Bäcklund Transformation

for

Discrete Painlevé Equations

M.Sc. Thesis

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Dedication

To ...

My Parents,

My Teachers,

And My Brothers.

Contents

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ABSTRACT

In this thesis we studied an algorithmic method for investigating transformation properties of discrete Painlevé equations. This method yields explicit transformations which relates the solutions of a given discrete Painlevé equation with either the solutions of another discrete Painlevé equation or the same equation but with different value of parameters. We applied the method to the second, third, fourth, and fifth discrete Painlevé equation. Moreover we studied special solutions of these equations. Some of these special solutions are given in terms of discrete analogue of the classical special functions.

Chapter 1

Introduction

1.1 Difference equation

Definition: 1.1

Let y be a given function, x in the domain of y and let h be any constant for which $x + h$ is in the domain of y. We define the first difference of y, denoted by $\Delta y(x)$, by $\Delta y(x) = y(x + h) - y(x)$ [2].

Definition: 1.2

Suppose that a function y and its first difference Δy are given . Then the second difference of y, denoted by $\Delta^2 y$, is the difference of the first difference of y, that is $\Delta^2 y(x) = \Delta(\Delta y)(x) = \Delta y(x + h) - \Delta y(x)$. In general, the n-th difference of y is $\Delta^n y = \Delta(\Delta^{n-1} y)$ [2].

Definition: 1.3

An equation relating the values of a function $y(x)$ and one or more of its differences $\Delta y(x), \Delta^2 y(x)$, ..., for each x-value is called a difference equation [2].

In particular, we consider difference equations over special sets of values of x characterized as follows. Let S be the set of all values contains some number x_0 and either finite or infinite set of equally spaced x-values

$x_0,\ x_0 + h,\ x_0 + 2h,\ ...$

Now if we define a new variable k related to x by the equation $k = \frac{x-(x_0-ah)}{h}$, where a is any nonnegative integer, then $k = a$ when $x = x_0,\ k = a+1$ when $x = x_0 + h$. In general the set S of x-values

x_0, $x_0 + h$, $x_0 + 2h$, ... can be transformed into the set of consecutive integers $k : a,\ a+1,\ a+2,\ ...$

Given any k value we find the corresponding x by solving the equation

$x = (x_0 - ah) + kh.$

To emphasize the restriction of the set S to consecutive integer, we define $y_k = y(x)$. Then $\Delta y(x) = y_{k+1} - y_k$.

Example: The equation $\Delta y(x) + 3y(x) = 0$ is a difference equation, this equation can be written as $y_{k+1} - y_k + 3y_k = 0$ then $y_{k+1} + 2y_k = 0$.

Definition: 1.4

A difference equation over a set S of y is linear over S if it can be written in the form

$$f_0(k)y_{k+n} + f_1(k)y_{k+n-1} + ... + f_{n-1}(k)y_{k+1} + f_n y_k = g(k), \qquad (1.1)$$

where $f_0, f_1, ..., f_{n-1}, f_n$, and g are , functions of k (but not of y_k) defined for all values of k in the set S [2].

Definition: 1.5

The order of a difference equation is the difference between the largest and smallest arguments (subscripts) appearing in the equation.

Example(1): $y_{k+1} + y_k = k$ is a first order difference equation.

Example(2): $y_{k+2}y_k + ky_{k-1} = 0$ is a third order difference equation.

Given that α and β are constants, the difference equation of the form $x_{n+1} = \alpha x_n + \beta, \quad n = 0,\ 1,\ 2,\ ...$ is a first-order linear difference equation. To solve this equation we do the following,

$$\begin{aligned}
x_n &= \alpha x_{n-1} + \beta \\
&= \alpha(\alpha x_{n-2} + \beta) + \beta \\
&= \alpha^2 x_{n-2} + \beta(\alpha + 1) \\
&= \alpha^2(\alpha x_{n-3} + \beta) + \beta(\alpha + 1) \\
&= \alpha^3 x_{n-3} + \beta(\alpha^2 + \alpha + 1) \\
&\vdots \\
&= \alpha^n x_0 + \beta(\alpha^{n-1} + \alpha^{n-2} + \ldots + \alpha^2 + \alpha + 1).
\end{aligned}$$

Note that if $\alpha = 1$, this give us

$$x_n = x_0 + n\beta.$$

If $\alpha \neq 1$, we know

$$\alpha^{n-1} + \alpha^{n-2} + \ldots + \alpha^2 + \alpha + 1 = \frac{1-\alpha^n}{1-\alpha}.$$

Hence

$$x_n = \alpha^n x_0 + \beta(\frac{1-\alpha^n}{1-\alpha}) \tag{1.2}$$

Definition: 1.6

A function of a complex variable z is said to be *analytic* at a point z_0 if its derivative exists at each point z in some neighborhood of z_0 [3].

Definition: 1.7

If a function fails to be analytic at a point z_0 but analytic at some points in every neighborhood of z_0, then the point z_0 is called a singular point of the function. If a function f is analytic everywhere in the neighborhood of a point z_0 except at z_0 itself, then the point z_0 is called an isolated singular point of f [3].

Theorem: 1.1

If a function f has an isolated singular point z_0, then the function can be represented by a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad ,$$

where $0 < |z - z_0| < r$ for some positive number r. The coefficients of the series are given by the formulas

$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)ds}{(s-z_0)^{n+1}} \qquad n = 0, 1, 2, ... \ ,$$

$$b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)ds}{(s-z_0)^{-n+1}} \qquad n = 1, 2, 3, ... \ ,$$

where C_1 and C_2 are two concentric circles centered at the point z_0 and with radii r_1 and r_2 respectively where $r_2 < r_1$. The portion of the series involving negative powers of $z - z_0$ is called the principal part of f at z_0 [3].

Definition: 1.8

If the principal part of the Laurent series expansion of f about an isolated singular point z_0 contains at least one nonzero term but the number of such terms is finite, that is

$$f(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + ... + \frac{b_m}{(z-z_0)^m}, \quad (0 < |z-z_0| < r)$$

where $m \in \mathbb{N}$, $b_m \neq 0$ and $b_{m+1} = b_{m+2} = ... = 0$, then z_0 is called a pole of of order m of the function f. A pole of order $m = 1$ is called a simple pole [3].

Definition: 1.9

When the principal part of Laurent series expansion of $f(z)$ a bout an isolated singular point z_0 has an infinite number of terms, then the point z_0 is called an essential singular point [3].

Definition: 1.10

A multiple-valued function F defined on a set $S \subset \mathbb{C}$ is said to have a branch point at $z_0 \in \mathbb{C}$ if, when z describes an arbitrary small circle about z_0, then for every branch f of F, $f(z)$ doesn't return to its original value. By a branch of F is mean a single-valued function f defined and analytic on some domain $D \subset S$ obtained from F in such away that at each point of D, f assumes exactly one of the possible values of F [4].

