THE ISLAMIC UNIVERSITY OF GAZA
DEANERY OF HIGHER STUDIES
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

On G-Cyclicity Of Operators

PRESENTED BY
Atef A. Salman

SUPERVISED BY
Associate Professor: As'ad Y. As'ad

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF MATHEMATICS

November, 2007

Dedicated

To spirit of my father...

To my mother...

To my wife...

To My kids

And to all knowledge seekers...

Acknowledgement

I would like to express my sincere thanks and gratitude to my supervisor Associate Professor As'ad Y. As'ad for helping and supervising me throughout this work.

I would like to thank the department of mathematics in the Islamic University of Gaza for offering me the opportunity to get the master degree.

Abstract

In this thesis, we focus our study on a part of cyclic phenomena, namely G-cyclic operators on an infinite dimensional separable complex Hilbert space. We study some properties of cyclic, supercyclic, and hyupersyclic operators, then we give some examples that explain the relationship between them, where we find that, supercyclicity stands in the midway between hypercyclicity and cyclicity.

In the first step we give a necessary and sufficient conditions for an operator to be G-cyclic, we show that every G-cyclic operator is supercyclic but the converse need not be true in general. Then we discuss some of the properties of the spectrum of G-cyclic operators.

In the second step, as examples of G-cyclic operators we define disk-cyclic and codisk-cyclic operators, and state and prove the Disk-Codisk cyclicity criterion. Finally we give applications of this result.

Contents

INTRODUCTION

Operator theory is an important branch in functional Analysis .It has many applications in science and engineering . Cyclic phenomena is considered as an important phenomena in operator theory, where it discus the answer of the important question .
Does every operator have a non − trivial invariant subspace?.
Cyclic, Hypercyclic and Supercyclic operators have been studied by many mathematician. The study of the phenomena of the cyclicity originates in the papers by Birkhoff, 1929. The first example of a hypercyclic operator on a Hilbert space was constructed by Rolewicz in 1969 . He was the first to isolate the concept of hypercyclicity, he showed that if B is the backward shift on the Hilbert space $l^2(N)$, defied by $B(a_0, a_1, a_2, ...) = (a_1, a_2, a_3, ...)$,then λB is hypercyclic for all $\lambda \in \mathbb{C}$ *with* $\mid \lambda \mid > 1$.
Later on, starting from the eighties the subject of cyclicity has been widely explored. It was studied by C.Kitai, J.Shapiro, S. Fledman and others .
We focus our study on a part of cyclic phenomena, namely G-cyclic operators on an infinite dimensional separable complex Hilbert space, where it is defined and introduced in[18] and[19] by A.Naoum and Z. Jamil [2006].
This thesis consists of four chapters.
In chapter 1. We introduce some basic concepts and definitions on a Topological Metric Space, Normed Vector space, bounded linear operators, and algebra that are necessary for understanding the thesis.
In chapter 2. We study cyclic, hypercyclic and supercyclic operators on an infinite dimensional separable complex Hilbert space and give some properties of their spectrum. This chapter consists of tow sections.
In section 2.1. We stat some properties of cyclic, supercyclic and hypercyclic operators and give some examples which explain the relationship between them.
In section 2.2. We stat some properties of the spectrum of cyclic, hypercyclic and supercyclic operators and their Hilbert-adjoint.
In chapter 3. We study G-cyclic operators and some of their properties

and give necessary and sufficient conditions for an operator to be G-cyclic. This chapter consists of two sections.
In section 3.1. We define the G-cyclic operator and introduce another equivalent forms.
In section 3.2. We study some of the spectral theory of G-cyclic operators and give a simpler condition to characterize the G-cyclic operators over a bounded semigroup .
In chapter 4. As examples of G-cyclic operators we study Disk-cyclic operators and Codisk cyclic operators and state and prove disk cyclic and codisk cyclic criterion.
This chapter consists of two sections.
In section 4.1. We define the Disk-cyclic operator and give examples of Diskcyclic operators which are not hypercyclic.
In section 4.2. We define the Codisk-cyclic operator and give examples of Codisk-cyclic operator which is not Disk-cyclic.
Throughout our study, all vector spaces and algebras are assumed to be over $\mathbb{C}$, the complex field.
$\mathbb{R}, , \mathbb{N}$ *and* $\mathbb{Z}$ denotes the set of; real numbers, natural numbers and integers, respectively.
$l^2(N), l^2(Z)$ denotes the Hilbert sequences spaces, where each element of $l^2(N)$ is a sequence $x = (x_i)_{i\in N}$ such that

$$\sum_{i\in N} | x_i |^2 < \infty.$$

and each element of $l^2(Z)$ is a sequence $x = (x_i)_{i\in Z}$ such that

$$\sum_{i\in Z} | x_i |^2 < \infty.$$

Finally, H denotes an infinite dimensional separable complex Hilbert space, and B(H) denotes the Banach algebra of all bounded linear operators on H.
It would be noted here that we deal (in our study) with infinite dimensional separable complex Hilbert space or with Banach space.
The reason of the above is that:
1) In the case of nonseparable it is clear from the definition of cyclic,hypercyclic,supercyclic, G-cyclic that there is no cyclic,hypercyclic,supercyclic,G-cyclic operators.
2) In the case of finite dimensional there is no hypercyclic operators, and there exist operators having supercyclic vectors in H if and only if
$\dim H \in \{0, 1, 2\}$ or $\dim H = \infty$.

Chapter 1

Preliminaries

In this chapter we introduce some basic concepts of a Topological Metric Space, normed Vector space, bounded linear operators, and algebra that are necessary for understanding the thesis.

1.1 Topological Spaces and Metric Spaces

In this section we give some facts about topological space and a metric space.

Definition 1.1.1. [25] *A topology on a set X is a collection τ of subset of X called the open sets satisfying :*
1) Any union of elements of τ belongs to τ.
2) Any finite intersection of elements of τ belongs to τ.
3) ϕ and X belongs to τ.
*We say **(X; τ)** is a topological Space.*

Note: The closure of a set A is denoted by $\overline{A}$

Definition 1.1.2. [25] *If (X, τ) is a topological space, a **base for** τ is a collection $\mathbb{B} \subset \tau$ such that*

$$\tau = \{ \bigcup_{B \in \ \mathbb{M}} B : \mathbb{M} \subset \mathbb{B} \}.$$

Definition 1.1.3. [25] *Let A be any subset of a topological space, then $x \in A$ is called an **isolated point** of A if and only if there exists an open set U of X such that*

$$U \bigcap A = \{x\}$$

Definition 1.1.4. [25] *A subset of a topological space X whose closure is X is said to be* ***dense*** *in X.*

Lemma 1.1.5. [7] *Let A subset of a topological space X, then the the following statements are equivalent:*
1) A is dense in X.
2)For all open U in X, $U \bigcap A \neq \emptyset$
3) For every element x in X there is a sequence x_n *in A such that* $x_n \longrightarrow x$.

Theorem 1.1.6. [7] ***(Baire's Theorem)***
Let X be a topological space and $\{A_n : n \in N\}$ *be a countable family of open subsets of X each of which is dense in X, then*

$$\bigcap_{n=1}^{\infty} A_n \quad is \;\; dense \;\; in \; X.$$

Definition 1.1.7. [25] *A subset of a topological space X is called a* $\boldsymbol{G_\delta}$ ***-set*** *if and only if it is a countable intersection of open sets.*

Definition 1.1.8. [25] *Let X and Y be topological spaces and let* $f : X \longrightarrow Y$*. Then f is continuous at* $x \in X$ *if and only if for each neighborhood V of f(x) in Y, there is a neighborhood U of x in X such that* $f(U) \subset V$*. We say f is continuous on X if and only if f is continuous at each* $x \in X$.

Theorem 1.1.9. [25] *If X and Y are topological spaces and* $f : X \longrightarrow Y$*, then the following are equivalent:*
a) f is continuous.
b) for each open set V in Y, $f^{-1}(V)$ *is open in X.*
c) for each $E \subset X, f(\overline{E}) \subset \overline{f(E)}$.

Definition 1.1.10. [25] *Suppose that X is a nonempty set. By a metric we mean a real valued function* $d : X \times X \longrightarrow R$ *which satisfies the following conditions:*
1) $d(x; y) \geq 0$*for all* $x; y \in X$.
2) d(x; y) = 0 if and only if x = y.
3) d(x; y) = d(y; x) for all $x; y \in X$.
4) $d(x; z) \leq d(x; y) + d(y; z)$ *for all* $x; y; z \in X$*. By a* ***metric space (X,d)****, we mean a non empty set X together with a metric d.*

Note [25] *A metric space is a topological space.*

Definition 1.1.11. [25] *The metric space X is said to be* ***separable*** *if and only if there is a countable subset of X that is dense in H.*

Theorem 1.1.12. [25] *For a metric space X, the following are equivalent:*
a) X has a countable base.
b) X is separable.

Definition 1.1.13. [25] *Let (X, d) be a metric space. Then X is called* ***connected*** *if there are no open - closed subset of X other than* ϕ *and X. Anon empty subset Y of X is said to be connected if and only if the subspace Y is a connected subspace.*

Definition 1.1.14. [25] *Suppose that (X, d) is a metric space, and that* $E \subset X$*. A set C is called a* ***component*** *of E if it is a maximal connected subset of E. ie) for every connected set D satisfying* $C \subseteq D \subseteq E$*, we must have D=C.*

1.2 Normed Vector Spaces

In this section we give some facts about Normed Vector space.

Definition 1.2.1. [16] *Let X be a vector space over the complex field* $\mathbb{C}$ *or(a real filed* $\mathbb{R}$ *). If for any* $x \in X$ *there exists a real number* $\| x \|$ *satisfying the following (1),(2), and (3), then* $\| x \|$ *is said to be the* ***norm*** *of x:*
1- $\| x \| \geq 0$ *for all x in X and* $\| x \| = 0$ *if and only if x=0.*
2- $\| x + y \| \leq \| x \| + \| y \|$ *for all x and y in X.*
3- $\| cx \| = | c | \| x \|$ *for all x in X and all complex (real) number c.*
A complex (real) vector space having the norm is said to be a normed space.

Note *A normed space X is a metric space which is given by* $d(x : y) = \| x - y \|$*, for all* $x, y \in X$*.*

Definition 1.2.2. [16] *A sequence* $\{x_n\}$ *in a normed space X is said to be a* ***Cauchy sequence*** *if*

$$\| x_m - x_n \| \longrightarrow 0 \qquad as \quad m \longrightarrow \infty \quad and \quad n \longrightarrow \infty.$$

Definition 1.2.3. [16] *A normed space X is said to be* ***complete*** *if every Cauchy sequence has a limit in X.*

Definition 1.2.4. [16] *A complete normed space is said to be **Banach space**.*

Definition 1.2.5. [16] *A total set(or fundamental set) in a normed space X is a subset $M \subset X$ whose span is dense in X.*

Definition 1.2.6. [7] *A linear combination of vectors $x_1, x_2, ...x_m$ of a vector space X is an expression of the form $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m$ where the coefficients $\alpha_1, \alpha_2.......\alpha_m$ are any scalars. For any nonempty subset $M \subset X$, the set of all linear combinations of vectors of M is called the span of M.*

Definition 1.2.7. [7] *A vector space X is saide to be finite dimension if there is a positive integer n such that X contains a linearly independent set of n vector where any set of n+1 or mor vectors of X is linearly dependent, n is called the dimension of X.*

Definition 1.2.8. [7] *A subset B of a vector space X is called a Hamal base for X if and only if B is a linearly independent set and span B= X.*

Definition 1.2.9. [7] *A sequence (x_i) in an infinite dimensional Banach space X is called a Schauder basis for X if for each $x \in X$ there exists a unique sequence of scalars (a_i) depending on x, such that*

$$\lim_{n \longrightarrow \infty} \| x - \Sigma_{i=1}^{n} a_i x_i \| = 0.$$

Definition 1.2.10. [7] *A sequence (x_i) in Banach space X is said to be a basic sequence if (x_i) is a basis for the closed linear subspace spanned by (x_i).*

1.3 Hilbert Spaces

In this section we give some facts about Hilbert space, and define the direct sum of Hilbert spaces.

Definition 1.3.1. [7] *A vector space V over a field F,(F is a real field $\mathbb{R}$ or a complex field $\mathbb{C}$) together with a function*

$$\langle , \rangle : V \times V \to F$$

is called an ***inner product space*** *if* $\langle , \rangle$ *satisfying the following conditions: for* $x, y, z \in V$ *and* $c \in F$
1- $\langle x, x\rangle \geq 0$*, and* $\langle x, x\rangle = 0$ *if and only if* $x{=}0$.
2- $\langle x, y\rangle = \overline{\langle y, x\rangle}$. *Where* $\overline{\langle y, x\rangle}$ *is the complex conjugates of* $\langle y, x\rangle$.
3- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.
4- $\langle cx, y\rangle = c\langle x, y\rangle$.

Theorem 1.3.2. [7] *An inner product space is a normed space.*

Definition 1.3.3. [7] *A complete inner product space is said to be a* ***Hilbert space****.*

Definition 1.3.4. [7] *An* ***orthonormal*** *subset of a Hilbert space H is a subset E having the properties*
a) For each $e \in E$, $\| e \| = 1$.
b) If $e_i, e_j \in E$ *and* $i \neq j$ *then* $\langle e_i, e_j\rangle = 0$.

Theorem 1.3.5. [16] *Let M be a subset of an inner product space X. If M is total in X, then there dose not exist a nonzero* $x \in X$ *which is orthogonal to every element of M, that is*

$$x \perp M \qquad \longrightarrow \quad x = 0.$$

Theorem 1.3.6. [16] ***(Orthogonal decomposition).*** *Let M be a closed subspace of a Hilbert space H . Any vector x in H has the unique representation as follows :* $x = y \oplus z$. *where* $y \in M$ *and* $z \in M^{\perp}$.
Briefly H can be expressed as $H = M \oplus M^{\perp}$.

Theorem 1.3.7. [9] *Let* $\{H_i\}_{i\in N}$ *be a family of Hilbert spaces, and let*

$$H = \{(h_i) : h_i \in H_i \quad for\ all \quad i \in N, and \quad \Sigma_{i=1}^{\infty} \| h_i \|^2 < \infty.\}$$

For $h = (h_i)$ *and* $g = (g_i)$ *in H, define*

$$\langle h, g\rangle = \Sigma_{i=1}^{\infty} \langle h_i, g_i\rangle.$$

Then, $\langle , \rangle$ *is an inner product on H, and the norm relative to this inner product is*

$$\| h \|^2 = \Sigma_{i=1}^{\infty} \| h_i \|^2 .$$

With this inner product H is a Hilbert space.

Definition 1.3.8. [16] *Let* $\{H_i\}_{i\in N}$ *be a family of Hilbert space .Then the space H of Theorem (1.2.30) is called the direct sum of* $H_1, H_2, ...$ *and it is denoted by*

$$H = \bigoplus_{i=1}^{\infty} H_i.$$

1.4 Bounded Linear Operators

In this section we give some facts about linear operator, bounded linear operator, and linear functional.

Definition 1.4.1. [16] *Let X and Y be two vector spaces over a scalar filed K. A mapping T from X to Y is said to be **linear operator** if T satisfies the following :*
i) T(x+y) = Tx+ Ty *for all* $x, y \in X$.
ii) $T(\alpha x) = \alpha Tx$ *for all* $x \in X$ *and* $\alpha \in \mathbb{K}$.

Definition 1.4.2. [16] *A linear operator T from a vector space X to a vector space Y is said to be **bounded** if there exist* $c > 0$ *such that* $\| Tx \| \leq c \| x \|$ *for all* $x \in X$.

$\| T \|$ *is defined by* $\| T \| = \inf\{c > 0 : \| Tx \| \leq c \| x \|\}$ *for all* $x \in X$, *and* $\| T \|$ *is said to be the **operator norm** .*

Definition 1.4.3. [16] *A linear functional f is a linear operator with domain in a vector space X and range in the scalar field K of X ; thus,*

$$f : D(f) \longrightarrow K,$$

where K=R if X is real and K=C if X is complex.

Definition 1.4.4. [16] *The set of all bounded linear operators from a vector space X to a vector space Y is denoted by B(X,Y).*
And the set of all bounded linear operators from a Hilbert space H to H is denoted by B(H).

Definition 1.4.5. [16] *The set of all bounded linear functional defined on a vector space X can itself be made into a vector space. This space is denoted by* X^* *and is called the algebraic dual space of X.*

Theorem 1.4.6. [16] *Let* $T : D(T) \longrightarrow Y$ *be a linear operator, where* $D(T) \subset X$ *and X,Y are normed spaces. Then T is continuous if and only if T is bounded .*

Definition 1.4.7. [16] *An isomorphism of a normed space X onto a normed space Y is a bijective linear operator* $T : X \longrightarrow Y$ *which preserves the norm, that is, for all* $x \in X$, $\| Tx \| = \| x \|$ *. X is then called isomorphic with Y.*

Theorem 1.4.8. [9]*(Hahn-Banach Theorem) If H is a normed space,* $\{x_1, x_2, ...x_i\}$ *is a linearly independent subset of H, and* $a_1, a_2, ...a_i$ *are arbitrary scalars, then there is an f in* H^* *such that* $f(x_k) = a_k$ *for all* $1 \leq k \leq i$.

Theorem 1.4.9. [16] *(Riesz-Frechet) Suppose that H is a Hilbert space over a filed (real or complex) F. Then for every continues linear functional* $f : H \longrightarrow F$, *there exists a unique* $x_0 \in H$ *such that* $\| f \| = \| x_0 \|$ *and* $f(x) = \langle x, x_0 \rangle \quad for\ every \ \ x \in H.$

Note *By Riesz-Frechet Theorem, for every Hilbert space H, H is isomorphic with* H^*.

Definition 1.4.10. [16] *Let* $T : H_1 \longrightarrow H_2$ *be a bounded linear operator, where* H_1 *and* H_2 *are Hilbert spaces. Then the Hilbert -adjoint operator* T^* *of T is the operator*

$$T^* : H_2^* \longrightarrow H_1^*$$

such that, for all $x \in H_1,$ *and* $y \in H_2, \langle Tx, y \rangle = \langle x, T^*y \rangle$.

Note: *Since H isomorphic to* H^* *so we can say that* $T^* : H_2 \longrightarrow H_1$

Definition 1.4.11. [16] *A linear operator* $T \in B(H)$ *is said to be **invertible** if there exists a linear operator* $S \in B(H)$ *such that,* $ST = I = TS$ *where I is the identity operator on H .*

Theorem 1.4.12. [16] $T \in B(H)$ *is invertible if and only if T is one-to-one and onto.*
Proof. *See*

Definition 1.4.13. [11] *Let* $T, F \in B(H)$, *then T and F are said to be **similar operators** if there is an invertible operator U such that ,* $T = UFU^{-1}$. *i.e TU=UF.*

Definition 1.4.14. [16] *Let M be a closed subspace of a Hilbert space H. Let* $P : H \longrightarrow M$ *be defined by* $Px = P(y \oplus z) = y$ *where* $y \in M$ *and* $z \in M^{\perp}$. *Then P is said to be an **orthogonal projection** of H onto M, denoted by* P_M.

Theorem 1.4.15. [16] *Let T be an operator on a Hilbert space H ,and M be a closed subspace of H. Then*
(1)PT=PTP.
(2) $(PT)^n = PT^n$. *where P is the projection onto M.*

Definition 1.4.16. [9] ***(Direct sum of operators)***
Let $\{H_i\}_{i\in N}$ *be a family of Hilbert space, and let* $H = \bigoplus_{i=1}^{\infty} H_i$*. If* $T_i \in B(H_i)$ *for all* $i \in N$ *and* $sup_{i\in N} \parallel T_i \parallel < \infty$*, then* $T = \bigoplus T_i$ *is defined by* $T\mid_{H_i} = T_i$*; that is, if* $x = (x_i) \in H$*, then* $Tx = (\bigoplus T_i)x = \bigoplus(T_i x_i)$*.*
In this case we define

$$\parallel T \parallel = sup_{i\in N} \parallel T_i \parallel .$$

Definition 1.4.17. [14] *Let T be an operator on a Hilbert space H.*
A closed subspace M of a Hilbert space H is said to be ***invariant*** *under T if* $TM \subset M$*, that is* $Tx \in M$ *whenever* $x \in M$*.*

1.5 An Algebra

In this section we define an algebra, a semigroup and prove the related lemma.

