,*On bi-ideals and quasi-ideals in semigroups* ,Publ. Math .Debrecen **16**, 179-185 (1969).

[10] Kandasamy, W.B. , *Smarandache Semirings, Semifields, and Semivector spaces*, A merican Research Press, Rehoboth (2002).

[11] Kemprasit, Y. and Baupradist, S. , *Existence of Bi-ideal Which Are Not Quasi-ideals in Semigroup of Continuous Functions and Semigroups of Differentiable Functions*, (2000) 133-142.

[12] Kemprasit, Y. , *Some Transformation Semigroups Whose Sets of Bi-ideals and Quasi-Ideals Conincide* , Communications in Algebra Vol. 30 No. 9. pp. 4499-4506, (2002).

[13] Kemprasit, Y. and Baupradist, S. , *A Note on the Multiplicative Semigroup* $\mathbb{Z}_n$ *Whose Bi-ideals Are Quasi-ideals* , Southeast Asian Bulletin of Mathematics (2004) **27** : 269-271.

[14] Kemprasit, Y. and Baupradist, S. , *Interval Semigroup on* $\mathbb{R}$ *Whose Sets of Bi-ideals and Quasi-ideals Conincide*, Southeast Asian Bulletin of Mathematics (2004) **27** : 829-832.

[15] Lajos, S. ,*Generalized Ideals in Semigroups*, Acta Sci. Math. (Szeged) Debrecen **22**, 217-222 (1961).

[16] Shabir, M. , Ali, A. and Batool, S. , *A Note on Quasi-Ideals in Semirings*, Southeast Asian Bulletin of Mathematics (2004) **27** : 923-928.

[17] Smith,.D , Eggen, M. and Andre, R.S. , *A Transition to Advanced Mathematics* , Fourth Edition (1996).

[18] Steinfeld, O. ,*On quotients and prime ideals*, Acta Math Acad.Sci. Hung. **4** , 289-298 (1953).

[19] Steinfeld, O. ,*Quasi-ideals in rings and semigroups*, Akademiaikiado, Budapest,(1978).