Definition: 1.11

A critical point is a branch point or an essential singularity in the solution of the ordinary differential equation. It is movable if its location in the complex plane depends on the constant of the integration of the ordinary differential equation [5].

1.2 Painlevé equations

Painlevé and his school classified the integrable second-order differential equations of the form

$$y^{''} = f(x, y, y^{'}), \tag{1.3}$$

where f is rational in y and $y^{'}$ and analytic in x.

Definition: 1.12

A family of solutions of the ordinary differential equations without movable critical points is said to have the Painlevé property or to be of Painlevé-type [5].

Painlevé and his colleagues were able to show that there are fifty canonical equations of the form (1.3) with this property. Six of these fifty equations, denoted by $P_I - P_{VI}$, cannot be solved in terms of the known functions and can be regarded as nonlinear analogues of the classical special functions. However, $P_{II} - P_{VI}$ have rational solutions and solutions expressible in terms of the classical special functions for certain values of parameters.
$P_I - P_{VI}$ also possess Bäcklund transformations which relate solutions of the same equation but with different values of parameters, or to solutions of another equation of Painlevé-type.

Discrete analogues of the Painlevé equations are mappings that are integrable in the same sense as the continuous Painlevé equations. The discrete Painlevé equations d-P_I–d-P_{VI} have the form

$$x_{n+1} = \frac{f_1(x_n; n) + x_{n-1} f_2(x_n; n)}{f_3(x_n; n) + x_{n-1} f_4(x_n; n)}, \tag{1.4}$$

where $f_j(x_n; n)$ are polynomials of degree at most four in x_n. In continuous limit, the discrete Painlevé equation yields a Painlevé equation. It is important to note that also some of the discrete Painlevé equations have limits more than one Painlevé equation.
Moreover, discrete Painlevé equations posses properties similar to the ones of the continuous Painlevé equations. For example, discrete Painlevé equations have rational solutions for certain values of parameters [11, 12, 13], have particular solutions for certain parameter values expressible in terms of discrete

analogue of special functions [10, 11, 15, 16, 17], and have Bäcklund transformations [11, 15, 17]. Discrete Painlevé equations also appears in physics. For example the computation of certain partition model of two-dimensional quantum gravity led to discrete P_I [18]. The only difference between continuous and discrete Painlevé equations is that the continuous Painlevé equations have unique canonical form up to a Möbius transformation, but there is more than one equivalent discrete equation which has the same Painlevé equation as its continuous limit.

In this thesis, we investigate the transformation properties of the discrete Painlevé equations by using an algorithmic method similar to the method developed by Fokas and Ablowitz [7] for investigating the transformation properties of the continuous Painlevé equations. The method of Fokas and Ablowitz can be summarized as follows. Consider a continuous Painlevé equation

$$v'' = P_1(v')^2 + P_2 v' + P_3, \tag{1.5}$$

where $P_1,\ P_2$ and P_3 depend on v, independent variable z and set of parameters θ. Consider transformation of the type

$$u(z;\hat{\theta}) = \frac{v' + av^2 + bv + c}{dv^2 + ev + f}, \tag{1.6}$$

where $a,\ b,\ c,\ d,\ e$, and f depend on z only and $u(z;\hat{\theta})$ solves some second-order equation of the Painlevé-type with set of parameters $\hat{\theta}$. If we solve

(1.6) for v', then we obtain

$$v' = (du - a)v^2 + (eu - b)v + (fu - c), \tag{1.7}$$

a Ricatti equation for v with the coefficients depending linearly on the solution u of a related Painlevé-type equation. Differentiating (1.7) to get v'', then substituting $v,\ v'\ ,\ v''$ in (1.5), we obtain an equation for u

$$u'' = \bar{P}_1(u')^2 + \bar{P}_2 u' + \bar{P}_3, \tag{1.8}$$

where $\bar{P}_1,\ \bar{P}_2$ and $\bar{P}_3$ are functions of $u,\ z$ and a set of parameters $\hat{\theta}$.

By following the similar argument, we have the following method to study the transformation properties of discrete Painlevé equation [1]. For a given discrete Painlevé equation (1.4) with parameter set α, we consider a discrete Riccati equation, that is

$$x_{n+1} = \frac{A_n x_n + B_n}{C_n x_n + D_n}, \tag{1.9}$$

where $A_n = A_{1,n} y_n + A_{0,1},\ B_n = B_{1,n} y_n + B_{0,n},\ C_n = C_{1,n} y_n + C_{0,n}$ and $D_n = D_{1,n} y_n + D_{0,n}$, such that y_n solves discrete equation of Painlevé-type with parameter-set $\hat{\alpha}$. We will find $A_{j,n}, ..., D_{j,n},\quad j = 0, 1$ requiring that (1.9) define one-to-one invertible map between the solutions x_n of a given discrete Painlevé equation, and solutions y_n of some second-order discrete equation of Painlevé-type. From equation (1.9) we have

$$x_{n-1} = -\frac{D_{n-1} x_n - B_{n-1}}{C_{n-1} x_n - A_{n-1}}. \tag{1.10}$$

Substituting x_{n+1} and x_{n-1} given by (1.9) and (1.10) respectively into a given discrete Painlevé equation (1.4), gives a polynomial for x_n with coefficients depending on $A_{j,n}, ..., D_{j,n},\ \ y_n$, and y_{n-1}. We choose $A_{j,n}, ..., D_{j,n}$ such that the polynomial for x_n reduces to a polynomial of degree one or of degree two. That is,

$$E(y_n, y_{n-1}; n)x_n + F(y_n, y_{n-1}; n) = 0, \tag{1.11}$$

or

$$E(y_n, y_{n-1}; n)x_n^2 + F(y_n, y_{n-1}; n)x_n + G(y_n, y_{n-1}; n) = 0. \tag{1.12}$$

If we solve (1.11) or (1.12) for x_n and substitute into (1.9) then we get a discrete equation of Painlevé-type for y_n.

This method yields explicit transformations between a given discrete Painlevé equation and the same discrete Painlevé equation but with different values of its parameters, and between two different discrete equations of Painlevé-type. We will also use this method to obtain particular solutions of discrete Painlevé equation in terms of discrete analogue of the classical functions. It turns out that, similar to the case of continuous Painlevé equations, discrete $P_{II} - P_V$ admit transformations of type (1.11). However, discrete P_{VI} does not admit a transformation of type (1.11). In this thesis we will consider the transformation of type (1.11) only.

Chapter 2

Discrete Painlevé II equation

Consider the discrete Painlevé II equation, d-P$_{II}$,

$$x_{n+1} + x_{n-1} = \frac{z_n x_n + a}{1 - x_n^2}, \tag{2.1}$$

where $z_n = \alpha n + \beta$, α, β, and a are constant. We will show that a continuous limit of (2.1) is the continuous Painlevé II equation

$$\omega'' = 2\omega^3 + t\omega + \mu.$$

Let $z_n = p + q\varepsilon^k n$, $x_n = r + s\varepsilon^m \omega(\varepsilon n)$, $t = \varepsilon n$ and $a = \varepsilon^l \mu$, where $p,\ q,\ r,\ s, k, m$ and l are constants to be determined.