Definition 1.5.1. [16] ***An algebra A*** *over a field K is a vector space A over K, such that for each ordered pair of elements* $x, y \in A$ *a unique product* $xy \in A$ *is defined such that, for all* $x, y, z \in A$ *and* $\alpha \in K$*,*
1) (xy)z=x(yz).
2a) x(y+z)=xy+xz. *2b) (x+y)z= xz+yz.*
3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$*.*
If $K = \mathbb{R}$ *or* $\mathbb{C}$*, then A is said to be a real or complex algebra, respectively.*

Definition 1.5.2. *A subset E of an algebra A is said to be* ***a subalgebra*** *of A if it is an algebra.*

Definition 1.5.3. [4] ***A semigroup*** *is a set with a binary operation* $(S, *)$ *such that*
i) $a * b \in S$ *for all a,b*$\in S$*.*
ii) $a * (b * c) = (a * b) * c$ *for all a,b and* $c \in S$*.*
*If there is an element e such that e*a = a*e = a for all a in S then,* $(S, *)$ *is said to be Multiplication semigroup with identity.*
(***e*** *is said to be the identity element in S.)*

Theorem 1.5.4. [16] *B(H) is a Banach algebra of all bounded linear operators over a Hilbert space H.*

Definition 1.5.5. [19] *Let* $T \in B(H)$*. A subalgebera generated by the identity operator I and the operator T is denoted by* ***A(T)****.*
i.e

$$A(T) = \{\alpha_0 + \alpha_1 T + \alpha_2 T^2 + ... + \alpha_n T^n : \alpha_i \in \mathbb{C}, n \geq 0\}.$$

Lemma 1.5.6. [19] *If $T \in B(H)$, then* ***G(T)*** *is a multiplication semigroup subset of A(T) consisting of the element of the form*

$$G(T) = \{\alpha T^k : \alpha \in \mathbb{C}, k \geq 0\}, \quad where \quad A(T) \quad as \quad in\ Definition\ (1.5.5)$$

Proof. Let αT^k *and* $\beta T^j \in G(T)$, where $\alpha, \beta \in \mathbb{C}$ *and* $j, k \in N$.
Hence, $\alpha\beta \in \mathbb{C}$ *and* $k + j \in N$, then $\alpha T^k \beta T^j = \alpha\beta T^{k+j} \in G(T)$.
Therefore, G(T) is a multiplication semigroup. □

Lemma 1.5.7. [19] *Let S be a multiplication semigroup of complex numbers with identity 1. Then* ***G(S,T)*** *is a semigroup subset of A(T) over S consisting of the element of the form.*

$$G(S, T) = \{\alpha T^k : \alpha \in S, k \geq 0\}.$$

Proof. Similarly as the proof of Lemma (1.5.6) □

Chapter 2

Cyclic, Supercyclic And Hypercyclic Operators

This chapter consists of two sections, where we introduce some basic concepts and theorems related to cyclic, hypercyclic and supercyclic Operators and give some examples that explain the relationship between them. Also, we study some properties of the spectrum of cyclic, hypercyclic and supercyclic operators and their Hilbert-adjoint.

2.1 Cyclic, Supercyclic And Hypercyclic Operators

2.1.1 Cyclic Operators

Definition 2.1.1. [12] *Let H be infinite dimensional separable complex Hilbert space and let $T \in B(H)$. Then the orbit of $x \in H$ under T is defined by*

$$\textbf{\textit{orbt(T,x)}} = \{T^n x : n \geq 0\} = \{x, Tx, T^2 x, ...\}$$

Note: *The closure of a set A is denoted by $\overline{A}$.*

Definition 2.1.2. [23] *An operator $T \in B(H)$ is said to be a* ***cyclic operator*** *if there is a vector $x \in H$ such that span$\{T^n x : n \geq 0\}$ is dense in H or span orbt(T,x) is dense in H. In this case x is said to be a* ***cyclic vector*** *for T.*

Example 2.1.3. [15] *Let$\{e_n\}_{n=1}^{\infty}$ be the canonical basis for $l^2(N)$, where $e_k = (0, 0, ..., 1, 0, 0, ...)$ where the 1 in the k-th position.*

Define $T : l^2(N) \longrightarrow l^2(N)$ *by*
$T(e_n) = e_{n+1} \qquad , n \geq 0.$
Then T is a cyclic operator.

Proof. First note that

$$T^n e_1 = e_{n+1} \qquad for\ all\ n \in N.$$

Now, since $\{e_n\}_{n=1}^{\infty}$ is a basis for $l^2(N)$, then

$$l^2(N) = \overline{span\{e_n\}_{n=1}^{\infty}} = \overline{span\{T^n e_1 : n \geq 0\}}.$$

Therefore, T is cyclic. □

Proposition 2.1.4. [11] *If* $T \in B(H)$*, let* Λ *be the algebra of all polynomials in T, and let* $\{U_n\}$ *be a basis for the topology on H. Then the set of all cyclic vectors of T,*

$$C(T) = \bigcap_{n \geq 0} \Big(\bigcup_{P \in \Lambda} P^{-1}(U_n) \Big).$$

Proof. Let $x \in C(T)$ then,

$$span\{T^n x, n \geq 0\} \quad is\ dense\ in\ H.$$

But

$$\Lambda = \{a_0 + a_1 T + a_2 T^2 + ... + a_n T^n : a_i \in \mathbb{C}, n \geq 0\}.$$

Hence $\{Px : P \in \Lambda\}$ *is dense in H.*
Since U_n is open for all $n \in N$ then, by Lemma(1.1.5), $Px \in U_n$ for some $P \in \Lambda$, then for all $n \in N$ there is $P \in \Lambda$ such that $x \in P^{-1}U_n$.
Hence, $x \in (\bigcup_{P \in \Lambda} P^{-1}(U_n))$ *for all* $n \in N$, that is $x \in \bigcap_n (\bigcup_{P \in \Lambda} P^{-1}(U_n))$.
Therefore, $C(T) \subseteq \bigcap_{n \geq 0} (\bigcup_{P \in \Lambda} P^{-1}(U_n))$.
Conversely, let $x \in \bigcap_{n \geq 0} (\bigcup_{P \in \Lambda} P^{-1}(U_n))$, hence $x \in (\bigcup_{P \in \Lambda} P^{-1}(U_n)$ for all $n \in N$, and so $x \in P^{-1}(U_n)$ for all $n \in N$ and some $P \in \Lambda$, that is $Px \in U_n$ for all $n \in N$ and for some $P \in \Lambda$. Therefore, by Lemma (1.1.5)

$$\{Px : P \in \Lambda\} \quad is\ dense\ in\ H.$$

Hence T is cyclic and $x \in C(T)$, that is $\bigcap_{n \geq 0} (\bigcup_{P \in \Lambda} P^{-1}(U_n)) \subseteq C(T)$.
Therefore, $C(T) = \bigcap_{n \geq 0} (\bigcup_{P \in \Lambda} P^{-1}(U_n))$. □

Theorem 2.1.5. [11] *Let $T \in B(H)$, then the following statements are equivalent:*
1. C(T) is dense in H.
2. For each non-empty open sets U,V in H there is a polynomial P in T such that, $P(U) \bigcap V \neq \emptyset$.

Proof. $1 \Longrightarrow 2$: Let U,V be non-empty open sets of H, Λ be the algebra of all polynomials in T, and $\{U_n\}$ be a basis for the topology on H.
Then by Proposition (2.1.4)

$$C(T) = \bigcap_n (\bigcup_{P \in \Lambda} P^{-1}(U_n)).$$

Since C(T) is dense in H, then
$\bigcup_{P \in \Lambda} P^{-1}(U_n)$ is dense in H for all n in N.
Now assume for all $P \in \Lambda, \quad P(U) \bigcap V = \emptyset$.
Thus

$$U \bigcap P^{-1} V = \emptyset.$$

But $V = \bigcup (U_m); U_m \in \{U_n\}$, hence $U \bigcap P^{-1} U_m = \emptyset$.
Therefore,

$$U \bigcap (\bigcup_{P \in \Lambda} P^{-1}(U_m)) = \emptyset.$$

which is a contradiction with the density of $\bigcup_{P \in \Lambda} P^{-1}(U_n)$.
$2) \Longrightarrow 1)$: Let Λ be the algebra of all polynomial in T, and let $\{U_n\}$ be a basis for the topology on H. Then we want to show that
C(T)=$\bigcap_n (\bigcup_{P \in \Lambda} P^{-1}(U_n))$ is dense in H.
By Baire's theorem (1.1.6)it is enough to prove

$$(\bigcup_{P \in \Lambda} P^{-1}(U_n)) \quad \textit{is dense in H for all n in N.}$$

Let W be open set in H, then by (2)there is a polynomial $P \in \Lambda$ such that $PW \bigcap U_n \neq \emptyset$ for all n in N.That is $W \bigcap P^{-1} U_n \neq \emptyset$ for all n in N.
Thus $W \bigcap \bigcup_{P \in \Lambda} P^{-1} U_n \neq \emptyset$ for all n in N.
But W was arbitrary. Therefore,

$$(\bigcup_{P \in \Lambda} P^{-1}(U_n)) \textit{ is dense in H.}$$

□

Theorem 2.1.6. [18] *(The cyclicity Criterion)*
Suppose that $T \in B(H)$. If there exist two dense subsets Y and Z in H and a map $S : Z \longrightarrow Z$ such that there are sequences of polynomials (P_k) and (Q_k) in T and S respectively, satisfying :
1- $P_k(T)Q_k(S) = I_y$ $for\ all\ k$, where I_y is the identity map on Y.
2-$\| P_k(T)y \| \longrightarrow 0$ for all $y \in Y$.
3- $\| Q_k(S)z \| \longrightarrow 0$ for all $z \in Z$.
Then T is a cyclic operator.

Proof. Suppose that U and V are two nonempty open sets in H.
Since Y,Z are dense in H, then by Lemma (1.1.5) we can assume
$x \in Y \bigcap U$ and $z \in Z \bigcap V$.
Consider the vectors $z_k = x + Q_{n_k}z$, then by (3)

$$z_k \longrightarrow x \qquad as \quad k \longrightarrow \infty$$

and from (1) and (2)we get

$$P_{n_k}z_k = P_{n_k}x + P_{n_k}Q_{n_k}z \longrightarrow 0 + z = z \quad as \ k \quad \longrightarrow \infty.$$

so for large values of k, $z_k \in U$ and $P_{n_k}z_k \in V$.
Thus, $P_{n_k}z_k \in P_{n_k}U$ *and* $P_{n_k}z_k \in V$ for large values of k.
Then, $P_{n_k}(U) \bigcap V \neq \emptyset$ for large values of k,
Thus, by Theorem (2.1.5) C(T) is dense in H. Hence, there exist x in C(T).
Therefore T is a cyclic operator . □

Notation : we denote $L^2[0,1]$ to be the space of all Lebesgue measurable functions that are square integrable on [0,1] with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt \qquad and \quad the\ norm$$

$$\| f \| = (\int_0^1 | f(t) |^2 \, dt)^{\frac{1}{2}} \quad for\ all \quad f, g \in L^2[0,1]$$

Definition 2.1.7. [17] *An operator $V : L^2[0,1] \longrightarrow L^2[0,1]$ defined by $Vf(x) = \int_0^x f(t)dt$, is said to be* ***Volterra*** *operator, where $f(x) \in L^2[0,1]$.*

Notation : We denote C[0,1] to be the space of all continuous complex valued functions on [0,1]. For every functions $f \in C[0,1]$ the norm of f(t) is

$$\| f \| = (\int_0^1 | f(t) |^2 \, dt)^{\frac{1}{2}}.$$

Theorem 2.1.8. ***Weierstrass***[20]
The set of all polynomials is dense in C[0,1].

Example 2.1.9. [22] *Volterra operator is cyclic on* $L^2[0,1]$.

Proof. Let $L^2_R[0,1] = \{f \in L^2[0,1] : f[0,1] \subset R\}$.

$$Since \quad orb(V,1) = \{1, x, \frac{x^2}{2}, ..., \frac{x^n}{n!}, ...\}, \ \ then$$

by Weierstrass,s Theorem (2.1.8), the linear span of orb(V,1) is dense in $L^2_R[0,1]$. because $C[0,1]$ *is dense in* $L^2_R[0,1]$ see (15)
That is, V is cyclic on $L^2_R[0,1]$ ....(1)
To show V is cyclic on $L^2[0,1]$. Let $f \in L^2[0,1]$, $\epsilon > 0$ then f=u+iv with $u, v \in L^2_R[0,1]$ so from (1) we can find a polynomials $p_u(x), p_v(x) \in$ *span* $orb(V,1)$
where $p_u(x) = u_0 + u_1x + u_2x^2..... + u_nx^n \longrightarrow u$ and
$p_v(x) = v_0 + v_1x + v_2x^2 + ... + v_mx^m \longrightarrow v$ with
$u_i, v_i \in R$ such that
$\| f - p(V)(1) \|^2 =$
$$\| u + iv - p_u(V)(1) - ip_v(V)(1) \|^2 =$$
$$\| u - p_u(V)(1) \|^2 + \| v - p_v(V)(1) \|^2 < \epsilon.$$
Hence, $f \in \overline{P(V)(1)}$.
That is $L^2[0,1] \subseteq \overline{P(V)(1)}$, but $\overline{P(V)(1)} \subseteq L^2[0,1]$.
Thus $\overline{P(V)(1)} = L^2[0,1]$, that is

$$P(V)(1) \quad is \ dense \ in \ L^2[0,1].$$

Therefore, 1 is a cyclic vector for V. □

2.1.2 Supercyclic operators

Definition 2.1.10. [23] *An operator $T \in B(H)$ is said to be* ***supercyclic*** *if there exists a vector $x \in H$, such that the*

$$\mathbb{C}Orb(T,x) = \{\gamma T^n x : \gamma \in \mathbb{C}, n \geq 0\} \qquad is \ dense \ in \ H.$$

In this case x is said to be a ***supercyclic vector*** *for T.*

Proposition 2.1.11. [6] *The range of a supercyclic operators $T \in B(H)$ is dense in the Hilbert space H.*

Proof. Let T be a supercyclic operator on H, then there exists $x \in H$ such that,

$$M = \{\alpha T^n x : \alpha \in \mathbb{C}, n \geq 0\} \quad \textit{is dense in } H \ .$$

Denote the range of T by $R(T) = \{z : Ty = z, y \in H\}$.
Now let $z \in M - \{\alpha x : \alpha \in \mathbb{C}\}$, then, there exists $\gamma \in \mathbb{C}$ and $k > 0$ such that,
z $= \gamma T^k x = T(\gamma T^{k-1} x)$ which is in R(T)
Then, $M - \{\alpha x : \alpha \in \mathbb{C}\} \subseteq R(T)$.................. (1)
But M is dense in H, and $\alpha x \in H$ for all $\alpha \in \mathbb{C}$, then for any $\alpha \in \mathbb{C}$, there exists a sequences $n_k \in N, \alpha_k \in \mathbb{C}$ such that,
$\alpha_k T^{n_k} x \longrightarrow \alpha x$ where, $\alpha_k T^{n_k} x \in M$.
However, $\alpha_k T^{n_k} x \in R(T)$, therefore, $\alpha x \in \overline{R(T)}$............(2).
From (1)(2) we get $M \subseteq \overline{R(T)}$, but $\overline{M} = H$.
Thus, $\overline{M} = \overline{R(T)} = H$. Therefore, R(T) is dense in H. □

Proposition 2.1.12. [13] *If $T \in B(H)$, then T is supercyclic if and only if for any two open sets U, V in H, there exist an $n \geq 0$, and $\alpha \in \mathbb{C}$ such that,*

$$T^n(\alpha V) \bigcap U \neq \emptyset.$$

Proof. Let $T \in B(H)$. Assume T is a supercyclic operator, then there exists $x \in H$ such that

$$M = \{\alpha T^n x : n \geq 0, \alpha \in \mathbb{C}\} \quad \textit{is dense in } H.$$

Let $U, V \neq \emptyset$ be open sets in $H = \overline{M}$. Then by Lemma (1.1.5),

$$U \bigcap M \neq \phi \quad \textit{and} \ \ V \bigcap M \neq \emptyset.$$

Then there exist $r, m \in N, r \geq m$, *and* $\alpha, \gamma \in \mathbb{C}$ such that

$$\alpha T^r x \in M \bigcap U, \quad \gamma T^m x \in M, \bigcap V.$$

Since $\alpha T^r x = (\alpha \frac{\gamma}{\gamma} T^m T^{r-m} x) = \frac{\alpha}{\gamma} T^{r-m}(\gamma T^m x) \in \frac{\alpha}{\gamma} T^{r-m}(V)$
But $\alpha T^r x \in U$, then $\frac{\alpha}{\gamma} T^{r-m}(V) \bigcap U \neq \emptyset$.
Now, since $r \geq m$, then there is $n = r - m \geq 0$, and $\beta = \frac{\alpha}{\gamma} \in \mathbb{C}$ such that

$$T^n(\beta V) \bigcap U \neq \emptyset.$$

Conversely, let $T^n(\beta V) \bigcap U \neq \emptyset$, where U and V are open sets in H, and $\beta \in \mathbb{C}/\{0\}$...........(1)

Since H is separable then by Theorem (1.1.12) there is a countable base of H, say it $\{U_i\}_{i\in N}$.
To show T is supercyclic operator we first need to show, the set

$$\cup_{n=1}^{\infty} T^{-n}(1/\beta)U_k \textit{ is dense in H.}$$

Now since $T \in B(H)$ then T is continues, and so T^n is continues.
Hence $(1/\beta)T^{-n}U_k = T^{-n}(1/\beta)U_k$ is open in H for all $k \in N$ *and all* $\beta \in \mathbb{C}/\{0\}$, and so $\cup_{n=1}^{\infty} T^{-n}(1/\beta)U_k$ is open in H.
Let $U_i \in U_{k\in N}$ be fixed, then by (1)

$$T^m U_i \cap \cup_{n=1}^{\infty} T^{-n}(1/\beta)U_k \neq \phi, k \in N, \beta \in \mathbb{C}/\{0\}$$

Then there exist $j > 0$ and $x \in U_i$,such that,

$$T^m x \in T^{-j}(1/\beta)U_k.$$

That is, $x \in T^{-(m+j)}(1/\beta)U_k$, and so $U_i \cap T^{-(m+j)}(1/\beta)U_k \neq \phi$ for all $k \in N$.
Therefore,

$$U_i \cap (\cup_{n=1}^{\infty} T^{-n}(1/\beta)U_k) \neq \emptyset \quad \textit{for all } k \in N.$$

But any open set U in H can be written as a union of elements from the base $\{U_i\}_{i\in N}$ of H, then $U \cap (\cup_{n=1}^{\infty} T^{-n}(1/\beta)U_k) \neq \emptyset \quad \textit{for all } k \in N.$
Therefore, by Lemma (1.1.5) $\cup_{n=1}^{\infty} T^{-n}(1/\beta)U_k$ is dense in H, for all $k \in N$
Hence, by Baire Theorem (1.1.6)

$$\cap_{k=1}^{\infty} (\cup_{n=1}^{\infty} T^{-n}(1/\beta)U_k) \textit{ is dense in H.}$$

Hence, there exist $x \in \cap_{k=1}^{\infty}(\cup_{n=1}^{\infty} T^{-n}(1/\beta)U_k)$, that is $x \in (\cup_{n=1}^{\infty} T^{-n}(1/\beta)U_k)$ for all $k \in N$.
Hence there exists $n \in N$, such that $x \in T^{-n}(1/\beta)U_k$ for all $k \in N$
Hence, $\beta T^n x \in U_k$ for all $k \in N$, and this implies that $\{\beta T^n x : n \geq 0, \beta \in \mathbb{C}/\{0\}\} \bigcap U_k \neq \emptyset$. Thus, $\{\beta T^n x : n \geq 0, \beta \in \mathbb{C}/\{0\}\}$ is dense in H. Therefore, T is a supercyclic operator. □

Theorem 2.1.13. [13] *(The Supercyclicity Criterion)*
Suppose that $T \in B(H)$, and there exist a sequence $n_k \longrightarrow \infty$ and two dense sets Y and Z such that :
i- There exists a function $B : Z \longrightarrow Z$ such that, TBz=z for all $z \in Z$ and
ii- If $y \in Y$ and $z \in Z$, then $\| T^{n_k} y \| \| B^{n_k} z \| \longrightarrow 0$ as $k \longrightarrow \infty$.
Then T is a supercyclic operator.