Then $x_{n+1} = r + s\varepsilon^m \omega(\varepsilon n + \varepsilon)$ and $x_{n-1} = r + s\varepsilon^m \omega(\varepsilon n - \varepsilon)$.

By Taylor expansion at εn we have

$$\omega(\varepsilon n + \varepsilon) = \omega(\varepsilon n) + \omega'(\varepsilon n)\varepsilon + \omega''(\varepsilon n)\frac{\varepsilon^2}{2!} + ... \ \ , \tag{2.2}$$

and

$$\omega(\varepsilon n - \varepsilon) = \omega(\varepsilon n) - \omega'(\varepsilon n)\varepsilon + \omega''(\varepsilon n)\frac{\varepsilon^2}{2!} + ... \ \ , \tag{2.3}$$

Substituting x_{n+1} and x_{n-1} from (2.2) and (2.3) into (2.1) we get

$$\begin{aligned} s\varepsilon^{m+2}[1-r^2-2rs\varepsilon^k\omega-s^2\varepsilon^{2m}\omega^2]\omega'' &= 2s^3\varepsilon^{3m}\omega^3+6rs^2\varepsilon^{2m}\omega^2 \\ &+ s\varepsilon^m[6r^2-2+p+q\varepsilon^k n]\omega \\ &- 2r+2r^3+pr+rq\varepsilon^k n+\varepsilon^l\mu \\ &- s\varepsilon^m\mathcal{O}(\varepsilon^4)[1-r^2-2rs\varepsilon^m\omega-s^2\varepsilon^{2m}\omega^2]. \end{aligned} \tag{2.4}$$

Let $m=1$, $r=0$, $s=1$, $p=2$, $q=1$, $l=3$, and $k=3$. Thus (2.4) becomes

$$[1-\varepsilon^2\omega^2]\omega'' = 2\omega^3+t\omega+\mu-\mathcal{O}(\varepsilon^2)[1-\varepsilon^2\omega^2]. \tag{2.5}$$

Now the limit of (2.5) as $\varepsilon \longrightarrow 0$ gives

$$\omega'' = 2\omega^3+t\omega+\mu. \tag{2.6}$$

2.1 Bäcklund transformations for d-P_{II}

We will apply the method introduced in the introduction to d-P_{II}.

Substituting x_{n+1} and x_{n-1} from (1.9) and (1.10) respectively into (2.1) we get

$$\frac{A_nx_n+B_n}{C_nx_n+D_n}-\frac{D_{n-1}x_n-B_{n-1}}{C_{n-1}x_n-A_{n-1}}=\frac{z_nx_n+a}{1-x_n^2}$$

which give the following polynomial for x_n,

$$\begin{aligned} &(z_nx_n+a)(C_nx_n+D_n)(C_{n-1}x_n-A_{n-1}) = \\ &\quad(1-x_n^2)[(A_nx_n+B_n)(C_{n-1}x_n-A_{n-1}) \\ &\quad-(C_nx_n+D_n)(D_{n-1}x_n-B_{n-1})]. \end{aligned} \tag{2.7}$$

Now our aim is to choose $A_n, ..., B_n$ such that equation (2.7) become linear for x_n. Let $A_n = C_n = D_n = 1$. The equation (2.1) can be written as

$$(B_n + B_{n-1} + z_n - 2)x_n = B_n - B_{n-1} - a \tag{2.8}$$

where $B_n = B_{1,n}y_n + B_{0,n}$, then

$$B_n + B_{n-1} + z_n - 2 = B_{1,n}y_n + B_{1,n-1}y_{n-1} + B_{0,n} + B_{0,n-1} + z_n - 2.$$

Let $B_{1,n} = 1$, $B_{0,n} + B_{0,n-1} = -z_n + 2$. Now if $B_{0,n} = An + c$, then

$$B_{0,n} + B_{0,n-1} = 2An + 2c - A = -\alpha n - \beta + 2$$

which implies $A = -\frac{\alpha}{2}$, $c = -\frac{\beta}{2} - \frac{\alpha}{4} + 1$, then without loss of generality we can choose $B_n = y_n - z_n - \frac{\alpha}{4} + 1$, and equation (1.9) becomes

$$x_{n+1} = \frac{x_n + y_n - \frac{1}{2}z_n - \frac{1}{4}\alpha + 1}{x_n + 1}. \tag{2.9}$$

Solving equation (2.9) for y_n we get

$$y_n = (x_n + 1)(x_{n+1} - 1) + \frac{1}{2}z_n + \frac{1}{4}\alpha. \tag{2.10}$$

Equation (2.8) becomes

$$(y_n + y_{n-1} + \frac{1}{2}z_n - \frac{1}{2}z_{n-1} - \frac{\alpha}{2})x_n = y_n - y_{n-1} - \frac{\alpha}{2} - a. \tag{2.11}$$

Let $\nu = -\frac{\alpha}{2} - a$. Then equation (2.11) gives

$$x_n = \frac{y_n - y_{n-1} + \nu}{y_n + y_{n-1}} \tag{2.12}$$

Now we eleminate x_n between (2.10) and (2.12). Thus we obtain

$$y_n = (\frac{y_n - y_{n-1} + \nu}{y_n + y_{n-1}} + 1)(\frac{y_{n+1} - y_n + \nu}{y_{n+1} + y_n} - 1) + \frac{1}{2}z_n + \frac{1}{4}\alpha,$$

or

$$(y_n + y_{n-1})(y_n + y_{n+1}) = \frac{\nu^2 - 4y_n^2}{y_n - \frac{1}{2}z_n - \frac{1}{4}\alpha}. \tag{2.13}$$

Equation (2.13) is a discrete form of P_{34} [15]. The 34-th continuous Painlevé equation P_{34} is

$$\omega'' + 2\omega^2 - t\omega + \frac{\omega' - (\omega')^2 + \lambda}{2\omega - t} = 0,$$

where λ is a parameter [7].

Thus we obtained the Bäcklund transformation (2.10) and (2.12) between d-P_{II} and d-P_{34}.

2.2 Auto-Bäcklund transformation for d-P$_{II}$

Equation (2.13) is invariant under the change of parameters $\bar{\nu} = -\nu$ that is $y_n(\nu) = y_n(-\nu)$. Let $\bar{a} = -\bar{\nu} - \frac{\alpha}{2}$ and $\bar{x}_n = x_n(\bar{a})$. Then equation (2.12) gives

$$\begin{aligned} \bar{x}_n &= \frac{y_n(-\nu) - y_{n-1}(-\nu) - \nu}{y_n(-\nu) + y_{n-1}(-\nu)} \\ &= \frac{y_n(\nu) - y_{n-1}(\nu) - \nu}{y_n(\nu) + y_{n-1}(\nu)} \\ &= \frac{y_n(\nu) - y_{n-1}(\nu) + \nu}{y_n(\nu) + y_{n-1}(\nu)} - \frac{2\nu}{y_n(\nu) + y_{n-1}(\nu)}. \end{aligned}$$

That is

$$\bar{x}_n = x_n - \frac{2\nu}{y_n(\nu) + y_{n-1}(\nu)}. \tag{2.14}$$

Using (2.1) and (2.10) to eliminate y_n, we get

$$\begin{aligned}\bar{x}_n &= x_n - \frac{2\nu}{(x_n+1)(x_{n+1}-1)+\frac{1}{2}z_n+\frac{\alpha}{4}+(x_{n-1}+1)(x_n-1)+\frac{1}{2}z_{n-1}+\frac{\alpha}{4}} \\ &= x_n - \frac{2\nu}{x_n(x_{n+1}+x_{n-1})+(x_{n+1}-x_{n-1})+\alpha n+\beta-2} \\ &= x_n - \frac{2\nu}{x_n(\frac{z_n x_n+a}{1-x_n^2})+2x_{n+1}-(\frac{z_n x_n+a}{1-x_n^2})+z_n-2}. \end{aligned} \tag{2.15}$$

So that

$$\bar{x}_n = x_n - \frac{2\nu(x_n+1)}{2(x_{n+1}-1)(x_n+1)+z_n-a}; \quad \bar{a} = -a-\alpha. \tag{2.16}$$

Equation (2.16) is an auto-Bäcklund transformation for d-P_{II}. The auto-Bäcklund transformation (2.16) was given in [15, 17, 19].

2.3 Special solutions

The transformation given in (2.12) breaks down if

$$y_n + y_{n-1} = 0 \tag{2.17}$$

and

$$y_n - y_{n-1} + \nu = 0. \tag{2.18}$$

If we solve (2.17) and (2.18), then we get $y_n = \nu = 0$. Substituting into (2.10) we have

$$x_{n+1} = \frac{2x_n - z_n + a + 2}{2(x_n+1)}. \tag{2.19}$$

Eqution (2.19) is a discrete Riccati equation . Therefor equation (2.19) can be linearized by Cole-Hopf transformation $x_n = \frac{w_n}{w_{n-1}} - 1$ to get the discrete analogue of the Airy equation [15, 16, 17]:

$$2w_{n+1} + (z_n - a)w_{n-1} - 4w_n = 0. \tag{2.20}$$

The Auto-Bäcklund transformation (2.16) can be used to generate a hierachy of special solutions of d-P_{II}.

For example $x_n = 0$ is a solution of (2.1) with $a = 0$. Applying the auto-Bäcklund transformation (2.16) to this solution, we obtain $\bar{a} = -\alpha$ and

$$\bar{x}_n = \frac{\alpha}{z_n - 2}.$$

Substituting by $\bar{x_n}$ in (2.1), we have:

The left hand side is

$$\frac{\alpha}{z_{n+1} - 2} + \frac{\alpha}{z_{n-1} - 2} = \frac{\alpha}{(2z_n - 2) + \alpha} = \frac{\alpha}{(2z_{n-2} - 2) - \alpha} = \frac{4\alpha(z_n - 1)}{(2z_n - 2)^2 - \alpha^2}.$$

And the right hand side is

$$\frac{z_n(\frac{\alpha}{z_n-2}) - \alpha}{1 - (\frac{\alpha}{z_n-2})^2} = \frac{\alpha z_n(z_n - 2) - \alpha(z_n - 2)^2}{(2z_n - 2)^2 - \alpha^2} = \frac{4\alpha(z_n - 1)}{(2z_n - 2)^2 - \alpha^2}.$$

So that $\bar{x_n} = \frac{\alpha}{z_n-2}$ is also a solution of (2.1) with parameter $\bar{a} = -\alpha$.

As another example, let us start from the solution $x_n = \frac{-\alpha}{z_n-2}$, $a = \alpha$. Applying the auto-Bäcklund transformation (2.16) to this solution, we obtain another solution

$$\bar{x} = \frac{-\alpha}{z_n - 2} + \frac{3\alpha[(z_n - 2)^2 - \alpha^2}{[8 + \alpha - \alpha^2]z_n - [2 + \alpha]z_n^2 + 2\alpha + 5\alpha^2 - 8}\,, \qquad \bar{a} = -2\alpha.$$

Chapter 3

q-Discrete Painlevé II equation

Discrete Painlevé equations are of two distinct types [14]. The first one is that of differece equations where the independent varaible enters in an additive way. The second type is that of q-equation where the independent variable enters in a multiplicative way. In this chapter, we will consider an equation of the second type, namely the discrete Painlevé II equation, q-P_{II},

$$x_{n+1}x_{n-1} = \frac{a_n(x_n - b_n)}{x_n(x_n - 1)} \tag{3.1}$$

where $b_n = e^{n\alpha}$ and $a_n = ae^{n\alpha}$. The continuous limit of (3.1) gives P_{II} equation.

3.1 Bäcklund transformation of q-P$_{II}$

Now if we substitute x_{n+1} and x_{n-1} from (1.9) and (1.10) to (3.1), we have

$$(\frac{A_n x_n + B_n}{C_n x_n + D_n})(-\frac{D_{n-1}x_n - B_{n-1}}{C_{n-1}x_n - A_{n-1}}) = \frac{a_n(x_n - b_n)}{x_n(x_n - 1)}. \tag{3.2}$$

Equation (3.2) becomes linear for x_n by letting $A_n = 0$ and $C_n = -D_n = 1$. Thus we have

$$x_n(B_n - a_n) = -(B_n B_{n-1} + a_n b_n)$$

and

$$x_n = \frac{-(B_n B_{n-1} + a_n b_n)}{B_n - a_n}. \tag{3.3}$$

From (1.9) we have

$$B_n = x_{n+1}(x_n - 1). \tag{3.4}$$

Now by eliminating x_n between (3.4) and (3.3) we get

$$B_n = (\frac{B_n B_{n+1} + a_{n+1}b_{n+1}}{B_{n+1} - a_{n+1}})(\frac{B_n B_{n-1} + a_n b_n + B_n - a_n}{B_n - a_n}). \tag{3.5}$$

Equation (3.5) can be written as

$(B_n B_{n+1} + ae^{2(n+1)\alpha} - B_n ae^{(n+1)\alpha} - ae^{2(n+1)\alpha})(B_n - ae^{n\alpha}) =$

$(B_n B_{n+1} + ae^{2(n+1)\alpha})(B_n B_{n-1} + ae^{n\alpha} + B_n - ae^{n\alpha})$.