Proof. Let $T \in B(H)$ and suppose that U and V are two nonempty open sets in H. Since Y and Z are two dense sets in H, then by Lemma (1.1.5) there exist $y \in Y, z \in Z, u \in U,\ and\ v \in V$, and $0 < \epsilon < 1$ such that

$$\| y - u \| < \epsilon/2 \qquad and \qquad \| z - v \| < \epsilon/2.$$

From (ii) there exists $k \in N$ such that
$\| T^n y \| \| B^n z \| < \epsilon \ \ for\ all\ n \geq k.$
Hence, Choose $\alpha \in \mathbb{C}$ such that, $\| \frac{1}{\alpha} B^n z \| = \frac{\epsilon}{2}$, and let $x_n = y + \frac{1}{\alpha} B^n z$.
Then

$$\| u - x_n \| \leq \| u - y \| + \| y - x_n \| < \frac{\epsilon}{2} + \| \frac{1}{\alpha} B^n z \| = \epsilon$$

for all $n \geq k$. Thus $x_n \in U \ \ for\ all\ n \geq k.$
That is, $\alpha T^n x_n \in \alpha T^n U$ for all $\alpha \in \mathbb{C}$, *and all* $n \geq k$............(2)
But $x_n = y + \frac{1}{\alpha} B^n z$ then, $T^n x_n = T^n y + \frac{1}{\alpha} T^n B^n z$ and by (i),
$T^n x_n = T^n y + \frac{1}{\alpha} z$. Hence,

$$\| v - \alpha T^n x_n \| \leq \| v - z \| + \| z - \alpha T^n x_n \| < \frac{\epsilon}{2} + \| \alpha T^n y \| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\alpha T^n x_n \in V$ for all $n \geq k$..............(3).
Then by (2) and (3) we get, $\alpha T^n x_n \in \alpha T^n U \cap V$ for all $n \geq k$;
that is,

$$\alpha T^n U \cap V \neq \emptyset.$$

Therefore, by Proposition (2.1.12) T is a supercyclic operator. □

Definition 2.1.14. [23] *An operator $T : l^2(Z) \longrightarrow l^2(Z)$ on the Hilbert space $l^2(Z)$, is said to be a unilateral weighted backward shift with respect to the canonical bases $(e_n)_{n \in Z}$ if there is a bounded positive weight sequence $\{w_n > 0 : n \in Z\}$ such that*

$$Te_n = w_n e_{n+1}.$$

Theorem 2.1.15. [13] *Suppose that $T : l^2(Z) \longrightarrow l^2(Z)$ is a backward weighted shift with weight sequence $\{w_n\}_{n \in Z}$ and either $w_n \geq m > 0 \ \ for\ all$ $n < 0$ or $w_n \leq m \ for\ all\ n > 0$. Then T is a supercyclic operator if and only if there exists a sequence*
$\{n_r\}$ in $Z; n_r \longrightarrow \infty$ such that

$$\lim_{r \longrightarrow \infty} (\prod_{k=1}^{n_r} w_k)(\prod_{k=1}^{n_r} \frac{1}{w_{-k}}) = 0.$$

Example 2.1.16. [18] *Suppose $T : l^2(Z) \longrightarrow l^2(Z)$ is a backward weighted shift with weight sequence* $w_n = \begin{cases} \frac{1}{a^2} & , \quad n \geq 0 \\ \frac{1}{a} & , \quad n < 0 \end{cases}$
where $a > 1$ .Then T is supercyclic.

Proof. clearly $w_n = \frac{1}{a} > \frac{1}{a+1}$ for all $n < 0$. Hence

$$\lim_{n \longrightarrow \infty} (\prod_{k=1}^{n} w_k)(\prod_{k=1}^{n} \frac{1}{w_{-k}}) = \lim_{n \longrightarrow \infty} (\frac{1}{a^n}) = 0.$$

Hence, by Theorem (2.1.15) T is supercyclic operator. □

Example 2.1.17. *Volterra operator is not supercyclic on $L^2[0, 1]$.*
The proof needs more study far from our attention in this thesis, see [22]

2.1.3 Hypercyclic Operators

Definition 2.1.18. [23] *An operator $T \in B(H)$ is said to be a **hypercyclic** operator if there exists $x \in H$ such that $\{T^n x : n \geq 0\}$ is dense in H or orbt(T,x) is dense in H.*
*In this case x is said to be a **hypercyclic vector** for T.*

Remark 2.1.19. [18] *Every hypercyclic operator is supercyclic,*
hence it is cyclic.
The converse is false as we will see in Remark (2.1.27).

Proposition 2.1.20. [6] *The range of a hypercyclic operators is dense in a Hilbert space H.*

Proof. Similarly as the proof of proposition (2.1.11) □

Proposition 2.1.21. [12] *If $T \in B(H)$, then T is hypercyclic if and only if for any two open sets U, V in H there exists an $n \geq 0$ such that*

$$T^n(U) \bigcap V \neq \emptyset.$$

Proof. Let $T \in B(H)$, be a hypercyclic operator. Then there exists $x \in H$ such that

$$M = \{T^n x : n \geq 0\} \qquad \textit{is dense in } H.$$

Let $U, V \neq \Phi$ be open sets in $H = \overline{M}$. Then by Lemma (1.1.5)

$$U \bigcap M \neq \phi \qquad and \quad V \bigcap M \neq \phi.$$

Then there exist $r, m \in N, r \geq m$ such that

$$T^r x \in M \bigcap U, \quad T^m x \in M \bigcap V.$$

Since $T^r x = T^{r-m}(T^m x) \in T^{r-m}(V)$ and $T^r x \in U$, then

$$T^{r-m}(V) \bigcap U \neq \phi.$$

Now,since $r \geq m$, then there is $n = r - m \geq 0$ such that,

$$T^n(U) \bigcap V \neq \phi \ .$$

Conversely, let $T^m(U) \bigcap V \neq \phi$ where U,V open set in H..........(1)
Since H is separable then by Theorem (1.1.10) there is a countable base of H, say it is $\{U_i\}_{i \in N}$.
To show T is hypercyclic, we first need to show the set
$\cup_{n=1}^{\infty} T^{-n} U_k$ is dense in H for all $k \in N$.

Now let $T \in B(H)$ then T is continues, and so T^n is continues.
Hence $T^{-n} U_k$ is open in H for all $n, k \in N$, and so $\cup_{n=1}^{\infty} T^{-n} U_k$ is open in H for all $k \in N$.
Let $U_i \in \{U_k\}_{k \in N}$ be arbitrary fixed , then from (1) [for $U = U_i \quad and \quad V = \bigcup_{n=1}^{\infty} T^{-n} U_k$],

$$T^m U_i \cap \cup_{n=1}^{\infty} T^{-n} U_k \neq \phi \qquad for\ all\ k \in N.$$

Hence there exists $x \in U_i$ such that

$$T^m x \in \cup_{n=1}^{\infty} T^{-n} U_k \qquad for\ all\ k \in N.$$

Hence, for all $k \in N$ there exists $j \in N$ such that $T^m x \in T^{-j} U_k$ that is $x \in T^{-(m+j)} U_k$ for all $k \in N$.
Hence $U_i \cap T^{-(m+j)} U_k \neq \phi$ for all $k \in N$.
Therefore, for any fixed $i \in N$

$$U_i \cap (\cup_{n=1}^{\infty} T^{-n} U_k) \neq \phi \ for\ all\ k \in N.$$

But any open set U in H can be written as a union of elements from the base $\{U_i\}_{i \in N}$ of H,then

$$U \cap (\cup_{n=1}^{\infty} T^{-n} U_k) \neq \emptyset \ for\ all\ k \in N, U\ is\ open\ in\ H.$$

Therefore, by Lemma (1.1.5) $\cup_{n=1}^{\infty} T^{-n}U_k$ is dense in H, for all $k \in N$
Seconed by Baire's Theorem (1.1.6) ,

$$\cap_{k=1}^{\infty} (\cup_{n=1}^{\infty} T^{-n}U_k) \textit{ is dense in H.}$$

Hence there exist $x \in \cap_{k=1}^{\infty}(\cup_{n=1}^{\infty} T^{-n}U_k)$, then there exists $n \in N$, such that $x \in T^{-n}U_k$ for all $k \in N$. Hence $T^n x \in U_k$ for all $k \in N$.
Therefore, $\{T^n x : n \geq 0\} \bigcap U_k \neq \emptyset$ for all $k \in N$ and as above $\{T^n x : n \geq 0\} \bigcap U \neq \emptyset$ for all open set U in H.
Hence by Lemma(1.1.5) $\{T^n x : n \geq 0\}$ is dense in H .
Therefore, T is a hyercyclic operator. □

Theorem 2.1.22. [12] *(The Hypercyclicity Criterion)*
Suppose that $T \in B(H)$, and there exist two dense subsets Y and Z in H and a sequence (n_k) such that:
1- $T^{n_k}x \longrightarrow 0$ for every $x \in Y$, and
2-there exist functions $B : Z \longrightarrow H$ such that for every $x \in Z$,
$B^{n_k}x \longrightarrow 0$ and $T^{n_k}B^{n_k}x \longrightarrow x$. Then T is hypercyclic.

Proof. Suppose that U and V are two nonempty open sets in H.
Since Y,Z is dense in H, then by Lemma (1.2.7) we can assume $x \in Y \bigcap U$ and $z \in Z \bigcap V$.
Consider the vectors $z_k = x + B^{n_k}z$, then
from (2) $z_k \longrightarrow x$, and
$T^{n_k}z_k = T^{n_k}x + T^{n_k}B_{n_k}z \longrightarrow 0 + z = z \quad as \;\; k \quad \longrightarrow \infty$.
So for all large values of k

$$z_k \in U \qquad and \qquad T^{n_k}z_k \in V.$$

Thus $T^{n_k}z_k \in T^{n_k}U \quad and \quad T^{n_k}z_k \in V$.
Then $T^{n_k}U \bigcap V \neq \emptyset$ for all large k.
Thus by proposition (2.1.21) T is a hypercyclic operator. □

Definition 2.1.23. [12] *An operator $B : l^2(N) \longrightarrow l^2(N)$ on the Hilbert space $l^2(N)$, is said to be a unilateral weighted backward shift if there is a bounded positive weight sequence $\{w_n > 0 : n \geq 1\}$ such that*
$T(a_0, a_1, a_2, ...) = (w_1a_1, w_2a_2, ...)$.

Example 2.1.24. [12] ***(Rolewicz)*** *If T=tB, where B denotes the Backward shift with weight $w_n = 1$ for all $n \in N$ on $l^2(N)$, and $\mid t \mid > 1$. Then T is a hypercyclic operator.*

Proof. First consider that $B^n(x) = B^n(x_1, x_2, ...x_n, ...) = (x_{n+1}, x_{n+2}, ...)$ for all $(x_i)_{i\in N} \in l^2(N)$.
Now we will use the hypercyclic criterion.
Let Y be the set of all vectors in $l^2(N)$ with only finitely many non-zero coordinates, that is

$$Y = \{(y_1, y_2, ..., y_n, 0, 0...) \in l^2(N) : n \geq 0\}.$$

We need to show Y is dense in $l^2(N)$............... (3)
Let $x \in l^2(N)$,where $x = (x_i)_{i\in N}$.
Then , $(x)_n = (x_0, x_1, ..., x_n, 0, 0....) \in Y$, for all n, and
$\| x - (x)_n \| \longrightarrow 0 \quad as \quad n \longrightarrow \infty$.
Hence, $x \in \overline{Y}$.
Therefore, $\overline{Y} = l^2(N)$; that is Y is dense in $l^2(N)$ and so (3) holds.
Define $D^n : l^2(N) \longrightarrow l^2(N)$ by

$$D^n(x_0, x_1, ...) = \frac{1}{t^n}(0, 0, ..., x_0, x_1, ...)$$

where there are n zeros in front of x_0
Then $(tB)^n x = t^n(B^n x) = (t^n)(0) = 0$ for every $x \in Y$.
Hence (1) in Theorem (2.1.22) holds. And
$D^n z = \frac{1}{t^n}(0, 0, ..., z_0, z_1, ...) \longrightarrow 0 \;\; as \;\; n \longrightarrow \infty$, since$| t |> 1$.
Morever, $((tB)^n D^n z) = t^n B^n(D^n)z = t^n B^n(\frac{1}{t^n}(0, 0, ..., z_0, z_1, ...)) = B^n(0, 0, ...z_0, z_1, ...) = (z_0, z_1, ...) = z$.
Hence (2) in Theorem (2.1.22) holds.
Therefore, by Theorem (2.1.22) T=(tB) is a hypercyclic operator. □

Theorem 2.1.25. [13] *Suppose that $T : l^2(Z) \longrightarrow l^2(Z)$ is a backward weighted shift with weight sequence $\{w_n\}_{n\in Z}$ and either $w_n \geq m > 0$ for all $n < 0$ or $w_n \leq m$ for all $n > 0$. Then T is a hypercyclic operator if and only if there exists a sequence $\{n_r\}$ in Z such that $n_r \longrightarrow \infty$ with*

$$\lim_{r\longrightarrow\infty}(\prod_{k=1}^{n_r} w_k) = 0 \qquad and \qquad \lim_{r\longrightarrow\infty}(\prod_{k=1}^{n_r}\frac{1}{w_{-k}}) = 0.$$

Example 2.1.26. [18] *Suppose that $T : l^2(Z) \longrightarrow l^2(Z)$ is a backward weighted shift with weight sequence* $w_n = \begin{cases} 2 & , \; n < 0 \\ \frac{1}{n+1} & , \; n \geq 0 \end{cases}$

Then T is hypercyclic, and hence it is supercyclic .

Proof. Clearly $w_n = 2 > 1$ for all $n < 0$. Hence,

$$\lim_{n\longrightarrow\infty}(\prod_{k=1}^{n}\frac{1}{w_{-k}}) = \lim_{n\longrightarrow\infty}(\frac{1}{2^n}) = 0,$$

and

$$\lim_{n\longrightarrow\infty}(\prod_{k=1}^{n}w_k) = \lim_{n\longrightarrow\infty}(\frac{1}{(n+1)!}) = 0.$$

Therefore, by Theorem (2.1.25) T is a hypercyclic operator.

$$\lim_{n\longrightarrow\infty}(\prod_{k=1}^{n}w_k)(\prod_{k=1}^{n}\frac{1}{w_{-k}}) = \lim_{n\longrightarrow\infty}\frac{1}{2^n(n+1)!} = 0.$$

Hence by Theorem (2.1.15), T is a supercyclic operator. □

Remark 2.1.27. [18] *Suppose that* $T : l^2(Z) \longrightarrow l^2(Z)$ *as in Example (2.1.16). Then T is supercyclic but it is not hypercyclic.*

Proof. By Example (2.1.16) T is supercyclic. Now since, $a > 1$ and,

$$\lim_{n\longrightarrow\infty}(\prod_{k=1}^{n}\frac{1}{w_{-k}}) = \lim_{n\longrightarrow\infty}a^n = \infty.$$

Then, by Theorem (2.1.25) T is not hypercyclic. □

Proposition 2.1.28. [18] *Let* $\{H_i\}_{i\in N}$ *be family of Hilbert spaces. Let* $T_i \in B(H_i)$ *for all i. If* $T = \bigoplus T_i$ *is a hypercyclic operator, then* T_i *is hypercyclic for all* $i \in N$.

Proof. First, note that, by Theorem (1.3.7), $H = \bigoplus H_i$ is a Hilbert space. To prove that T_i is hypercyclic for all $i \in N$, we need to show that for any fixed $i \in N$, there is $x_i \in H_i$ such that

$$\{T_i^n x_i : n \geq 0\}$$

is dense in H_i for all i.
Since $T = \bigoplus T_i$ is hypercyclic, then there is $x = (x_i)_{i\in N} \in H = \bigoplus_{i=1}^{\infty} H_i$, such that the set

$$M = \{(\bigoplus T_i)^n x : n \geq 0,\} \textit{ is dense in } \bigoplus H_i \quad .$$

Let y_i be an element in H_i for any $i \in N$, then $y = (y_i) \in \bigoplus H_i$, and so, there exists a sequence $\{(\bigoplus T_i)^{n_k} x\}_{k \in N}$ in M such that

$$(\bigoplus T_i)^{n_k} x \longrightarrow y.$$

Then ,for all $\epsilon > 0$ there exists $N_0 \in N$ such that

$$\| (\bigoplus T_i)^{n_k} x - y \| < \epsilon \qquad for\ all \quad k > N_0.$$

Then,

$$\| \bigoplus (T_i^{n_k} x_i - y_i) \|^2 < \epsilon^2 \qquad for\ all \quad k > N_0.$$

Then by Theorem (1.3.7) we have,

$$\Sigma_{i=1}^{\infty} \| T_i^{n_k} x_i - y_i \|^2 < \epsilon^2 \quad for\ all\ k > N_0.$$

And so,

$$\| T_i^{n_k} x_i - y_i \| < \epsilon \quad for\ all\ k > N_0.$$

This means that, for any fixed $y_i \in H_i$, there is a sequence $(T_i^{n_k} x_i)_{k \in N}$ in

$$M_i = \{T_i^{n_k} x_i : n_k > 0\}, \quad such\ that$$

$$T_i^{n_k} x_i \longrightarrow y_i \quad , \quad as \quad k \longrightarrow \infty$$

Then, $y_i \in \overline{M_i}$, hence, $H_i \subset \overline{M_i}$.
But $\overline{M_i} \subset H_i$. Then $\overline{M_i} = H_i$.
Therefore, x_i is a hypercyclic vector in H_i, so that, T_i is a hypercyclic operator for all $i \in N$. □

2.2 Spectral Properties of Cyclic, Supercyclic and Hypercyclic Operators

Definition 2.2.1. [16] *Let $T \in B(H)$ with identity I, then the spectrum of T, denoted by $\sigma(T)$, is defined by*

$$\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)^{-1}\ dose\ not\ exist\ in\ B(H)\}.$$

The complement of $\sigma(T)$ in $\mathbb{C}$ is called the resolvent of T and it is denoted by $\rho(T)$.

Definition 2.2.2. [16] *Let $T \in B(H)$, then the scaler λ is called an eigenvalue of T if there exists a nonzero vector $x \in H$ such that $Tx = \lambda x$.*
The set of all eigenvalues of T is denoted by $\sigma_p(T)$.

Lemma 2.2.3. *If $\langle x_1, y\rangle = \langle x_2, y\rangle$ for all y in the complex inner product X. Then $x_1 = x_2$.*
proof. *See* [16].

Proposition 2.2.4. [16] *Let $T \in B(H)$. Then $T^{*^n} = T^{n^*}$*

Proof. The proof is by induction on n.
Clearly the proposition is true for n=1.
Assume that it is true for m, then $T^{*^m} = T^{m^*}$, we show that it is true for m+1. Now, let $x, y \in H$, then by the definition of T^* we have
$\langle x, (T^{m+1})^* y\rangle = \langle T^{m+1}x, y\rangle = \langle T(T^m)x, y\rangle = \langle T^m x, T^* y\rangle =$

$$\langle x, (T^m)^* T^* y\rangle = \langle x, (T^*)^m T^* y\rangle = \langle x, (T^*)^{m+1} y\rangle.$$

Hence, $(T^{m+1})^* y = (T^*)^{m+1} y$. But y was arbitrary, hence, $(T^{m+1})^* = (T^*)^{m+1}$. Therefore, the proof is complete. □

Theorem 2.2.5. *Let H_1 and H_2 be Hilbert spaces.*
i) If $S, T \in B(H_1, H_2)$, then $(S + T)^* = S^* + T^*$.
ii) If $I \ :H_1 \longrightarrow H_1$ is the identity operator, then $I^* = I$.
Proof. *see[16].*

Proposition 2.2.6. [18] *Let $T \in B(H)$. Then T is cyclic, if and only if $T + \lambda I$ is cyclic for all $\lambda \in \mathbb{C}$.*

Proof. Let T be a cyclic operators on H. Let $\lambda \in \mathbb{C}$, then there exists $x \in H$ such that

$$span\ \{T^n x : n \geq 0\} \qquad is\ dense\ in\ H.$$

Hence, $\lambda T^k x \in\ span\ \{T^n x : n \geq 0\}\ for\ all\ k \geq 0, \lambda \in \mathbb{C}$.
Thus, $\lambda T^k x + \lambda x \in\ span\ \{T^n x : n \geq 0\}$ for all $k \geq 0, \lambda \in \mathbb{C}$.
Therefore,

$$span\ \{(T+\lambda I)^n x : n \geq 0\} \subseteq span\{T^n x : n \geq 0\}, for\ all\ \lambda \in \mathbb{C}...............(1).$$

However, it is clear that

$$span\{T^n x : n \geq 0\} \subseteq span\ \{(T + \lambda I)^n x : n \geq 0\}..............(2).$$

from (1)and(2) we get

$$span\{T^n x : n \geq 0\} = span\ \{(T + \lambda I)^n x : n \geq 0\}.$$

Therefore, $T + \lambda I$ is cyclic for all $\lambda \in \mathbb{C}$. □

Proposition 2.2.7. *Let T be a cyclic operator, and let $S \in B(H)$ such that ST=TS. If the range of S, R(S) is dense in H, then S is a cyclic operator.*
Proof. *See* [18].

Proposition 2.2.8. [18] *If T is a cyclic operator, and $\sigma_P(T^*) \neq \emptyset$, then for any $\lambda \in \sigma_P(T^*)$, $R(T - \overline{\lambda})$ is not dense in H.*
(In Particular, if T is cyclic and $0 \in \sigma_P(T^*)$, then $R(T)$ is not dense in H.)