That is

$$(B_n B_{n+1} + ae^{2(n+1)\alpha})(B_n B_{n-1} + ae^{2n\alpha}) = -ae^{(n+1)\alpha}(B_n - ae^{n\alpha})(B_n + e^{(n+1)\alpha}).$$

Now let $B_n = \nu e^{(n+\frac{1}{2})\alpha} y_n$, and $\nu^2 = -a$. Then we get

$$e^{n\alpha}(y_n y_{n+1} - 1)(y_n y_{n-1} - 1) = (y_n + \nu e^{\frac{-1}{2}\alpha})(y_n + \frac{e^{\frac{\alpha}{2}}}{\nu}) \tag{3.6}$$

If we let $c = \frac{-\nu}{e^{\frac{\alpha}{2}}}$, then we obtain the $q - P_{34}$ equation [6]

$$(y_n y_{n+1} - 1)(y_n y_{n-1} - 1) = \frac{1}{b_n}(y_n - \frac{1}{c})(y_n - c). \qquad (3.7)$$

3.2 Auto Bäcklund transformation for q-P$_{II}$

Since (3.7) is invariant under the change of parameter $\bar{c} = \frac{1}{c}$, from (3.4) and (3.3) we have

$$y_n = \frac{c}{ab_n} x_{n+1}(x_n - 1) \qquad (3.8)$$

and

$$x_n = cb_n(\frac{y_n y_{n-1} - 1}{y_n - c}). \qquad (3.9)$$

Let $\bar{x}_n = x_n(\frac{1}{c})$. Then equation (3.9) gives

$$\bar{x}_n = \bar{c}b_n(\frac{y_n y_{n-1} - 1}{y_n - \bar{c}})$$

and we get

$$\bar{x}_n = \frac{x_n(x_{n-1}x_n - (x_n - b_n))}{(x_n - b_n)a\lambda + x_{n-1}x_n}, \qquad \bar{a} = \frac{e^{2\alpha}}{a} \qquad (3.10)$$

which is an auto-Bäcklund transformation for q-P_{II}.

As an example consider $x_n = \sqrt{b_n}$ which is a solution of (3.1) with $a = -1$. Then (3.10) gives

$$\bar{x}_n = \frac{\sqrt{b_n}(\sqrt{b_{n-1}}\sqrt{b_n} - (\sqrt{b_n} - b_n))}{-\lambda(\sqrt{b_n} - b_n)\sqrt{q_{n-1}}\sqrt{q_n}}$$

which gives the next rational solution

$$\bar{x}_n = \bar{c}\sqrt{b_n}(\frac{\bar{c} - \sqrt{b_n}(\bar{c} + 1)}{1 - \sqrt{b_n}(\bar{c} + 1)}), \qquad \bar{a} = -e^{2\alpha}.$$

Chapter 4

q-Discrete Painlevé III equation

In this chapter, we consider the discrete Painlevé III equation, q-P_{III}, [9]

$$x_{n+1}x_{n-1} = \frac{ab(x_n - p_n)(x_n - q_n)}{(x_n - a)(x_n - b)}, \tag{4.1}$$

where $p_n = p_0\lambda^n$, $q_n = q_0\lambda^n$, p_0, q_0, a, b, c and d are constants.

The continuous Painlevé III equation reads [7]

$$\omega'' = \frac{\omega'^2}{\omega} - \frac{1}{t}\omega' + \frac{1}{t}(\alpha\omega^2 + \beta) + \gamma\omega^3 + \frac{\delta}{\omega}$$

where α, β, γ, and δ are parameters.

4.1 Bäcklund transformation of $\mathbf{P}_{III}$

Substituting x_{n+1} and x_{n-1} from (1.9) and (1.10) into equation (4.1), we obtain

$$\frac{A_n x_n + B_n}{C_n x_n + D_n} \cdot \frac{D_{n-1} x_n - B_{n-1}}{C_{n-1} x_n - A_{n-1}} = -\frac{ab(x_n - p_n)(x_n - q_n)}{(x_n - a)(x_n - b)}. \tag{4.2}$$

Now our aim is to reduce the equation (4.2) to linear equation for x_n.

Choose $D_n = -aC_n$, $A_{n-1} = bC_{n-1}$. Then we get

$$(bC_{n-1} x_n B_n)(aC_{n-1} x_n + B_{n-1}) = abC_n C_{n-1}(x_n - p_n)(x_n - q_n),$$

which can be simplified to the following linear equation of x_n

$$\begin{gathered}[bC_{n-1}B_{n-1} + aC_{n-1}B_n + (q_n + p_n)abC_n C_{n-1}]x_n = \\ abC_n C_{n-1} p_n q_n - B_n B_{n-1}.\end{gathered} \tag{4.3}$$

Now from (1.9) we get

$$x_{n+1} = \frac{bC_n x_n + B_n}{C_n x_n - aC_n} = \frac{bx_n + \frac{B_n}{C_n}}{x_n - a}. \tag{4.4}$$

Without loss of generality, let $C_n = 1$, $B_n = \mu\lambda^{n+\frac{1}{2}} y_n$. Then (4.4) yields

$$x_{n+1} = \frac{bx_n + \mu\lambda^{n+\frac{1}{2}} y_n}{x_n - a} \tag{4.5}$$

and equation (4.3) becomes

$$x_n = \frac{abp_n q_n - \mu^2 \lambda^{2n} y_n y_{n-1}}{a\mu\lambda^{n+\frac{1}{2}} y_n + b\mu\lambda^{n-\frac{1}{2}} y_{n-1} + ab(p_n q_n)}. \tag{4.6}$$

Now we eliminate x_n between (4.5) and (4.6). After some calculation we get

$$bx_n + \mu\lambda^{n+\frac{1}{2}} y_n = \frac{a(\mu\lambda^{n+\frac{1}{2}} y_n + bq_n)(\mu\lambda^{n+\frac{1}{2}} y_n + bp_n)}{a\mu\lambda^{n+\frac{1}{2}} y_n + b\mu\lambda^{n-\frac{1}{2}} y_{n-1} + ab(p_n + q_n)}$$

and

$$x_n - a = \frac{ab(p_nq_n - a(p_n+q_n) - \mu\lambda^{n-\frac{1}{2}}y_{n-1}(\mu\lambda^{n+\frac{1}{2}}y_n + ab) - a^2\mu\lambda^{n+\frac{1}{2}}y_n}{a\mu\lambda^{n+\frac{1}{2}}y_n + b\mu\lambda^{n-\frac{1}{2}}y_{n-1} + ab(p_n+q_n)}.$$