Proof. Let $y \in H$, and let $\lambda \in \sigma_P(T^*)$.
Then, there exists a non-zero vector $z \in H$ such that $T^*z = \lambda z$,
hence, $(T^* - \lambda I)z = 0$. But by Theorem (2.2.5)

$$(T^* - \lambda I) = (T - \overline{\lambda} I)^*.$$

Thus $(T - \overline{\lambda I})^* z = 0$, hence, $(T - \overline{\lambda I})^{n^*} z = 0$ for all $n \in N$.
Now, for all $n \in N$
$\langle (T - \overline{\lambda I})^n y, z\rangle = \langle y, (T - \overline{\lambda I})^{n^*} z\rangle = 0.$
Thus $(T - \overline{\lambda I})^n y = 0$, for all $n \in N$.
That is $(T - \overline{\lambda I})$ is not a cyclic operator. However, $T(T - \overline{\lambda} I) = (T - \overline{\lambda} I)T$, then by Proposition (2.2.7), $R(T - \overline{\lambda})$ is not dense in H. □

Notation [10] Let $T \in B(H)$, then we use the following notation
1) The kernel of T, denoted by $Ker\ T = \{x \in H : Tx = 0\}$.
2) The dimension of ker (T) = nul (T).
3) The index of T, ind (T) = nul (T) - nul (T^*).
4) $\mathbb{D} = \{\lambda \in \mathbb{C} : \mid \lambda \mid < 1\}$.
5) $\partial\mathbb{D} = \{\lambda \in \mathbb{C} : \mid \lambda \mid \ = 1\}$.

Definition 2.2.9. [10] *An operator $T \in B(H)$ is called a semi-Fredholm if T has a closed range, and either dim ker T or dim ker T^* is finite.*

Definition 2.2.10. [10] *Let $T \in B(H)$, then the set*

$$P_{s-F}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \quad is\ a\ semi - Fredholm\}.$$

is called the semi-Fredholm domain of T.

Theorem 2.2.11. *Let T be a cyclic operator. Then*

$$ind(T - \lambda I) \geq -1 \qquad\qquad for\ all\ \lambda \in P_{s-F}(T).$$

proof *See*[18].

Lemma 2.2.12. [19] *Let x be a supercyclic vector for T, then for all $y \in H$, the set $M = \{\langle \alpha T^n x, y\rangle : n \geq 0, \alpha \in \mathbb{C}\}$ is dense in $\mathbb{C}$.*

Proof. Let $B \in \mathbb{C}$, let $y \in H$ and $y \neq 0$, then $\frac{By}{\|y\|^2} \in H$.
But T is supercyclic, then there are sequences $n_k \in N, \alpha_k \in \mathbb{C}$, such that

$$\alpha_k T^{n_k} x \longrightarrow \frac{By}{\| y \|^2}.$$

Then use the continuity of the inner product to get

$$\langle \alpha_k T^{n_k} x, y\rangle \longrightarrow \langle \frac{By}{\| y \|^2}, y\rangle = \frac{B}{\| y \|^2}\langle y, y\rangle = B.$$

Hence, $B \in \overline{M}$
Therefore, $\overline{M} = \mathbb{C}$, that is, M is dense in $\mathbb{C}$. □

Theorem 2.2.13. [15] *Let $T \in B(H)$, If T is a supercyclic operator, then T^* has at most one eigenvalue.*

Proof. Let γ_1 *and* γ_2 be two eigenvalues of T^*, where $\gamma_1 \neq \gamma_2$..........(1).
Then there exists $z_1, z_2 \in H - \{0\}$, such that

$$T^* z_1 = \gamma_1 z_1 \quad and \quad T^* z_2 = \gamma_2 z_2.$$

Now, let x be a supercyclic vector for T, then

$$M = \{cT^n x : n \geq 0, c \in \mathbb{C}\} \quad is\ dense\ in\ H.$$

claim $\langle x, z_i\rangle \neq 0$ *and* $\gamma_i \neq 0$ *for* $i = 1, 2$.
proof of the claim Let $\langle x, z_i\rangle = 0$ *or* $\gamma_i = 0$ *for some* $i = 1, 2$, then by proposition(2.2.4)

$$\langle T^n x, z_i\rangle = \langle x, T^{*^n} z_i\rangle = \langle x, \gamma_i^n z_i\rangle = \overline{\gamma_i}^n \langle x, z_i\rangle = 0.$$

This is a contradiction withe Lemma(2.2.12).Therefore, the proof of the claim is complete.

Also z_1, z_2 are linearly independent, otherwise if there is $\alpha \in C; z_1 = \alpha z_2,\ then\ T^* z_1 = \alpha T^* z_2$.
So $\gamma_1 z_1 = \alpha\gamma_2 z_2 = \gamma_2 z_1\ , \gamma_1 = \gamma_2$, a contradiction with (1).
Hence by [Hahn-Banach Theorem (1.4.8) and Riesz-Frechet Theorem (1.4.9)] there is $y \in H$ such that

$$f(z_1) = \langle z_1, y\rangle = 1, \quad and\ f(z_2) = \ \langle z_2, y\rangle = 0.$$

Now, $y \in H$, thus there are sequences $(c_i) \in \mathbb{C}, (n_i) \in N$ such that $c_i T^{n_i} x \longrightarrow y$. Thus

$$\langle z_1, c_i T^{n_i} x\rangle \longrightarrow \langle z_1, y\rangle = 1 \quad and \ \ \langle z_2, c_i T^{n_i} x\rangle \longrightarrow \langle z_2, y\rangle = 0.$$

Hence,

$$\frac{\langle z_2, c_i T^{n_i} x\rangle}{\langle z_1, c_i T^{n_i} x\rangle} \longrightarrow 0.$$

That is

$$\frac{\langle T^{n_i^*} z_2, x\rangle}{\langle T^{n_i^*} z_1, x\rangle} = \frac{\langle \gamma_2^{n_i} z_2, x\rangle}{\langle \gamma_1^{n_i} z_1, x\rangle} \longrightarrow 0.$$

Therefore,

$$\mid \frac{\gamma_2}{\gamma_1} \mid^{n_i} \frac{\langle z_2, x\rangle}{\langle z_1, x\rangle} \longrightarrow 0.$$

Thus,

$$\mid \gamma_2 \mid < \mid \gamma_1 \mid \ldots\ldots \qquad (2)$$

Analogously we can choose $h \in H$, such that ,

$$f(z_1) = \langle z_1, h\rangle = 0, and\ f(z_2) = \langle z_2, h\rangle = 1.$$

And by the same argument we showing that,

$$\mid \gamma_1 \mid < \mid \gamma_2 \mid \ldots\ldots \qquad (3).$$

So by (2)and(3) we get a contradiction.
Therefore, T^* has at most one eigenvalue . □

Lemma 2.2.14. [1] *Let x be a hypercyclic vector for T, then for all $y \in H$, the set*

$$M = \{\langle T^n x, y\rangle : n \geq 0\} \quad is\ dense\ in\ \mathbb{C}.$$

proof *Similar to the proof of Lemma (2.2.12)*

Theorem 2.2.15. [15] *If $T \in B(H)$ is a hypercyclic operator, then*

$$\sigma_p(T^*) = \emptyset.$$

Proof. Assume on the contrary that $\sigma_p(T^*) \neq \emptyset$, then there is $\lambda \in \sigma_p(T^*)$. Hence, there is a nonzero vector $y \in H$ such that

$$T^* y = \lambda y.$$

Now let x be a hypercyclic vector for T.
so by Lemma(2.2.14) the set

$$M = \{\langle T^n x, y\rangle : n \geq 0,\} \quad \text{is dense in } \mathbb{C}.$$

Now, let $r \in \mathbb{R}^+ \bigcup \{0\}$, then $r \in \mathbb{C}$, there exists a sequence $\langle T^{n_k} x, y\rangle$ in M, such that

$$\langle T^{n_k} x, y\rangle \longrightarrow r.$$

Hence,

$$\mid \langle T^{n_k} x, y\rangle \mid \longrightarrow \mid r \mid = r.$$

Therefore, the set

$$K = \{\mid \langle T^n x, y\rangle \mid : n \geq 0,\} \quad \text{is dense in } \mathbb{R}^+ \bigcup \{0\}.$$

But for all $n \geq 0$,

$$\mid \langle T^n x, y\rangle \mid = \mid \langle x, {T^*}^n y\rangle \mid = \mid \langle x, \lambda^n y\rangle \mid = \mid \lambda \mid^n \mid \langle x, y\rangle \mid .$$

Then the set

$$K = \{\mid \lambda \mid^n \mid \langle x, y\rangle \mid : n \geq 0\} \quad \text{is dense in } \mathbb{R}^+ \bigcup \{0\}.$$

Now, we shall get a contradiction by proving that K can not be dense in $\mathbb{R}^+ \bigcup \{0\}$.
Case(1) :if $\lambda \geq 1$.
Let r= $\frac{1}{2} \mid \langle x, y\rangle \mid$ which belongs to $\mathbb{R}^+ \bigcup \{0\}$.
Take $\epsilon = \frac{1}{4} \mid \langle x, y\rangle \mid$ to get an open ball $B = B(r, \epsilon)$ such that, $B \bigcap K = \emptyset$.
Hence, $r \notin \overline{K}$, that is, K is not dense in $\mathbb{R}^+ \bigcup \{0\}$.
Case (2):if $\lambda \leq 1$.
Let r= $2 \mid \langle x, y\rangle \mid$ which belongs to $\mathbb{R}^+ \bigcup \{0\}$.
Take $\epsilon = \frac{1}{2} \mid \langle x, y\rangle \mid$ to get an open ball $B^* = B(r, \epsilon)$ such that, $B^* \bigcap K = \emptyset$.
Hence $r \notin \overline{K}$, that is, K is not dense in $\mathbb{R}^+ \bigcup \{0\}$.
Therefore, $\sigma_p(T^*) = \emptyset$. □

Remark 2.2.16. [18] *There is a non- hypercyclic operator B, and*

$$\sigma_p(B^*) = \emptyset.$$

Proof. Let $B \in l^2(N)$ be the backward shift in Example (2.1.24), then by this example tB is hypercyclic for all $t \in \mathbb{C}, \mid t \mid > 1$.
Hence $\sigma_p((tB)^*) = \phi$.............(1).
Suppose that $\sigma_p(B^*) \neq \emptyset$.Then there is $\lambda \in \sigma_p(B^*)$.
Hence there is $x \in l^2(N)$ such that, $B^*x = \lambda x$.
Hence $\bar{t}B^*x = \bar{t}\lambda x$. Therefore, $\bar{t}\lambda \in \sigma_p((tB)^*)$, a contradiction with (1).
Therefore, $\sigma_p(B^*) = \emptyset$. Finally, B is not hypercyclic. □

Definition 2.2.17. [10] *$T \in B(H)$ is said to be a Fredholm operator if its range R(T) is closed and both nul(T) and nul(T^*) are finite.*
F_0 will denote the set of all Fredholm operator with zero index.

Definition 2.2.18. [5] *Let $T \in B(H)$, then*
1) the weyl spectrum $\sigma_w(T)$ of T is defined as

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \notin F_0\}.$$

2) The finite point spectrum $\sigma_{pf}(T)$ of T is defined as

$$\sigma_{pf}(T) = \{\lambda \in \sigma_p(T) : nul(T - \lambda I) < \infty\}.$$

Theorem 2.2.19. [5] *Let $\sigma(T)$ be the spectrum of an operator T. Then $\sigma(T^*) = \overline{\sigma(T)}$, where $\overline{\sigma(T)}$ is the conjugates of $\sigma(T)$.*

Proof. Let $\lambda \notin \sigma(T)$, then $\overline{\lambda} \notin \overline{\sigma(T)}$ *and* $(T - \lambda I)$ is invertible, that is $(T - \lambda I)(T - \lambda I)^{-1} = I$.
Hence, $I^* = I = ((T - \lambda I)(T - \lambda I)^{-1})^* = (T - \lambda I)^{-1^*}(T - \lambda I)^*$.
That is, $(T - \lambda I)^*$ is invertible.
But by Theorem (2.2.5) $(T - \lambda I)^* = (T^* - \overline{\lambda} I)$ is invertible. Hence, $\overline{\lambda} \notin \sigma(T^*)$.
Therefore, $\sigma(T^*) \subseteq \overline{\sigma(T)}$......................(1).
Replace T by T^* in (1) and take a conjugates to get $\overline{\sigma(T)} \subseteq \sigma(T^*)$.........................(2).
Therefore, from (1) and (2) we get $\sigma(T^*) = \overline{\sigma(T)}$. □

Theorem 2.2.20. *Let $T \in B(H)$. Then $T \in F_0$, if and only if $T^* \in F_0$*
Proof . *See* [5].

Proposition 2.2.21. [5] *Let $T \in B(H)$. Then*
1) $\sigma(T) - \sigma_w(T) \subseteq \sigma_{pf}(T)$.
2) $\sigma_w(T^*) = \overline{\sigma_w(T)}$.
Where $\overline{\sigma_w(T)}$ is the conjugates of $\sigma_w(T)$.

Proof. (1) Let $\lambda \in \sigma(T) - \sigma_w(T)$, then $\lambda \notin \sigma_w(T)$....................(1).
Hence by Definition (2.2.18)part (1), $(T - \lambda I) \in F_0$,and so by Definition (2.2.17) ind$(T - \lambda I) = nul(T - \lambda I) - nul(T - \lambda I)^* = 0$, that is nul$(T - \lambda I) < \infty$.
Hence, by Definition (2.2.18) part (2), $\lambda \in \sigma_{pf}(T)$.
Therefore, $\sigma(T) - \sigma_w(T) \subseteq \sigma_{pf}(T)$.
(2) Let $\lambda \notin \sigma_w(T^*)$, then by Definition (2.2.18)part (1), $(T^* - \lambda I) \in F_0$.
Hence by Theorem (2.2.20) $(T - \overline{\lambda} I) = (T^* - \lambda I)^* \in F_0$. That is $\overline{\lambda} \notin \sigma_w(T)$, hence $\lambda \notin \overline{\sigma_w(T)}$.
Therefore, $\overline{\sigma_w(T)} \subseteq \sigma_w(T^*)$ (3).
Replace T by T^* and take a conjugate in (3) to get $\sigma_w(T^*) \subseteq \overline{\sigma_w(T)}$ (4).
Therefore, from (3)(4) we have $\sigma_w(T^*) = \overline{\sigma_w(T)}$. □

Proposition 2.2.22. [15] *Let T be a hypercyclic operator, then*

$$\sigma(T) = \sigma_w(T).$$

Proof. Let T be hypercyclic operator, then by Theorem (2.2.15)

$$\sigma_p(T^*) = \emptyset.$$

Hence, by Proposition (2.2.21) part (1),

$$\sigma(T^*) - \sigma_w(T^*) \subseteq \sigma_{pf}(T^*) \subseteq \sigma_p(T^*) = \emptyset.$$

Hence, $\sigma(T^*) = \sigma_w(T^*)$, that is $\overline{\sigma(T^*)} = \overline{\sigma_w(T^*)}$.
Therefore, by Theorem (2.2.19) and Proposition (2.2.21) $\sigma(T) = \sigma_w(T)$. □

Proposition 2.2.23. [15] *Let T be a superccyclic operator, then Weyl-spectrum of T is the spectrum of T except possibly one element.*

Proof. Let T be a supercyclic operator, then by Theorem(2.2.13)

$$\sigma_p(T^*) = \emptyset.........(1)$$

Or for some one $\lambda \in \mathbb{C}$,

$$\sigma_p(T^*) = \lambda..........(2)$$

If (1) holds then by the same argument of proving Proposition (2.2.22), $\sigma(T) = \sigma_w(T)$.
If (2) holds then,

$$\sigma(T^*) - \sigma_w(T^*) \subseteq \sigma_{pf}(T^*) \subseteq \sigma_p(T^*) = \{\lambda\}.$$

Therefore, $\sigma_w(T^*) = \sigma(T^*) - \{\lambda\}$.
Since the conjugate of difference from two sets is the difference of its conjugates , hence, $\overline{\sigma_w(T^*)} = \overline{\sigma(T^*)} - \overline{\{\lambda\}}$
Therefore, by Theorem (2.2.19) and Proposition (2.2.21) $\sigma_w(T) = \sigma(T) - \overline{\{\lambda\}}$.
Then the Weyl-spectrum of T is the spectrum of T except possibly one element. □

Theorem 2.2.24. *Let $T \in B(H)$, then $\sigma(T) \neq \emptyset$ and $\sigma(T)$ is closed.*
proof *see* [16].

Theorem 2.2.25. ***(The Riesz Decomposition Theorem)***
Let $T \in B(H)$, let $\sigma(T) = \sigma_1 \bigcup \sigma_2$ where σ_1, σ_2 are disjoint non empty closed sets. Then T has a complementary pair $\{M_1, M_2\}$, of non -trivial invariant subspaces such that, $\sigma(T_1) = \sigma_1$ and $\sigma(T_2) = \sigma_2$, where $T_1 = T_{|_{M_1}}$ and $T_2 = T_{|_{M_2}}$.
proof *see* [21].

Theorem 2.2.26. *Let T be hypercyclic operator. Then, $\sigma(T) \bigcap \partial\mathbb{D} \neq \emptyset$.*
proof *see* [15].

Proposition 2.2.27. [18] *If T is a hypercyclic operator, then every component of $\sigma(T)$ has a non-empty intersection with $\partial\mathbb{D}$.*

Proof. (case 1.) If $\sigma(T)$ is connected, then $\sigma(T)$ is the only component of $\sigma(T)$. Hence by Theorem(2.2.26) $\sigma(T) \bigcap \partial\mathbb{D} \neq \emptyset$.
(case2.) If $\sigma(T)$ is not connected, then there is $\sigma_1 \subset \sigma(T)$ which is open and closed. Let σ_1 be component of $\sigma(T)$. Since $\sigma(T)$ is not connected, then $\sigma_1 \neq \emptyset$.
But $\sigma(T) = \sigma_1 \bigcup \{\sigma(T) - \sigma_1\}$, and by Theorem (2.2.24)

$$\sigma(T) - \sigma_1 \ is \ \ closed \ with \ \sigma(T) - \sigma_1 \neq \emptyset,$$

then by Theorem (2.2.25),

$$T = T_1 + T_2 \ \ and \ \ \sigma(T_1) = \sigma_1.$$

Hence, by Proposition (2.1.28), T_1 is a hypercyclic operator on H.
Thus from Theorem (2.2.26) $\sigma(T_1)\bigcap\partial\mathbb{D}\neq\emptyset$; that is $\sigma_1\bigcap\partial\mathbb{D}\neq\emptyset$.
Therefore, every component of $\sigma(T)$ has non-empty intersection with $\partial\mathbb{D}$.
$\square$

Chapter 3

G-Cyclicity

This chapter is the main one in this thesis, where we introduce the concept of G-cyclic operator. This chapter consists of two section.
In section 3.1 we define the G-cyclic operator and introduce another equivalent forms.
We also, give the necessary and sufficient conditions for G-cyclicity of operators.
In section 3.2 we study some properties of the spectral theory of G-cyclic operators and give a simpler condition for characterizing the G-cyclic operators over a bounded semigroup .

3.1 G- Cyclic operators and Their properties

In this section we give the necessary and sufficient conditions for a bounded linear operator to be G-cyclic, and we give some properties of G-cyclic operators. We show that a G-cyclicity stands in the midway between hypercyclicity and supercyclicity. Finally we characterize the set of all G-cyclic vectors.

Definition 3.1.1. [19] *Let S be a multiplication semigroup of $\mathbb{C}$ with identity, and let H be a separable complex Hilbert space.*
Then an operator $T \in B(H)$ is said to be G-cyclic over S if there exists, $x \in H$ such that $\{\alpha T^n x : \alpha \in S, n \geq 0\}$ is dense in H.
In this case x is said to be a G-cyclic vector for T over S.

Notation 2-1-2[19] Let S be A multiplication semigroup of $\mathbb{C}$ with identity 1 and let $T \in B(H)$. Then through of this thesis we use the

following notations
1- $\mathbf{GC_s(T)} = \{x \in H$: x is a G-cyclic vector for T over S$\}$.
2- $\mathbf{GC_s(H)} = \{T \in B(H)$: T is a G-cyclic operator over S$\}$.
3-The orbt of x for T over S is denoted by,
Sorbt(T,x)$=\{\alpha T^n x : \alpha \in S, n \geq 0\}$.
4-$| S |= \{| s |: s \in S\}$.
5- $S^{-1} = \{s^{-1} : s \in S\}$.

Proposition 3.1.2. [19] *(a) Every hypercyclic operator is G -cyclic.*
(b) Every G-cyclic operator over a semigroup S, is supercyclic.

Proof. (a) Let T be a hypercyclic operator, and let S be any semigroup of $\mathbb{C}$ with 1. Then there exists $x \in H$ such that,

$$\{T^n x : n \geq 0\} \;\; \textit{is dense in } H.$$

However, $\{T^n x : n \geq 0\} \subseteq \{\alpha T^n x : \alpha \in S, n \geq 0\}$, hence $\{\alpha T^n x : \alpha \in S, n \geq 0\}$ *is dense in* H. Therefore, T is G-cyclic over S .
(b) Let T be a G-cyclic operator over a semigroup S. Then, there exists $x \in H$ such that

$$\{\alpha T^n x : \alpha \in S, n \geq 0\} \qquad \textit{is dense in } H \;.$$

But,

$$\{\alpha T^n x : \alpha \in S, n \geq 0\} \subseteq \{\beta T^n x : \beta \in \mathbb{C}, n \geq 0\}$$

Hence,

$$\{\beta T^n x : \beta \in \mathbb{C}, n \geq 0\} \quad \textit{is dense in } H.$$

Therefore, T is a supercyclic operator. □

Proposition 3.1.3. [19] *Let $T \in B(H)$. Then $x \in GC_s(T)$ if and only if $Sorbt(T,x)^{\perp} = \{0\}$.*

Proof. Let $x \in GC_s(T)$, then

$$Sorbt(T,x) = \{\alpha T^n x : \alpha \in S, n \geq 0\} \qquad \textit{is dense in } H.$$

So by Definition (1.2.5) Sorbt(T,x) is a total set.
Hence by Theorem (1.3.5) Sorbt $(T,x)^{\perp} = \{0\}$.
Conversely, Suppose that Sorbt$(T,x)^{\perp} = \{0\}$. Let M=$\overline{Sorbt(T,x)}$, then $M^{\perp} = \overline{Sorbt(T,x)}^{\perp} \subseteq Sorb(T,x)^{\perp} = \{0\}$ and so $M^{\perp} = \{0\}$.
Let $y \in H$, hence by Orthogonal decomposition Theorem,(1.3.6)

$y = m + m^{\perp}$, where $m \in M, m^{\perp} \in M^{\perp}$ *then* $m^{\perp} = 0$.
That is $y = m \in M$. Hence, $H \subseteq M$, but $M \subseteq H$. Therefore, $H = M$
That is Sorbt(T,x) is dense in H. Therefore, $T \in GC_s(H)$.

□

Proposition 3.1.4. [19] *If $T \in GC_s(H)$, then the range of T is dense in H.*

Proof. Let $T \in GC_s(H)$, then there exists $x \in H$ such that,

$$M = \{\alpha T^n x : \alpha \in S, n \geq 0\} \quad \textit{is dense in } H\ .$$

Let $R(T) = \{z : Ty = z, y \in H\}$.
Now, let $z \in M - \{\alpha x : \alpha \in S\}$, then, there exists $\gamma \in S$ and $k > 0$ such that,
z = $\gamma T^k x = T(\gamma T^{k-1} x)$ which is in R(T)
Then, $M - \{\alpha x : \alpha \in S\} \subseteq R(T)$.................. (1)
But M is dense in H, and $\alpha x \in H$ for all $\alpha \in S$, then for all $\alpha \in S$,
there exists a sequences $n_k \in N, \alpha_k \in S$ such that,

$$\alpha_k T^{n_k} x \longrightarrow \alpha x$$

where, $\alpha_k T^{n_k} x \in M$.
However, $\alpha_k T^{n_k} x \in R(T)$, therefore,

$$\alpha x \in \overline{R(T)} \qquad\qquad (2).$$

From (1)(2) we get $M \subseteq \overline{R(T)}$, but $\overline{M} = H$.
Thus, $\overline{M} = \overline{R(T)} = H$. Therefore, R(T) is dense in H. □

Proposition 3.1.5. [19] *If $x \in GC_s(T)$, then*
$\inf\{\gamma \parallel T^n x \parallel : n \geq 0, \gamma \in \mid S \mid\} = 0,$ *and*
$\sup\{\gamma \parallel T^n x \parallel : n \geq 0, \gamma \in \mid S \mid\} = \infty.$

Proof. Let $x \in GC_s(T)$.
First, we show that $\inf\{\gamma \parallel T^n x \parallel : n \geq 0, \gamma \in \mid S \mid\} = 0$
Assume not, i.e, $\inf\{\gamma \parallel T^n x \parallel : n \geq 0, \gamma \in \mid S \mid\} = m > 0$
Then $\gamma \parallel T^n x \parallel \geq m$ for all n, and $\gamma \in \mid S \mid$...............(1).
Since $x \in GC_s(T)$, then Sorb (T,x) is dense in H.
Now $0 \in H$, then there are sequences $\{\alpha_k\}$ in S and $\{n_k\}$ in N such that $\alpha_k T^{n_k} x \to 0$ as $k \longrightarrow \infty$.
Then, for $\epsilon = m$ there exists $K_0 \in N$ such that

$$\parallel \alpha_k T^{n_k} x \parallel < m \qquad for \;\; all \;\; k > K_0,$$

Then

$$| \alpha_k ||| T^{n_k}x \|< m, \quad and \quad | \alpha_k |\in| S |$$

This is a contradiction with (1).
Therefore,

$$\inf\{\gamma \parallel T^n x \parallel: n \geq 0, \gamma \in| S |\} = 0.$$

To show that $\sup\{\gamma \parallel T^n x \parallel: n \geq 0, \gamma \in| S |\} = \infty$, assume not
'i.e $\sup\{\gamma \parallel T^n x \parallel: n \geq 0, \gamma \in| S |\} = t < \infty$
Let $y \in H$ such that $\parallel y \parallel> t$................(2)
Since $x \in GC_s(T)$ then, the set

$$\mathbf{K} = \{\gamma T^n x : \gamma \in S, n \geq 0 \} \quad is \;\; dense \;\; in \; H.$$

However, $y \in H$, then there exist sequences $\{\alpha_k\}$ in S and $\{n_k\}$ in N such that

$$\alpha_k T^{n_k} x \longrightarrow y \qquad as \quad k \longrightarrow \infty.$$

Hence,

$$| \alpha_k ||| T^{n_k} x \| \rightarrow \| y \| \;.$$

However, $\sup\{\gamma \parallel T^n x \parallel: n \geq 0, \gamma \in| S |\} = t$, then
$| \alpha_k ||| T^{n_k} x \|\leq t \; for \; all \; k \in N$. Hence $\parallel y \parallel\leq t$.........(3)
From (2)(3) we get a contradiction.
Therefore, $\sup\{\gamma \parallel T^n x \parallel: n \geq 0, \gamma \in| S |\} = \infty$. □

Corollary 3.1.6. [19] *If S is a bounded semigroup, and $x \in GC_s(T)$, then*

$$\sup\{\parallel T^n x \parallel: n \geq 0\} = \infty.$$

Proof. Since S is bounded, then $| S |\leq m$ for some $m \in R^+$
Then by proposition (3.1.5)
$\infty = \sup\{\alpha \parallel T^n x \parallel: \alpha \in| S |, n \geq 0\} \leq$

$$\sup\{m \parallel T^n x \parallel: n \geq 0\} = m \sup\{\parallel T^n x \parallel: n \geq 0\}.$$

Hence $\quad \sup\{\parallel T^n x \parallel: n \geq 0\} = \infty.$ □

Corollary 3.1.7. [19] *Let $T \in B(H)$, and S be a bounded semigroup. If $\parallel T \parallel\leq 1$ then, $T \notin GC_s(H)$*

Proof. Assume not i.e $T \in GC_s(H)$.
Since S is bounded then by Corollary (3.1.6) there exists
$x \in GC_s(T)$ such that

$$\sup\{\| T^n x \| : n \geq 0\} = \infty.$$

That is for all $m \in \mathbb{R}^+$ there exists $k \in N$ such that $\| T^k x \| > m$.
Since $\| T \| < 1$ then,

$$m < \| T^k x \| \leq \| T^k \| \| x \| \leq \| T \|^k \| x \| \leq \| x \| .$$

That is $\| x \| > m$ for all $m \in \mathbb{R}^+$. Thus $\| x \| = \infty$.
A contradiction with $x \in GC_s(T)$. Therefore, $T \notin GC_s(H)$. □

Example 3.1.8. [19] *Let $T : l^2(N) \longrightarrow l^2(N)$, where $T(a_1, a_2, ...) = (a_2, a_3, ...)$. Then T cannot be a G-cyclic operator over any bounded semigroup.*

Proof. By Corollary (3.1.7) it is enough to prove that, $\| T \| \leq 1$.
Now let $x = (a_1, a_2, ...) \in l^2(N)$, then

$$\| Tx \|^2 = \Sigma_{k=2}^{\infty} a_k^2 \leq \Sigma_{k=1}^{\infty} a_k^2 = \| x \|^2$$

Hence $\| Tx \| \leq \| x \|$ thus

$$\| T \| = \sup_{x \in l^2(N),\ x \neq 0} \frac{\| Tx \|}{\| x \|} \leq 1.$$

Therefore, $T \notin GC_s(H)$. □

Proposition 3.1.9. [19] *Let $T, F \in B(H)$, such that FT=TF and the range of F, R(F) is dense in H. If $x \in GC_s(T)$, then $Fx \in GC_s(T)$.*

Proof. First we need to show if FT=TF then, $FT^n = T^n F \quad for\ all\ n \in N$.
By induction. The statement is clear true for n=1.
Suppose that it is true for n=m, hence $FT^m = T^m F$.
Then, $FT^{m+1} = FT^m T = T^m FT = T^m TF = T^{m+1} F$.
Therefore, $FT^n = T^n F \quad for\ all\ n \in N$.
Now since F is bounded, then F is continuous, hence by Theorem(1.1.9)

$$F\overline{\{\alpha T^n x, \alpha \in S, n \geq 0\}} \subset \overline{F\{\alpha T^n x, \alpha \in S, n \geq 0\}}.$$

But the range of F, R(F) is dense in H and $x \in GC_s(T)$, then

$$R(F) = F(H) = F\overline{\{\alpha T^n x, \alpha \in S, n \geq 0\}} \subset \overline{F\{\alpha T^n x, \alpha \in S, n \geq 0\}}.......(1)$$

Hence $\overline{Sorbt(T, Fx)} = \overline{\{\alpha T^n Fx : \alpha \in S, n \geq 0\}} = \overline{\{\alpha F T^n x : \alpha \in S, n \geq 0\}}$
$= \overline{F\{\alpha T^n x, \alpha \in S, n \geq 0\}}$.............(2)
From (1)(2) $R(F) \subset \overline{Sorbt(T, Fx)}$. Hence $H = \overline{R(F)} \subset \overline{Sorbt(T, Fx)}$.
But $\overline{Sorbt(T, Fx)} \subset H$. Therefore, $\overline{Sorbt(T, Fx)} = H$.
Therefore, $F(x) \in GC_s(T)$. □

Corollary 3.1.10. [19] *Let $T \in B(H)$ and $x \in GC_s(T)$.
Then $\alpha T^k x \in GC_s(T)$ for all $\alpha \in S$, $k \geq 0$.*

Proof. We need to show by induction on k, that $R(\alpha T^k)$ is dense in H for all $\alpha \in S$, $\alpha \neq 0$, $k \in N$.
Now let α be any fixed element in S.
Hence, $\overline{\{\alpha T y : y \in H\}} = \overline{\{T \alpha y : y \in H\}} = \overline{T\alpha(H)} = \overline{TH}$......(1).
Since $x \in GC_s(T)$, then by Proposition (3.1.4) the range of T, R(T) is dense in H. Now by (1) $H = \overline{R(T)} = \overline{T(H)} = \overline{\{\alpha T y : y \in H\}} = \overline{R(\alpha T)}$.
That is, $R(\alpha T)$ is dense in H.
Assume that $R(\alpha T^k)$ is dense in H, we show that $R(\alpha T^{k+1})$ is dense in H.
Hence, by Theorem (1.1.9)

$$\overline{R(\alpha T^{k+1})} = \overline{\{\alpha T^{k+1} y : y \in H\}} = \overline{T\{\alpha T^k y : y \in H\}} \supset T\overline{\{\alpha T^k y : y \in H\}} = T(H).$$

Since R(T) is dense,then $H = \overline{T(H)} \subset \overline{R(\alpha T^{k+1})}$
That is, $R(\alpha T^{k+1})$, is dense in H for all $\alpha \in S, \alpha \neq 0, k \in N$.
Hence the induction is complete.
Now, let $F = \alpha T^k$, then $FT = \alpha T^k T = T \alpha T^k = TF$, and R(F) is dense in H.
Since $x \in GC_s(T)$. Then by Proposition (3.1.9), $Fx \in GC_s(T)$.
Therefore, $\alpha T^k x \in GC_s(T)$ for all $\alpha \in S, k \geq 0$. □

Proposition 3.1.11. [19] *Let H and K be Hilbert spaces, let $T \in B(H)$ and $F \in B(K)$, and let $X : H \longrightarrow K$ be abounded linear operator such that R(X) is dense in K and $FX = XT$. If $T \in GC_s(H)$ then, $F \in GC_s(K)$.
In particular, if $T, F \in B(H)$ are similar operators, then $T \in GC_s(H)$, if and only if $F \in GC_s(H)$.*

Proof. First we need to show by induction that , $F^n X = XT^n$ for all $n \in N$
Since $FX = XT$ then the statement is true for n=1.
Assume that $F^m X = XT^m$. Then

$$F^{m+1} X = F(F^m X) = F(XT^m) = FX(T^m) = XT(T^m) = XT^{m+1}.$$

Therefore, the induction is complete.
Now since $T \in GC_s(H)$, then there is $y \in H$ such that Sorbt(T,y) is dense in H. Hence, $\overline{Sorbt(F, Xy)} = \overline{\{\alpha F^n Xy : n \geq 0, \alpha \in S\}}$ $=\overline{\{X\alpha T^n y : n \geq 0, \alpha \in S\}} = \overline{X\{\alpha T^n y : n \geq 0, \alpha \in S\}}$........(1).
Since the range of X, R(X) is dense in K, then

$$R(X) = X(H) = X\overline{\{\alpha T^n y, \alpha \in S, n \geq 0\}}.$$

But X is bounded, then X is continuous, hence by Theorem(1.1.9)

$$X\overline{\{\alpha T^n y, \alpha \in S, n \geq 0\}} \subset \overline{X\{\alpha T^n y, \alpha \in S, n \geq 0\}}.$$

That is $R(X) \subset \overline{X\{\alpha T^n y, \alpha \in S, n \geq 0\}}$...........(2).
From (1)(2) $R(X) \subset \overline{Sorbt(F, Xy)}$. Hence $K = \overline{R(X)} \subset \overline{Sorbt(F, Xy)}$.
But $\overline{Sorbt(F, Xy)} \subset K$. Therefore, $\overline{Sorbt(F, Xy)} = K$.
Therefore, $F \in GC_s(K)$.
So if $T, F \in B(H)$ are similar operators, then there is an invertible U such that TU=UF, and U is one to one and onto, hence U(H)=H.
Thus R(U) is dense in H.
Hence, If $T \in GC_s(H)$, then $F \in GC_s(H)$. □

Proposition 3.1.12. [19] *Let $\{H_i\}_{i \in N}$ be a family of Hilbert spaces. Let $T_i \in B(H_i)$ for all i. If $\bigoplus T_i \in GC_s(\bigoplus H_i)$, then $T_i \in GC_s(H_i)$ for all $i \in N$.*

Proof. First, note that, by Theorem (1.3.7), $\bigoplus H_i$ is a Hilbert space. To prove That $T_i \in GC_s(H_i)$ for all $i \in N$, we need to show that for all $i \in N$, there is $x_i \in GC_s(T_i)$.
Now let $\bigoplus T_i \in GC_s(\bigoplus H_i)$, then there is $x = (x_i)_{i \in N} \in GC_s(\bigoplus T_i)$, such that the set

$$M = \{\alpha(\bigoplus T_i)^n x : n \geq 0, \alpha \in S\} \textit{ is dense in } \bigoplus H_i.$$

For each $i \in N$,let $y_i \in H_i$. Then $y = (y_i) \in \bigoplus H_i$, and so, there exists a sequence $\{\alpha_k(\bigoplus T_i)^{n_k} x\}_{k \in N}$ in M such that

$$\alpha_k(\bigoplus T_i)^{n_k} x \longrightarrow y.$$

Then, for all $\epsilon > 0$ there exists $N_0 \in N$ such that,

$$\| \alpha_k(\bigoplus T_i)^{n_k} x - y \| < \epsilon \qquad \textit{for all} \;\; k > N_0.$$

Then,

$$\parallel \alpha_k \bigoplus T_i^{n_k} x_i - y_i \parallel^2 < \epsilon^2 \qquad for\ all \ \ k > N_0.$$

Then by Theorem (1.3.7) we have,

$$\Sigma_{i=1}^{\infty} \parallel \alpha_k T_i^{n_k} x_i - y_i \parallel^2 < \epsilon^2 \quad for\ all\ k > N_0.$$

Hence,

$$\parallel \alpha_k T_i^{n_k} x_i - y_i \parallel < \epsilon \quad for\ all\ k > N_0.$$

This means that, each $y_i \in H_i$, there is a sequence $(\alpha_k T_i^{n_k} x_i)_{k\in N}$ in

$$M_i = \{\alpha_k T_i^{n_k} x_i : n_k > 0, \alpha_k \in S\},\ such\ that$$

$$\alpha_k T_i^{n_k} x_i \longrightarrow y_i, \qquad as \quad k \longrightarrow \infty.$$

Then, $y_i \in \overline{M_i}$, hence, $H_i \subset \overline{M_i}$.But $\overline{M_i} \subset H_i$, then $\overline{M_i} = H_i$.
Therefore, $x_i \in GC_s(T_i)$, that is, $T_i \in GC_s(H_i)$ for all $i \in N$. □

Proposition 3.1.13. [19] *Let $T \in B(H)$. Then*

$$GC_s(T) = \bigcap_k (\bigcup_{\alpha\in S} \bigcup_n T^{-n}(\frac{1}{\alpha} U_k)) \ ,$$

where $\{U_k\}_{k=1}^{\infty}$ is a countable base for the topology on H.

Proof. Since H is separable, then we can assume that there is $\{U_k\}_{k=1}^{\infty}$ be a countable base for the topology on H.
$x \in GC_s(T)$, iff $sorb(T, x) = \{\alpha T^n x : n \geq 0, \alpha \in S\}$ is dense in H.
Iff for all $k \geq 1$, there is $\alpha \in S,\ n \in N$ such that,
$\alpha T^n x \in U_k$, [by Lemma (1.1.5)]
Iff for all, $k \geq 1$, there is $\alpha \in S,\ n \in N$ such that, $T^n x \in \frac{1}{\alpha} U_k$.
Iff for all $k \geq 1$, there is $\alpha \in S,\ n \in N$ such that, $x \in T^{-n}(\frac{1}{\alpha} U_k)$,
Iff $x \in (\bigcup_{\alpha\in S} \bigcup_n T^{-n}(\frac{1}{\alpha} U_k))$, for all $k \geq 1$,
Iff $x \in \bigcap_k (\bigcup_{\alpha\in S} \bigcup_n T^{-n}(\frac{1}{\alpha} U_k))$.
Therefore, $GC_s(T) = \bigcap_k (\bigcup_{\alpha\in S} \bigcup_n T^{-n}(\frac{1}{\alpha} U_k))$. □

Corollary 3.1.14. [19] *Let $T \in B(H)$. Then, if the set of G-cyclic vectors over a semigroup S is not empty, then it is a G_δ -set in H.*

Proof. By Proposition (3.1.13) if $GC_s(T) \neq \emptyset$, then

$$GC_s(T) = \bigcap_k (\bigcup_{\alpha\in S} \bigcup_n T^{-n}(\frac{1}{\alpha} U_k)).$$

Where $\{U_k\}_{k=1}^{\infty}$ is a countable base for the topology on H.
And since T is continuous, hence T^k is continuous, and
$\frac{1}{\alpha}U_k$ is open for all k in N and all $\alpha \in S$, then

$$(\bigcup_{\alpha \in S} \bigcup_{n} T^{-n}(\frac{1}{\alpha}U_k)) \;\; is \;\; open.$$

Hence, $GC_s(T)$ is a countable intersection of open sets.
Therefore, by Definition (1.1.7) $GC_s(T)$ is a G_δ -set □

Theorem 3.1.15. [19] *Let $T \in B(H)$. then the following statements are equivalent:*
1- $T \in GC_s(H)$.
2- For each non-empty open sets U, V, there are $\alpha \in S, n \in N$ such that $T^n(\alpha U) \bigcap V \neq \phi$.
3- For each $x, y \in H$,there are a sequences$\{x_k\}_{k=1}^{\infty}$ in H ,$\{n_k\}_{k=1}^{\infty}$ in N ,$\{\alpha_k\}$in S such that $x_k \longrightarrow x$ and $T^{n_k}\alpha_k x_k \longrightarrow y$.
4- For each $x, y \in H$,and each neighborhood W for zero in H ,there are $z \in H, n \in N, \alpha \in S$ such that $x - z \in W$ and $T^n\alpha z - y \in W$.