By substituting in (4.5) we have

$$\begin{aligned}&\frac{abp_{n+1}q_{n+1} - \mu^2\lambda^{2n+2}y_ny_{n+1}}{a\mu\lambda^{n+\frac{3}{2}}y_{n+1} + b\mu\lambda^{n+\frac{1}{2}}y_n + ab(p_{n+1}+q_{n+1})} = \\ &\frac{a(\mu\lambda^{n+\frac{1}{2}}y_n + bq_n)(\mu\lambda^{n+\frac{1}{2}}y_n + bp_n)}{ab(p_nq_n - a(p_n+q_n)) - \mu\lambda^{n-\frac{1}{2}}y_{n-1}(\mu\lambda^{n+\frac{1}{2}}y_n + ab) - a^2\mu\lambda^{n+\frac{1}{2}}y_n}\end{aligned} \tag{4.7}$$

from which we get

$$\begin{aligned}&a(\mu\lambda^{n+\frac{1}{2}}y_n + bq_n)(\mu\lambda^{n+\frac{1}{2}}y_n + bp_n)[a\mu\lambda^{n+\frac{3}{2}}y_{n+1} + b\mu\lambda^{n+\frac{1}{2}}y_n + ab(p_{n+1}+q_{n+1})] = \\ &\mu\lambda^{n+\frac{3}{2}}y_{n+1}\mu^2\lambda^{2n}y_ny_{n-1}(\mu\lambda^{n+\frac{1}{2}}y_n + ab) + \\ &a\mu^2\lambda^{2n+2}y_ny_{n+1}(a\mu\lambda^{n+\frac{1}{2}}y_n + ab(p_n+q_n) - bp_nq_n) - \\ &abp_{n+1}q_{n+1}[\mu\lambda^{n-\frac{1}{2}}y_{n-1}(\mu\lambda^{n+\frac{1}{2}}y_n + ab) + a^2\mu\lambda^{n+\frac{1}{2}}y_n + a^2b(p_nq_n) - abp_nq_n].\end{aligned} \tag{4.8}$$

Equation (4.8) can be simplified to obtain the equation

$$\begin{aligned}&ab(\mu\lambda^{n+\frac{1}{2}}y_n + bq_n)(\mu\lambda^{n+\frac{1}{2}}y_n + bp_n)[\mu\lambda^{n+\frac{1}{2}}y_n + ap_{n+1} + aq_{n+1}] = \\ &(\mu^2\lambda^{2n+2}y_ny_{n+1} - abp_{n+1}q_{n+1})(\mu^2\lambda^{2n}y_ny_{n-1} - abp_nq_n) + \\ &\tfrac{ab}{\mu\lambda^{n+\frac{1}{2}}y_n}\{(\mu^2\lambda^{2n}y_ny_{n-1} - abp_nq_n)(\mu^2\lambda^{2n+2}y_ny_{n+1} - abp_{n+1}q_{n+1}) - \\ &a^2p_{n+1}q_{n+1}(\mu\lambda^{n+\frac{1}{2}}y_n + bq_n)(\mu\lambda^{n+\frac{1}{2}}y_n + bp_n)\}.\end{aligned} \tag{4.9}$$

Equation (4.9) gives

$$\begin{aligned}&ab(\mu\lambda^{n+\frac{1}{2}}y_n + bq_n)(\mu\lambda^{n+\frac{1}{2}}y_n + bp_n)[(\mu\lambda^{n+\frac{1}{2}}y_n + ap_{n+1})(\mu\lambda^{n+\frac{1}{2}}y_n + aq_{n+1})] = \\ &(\mu^2\lambda^{2n+2}y_ny_{n+1} - abp_{n+1}q_{n+1})(\mu^2\lambda^{2n}y_ny_{n-1} - abp_nq_n)(u_n + ab).\end{aligned} \tag{4.10}$$

Let $\mu^2 = abp_0q_0$. Then equation (4.6) becomes

$$x_n = \frac{\mu^2\lambda^{n+\frac{1}{2}}[1 - y_ny_{n-1}]}{a\lambda\mu y_n + b\mu y_{n-1} + ab\sqrt{\lambda}(p_0+q_0)} \tag{4.11}$$

and equation (4.5) becomes

$$y_n = \frac{x_{n+1}(x_n - a) - bx_n}{\mu\lambda^{n+\frac{1}{2}}}. \tag{4.12}$$

Therefor equation (4.10) can be simplified as follows

$$\begin{aligned} &ab\lambda^{4n}[(\mu\sqrt{\lambda}y_n + bq_0)(\mu\sqrt{\lambda}y_n + bp_0) \\ &\qquad (\mu\sqrt{\lambda}y_n + ap_0\lambda)(\mu\sqrt{\lambda}y_n + aq_0\lambda)] = \\ &\mu^4\lambda^{4n+2}(\mu\lambda^{n+\frac{1}{2}}y_n + ab)(y_ny_{n+1} - 1)(y_ny_{n-1} - 1). \end{aligned} \tag{4.13}$$

Equation (4.13) implies

$$\begin{aligned} &(y_ny_{n+1} - 1)(y_ny_{n-1}) = \\ &\frac{\frac{\mu^2}{p_0q_0}(y_n + \frac{\mu}{ap_0\sqrt{\lambda}})(y_n + \frac{\mu}{aq_0\sqrt{\lambda}})(y_n + \frac{ap_0\sqrt{\lambda}}{\mu})(y_n + \frac{aq_0\sqrt{\lambda}}{\mu})}{\mu\lambda^{n+\frac{1}{2}}(y_n + \frac{\mu}{p_0q_0\sqrt{\lambda}}\lambda^{-1})}. \end{aligned} \tag{4.14}$$

Now let $\alpha = \frac{-\mu}{ap_0\sqrt{\lambda}}$, $\beta = \frac{-\mu}{aq_0\sqrt{\lambda}}$, $\varphi_n = -\frac{\mu}{p_0q_0\sqrt{\lambda}}\lambda^{-n}$. Then equation (4.14) can be written as

$$(y_ny_{n+1} - 1)(y_ny_{n-1}) = \frac{-\varphi_n(y_n - \alpha)(y_n - \frac{1}{\alpha})(y_n - \beta)(y_n - \frac{1}{\beta})}{y_n - \varphi_n} \tag{4.15}$$

Equation (4.15) is the special case, $\delta = 0, \quad \gamma = 1$, of the discrete Painlevé equation q-P_V [9]

$$(y_ny_{n+1} - 1)(y_ny_{n-1}) = \frac{\varphi_n^2(y_n - \alpha)(y_n - \frac{1}{\alpha})(y_n - \beta)(y_n - \frac{1}{\beta})}{(\gamma y_n - \varphi_n)(\delta y_n - \varphi_n)}. \tag{4.16}$$

Thus there exists one-to-one correspondence given by (4.11) and (4.12) between the solutions of q-P_{III} and the solutions of (4.15)