Proof. $1 \Longrightarrow 2$
Let $T \in GC_s(H)$ and let $\{U_k\}$ be a countable base for the topology on H .
Hence by Proposition (3.1.13)

$$GC_s(T) = \bigcap_k (\bigcup_{\alpha \in S} \bigcup_{n} T^{-n}(\frac{1}{\alpha}U_k)).$$

Let

$$M_k = (\bigcup_{\alpha \in S} \bigcup_{n} T^{-n}(\frac{1}{\alpha}U_k)), \quad k \geq 1.$$

Then $GC_s(T) = \bigcap_k M_k$. Hence $GC_s(T) \subseteq M_k$ for all $k \geq 1$
But by corollary (3.1.10), $\alpha T^n x \in GC_s(T)$ for all $n \geq 0,\ \alpha \in S$.
Hence, $Sorbt(T, x) \subseteq GC_s(T) \subseteq M_k$.
Then M_k is dense in H.
Let U be open set in H, then by Lemma (1.1.5) for all $k \geq 1$, $U \bigcap M_k \neq \phi$.
That is

$$U \bigcap (\bigcup_{\alpha \in S} \bigcup_{n} T^{-n}(\frac{1}{\alpha}U_k)) \neq \emptyset \qquad for\ all \;\; k \in N.$$

Let V be any open set in H, since $\{U_k\}_{k=1}^{\infty}$ is a base for the topology on H. Hence V can be written as a union of elements from the base $\{U_k\}_{k=1}^{\infty}$.That is $V = \bigcup_i U_i, \;\; U_i \in \{U_k\}_{k=1}^{\infty}$.
Hence, there are $n \in N, \; \alpha \in S$ such that $U \bigcap T^{-n}\{\frac{1}{\alpha}V\} \neq \emptyset$
Therefore, $T^n(\alpha U) \bigcap V \neq \phi$.

$2 \Longrightarrow 3$**:** Let $x, y \in H$. For all $k \geq 1$, let $B_k = \mathbb{B}(x, \frac{1}{k})$, $B_k^* = \mathbb{B}(y, \frac{1}{k})$. Since $\alpha_k B_k \quad and \quad B_k^*$ are open sets in H for all $k \geq 1$, then by (2), there exists $n = n_k \in N$ such that

$$T^{n_k}(\alpha_k B_k) \bigcap B_k^* \neq \emptyset \qquad\qquad for\ all\ k \geq 1.$$

Let $y_k \in T^n(\alpha_k B_k) \bigcap B_k^* \qquad for\ all\ k \geq 1$
Hence, we get a sequence $\{y_k\}_{k=1}^{\infty}$ such that $y_k = \alpha_k T^{n_k} x_k$ for some $x_k \in B_k$. Thus $\{n_k\}$ in N, $\{\alpha_k\}$ in S, $x_k \in B_k$ and $T^{n_k}(\alpha_k x_k) \in B_k^*$ for all $k \geq 1$.
That is, $\parallel x_k - x \parallel < \frac{1}{k}$, and $\parallel T^{n_k}(\alpha_k x_k) - y \parallel < \frac{1}{k}$ for all $k \geq 1$.
So, $x_k \longrightarrow x$ and $T^{n_k}\alpha_k x_k \longrightarrow y$, as $k \longrightarrow \infty$.
$3 \Longrightarrow 4$ Let $x, y \in H$. Let W be a neighborhood for zero in H.
By (3) there are a sequences $\{x_k\}_{k=1}^{\infty}$ in H, $\{n_k\}_{k=1}^{\infty}$ in N, $\{\alpha_k\}_{k=1}^{\infty}$ in S such that, $x_k \longrightarrow x$ and $T^{n_k}\alpha_k x_k \longrightarrow y$.
Hence there is $k \in N$ such that, $x_k - x \in W$ and $T^{n_k}\alpha_k x_k - y \in W$.
Take $z = x_k$ then $x - z \in W$ and $T^n \alpha z - y \in W$.

$4 \Longrightarrow 3$ Let $x, y \in H$.
For all $k \geq 1$, let $B_k = \mathbb{B}(0, \frac{1}{k})$.
By (4) we get sequences z_k in H, n_k in N, α_k in S, such that
$z_k - x \in B_k$ and $T^{n_k}\alpha_k z_k - y \in B_k$ for all $k \geq 1$.
Therefore, $\parallel z_k - x \parallel < \frac{1}{k}$ and $\parallel T^{n_k}\alpha_k z_k - y \parallel < \frac{1}{k}$ for all $k \geq 1$.
Let $k \longrightarrow \infty$ then,
$z_k \longrightarrow x$ and $T^{n_k}\alpha_k z_k \longrightarrow y$.

$3 \Longrightarrow 1$ Since H is separable, then there is a countable set, $\{x_j\}_{j \in N}$ is dense in H. Let $F(j, k) = B(x_j, \frac{1}{k})$ for all $j \in N, \; k \geq 1$.
Claim.
The collection $F(j, k)$ is a base topology of H.
Proof the Claim:
Let W be nonempty open set in H.
Since $\{x_j\}_{j \in N}$ is dense in H, then by Lemma (1.1.5) for all $w \in W$, there

exists $j, m \in N$ such that,

$$w \in B(x_j, \frac{1}{m}) = F(j, m).$$

Hence $w \in \bigcup F(j, k)$ $k \geq 1$.
That is

$$W \subseteq \bigcup F(j, k).$$

But W was arbitrary, therefore, by definition (1.1.2),

$$\{F(j, k) : j, k \in N\} \quad \textit{is a base topology of } H.$$

Therefore, the proof of claim is complete.
Now, let

$$m_{jk} = \bigcup_{\alpha \in S} \bigcup_{n} T^{-n}(\frac{1}{\alpha} B(x_j, \frac{1}{k})).$$

Then we need to show m_{jk} is dense in H for all $k \geq 1$, $j \in N$.
Let $y \in H$, then by (3) for any $j \in N$,
there are sequences z_k in H, $\{\alpha_k\} \in S$, $\{n_k\} \in N$ such that,

$$z_k \longrightarrow y \qquad \textit{and} \qquad T^{n_k} \alpha_k z_k \longrightarrow x_j.$$

Thus there is $m_0 \in N$ such that,
$\| T^{n_k} \alpha_k z_k - x_j \| < \frac{1}{k}$ for all $k > m_0$. Hence, for all $k > m_0$,there is $\alpha_k \in S, n_k \in N$ such that

$$T^{n_k} \alpha_k z_k \in B(x_j, \frac{1}{k})$$

That is $\quad z_k \in T^{-n}(\frac{1}{\alpha_k} B(x_j, \frac{1}{k}))$. Hence

$$z_k \in \bigcup_{\alpha \in S} \bigcup_{n} T^{-n}(\frac{1}{\alpha_k} B(x_j, \frac{1}{k})) = m_{jk} \qquad \textit{for all } k \in N.$$

But $\quad z_k \longrightarrow y$. Hence for all $y \in H$, there is a sequence $\{z_k\}$ in m_{jk} such that $z_k \longrightarrow y$. That is $H \subseteq \overline{m_{jk}}$, $\quad$ *but* $\quad \overline{m_{jk}} \subseteq H$.
Therefore, $m_{jk} = H$. Hence m_{jk} is dense in H.
Then by Baires Theorem (1.1.6),

$$\bigcap_j \bigcap_k mjk = \bigcap_j \bigcap_k [\bigcup_{\alpha \in S} \bigcup_n T^{-n}(\frac{1}{\alpha} B(x_j, \frac{1}{k}))] \quad \textit{is dense in } H.$$

Hence by proposition (3.1.13),
$GC_s(T) = \bigcap_j \bigcap_k [\bigcup_{\alpha \in S} \bigcup_n T^{-n}(\frac{1}{\alpha} B(x_j, \frac{1}{k}))] \neq \emptyset$. Therefore, $T \in GC_s(H)$. $\square$

Proposition 3.1.16. [19] *Let $T \in B(H)$ be an invertible operator. $T \in GC_s(H)$ if and only if $T^{-1} \in GC_{s^{-1}}(H)$.*

Proof. Let $x, y \in H$ and $T \in GC_s(H)$. Then by Theorem (3.1.15) part (4), for all neighborhood V of zero in H, there are $z \in H,\ n \in N,\ a \in S$ such that,

$$z - x \in V \text{ and } \alpha T^n z - y \in V.$$

Let $u = aT^n z$ then $u - y \in V$ and $\frac{1}{\alpha}T^{-n}u - x \in V$.
Since $\frac{1}{\alpha} \in S^{-1}$ and $T^{-n} = (T^{-1})^n$, then for all $x, y \in H$ and all neighborhood V of zero in H, there are $u \in H,\ \ n \in N, \frac{1}{\alpha} \in S^{-1}$ such that
$u - y \in V$ and $\frac{1}{\alpha}(T^{-1})^n u - x \in V$.
Hence from Theorem (3.1.15) part (4), $T^{-1} \in GC_{s^{-1}}(H)$.
Conversely sine $(S^{-1})^{-1} = S$ and $(T^{-1})^{-1} = T$, then if $T^{-1} \in GC_{s^{-1}}(H)$, then $T = (T^{-1})^{-1} \in GC_{(S^{-1})^{-1}}(H) = GC_s(H)$. □

Proposition 3.1.17. [19] *The operator $T \in GC_s(H)$ if and only if the set $\{(x, \alpha T^n x) : x \in H, n \geq 0, \alpha \in S\}$ is dense in $H \oplus H$.*

Proof. Let $(y, z) \in H \oplus H$, and let $\epsilon > 0$.
Since $T \in GC_s(H)$, then by Theorem (3.1.15) part (4),for all V neighborhood of zero in H, there are $w \in H, n \geq 0, \alpha \in S$ such that,
$\parallel w - y \parallel < \frac{\epsilon}{2}$ and $\parallel \alpha T^n w - z \parallel < \frac{\epsilon}{2}$.
Hence $\parallel (w, \alpha T^n w) - (y, z) \parallel^2 = \parallel w - y \parallel^2 + \parallel \alpha T^n w - z \parallel^2 < \frac{\epsilon^2}{2}$.
That is for all $(y, z) \in H \bigoplus H$ there is $w \in H$ such that
$\parallel (w, \alpha T^n w) - (y, z) \parallel < \frac{\epsilon}{\sqrt{2}}$.
Therefore, $\{(w, \alpha T^n w) : w \in H, n \geq 0, \alpha \in S\}$ is dense in $H \bigoplus H$.

Conversly, Let $z, y \in H$ and let $\epsilon > 0$.
Since $\{(w, \alpha T^n w) : w \in H, n \geq 0, \alpha \in S\}$ is dense in $H \bigoplus H$ for all $w \in H$.
Then, there is $k_0 \in N$ and sequences
$\{w_k\}$ *in* $H, \{\alpha_k\}$ *in* S , *and* $\{n_k\}$ *in* N, such that,

$$\parallel (w_k, \alpha_k T^n w_k) - (y, z) \parallel^2 < \epsilon^2 \qquad for\ all\ k > k_0.$$

Then, $\parallel y - w_k \parallel < \epsilon$ *and* $\parallel z - \alpha_k T^{n_k} w_k \parallel < \epsilon$ $\qquad for\ \ all\ \ k > k_0$.
Then, as $k \longrightarrow \infty$ we get $w_k \longrightarrow y$ and $\alpha_k T^{n_k} w_k \longrightarrow z$.
Therefore, by Theorem (3.1.15) part (4), $T \in GC_s(H)$ □

Proposition 3.1.18. [19] *Let $T \in B(H)$, let U and V be nonempty open sets in H, and let W be a neighborhood for zero in H.*
If there are $n \geq 0, \alpha \in S$ such that, $T^n \alpha U \cap W \neq \emptyset$ and $T^n \alpha W \cap V \neq \emptyset$, then $T \in GC_s(H)$.

Proof. Let $x, y \in H$.For all $k \geq 1$, let $B_k = \mathbb{B}(x, \frac{1}{k})$, $\quad B_k^* = \mathbb{B}(y, \frac{1}{k})$.
Hence $B_k \quad and \quad B_k^*$ are open sets in H. Then there exists a sequences $\{n_k\} \in N, \{\alpha_k\} \in S$ such that,

$$T^{n_k}\alpha_k B_k \cap W \neq \emptyset \ \ and \ \ T^{n_k}\alpha_k W \cap B^* \neq \emptyset \ \ for \ all \ k \geq 1 \ .$$

That is, there are sequences $\{w_k\}$ in W, $and \quad \{z_k\} \quad in \quad B_k$.
So we get $T^{n_k}\alpha_k z_k \in W \ \ and \ \ T^{n_k}\alpha_k w_k \in B_k^* \quad for \ all \ k \geq 1$.
Hence, $z_k \longrightarrow x \ \ and \ \ w_k \longrightarrow 0$. Also

$$T^{n_k}\alpha_k z_k \longrightarrow 0 \quad and \ \ T^{n_k}\alpha_k w_k \longrightarrow y.$$

Let $x_k = z_k + w_k \ \ for \ all \ k \geq 1$, then we get $x_k \longrightarrow x$. and

$$T^{n_k}\alpha_k x_k = T^{n_k}\alpha_k z_k + T^{n_k}\alpha_k w_k \longrightarrow 0 + y = y \quad as \ k \longrightarrow \infty.$$

Hence by Theorem (3.1.15) part (3), $T \in GC_s(H)$. □

3.2 Spectral Properties of G-cyclic Operators

In this section we discuss the properties of the spectrum of G-cyclic operators. We give a sufficient condition for a spectral of operator to be G-cyclic over a bounded semigroup. We show that quasinilpotent and compact operators are not G-cyclic over any bounded semigroup.

Proposition 3.2.1. [19] *Let $T \in GC_s(H)$ .Then T^* has at most one eigenvalue with modulus :*
1) Greater than one ,if S is bounded above.
2) Less than one , if S is bounded below.

Proof. Let $T \in GC_s(H)$, then by Proposition (3.1.2)part(b) T is supercyclic ,thus by Theorem (2.1.13), T^* has at most one eigenvalue.
Hence $\sigma_p(T^*) = \emptyset$ or $\sigma_p(T^*) = \{\beta\}$ for some one $\beta \in \mathbb{C}$.
Hence, there is $z \in H$ such that

$$T^* z = \beta z.$$

Let S be bounded above, then we need to show $\mid \beta \mid > 1$.
Let $x \in GC_s(T)$, then by Lemma(2.2.12),

$$M = \{\langle \alpha T^n x, z \rangle : n \geq 0, \alpha \in S\} \quad is \ dense \ in \ \mathbb{C}.$$

Let $r \in \mathbb{R}^+ \bigcup \{0\}$, then $r \in \mathbb{C}$, hence there exists a sequences $\langle \alpha_k T^{n_k} x, z \rangle$ in M, such that $\langle \alpha_k T^{n_k} x, z \rangle \longrightarrow r$.
But $| \langle \alpha_k T^{n_k} x, z \rangle | \longrightarrow | r | = r$.
Therefore, the set

$$K = \{| \langle \alpha_k T^{n_k} x, z \rangle | : n \geq 0, \alpha \in S\} \textit{ is dense in } \mathbb{R}^+ \bigcup \{0\}.$$

Now, for all $n \geq 0$, $| \langle \alpha T^n x, z \rangle | = | \alpha || \langle T^n x, z \rangle |$.
Since S is bounded above, then $| S | \leq d_1 \textit{ for some } d_1 \in \mathbb{R}^+ \bigcup \{0\}$.
Let$| \beta | \leq 1$, then we shall get a contradiction by proving that K can not be dense in $\mathbb{R}^+ \bigcup \{0\}$.
Hence for all $k \in K$, by Proposition (2.2.4)

$$k = | \langle \alpha T^n x, z \rangle | = | \alpha || \langle T^n x, z \rangle | \leq$$
$$d_1 | \langle T^n x, z \rangle | = d_1 | \langle x, T^{n^*} z \rangle | =$$
$$d_1 \beta^n | \langle x, z \rangle | \leq d_1 | \langle x, z \rangle |.$$

Hence $\quad k \leq d_1 | \langle x, z \rangle | \quad$ for all $k \in K$.
Let $m = 2d_1 | \langle x, z \rangle | \in \mathbb{R}^+ \bigcup \{0\}$, and $r = \frac{1}{2} d_1 | \langle x, z \rangle | \in \mathbb{R}^+ \bigcup \{0\}$.
Then $B(m, r) \bigcap K = \emptyset$.
Then by Lemma (1.1.5) K can not be dense in $\mathbb{R}^+ \bigcup \{0\}$, contradiction .
Therefor, $| \beta | > 1$.
Hence T^* has at most one eigenvalue with modulus greater than one.
2) Let S be bounded below, then we need to show $| \beta | < 1$.
Let $| \beta | \geq 1$, since S is bounded below then ,$| S | \geq d_2 \textit{ for some } d_2 \in \mathbb{R}^+ \bigcup \{0\}$.
So, we shall get a contradiction by proving that K can not be dense in $\mathbb{R}^+ \bigcup \{0\}$, hence for all $k \in K$

$$k = | \langle \alpha T^n x, z \rangle | \geq$$
$$d_2 | \langle T^n x, z \rangle | = d_2 | \langle x, T^{n^*} z \rangle | =$$
$$d_2 \beta^n | \langle x, z \rangle | \geq d_2 | \langle x, z \rangle |$$

thus,

$$k \geq d_2 | \langle x, z \rangle | \text{ for all } k \in K.$$

Let $m_2 = \frac{1}{2} d_2 | \langle x, z \rangle | \in \mathbb{R}^+ \bigcup \{0\}$,and $r = \frac{1}{4} d_2 | \langle x, z \rangle | \in \mathbb{R}^+ \bigcup \{0\}$.
Then $B(m_2, r) \bigcap K = \emptyset$.
Then by Lemma (1.1.5), K can not be dense in $\mathbb{R}^+ \bigcup \{0\}$, contradiction.
Therefore, $| \beta | < 1$.
Hence T^* has at most one eigenvalue of modulus less than one. □

Corollary 3.2.2. [19] *Let $T \in GC_s(T)$. Then $\sigma_w(T)$is the spectrum of T except possibly one element of modulus :*
a) Greater than one ,if S is bounded above .
b) Less than one ,if S is bonded below .

Proof. From Proposition(2.2.21) part (1),

$$\sigma(T) - \sigma_w(T) \subseteq \sigma_{pf}(T) \subseteq \sigma_p(T).$$

Replace T by T^* to get

$$\sigma(T^*) - \sigma_w(T^*) \subseteq \sigma_p(T^*).............(1).$$

(a) Let S be bounded above then by Proposition (3.2.1)

$$\sigma_p(T^*) = \emptyset \qquad or \qquad \sigma_p(T^*) = \{\lambda\},\ for\ some\ \lambda\ such\ that \quad |\lambda| > 1.$$

Then from (1)

$$\sigma_w(T^*) = \sigma(T^*) \qquad or \qquad \sigma_w(T^*) = \sigma(T^*) - \{\lambda\}.$$

Case (1), if $\sigma_w(T^*) = \sigma(T^*)$, then by Theorem (2.2.19) and Proposition (2.2.21) part (2), $\overline{\sigma_w(T^*)} = \sigma_w(T) = \overline{\sigma(T^*)} = \sigma(T)$.
Case (2)If $\sigma_w(T^*) = \sigma(T^*) - \{\lambda\},\ where\ \ |\lambda| > 1$.
Since the conjugate of difference from two sets is the difference of its conjugates .Then by Theorem (2.2.19) and Proposition (2.2.21),

$$\overline{\sigma_w(T^*)} = \sigma_w(T) = \overline{\sigma(T^*) - \{\lambda\}} = \sigma(T) - \overline{\{\lambda\}}.$$

Hence, $\sigma_w(T)$ is the spectrum of T except possibly one element of modulus greater than one.
(b) Let S bounded below then,by Proposition (3.2.1)

$$\sigma_p(T^*) = \emptyset \qquad or \qquad \sigma_p(T^*) = \{\lambda\}\ for\ some\ \lambda\ such\ that \quad |\lambda| < 1.$$

Then from (1)

$$\sigma_w(T^*) = \sigma(T^*) \qquad or \qquad \sigma_w(T^*) = \sigma(T^*) - \{\lambda\}.$$

Case (1), if $\sigma_w(T^*) = \sigma(T^*)$, then by Theorem (2.2.19) and Proposition (2.2.21) part (2), $\overline{\sigma_w(T^*)} = \sigma_w(T) = \overline{\sigma(T^*)} = \sigma(T)$.
Case (2)If $\sigma_w(T^*) = \sigma(T^*) - \{\lambda\}$.
Since the conjugate of difference from two sets is the difference of its conjugates .Then by Theorem (2.2.19) and Proposition (2.2.21),

$$\overline{\sigma_w(T^*)} = \sigma_w(T) = \overline{\sigma(T^*) - \{\lambda\}} = \sigma(T) - \overline{\{\lambda\}}.$$

Hence, $\sigma_w(T)$ is the spectrum of T except possibly one element of modulus less than one. $\square$

Lemma 3.2.3. [21] *Let $T \in B(H)$, then:*
1) If $\sigma(T) \subset \{\lambda \in \mathbb{C} :| \lambda |< 1\}$, then $\lim_{n\longrightarrow\infty} \| T^n x \|= 0$ for all $x \in H$.
2)If $\sigma(T) \subset \{\lambda \in \mathbb{C} :| \lambda |> 1\}$, then $\lim_{n\longrightarrow\infty} \| T^n x \|= \infty$ for all $x \in H$.

Theorem 3.2.4. *Let $T \in B(H)$ and $\sigma(T)$ be the spectrum of an invertible operator T.Then*

$$\sigma(T^{-1}) = \{\sigma(T)\}^{-1} = \{\frac{1}{\lambda} : \lambda \in \sigma(T)\}.$$

Proof. See[14]. □

Proposition 3.2.5. [19] *Let $T \in B(H)$, then $T \notin GC_s(H)$ if one of the following holds :*
a) S is bounded, and $\sigma(T)$ has a component σ such that $\sigma \subset \mathbf{B}(0, 1)$.
b) S^{-1} is bounded and $\sigma(T)$ has a component σ such that $\sigma \subset \{\lambda :| \lambda |> 1\}$.

Proof. (a)Assume $T \in GC_s(H)$................(1).
Case 1. If $\sigma(T)$ is connected, then by Definition (1.1.14) $\sigma(T)$ is the only component of $\sigma(T)$ and if
$\sigma(T) \subset \mathbf{B}(0, 1)$, then by Lemma (3.2.3),
$\lim_{n\longrightarrow\infty} \| T^n x \|\longrightarrow 0$ for all $x \in H$.
Thus

$$sup\{\| T^n x \|: n \geq 0\} \neq \infty.$$

But by Corollary (3.1.6),

$$sup\{\| T^n x \|: n \geq 0\} = \infty.$$

Therefore, we get a contradiction with (1).
Hence, $T \notin GC_s(H)$.
Case 2. If $\sigma(T)$ is not connected, then there exists an open and closed component $\sigma_1 \subset \sigma(T)$. Let $\sigma_1 \subset \mathbf{B}(0, 1)$.
Hence σ_1 is closed and by Theorem(2.2.24) $\sigma(T) - \sigma_1$ is closed.
Then by Risez decomposition Theorem (2.2.25), $T = T_1 + T_2$, such that $\sigma(T_1) = \sigma_1$. But since $T \in GC_s(H)$, then by Proposition (3.1.12), $T_1 \in GC_s(H)$.
And since $\sigma_1 \subset \mathbf{B}(0, 1)$, then by Lemma (3.2.3), $\lim_{n\longrightarrow\infty} \| T_1^n x \|\longrightarrow 0$ for all $x \in H$.
Thus $sup\{\| T_1^n x \|: n \geq 0\} \neq \infty$, a contradiction with Corollary (3.1.6)
Hence, we get a contradiction with (1).Therefore, $T_1 \notin GC_s(H)$.

b)We need to show if S^{-1} is bounded and $\sigma(T)$ has a component σ such that
$\sigma \subset \{\lambda : \mid \lambda \mid > 1\}$, then $T \notin GC_s(T)$.
Assume $T \in GC_s(T)$, then by Proposition(3.1.16),
$T^{-1} \in GC_{s^{-1}}(T)$...................(2).
But by Theorem (3.2.4),

$$\sigma^{-1}(T) = \sigma(T^{-1}) = \{\frac{1}{\lambda} : \lambda \in \sigma(T)\}.$$

Since $\sigma(T)$ has a component σ_1, such that $\sigma_1 \subseteq \{\lambda : \mid \lambda \mid > 1\}$, then $\sigma(T^{-1})$ has a component $\{\sigma_1\}^{-1} = \frac{1}{\sigma_1} \subset \mathbf{B}(0,1)$. But S^{-1} is bounded, hence By part(a) $T^{-1} \notin GC_s^{-1}H$.
A contradiction with (2) . Therefore, $T \notin GC_s(T)$. □

Corollary 3.2.6. [19] *Let $T \in GC_s(H)$.*
1)If S is bounded, then $\sigma(T) \bigcap r\mathbb{D}$ is connected for all $r \leq 1$.
2) If S^{-1} is bounded, then $\sigma(T) \bigcap (r\mathbb{B})^c$ is connected for all $r \geq 1$.

Proof. 1) Assume $\sigma(T) \bigcap r\mathbb{D}$ is not connected, for some $r \leq 1$.
Then there is a closed and open subset σ of $\sigma(T) \bigcap r\mathbb{D}$.
Hence $\sigma \subset r\mathbb{D}$.
But $r \leq 1$, then $\sigma \subset \mathbb{B}(0,1)$.
Therefore, by Proposition (3.2.5)part (1), $T \notin GC_s(H)$.
A contradiction with $T \in GC_s(H)$.
2)Assume $\sigma(T) \bigcap (r\mathbb{B})^c$ is not connected, for some, $r \geq 1$,
then there is a closed and open subset σ of $\sigma(T) \bigcap (r\mathbb{B})^c$.
Hence $\sigma \subset (r\mathbb{B})^c$.
But $r \geq 1$, then $\sigma \subset \{\lambda : \mid \lambda \mid \geq 1\}$.
But since σ is closed and open, then $\partial\sigma = \emptyset$.
Hence $\sigma \subset \{\lambda : \mid \lambda \mid > 1\}$. And since S^{-1} is bounded, then, by Proposition (3.2.5)Part (2), $T \notin GC_s(H)$. □

Definition 3.2.7. [3] *A bounded linear operator T on a Hilbert space H is said to be quasinilpotent if*

$$\lim_{n \longrightarrow \infty} \parallel T^n \parallel^{\frac{1}{n}} = 0.$$

Definition 3.2.8. [14] *Let $T \in B(H)$. The spectral radius of T, denoted by $r_\sigma(T)$, is defined by $r_\sigma(T) = \sup_{\lambda \in \sigma(T)} \mid \lambda \mid$.*

Theorem 3.2.9. *Let $T \in B(H)$, then*

$$r_\sigma(T) = \lim_{n \longrightarrow \infty} \| T^n \|^{\frac{1}{n}} .$$

Proof. *See* [16]

Theorem 3.2.10. [3] *Let $T \in B(H)$. Then T is quasinilpotent if and only if $\sigma(T) = \{0\}$.*

Proof. T is a quasinilpotent iff $\lim_{n\longrightarrow\infty} \| T^n \|^{\frac{1}{n}} = 0 = r_\sigma(T)$ iff $\sigma(T) = \{0\}$. □

Corollary 3.2.11. [19] *Let S be a bounded semigroup, then a quasinilpotent operator $T \in B(H)$ can not be G-cyclic over S.*

Proof. Since T is quasinilpotent, then by Theorem (3.2.10), $\sigma(T) = \{0\} \subset \mathbb{B}(0,1)$. But S is bounded, then by Proposition (3.2.5)part (1), $T \notin GC_s(H)$. □

Definition 3.2.12. [16] *A metric space X is said to be compact if every sequence in X has a convergent subsequence.*
A subset M of X is said to be compact if every sequence in M has a convergent subsequence whose limit is an element of M.

Definition 3.2.13. [16] *Let X and Y be normed spaces. An operator $T : X \longrightarrow Y$ is called a compact linear operator if T is linear and if for every bounded subset M of X, the image T(M) is compact, that is, the closure $\overline{T(M)}$ is compact.*

Theorem 3.2.14. [16] *Let $T : X \longrightarrow X$ be a compact linear operator on a Banach space X. Then every spectral value $\lambda \neq 0$ of T is an eigenvalue of T.*

Corollary 3.2.15. [19] *A compact operator can not be G-cyclic over any bounded semigroup S.*

Proof. Let T be a compact operator which is G-cyclic over some bounded semigroup S. Then by Theorem (3.2.14)

$$\sigma_p(T) = \sigma(T) - \{0\}.............(1)$$

Hence by Theorem (2.2.19),

$$\overline{\sigma_p(T^*)} = \overline{\sigma(T^*)} - \{0\}.$$

But by Proposition (3.2.1),since S is bounded

$$\sigma_p(T^*) = \emptyset \qquad or \qquad \sigma_p(T^*) = \{\overline{\lambda}\} \; for\ some\ \lambda\ with \ \mid \lambda \mid > 1.$$

Hence by Theorem (2.2.19),

$$\sigma_p(T) = \overline{\sigma_p(T^*)} = \emptyset \qquad or \qquad \sigma_p(T) = \overline{\sigma_p(T^*)} = \{\lambda\} \; and \ \mid \lambda \mid > 1.$$

Hence by (1)

$$\sigma(T) = \{0\} \qquad or \qquad \sigma(T) = \{0, \lambda\} \; and \ \mid \lambda \mid > 1.$$

If $\sigma(T) = \{0\}$ then $\sigma(T)$ is connected set. That is $\sigma(T)$ is component of $\sigma(T)$ and $\sigma(T) \subseteq \mathbf{B}(0, 1)$.
But S is bounded, then by Proposition (3.2.5) Part (1), $T \notin GC_s(H)$.
If $\sigma(T) = \{0, \lambda\}, \mid \lambda \mid > 1$.
Then, $\mathbb{B}(\lambda, 1) \bigcap \sigma(T) = \{\lambda\}$.
Therefore, by Definition (1.1.3), λ is a isolated point of $\sigma(T)$.
Hence, $\{0\}$ is a component of $\sigma(T)$.
Since S is bounded, then by Proposition (3.2.5), $T \notin GC_s(H)$. □

Chapter 4

Disk-Cyclicity and Codisk-cyclicity

This chapter consists of two sections.
In section 4.1 we define the Disk-cyclic operator and give a characterization of Disk-cyclic Criterion, and give an example of a Disk-cyclic operator which is not hypercyclic.
In section 4.2 we define the Codisk-cyclic operator and give a characterization of Codisk-cyclic Criterion, and give an example of a Codisk-cyclic operator which is not Disk-cyclic.

4.1 Disk-Cyclic operators and Their properties

Definition 4.1.1. [18] *An operator $T \in B(H)$ is said to be disk-cyclic if there exists, $x \in H$ such that*

$$\{\alpha T^n x : \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1, n \geq 0\} \text{ is dense in } H.$$

In this case x is said to be a disk-cyclic vector for T.

Next we fix notation required for the discussion.
Notation [18] Let $T \in B(H)$.
1-$\mathbb{D}C(T) = \{x \in H$: x is a disk-cyclic vector for T $\}$.
2-$\mathbb{D}C(H) = \{T \in B(H)$: T is disk-cyclic operator $\}$.
3-$\mathbb{D}orbt(T, x) = \{\alpha T^n x : \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1, n \geq 0\}$.

Theorem 4.1.2. [18] *Let $T \in B(H)$. then the following statements are equivalent:*
1- $T \in \mathbb{D}C(H)$.
2- For each non-empty open sets U, V, there are $\alpha \in \mathbb{D}, n \in N$ such that $T^n(\alpha U) \bigcap V = \phi$.
3- For each $x, y \in H$,there are a sequences$\{x_k\}_{k=1}^{\infty}$ in H ,$\{n_k\}_{k=1}^{\infty}$ in N ,$\{\alpha_k\} \in \mathbb{D}$ such that $x_k \longrightarrow x$ and $T^{n_k}\alpha_k x_k \longrightarrow y$.
4- For each $x, y \in H$,and each neighborhood W for zero in H ,there are $z \in H, n \in N, \alpha \in \mathbb{C}.\ |\ \alpha\ | \leq 1$ such that
$x - z \in W$ and $T^n\alpha z - y \in W$.
Proof : *Similarly as the proof of Theorem (3.1.15).*

Remark 4.1.3. [18] *Every hypercyclic operator is disk- cyclic.*
And every disk-cyclic operator is supercyclic.

Proposition 4.1.4. [18] ***(Disk-cyclic Criterion).***
Let $T \in B(H)$ Such that :
1) There are dense sets X, Y in H and right inverse S to T
(not necessary bounded) such that $S(Y) \subset Y$ and $TS = I_Y$.
2) There is a sequence n_k in N such that
a) $\lim_{k\longrightarrow\infty} \| S^{n_k}y \| = 0$ for all $y \in Y$.
b) $\lim_{k\longrightarrow\infty} \| T^{n_k}x \| \| S^{n_k}y \| = 0$ for all $x \in X, y \in Y$.
Then $T \in \mathbb{D}C(H)$.

Proof. To prove that $T \in \mathbb{D}C(H)$ we show that condition (4) in Theorem (4.1.2) is satisfied.
Let $z, w \in H$, let W be a neighborhood for zero in H, suppose $W = \mathbb{B}(0, \epsilon)$.
By condition (1) there are $x \in X, y \in Y$ such that $z - x \in \mathbb{B}(0, \frac{\epsilon}{2})$ *,and* $w - y \in \mathbb{B}(0, \frac{\epsilon}{2})$.
By condition (2)there are $t_1 \geq 0$ *and* $t_2 \geq 0$ such that,

$$\| S^{n_k}y \| \leq \frac{\epsilon}{3} \quad for\ all\ k > t_1,$$

and

$$\| T^{n_k}x \| \| S^{n_k}y \| \leq \frac{\epsilon}{6}\ for\ all\ k \geq t_2.$$

Put $t = max\{t_1, t_2\}$, and fix $k > t$ such that

$$\| S^{n_k}y \| \leq \frac{\epsilon}{3}$$

$$\| T^{n_k}x \| \| S^{n_k}y \| \leq \frac{\epsilon}{6}.$$

Choose $u = x + \frac{1}{c}S^{n_k}y$ where $0 < c \leq \epsilon < 1$.
case(1). If $\| S^{n_k}y \| \neq 0$, take $c = 3 \| S^{n_k}y \|$ hence $c \| T^{n_k}x \| \leq \frac{\epsilon}{2}$.
Then $\| z - u \| \leq \| z - x \| + \| x - u \| < \frac{\epsilon}{2} + \frac{\epsilon}{3} < \epsilon$ (1).
And since $cT^{n_k}u = cT^{n_k}x + y$, hence
$\| w - cT^{n_k}u \| \leq \| w - y \| + \| y - cT^{n_k}u \| < \frac{\epsilon}{2} + \| cT^{n_k}x \| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ (2)
Hence from (1) and (2)

$$z - u \in W \quad , \quad and \quad w - cT^{n_k}u \in W.$$

Then, Condition (4) in Theorem (3.1.2) is satisfied. Therefore, $T \in \mathbb{D}C(H)$.
case(2). If $\| S^{n_k}y \| = 0$,and since TSy=y for all $y \in Y$,
then $0 = \| T^{n_k}S^{n_k}y \| = \| y \|$. So y=0 and x=u ,hence $w \in \mathbb{B}(0, \frac{\epsilon}{2})$.
Therefore, $\| z - u \| = \| z - x \| \leq \frac{\epsilon}{2}$ (3).
By choosing c small enough such that $c \| T^{n_k}x \| < \frac{\epsilon}{2}$, we have,

$$\| w - cT^{n_k}u \| \leq \| w - y \| + \| y - cT^{n_k}u \| \leq \frac{\epsilon}{2} + \| cT^{n_k}x \| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad (4).$$

Then, from (3)(4)
$z - u \in W$, and $w - cT^{n_k}u \in W$.
Then, Condition (4) in Theorem (4.1.2) is satisfied. Therefore, $T \in \mathbb{D}C(H)$.
□

Theorem 4.1.5. [18] *Suppose $T : l^2(Z) \longrightarrow l^2(Z)$ is a forward weighted shift with weight sequence $\{w_n\}_{n \in Z}$ and either $w_n \geq m > 0$ for all $n < 0$ or $w_n < m$ for all $n > 0$. Then $T \in \mathbb{D}C(H)$ if and only if there exists a sequence*
$\{n_r\}$ in Z such that; $n_r \longrightarrow \infty$ with

$$1) \lim_{r \longrightarrow \infty} \left(\prod_{k=1}^{n_r} \frac{1}{w_{-k}}\right) = 0, \quad and$$

$$2) \lim_{r \longrightarrow \infty} \left(\prod_{k=1}^{n_r} w_k\right)\left(\prod_{k=1}^{n_r} \frac{1}{w_{-k}}\right) = 0.$$

Example 4.1.6. [18] *Let $T : l^2(Z) \longrightarrow l^2(Z)$ be the forward weighted shift with weight sequence $w_n = \begin{cases} \frac{1}{a^2} & , \quad n \geq 0 \\ \frac{1}{a} & , \quad n < 0 \end{cases}$*
where $a > 1$. Then T is supercyclic. But it is not a disk - cyclic operator.

Proof. Clearly $\frac{1}{a^2} < 1$ for all $n \geq 0$. Then

$$\lim_{n\longrightarrow\infty}(\prod_{k=1}^{n} w_k)(\prod_{k=1}^{n}\frac{1}{w_{-k}}) = \lim_{n\longrightarrow\infty}(\frac{1}{a})^{2n}a^n = \lim_{n\longrightarrow\infty}(\frac{1}{a})^n = 0 \quad .$$

Thus by Theorem (2.1.15), T is supercyclic. But

$$\lim_{n\longrightarrow\infty}(\prod_{k=1}^{n}\frac{1}{w_{-k}}) = \lim_{n\longrightarrow\infty} a^n = \infty$$

then by theorem (4.1.5), T is not disk- cyclic. □

Example 4.1.7. [18] *Let $T : l^2(Z) \longrightarrow l^2(Z)$ be the forward weighted shift with weight sequence* $w_n = \begin{cases} a & , \ n \geq 0 \\ a^2 & , \ n < 0 \end{cases}$
where $a > 1$. Then T is a disk-cyclic operator, but it is not a hypercyclic operator.

Proof. Clearly $w_n = a^2 > 1$ for all $n < 0$. Then

$$\lim_{n\longrightarrow\infty}(\prod_{k=1}^{n} w_k)(\prod_{k=1}^{n}\frac{1}{w_{-k}}) = \lim_{n\longrightarrow\infty} a^n(\frac{1}{a^2})^n = \lim_{n\longrightarrow\infty}\frac{1}{a^n} = 0,$$

and

$$\lim_{n\longrightarrow\infty}(\prod_{k=1}^{n}\frac{1}{w_{-k}}) = \lim_{n\longrightarrow\infty}(\frac{1}{a})^{2n} = 0.$$

Then by Theorem (4.1.5) T is a disk -cyclic. But

$$\lim_{n\longrightarrow\infty}(\prod_{k=1}^{n} w_k) = \lim_{n\longrightarrow\infty} a^n = \infty$$

then by theorem (2.1.25) T is not hypercyclic. □

Proposition 4.1.8. [18] *Let $T \in B(H)$. If there exists a number $\rho > 1$ such that for every $\epsilon > 0$, span $\{ker(T - \lambda I) : \rho <\mid \lambda \mid< \rho + \epsilon\}$ is dense in H, then $T \in \mathbb{D}C(H)$.*

Proof. To prove that $T \in \mathbb{D}C(H)$, we will use the disk-cyclic Criterion Proposition (4.1.4).
Let Y be the linear span of $\{ker(T - \lambda I) :\mid \lambda \mid> \rho\}$.