4.2 Auto-Bäcklund transformation for q-P$_{III}$

Equation (4.15) is invariant under the change of parameters $\bar{\alpha} = \frac{1}{\alpha}$, $\bar{\beta} = \frac{1}{\beta}$, $\bar{\varphi}_0 = \varphi_0$. Which implies $\mu\bar{\mu} = a\bar{a}q_0\bar{q}_0\lambda$, $\mu\bar{\mu} = a\bar{a}p_0\bar{p}_0\lambda$ and $\mu\bar{a}\bar{b} = ab\bar{\mu}$. Then we have $q_0\bar{q}_0 = p_0\bar{p}_0$, $a\bar{a}\lambda^2 = b\bar{b}$, $p_0\bar{b} = a\bar{q}_0\lambda$, $\bar{b} = \frac{a\bar{q}_0\lambda}{p_0}$ and $\bar{a} = \frac{b\bar{q}_0}{\lambda p_0}$. Therefor equation (4.11) gives

$$\bar{x}_n = \frac{\bar{\mu}^2\lambda^{n+\frac{1}{2}}[1 - y_n y_{n-1}]}{\bar{a}\bar{\mu}\lambda y_n + \bar{b}\bar{\mu}y_{n-1} + \bar{a}\bar{b}\sqrt{\lambda}(\bar{q}_0 + \bar{p}_0)}. \tag{4.17}$$

However, we have

$$\bar{a}\bar{b}(\bar{p}_0 + \bar{q}_0)\sqrt{\lambda} = \frac{\sqrt{\lambda}ab\bar{\mu}\bar{q}_0}{\mu p_0}(p_0 + q_0).$$

Thus

$$\bar{x}_n = \frac{abq_0\bar{q}_0\lambda^{n+\frac{1}{2}}(1 - y_n y_{n-1})}{\mu b y_n + \mu a\lambda y_{n-1} + ab\sqrt{\lambda}(p_0 + q_0)}. \tag{4.18}$$

Now from equations (4.11) and (4.12) we have

$$(1 - y_n y_{n-1}) = \frac{x_n}{\mu^2\lambda^{2n}}[ax_{n+1}(x_n - a) + bx_{n-1}(x_n - b) - 2abx_n + ab(p+q)]$$

and

$$\begin{aligned}\mu b y_n + \mu a\lambda y_{n-1} + ab\sqrt{\lambda}(p_0 + q_0) = \tfrac{1}{\lambda^{n+\frac{1}{2}}}[&bx_{n+1}(x_n - a) - b^2x_n + \\ &a\lambda^2x_{n-1}(x_n - b) - a^2\lambda^2x_n + ab\lambda^{n+1}(p_0 + q_0)].\end{aligned} \tag{4.19}$$

Substituting into equation (4.18), we get the following auto-Bäcklund transformation for q-P$_{III}$

$$\bar{x}_n = \frac{\frac{\lambda\bar{q}_0}{p_0}x_n[ax_{n+1}(x_n - a) + bx_{n-1}(x_n - b) - 2abx_n + ab(p+q)]}{[bx_{n+1}(x_n - a) + a\lambda^2x_{n-1}(x_n - b) - (a^2\lambda^2 + b^2)x_n + ab\lambda(p+q)}, \tag{4.20}$$

$\bar{p}_0 = \frac{\bar{q}_0 q_0}{p_0}$, $\bar{b} = \frac{a\bar{q}_0\lambda}{p_0}$, $\bar{a} = \frac{b\bar{q}_0}{\lambda p_0}$.

4.3 Special solutions

The transformation $x_n = \dfrac{\mu^2\lambda^{n+\frac{1}{2}}[1 - y_n y_{n-1}]}{a\lambda\mu y_n + b\mu y_{n-1} + ab\sqrt{\lambda}(p_0 + q_0)}$ breaks down if

$$1 - y_n y_{n-1} = 0 \tag{4.21}$$

and

$$a\lambda\mu y_n + b\mu y_{n-1} + ab\sqrt{\lambda}(p_0 + q_0) = 0. \tag{4.22}$$

By solving (4.21) and (4.22) we obtain

$$a\mu\lambda y_n^2 + b\mu + ab\sqrt{\lambda}(p_0 + q_0)y_n = 0.$$

Thus

$$y_n = \frac{-b}{2\mu\sqrt{\lambda}}[p_0 + q_0 \mp (p_0 - q_0)]. \tag{4.23}$$

Therefor we have two solutions $y_n = \dfrac{-bp_0}{\mu\sqrt{\lambda}}$ and $y_n = \dfrac{-bq_0}{\mu\sqrt{\lambda}}$. Now $y_n y_{n-1} = 1$ implies that

$b^2 p_0^2 = \mu^2\lambda = abp_0q_0\lambda$ and hence $bp_0 = \lambda a q_0$. Substituting into equation (4.12) we get

$$x_{n+1} = \frac{bx_n - \frac{bp_0}{\mu\sqrt{\lambda}}\mu\lambda^{n+\frac{1}{2}}}{x_n - a}. \tag{4.24}$$

Equation (4.24) leads to the following discrete Riccati equation

$$x_{n+1} = \frac{b(x_n - p_0)}{x_n - a}. \tag{4.25}$$

Equation (4.25) can be linearized by the Cole-Hopf transformation

$$x_n = a + \frac{w_n}{w_{n-1}},$$

to obtain the discrete analogue of the Bessel equation [10]:

$$w_{n+1} - b(a-p)w_{n-1} + (a-b)w_n = 0. \qquad (4.26)$$

Therefor, we have shown that if $bp_0 = \lambda a q_0$, then the discrete Painlevé III equation, q-P$_{III}$ admits special solutions given by $x_n = a + \frac{w_n}{w_{n-1}}$, where w_n is a solution of the discrete Bessel equation (4.26).

Chapter 5

Discrete Painlevé IV equation

In this chapter, we consider the discrete Painlevé IV equation, d-P$_{IV}$,

$$(x_{n+1} + x_n)(x_{n-1} + x_n) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n - z_n)^2 - c^2}, \tag{5.1}$$

where $z_n = \alpha n + \beta$, and a, b, α, β are constants. The continuous form of $d - P_{IV}$ is

$$\omega'' = \frac{\omega'^2}{2\omega} + \frac{3}{2}\omega^3 + 4t\omega^2 + 2(t^2 - \mu)\omega + \frac{\lambda}{\omega},$$

where μ and λ are parameters [7].