Since, span $\{ker(T-\lambda I):\rho<|\lambda|<\rho+\epsilon\}\subseteq Y$, then Y is dense in H.
Define $S:Y\longrightarrow Y$ as $Sy=S(\Sigma_{i=1}^{n}\alpha_i y_i)=\Sigma_{i=1}^{n}\frac{\alpha_i}{\lambda_i}y_i$,
where $y_i\in ker(T-\lambda_i I)$,and $|\lambda_i|>\rho$.
Then $(T-\lambda_i I)y_i=0$ $,and\ so\quad Ty_i=\lambda_i y_i$.
Therefore, for any $y=\Sigma_{i=1}^{n}\alpha_i y_i\in Y$ we have,

$$TSy=T\Sigma_{i=1}^{n}\frac{\alpha_i}{\lambda_i}y_i=y \quad ....(1).$$

And since $1<\rho<|\lambda_i|$, then $\frac{1}{|\lambda_i|}<1$, hence $\lim_{k\longrightarrow\infty}\frac{1}{|\lambda_i^k|}=0$.
So, $\lim_{k\longrightarrow\infty}\frac{|\alpha_i|}{|\lambda_i^k|}=0$ for all fix $1\leq i\leq n$. Hence,

$$\lim_{k\longrightarrow\infty}\parallel S^k y\parallel=\lim_{k\longrightarrow\infty}\parallel\Sigma_{i=1}^{n}\frac{\alpha_i}{\lambda_i^k}y_i\parallel\leq\lim_{k\longrightarrow\infty}(\Sigma_{i=1}^{n}(\frac{|\alpha_i|}{|\lambda_i|^k})\parallel y_i\parallel)=0\quad for\ all\ y\in Y\ ....(2).$$

Now, Let m $=\min\{|\lambda_i|:|\lambda_i|>\rho,1\leq i\leq n\}$, hence $m>\rho$.
Let X_y be the finite linear combination of the span of the set,

$$\{ker(T-\mu I):\rho<|\mu|<m\}.$$

Then, span $\{ker(T-\lambda I):\rho<|\lambda|<\rho+\epsilon\}\subseteq X_y$, hence X_y is dense in H. Let $x\in X_y, y\in Y$,hence $x=\Sigma_{i=1}^{t}\gamma_i x_i$ where $x_i\in\{ker(T-\mu_i I):\rho<|\mu_i|<m\}$, and $y=\Sigma_{i=1}^{n}\alpha_i y_i$ where
$y_i\in\{ker(T-\lambda_i I):\rho<|\lambda_i|\}$. Hence

$$\parallel T^k x\parallel=\parallel\Sigma_{i=1}^{t}\gamma_i T^k x_i\parallel=\parallel\Sigma_{i=1}^{t}\gamma_i\mu_i^k x_i\parallel\leq\Sigma_{i=1}^{t}|\gamma_i\mu_i^k|\parallel x_i\parallel \quad .$$

and

$$\parallel S^k y\parallel\leq\Sigma_{i=1}^{n}|\frac{\alpha_j}{\lambda_j^k}|\parallel y_j\parallel .$$

Therefore,

$$\parallel T^k x\parallel\parallel S^k y\parallel\leq\Sigma_{i=1}^{t}\Sigma_{j=1}^{n}c_{ij}^k\parallel x_i\parallel\quad\parallel y_j\parallel$$

where $c_{ij}^k=|\gamma_i\alpha_j||\frac{\mu_i^k}{\lambda_j^k}|$ for all i,j; $1\leq i\leq n, 1\leq j\leq t$.
But $|\mu_i|<m\ \ then\quad|\mu_i|<|\lambda_j|\quad for\ \ all\quad i;1\leq i\leq n,1\leq j\leq t.$
So, $\lim_{k\longrightarrow\infty}\frac{|\mu_i|^k}{|\lambda_j|^k}=0$.
Therefore, $\lim_{k\longrightarrow\infty}c_{ij}^k=0$ for all i,j. Then

$$\lim_{k\longrightarrow\infty}\parallel T^k x\parallel\parallel S^k y\parallel=0\quad ....(3)$$

Hence from (1)(2)and(3) and by Proposition (4.1.4) $T\in\mathbb{D}C(H)$. □

Lemma 4.1.9. [18] *Let $T \in B(H)$, M be T-invariant subspace of H, and $P : H \longrightarrow M^{\perp}$ orthogonal projection onto $M^{\perp}$. If $x \in \mathbb{D}C(T)$, then $Px \neq 0$.*

Proof. By Theorem (1.3.6)for all $x \in H$, x has the unique representation as
$x = m \oplus m^{\perp}$. where $m \in M$ *and* $m^{\perp} \in M^{\perp}$.
Assume $x \in \mathbb{D}C(T)$ *and* $Px = m^{\perp} = 0$, hence $x \in M$.
But M is T-invariant subspace of H, then

$$\overline{\{\alpha T^n x : \alpha \in \mathbf{C}, n \geq 0\}} \subseteq M...............(1).$$

Since $x \in \mathbb{D}C(T)$, then

$$\overline{\{\alpha T^n x : \alpha \in \mathbf{C}, \mid \alpha \mid \leq 1, n \geq 0\}} = H \neq M........(2).$$

From (1)(2) we get a contradiction. Therefore, $Px \neq 0$. □

Proposition 4.1.10. [18] *Let $T \in \mathbb{D}C(H)$, and let M be an invariant subspace of H under T. Then the operator $S : M^{\perp} \longrightarrow M^{\perp}$ defined as Sx=P(Tx) is a disk-cyclic operator, where P is the projection onto $M^{\perp}$.*

Proof. Let $x \in \mathbb{D}C(T)$, let M be an invariant subspace of H under T, and let

$$P : H \longrightarrow M^{\perp} \quad \textit{be the orthogonal projection.}$$

Since $x \neq 0$ *and* $H = M \oplus M^{\perp}$, then, by Lemma (4.1.9)
$Px = P(y \oplus z) = z \neq 0$, where $y \in M$ *and* $z \in M^{\perp}$.
we claim that z is a disk-cyclic vector for S.
By Theorem (1.4.15) $(PT)^n x = (PT^n)x$ for all $n \geq 0, x \in H$.
Thus

$$\begin{aligned}\overline{\mathbb{D}orbt(S, z)} &= \overline{\{\alpha(PT)^n x : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}} \\ &= \overline{\{\alpha PT^n x : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}} \\ &= P\overline{\{\alpha T^n x : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}} = P(H) = M^{\perp},\end{aligned}$$

because P is onto. Therefore, $z \in \mathbb{D}C(S)$. □

Proposition 4.1.11. [18] *If $T \in B(H)$ and M is an invariant subspace of H ,then the following statements are equivalent :*
(1)The operator $S : M^{\perp} \longrightarrow M^{\perp}$ such that Sx=PTx is a disk-cyclic operator, where P is the projection onto $M^{\perp}$.
(2) The operator $\overline{T} : H/M \longrightarrow H/M$ such that $\overline{T}(a + M) = Ta + M$ is a disk-cyclic operator.

Proof. 1)$\Longrightarrow$2): Let $x \in M^{\perp}$ be a disk-cyclic vector for S.
Thus, $\mathbb{D}orbt(S,x) = M^{\perp} = \overline{\{\alpha(PT)^n x : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}}$.
But M is invariant under T, and P is the projection onto $M^{\perp}$,
then by Theorem (1.4.15),
$(PT)^n x = PT^n x = T^n x$ for all $n \geq 0$.
Now, since $\overline{T}(x+M) = Tx + M$, then we can show by induction,
$\overline{T}^n(x+M) = T^n x + M$.
Hence,
$\overline{\{\alpha\overline{T^n}(x+M) : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}} = \overline{\{\alpha T^n x + M : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}} = \overline{\{\alpha PT^n x : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}} + M = M^{\perp}/M.$ (1).
Let $y \in H/M$, hence $y = z + M$ where $z \in H$, but $z = a + b$ where, $a \in M, b \in M^{\perp}$ that is $y = z + M = a + b + M = b + M \in M^{\perp}/M$.
Hence $H/M = M^{\perp}/M$.
Thus from (1),
$\overline{\{\alpha\overline{T^n}(x+M) : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}} = H/M$.
Therefore, $\overline{T}$ is a disk-cyclic operator of H/M.

2)$\Longrightarrow$1): Let $x + M \in \mathbb{D}C(\overline{T})$ where $x \in M^{\perp}$. Thus,

$$\overline{\{\alpha PT^n x : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}} + M = \overline{\{\alpha(T^n x + M) : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}} = \overline{\{\alpha\overline{T^n}(x+M) : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}} = H/M = M^{\perp}/M.$$

Therefore,
$M^{\perp} = \overline{\{\alpha PT^n x : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}} = \overline{\{\alpha S^n x : n \geq 0, \alpha \in \mathbb{C}, \mid \alpha \mid \leq 1\}}$.
Hence, $x \in \mathbb{D}C(S)$. □

4.2 Codisk- Cyclic operators and Their properties

Definition 4.2.1. [18] *An operator $T \in B(H)$ is said to be codisk-cyclic if there exists, $x \in H$ such that*

$$\{\alpha T^n x : \alpha \in \mathbb{C}, \mid \alpha \mid \geq 1, n \geq 0\} \textit{ is dense in } H.$$

In this case x is said to be a codisk-cyclic vector for T.

Next we fix notation required for the discussion .
Notation 2-1-2 [18] Let $T \in B(H)$.
1-$\mathbb{D}^c C(T) = \{x \in H$: x is a codisk-cyclic vector for T $\}$.
2-$\mathbb{D}^c C(H) = \{T \in B(H)$: T is a codisk-cyclic operator $\}$.

3-$\mathbb{D}^c orbt(T,x) = \{\alpha T^n x : \alpha \in \mathbb{C}, \mid \alpha \mid \geq 1, n \geq 0\}$.

Remark 4.2.2. [18] *Every hypercyclic operator is codisk- cyclic, and every codisk-cyclic operator is supercyclic.*

Example 4.2.3. [18] *Let $T : l^2(N) \longrightarrow l^2(N)$ be backward shift. Then $T \in \mathbb{D}^c C(H)$*

Proof. From Example (2.1.24) λT is a hypercyclic operator for all $\lambda \in \mathbb{C}, \mid \lambda \mid > 1$. Hence there exists $x \in l^2(N)$ such that,

$$\{(\lambda T)^n x, n \geq 0, \mid \lambda \mid > 1\} is\ dense\ in\ l^2(N).$$

But

$$\{(\lambda T)^n x, n \geq 0, \mid \lambda \mid > 1\} \subset \{\alpha T^n x : n \geq 0, \mid \alpha \mid \geq 1\}.$$

Therefore, $T \in \mathbb{D}^c C(H)$. □

Theorem 4.2.4. [16] *Let $T \in B(H)$. then the following statements are equivalent:*
1- $T \in \mathbb{D}^c C(H)$.
2- For each non-empty open sets U, V, there are $\alpha \in \mathbb{C}, \mid \alpha \mid \geq 1$, $n \in \mathbb{N}$ such that $T^n(\alpha U) \bigcap V = \emptyset$.
3- For each $x, y \in H$, there are a sequences $\{x_k\}_{k=1}^{\infty}$ in H, $\{n_k\}_{k=1}^{\infty}$ in N, $\{\alpha_k\} \in \mathbb{C}, \mid \alpha_k \mid \geq 1$, such that $x_k \longrightarrow x$ and $T^{n_k} \alpha_k x_k \longrightarrow y$.
4- For each $x, y \in H$,and each neighborhood W for zero in H, there are $z \in H, n \in N, \alpha \in \mathbb{C}, \mid \alpha \mid \geq 1$ such that $x - z \in W$ and $T^n \alpha z - y \in W$

Proof. Similarly as the proof of Theorem (3.1.15) □

Proposition 4.2.5. [18] ***(codisk-cyclic Criterion).***
Let $T \in B(H)$ Such that :
1) There are dense sets X, Y in H and a right inverse to T (not necessary bounded) S such that $S(Y) \subset Y$ and $TS = I_Y$.
2) There is a sequence n_k in N such that
a) $\lim_{k \longrightarrow \infty} \parallel T^{n_k} x \parallel = 0$ for all $x \in X$.
b) $\lim_{k \longrightarrow \infty} \parallel T^{n_k} x \parallel \parallel S^{n_k} y \parallel = 0$ for all $x \in X, y \in Y$.
Then $T \in \mathbb{D}^c C(H)$.

Proof. To prove that $T \in \mathbb{D}^c C(H)$ we show that condition (4) in Theorem (4.2.4) is satisfied.
Let $z, w \in H$, let W be a neighborhood for zero in H, suppose $W = \mathbb{B}(0, \epsilon)$.
By condition (1) there are $x \in X, y \in Y$ such that, $z - x \in \mathbb{B}(0, \frac{\epsilon}{2}),\ and$
$w - y \in \mathbb{B}(0, \frac{\epsilon}{2})$.
Let $u = x + cS^{n_k}y$ for some $k \in N\ and\ 0 < c \leq \epsilon \leq 1$.
By condition (2) there are $t_1 \geq 0\ and\ t_2 \geq 0$ such that,

$$\| T^{n_k}x \| \leq \frac{\epsilon}{3} \quad for\ all\ k > t_1,$$

and

$$\| T^{n_k}x \| \| S^{n_k}y \| \leq \frac{\epsilon}{6}\ for\ all\ k \geq t_2$$

Put $t = max\{t_1, t_2\}$, and fix $k > t$ such that,

$$\| T^{n_k}x \| \leq \frac{\epsilon}{3} \qquad and \qquad \| T^{n_k}x \| \| S^{n_k}y \| \leq \frac{\epsilon}{6}.$$

<u>case (1)</u> If $\| T^{n_k}x \| \neq 0$, take $c = 3 \| T^{n_k}x \|$.
Hence, $c \| S^{n_k}y \| < \frac{\epsilon}{2}$. However, $\| u - x \| = c \| S^{n_k}y \|$, then
$\| z - u \| \leq \| z - x \| + \| x - u \| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$ (3).
Since $\frac{1}{c}T^{n_k}u = \frac{1}{c}T^{n_k}x + y$, then,

$$\| w - \frac{1}{c}T^{n_k}u \| \leq \| w - y \| + \| y - \frac{1}{c}T^{n_k}u \| \leq \frac{\epsilon}{2} + \| \frac{1}{c}T^{n_k}x \| < \frac{\epsilon}{2} + \frac{\epsilon}{3} < \epsilon \qquad (4).$$

Hence from (3)(4), $z - u \in W,\ and\ w - cT^{n_k}u \in W$.
Therefore, by Theorem (4.2.4), $T \in \mathbb{D}^c C(H)$.
<u>case(2)</u> If $\| T^{n_k}x \| = 0$.
Hence choose $c > 0$, small enough such that, $c \| S^{n_k}y \| < \frac{\epsilon}{2}$, and let $\alpha = \frac{1}{c}$
Therefore, $\| z - u \| \leq \| z - x \| + c \| S^{n_k}y \| < \epsilon$ (5).
And

$$\| w - \alpha T^{n_k}u \| \leq \| w - y \| + \| y - \alpha T^{n_k}u \| \leq \frac{\epsilon}{2} + \| \alpha T^{n_k}x \| \leq \frac{\epsilon}{2} + 0 = \frac{\epsilon}{2} \qquad (6),$$

then, from (5)(6) $z - u \in W$, and $w - \alpha T^{n_k}u \in W$.
Hence by Theorem (4.2.4) $T \in \mathbb{D}^c C(H)$. □

Theorem 4.2.6. [18] *Suppose* $T : l^2(Z) \longrightarrow l^2(Z)$ *is a forward weighted shift with weight sequence* $\{w_n\}_{n \in Z}$ *and either* $w_n \geq m > 0$ *for all*

$n < 0$ or $w_n < m$ for all $n > 0$. Then, $T \in \mathbb{D}^c C(H)$ if and only if there exists a sequence $\{n_r\}$ in $Z; n_r \longrightarrow \infty$ such that:

$$1) \lim_{r \longrightarrow \infty} \prod_{k=1}^{n_r} w_k = 0.$$

$$2) \lim_{r \longrightarrow \infty} (\prod_{k=1}^{n_r} w_k)(\prod_{k=1}^{n_r} \frac{1}{w_{-k}}) = 0.$$

Example 4.2.7. [16] *Let $T : l^2(Z) \longrightarrow l^2(Z)$ be the forward weighted shift with weight sequence*

$$w_n = \begin{cases} \frac{1}{n+2} & , \quad n \geq 0 \\ \frac{1}{2} & , \quad n < 0 \end{cases}$$

Then, $T \in \mathbb{D}^c C(l^2(Z))$. But $T \notin \mathbb{D}C(l^2(Z))$.

Proof. Clearly, $w_n = \frac{1}{2} > \frac{1}{3}$ for all $n < 0$. Not that,

$$\lim_{n \longrightarrow \infty} (\prod_{k=1}^{n} w_k) = \lim_{n \longrightarrow \infty} (\prod_{k=1}^{n} \frac{1}{k+2}) = \lim_{n \longrightarrow \infty} \frac{2}{(n+2)!} = 0.$$

and

$$\lim_{n \longrightarrow \infty} (\prod_{k=1}^{n} w_k)(\prod_{k=1}^{n} \frac{1}{w_{-k}}) = \lim_{n \longrightarrow \infty} (\frac{2}{(n+2)!})(2^n) = 0.$$

Since $2^{i+1} < (i+2)!$, then by theorem (4.2.6), $T \in \mathbb{D}^c C(l^2(Z))$.
Since,

$$\lim_{n \longrightarrow \infty} (\prod_{k=1}^{n} \frac{1}{w_{-k}}) = \lim_{n \longrightarrow \infty} 2^n = \infty.$$

Therefore, by Theorem (4.1.5), $T \notin \mathbb{D}C(l^2(Z))$. □

Example 4.2.8. [18] *Let $T : l^2(Z) \longrightarrow l^2(Z)$ be the forward weighted shift with weight sequence*

$$w_n = \begin{cases} a^2 & , \quad n \leq 0 \\ a & , \quad n > 0 \end{cases}$$

where $a > 1$. Then $T \in \mathbb{D}C(l^2(Z))$. But $T \notin \mathbb{D}^c C(l^2(Z))$.

Proof. Clearly $w_n = a^2 \geq 1$ for all $n \geq 0$.

$$\lim_{n \longrightarrow \infty} (\prod_{k=1}^{n} w_k)(\prod_{k=1}^{n} \frac{1}{w_{-k}}) = \lim_{n \longrightarrow \infty} a^n (\frac{1}{a^2})^n = \lim_{n \longrightarrow \infty} \frac{1}{a^n} = 0,$$

and

$$\lim_{n\longrightarrow\infty}(\prod_{k=1}^{n}\frac{1}{w_{-k}}) = \lim_{n\longrightarrow\infty}(\frac{1}{a^2})^n = 0.$$

Then, by Theorem (4.1.5), $T \in \mathbb{D}C(l^2(Z))$. But

$$\lim_{n\longrightarrow\infty}(\prod_{k=1}^{n} w_k) = \lim_{n\longrightarrow\infty} a^n = \infty,$$

then, by theorem (4.2.6), $T \notin \mathbb{D}^c C(l^2(Z))$. □

Proposition 4.2.9. [18] *Let $T \in \mathbb{D}^c C(H)$, and let M be an invariant subspace of H under T. then the operator $S : M^{\perp} \longrightarrow M^{\perp}$, defined as Sx=P(Tx) is a codisk-cyclic operator, where P is the projection onto $M^{\perp}$.*

Proof. Similarly as the proof of Proposition (4.1.10). □

Proposition 4.2.10. [18] *If $T \in B(H)$ and M is an invariant subspace of H, then the following statements are equivalent :*
(1)The operator $S : M^{\perp} \longrightarrow M^{\perp}$, such that Sx=$PTx$ is codisk-cyclic operator, where P is the projection onto $M^{\perp}$.
(2) The operator $\overline{T} : H/M \longrightarrow H/M$ such that,
$\overline{T}(a + M) = Ta + M$ is a codisk-cyclic operator.

Proof. Similarly as as the proof of Proposition (4.1.11). □

Bibliography

[1] J.Al-Haj Ahmad, *Hypercyclic and Circle − Cyclic operator* ,Thesis .The Islamic University of Gaza, 2003.

[2] S.Ansari, Existence of hypercyclic operators on topological vector space, *J.Funct. Anal.* 1997, 384-390.

[3] A. Ashour, *Nilpotent and Quasinilpotent Operators*, University of Baghdad, 1996.

[4] Robert B.Ash, *Abstract Algebra*,J.math., 2000

[5] S.K.Berberian, The Weyl spectrum of an operator, *Indiana University. Math. J.* (6), 20 (1970), 529-544.

[6] Bourdon, P, S Orbits of hyponormal operators.*MichiganMath.* J,1997,345-353.

[7] A. L. Brown and A. Page *Elements of functional analysis*, Van Nostrand Reinhold company London, 1970.

[8] Wwl.Chen, *Linear functional analysis* math. mathnotes/analysis. 2002, 2003.

[9] J. B. Conway, *A course in functional analysis*, Springer-Verlag, New York, Heidelberg Berlin, 1985.

[10] M. Dostal, Closures of U+k orbits of essentially normal models 1998, preprint.

[11] N. S.Feldman Pure subnormal operators having cyclic adjoint, *J, FunctionalAnalysis*(2) ,1999, 379-399.

[12] N. S.Feldman, The Dynamics of Cohyponormal Operators, *Math. Subject Classification Primary*, 47B20,47A16, 2000.

[13] N. S. Feldman Hypercyclic, Supercyclic For Invertible Bilateral weighted Shifts, *j.Math.Analysisand Appl* , 2001, No .1,67-74.

[14] T.Furuta, *Invitation to Linear operator*, Taylor, and, Francis. 2001.

[15] D.A.Herrero, Limits of hypercyclic and supercyclic operators, *Journal Functional Analysis*, Vol.99.1991.

[16] E. Kreyszing, *Intoductory functional analysis with applications*, Wiley, New York, 1978.

[17] A.B.Lambert and S Petrovic, Extended eigenvalues and Volterra operator, *Math, sub, clasification*, 47A65, 2004.

[18] A.Naoum.and Z .Jamil, *Cyclic Phenomena of operators on Hilbert space*, Thesis, University of Bagdad , 2002.

[19] A. G.Naoum and Z.Jamil, G-cyclicity, *Proceeding of the first conference on Mathematical science Zarqa private univercity* , 2006,58-66

[20] Harald.Olsen,The Weierstrass density theorem, *TMA*. 4230, *functional analysis*, *Version*. 2005

[21] H. Radjavi. and p. Rosenthal, *Invariant subspace* ,Springer Verlag, New York, 1973.

[22] F. Saavedra and A.Piqueras, Cyclic Properties of Volterra operator *Pacific Journal of mathematics* , vol. 211, Num 1,145-149 (2003).

[23] N.Salas, Hypercyclic weighted shift, *Transactions of the American Mathematical Society*, Vol. 347, No. 3 (Mar., 1995),pp. 993-1004.

[24] H.Shapiro, Simple Connectivity and linear Chaos, *Science Foundation*. 1998

[25] S. Willered, *General Topology*, June 1970.