5.1 Bäcklund transformation of P_{IV}

Substituting x_{n+1} and x_{n-1} from equations (1.9) and (1.10) into equation (5.1) we get

$$\frac{(x_n^2-a^2)(x_n^2-b^2)}{(x_n-z_n)^2-c^2} = \frac{[C_nx_n^2+(D_n+A_n)x_n+B_n][C_{n-1}x_n^2-(D_{n-1}+A_{n-1})x_n+B_{n-1}]}{(C_nx_n+D_n)(C_{n-1}x_n-A_{n-1})}. \tag{5.2}$$

In order to reduce (5.2) to linear equation for x_n, let us choose $C_n = 1$, $B_n = ab$, $D_n + A_n = a + b$. Then (5.2) becomes

$$\frac{(x_n^2-a^2)(x_n^2-b^2)}{(x_n-z_n)^2-c^2} = \frac{(x_n+a)(x_n+b)(x_n-a)(x_n-b)}{(x_n+D_n)(x_{n-1}-A_{n-1})}. \tag{5.3}$$

Therefor we get

$$(x_n+D_n)(x_n-A_{n-1}) = (x_n-z_n)^2-c^2. \tag{5.4}$$

Equation (5.4) reduces to the following linear equation for x_n

$$(D_n-A_{n-1}+2z_n)x_n = D_nA_{n-1}+z_n^2-c^2. \tag{5.5}$$

Without loss of generality we may choose $A_n = -(y_n - z_n + \mu)$. Then

$D_n = y_n - z_n + \mu + a + b$, where $\mu = -\frac{1}{2}(a + b + \alpha)$. Now from (5.5) we have

$$x_n = \frac{-(y_n-z_n-(\mu+\alpha))(y_{n-1}-z_n+\mu+\alpha)+z_n^2-c^2}{y_n+y_{n-1}}. \tag{5.6}$$

Hence

$$x_n = \frac{-\{y_{n-1}y_n-y_{n-1}(z_n+\mu+\alpha)-y_n(z_n-\mu-\alpha)-(\mu+\alpha)^2+c^2\}}{y_n+y_{n-1}}. \tag{5.7}$$

Equation (1.9) becomes

$$x_{n+1} = \frac{-(y_n - z_n + \mu)x_n + ab}{x_n + y_n - z_n - \mu - \alpha}. \tag{5.8}$$

Then from (5.7) we have

$$\begin{aligned} x_n + y_n - z_n - \mu - \alpha &= \frac{-1}{y_n + y_{n-1}}\{-y_{n-1}y_n - y_{n-1}(z_n + \mu + \alpha) \\ &- y_n(z_n - \mu - \alpha) - (\mu + \alpha)^2 - c^2 \\ &- (y_n + y_{n-1})(y_n z_n - \mu - \alpha)\}. \end{aligned} \tag{5.9}$$

Equation (5.9) can be simplified to obtain the equation

$$x_n + y_n - z_n - \mu - \alpha = \frac{(y_n - \mu - \alpha)^2 - c^2}{y_n + y_{n-1}}. \tag{5.10}$$

Similarly

$$\begin{aligned} -(y_n - z_n + \mu)x_n + ab &= \frac{(y_n - z_n + \mu)}{y_n + y_{n-1}}[y_{n-1}y_n - y_{n-1}(z_n + \mu + \alpha) \\ &- y_n(z_n - \mu - \alpha) - (\mu + \alpha)^2 + c^2] + ab \\ &= \frac{1}{y_n + y_{n-1}}\{(y_n - z_n + \mu)[y_{n-1}(y_n - z_n - \mu - \alpha) \\ &- y_n(z_n - \mu - \alpha) - (\mu + \alpha)^2 + c^2] + ab(y_n + y_{n-1})\} \\ &= \frac{1}{y_n + y_{n-1}}\{y_{n-1}[(y_n - z_n - \mu - \alpha)(y_n - z_n + \mu) + ab] \\ &- y_n[(z_n - \mu - \alpha)(y_n - z_n + \mu) + ab] \\ &+ (c^2 - (\mu + \alpha)^2)(y_n - z_n + \mu)\}. \end{aligned} \tag{5.11}$$

Substituting into (5.8) and we obtain

$$x_{n+1} = \frac{-\{y_{n+1}y_n - y_n(z_{n+1} + \mu + \alpha) - y_{n+1}(z_{n+1} + \mu - \alpha) - (\mu + \alpha)^2 + c^2\}}{y_n + y_{n+1}}.$$

But since $z_{n+1} = z_n + \alpha$ and $z_{n-1} = z_n - \alpha$, we have

$$x_{n+1} = \frac{-\{y_{n+1}(y_n - z_n + \mu) - y_n(z_n + \mu + 2\alpha) - (\mu + \alpha)^2 + c^2\}}{y_n + y_{n+1}}. \quad (5.12)$$

Substituting (5.7) and (5.10) into (5.8), we obtain

$$\begin{aligned} x_{n+1} \quad = \quad & \frac{(y_n - z_n + \mu)\{y_{n-1}(y_n - z_n - \mu - \alpha) - y_n(z_n - \mu - \alpha) - (\mu + \alpha)^2 + c^2\}}{(y_n - \mu - \alpha)^2 - c^2} \\ & + \frac{(y_n + y_{n-1})ab}{(y_n - \mu - \alpha)^2 - c^2}. \end{aligned}$$

Then x_{n+1} can be written as

$$\begin{aligned} x_{n+1} = & \frac{y_{n-1}[(y_n - z_n - \mu - \alpha)(y_n - z_n + \mu) + ab]}{(y_n - \mu - \alpha)^2 - c^2} \\ - & \frac{y_n(z_n - \mu - \alpha)(y_n - z_n + \mu) + aby_n + (c^2 - (\mu + \alpha)^2)(y_n - z_n + \mu)}{(y_n - \mu - \alpha)^2 - c^2}. \end{aligned} \quad (5.13)$$

Eliminating x_n between (5.12) and (5.13) we get

$$\begin{aligned} & \frac{-\{y_{n+1}(y_n - z_n + \mu) - y_n(z_n + \mu + 2\alpha) - (\mu + \alpha)^2 + c^2\}}{y_n + y_{n+1}} \\ = \quad & \frac{1}{(y_n - \mu - \alpha)^2 - c^2}\{y_{n-1}[(y_n - z_n - \mu - \alpha)(y_n - z_n + \mu) + ab] \\ - \quad & y_n(z_n - \mu - \alpha)(y_n - z_n + \mu) + aby_n + (c^2 - (\mu + \alpha)^2)(y_n - z_n + \mu)\}. \end{aligned}$$

Therefor

$$\begin{aligned} & (y_n + y_{n+1})\{(y_{n-1} + y_n)[(y_n - z_n + \mu)(y_n - z_n - \mu - \alpha) + ab] \\ & -y_n[(y_n - z_n + \mu)(y_n - z_n - \mu - \alpha) + ab] \\ & -y_n[(y_n - z_n + \mu)(z_n - \mu - \alpha)] + aby_n + (c^2 - (\mu + \alpha)^2)(y_n - z_n + \mu)\} \quad (5.14) \\ & = -[(y_n - \mu - \alpha)^2 - c^2]\{(y_{n+1} + y_n)(y_n - z_n + \mu) - y_n(y_n - z_n + \mu) \\ & \qquad -y_n(z_n + \mu + 2\alpha) - (\mu + \alpha)^2 + c^2\}. \end{aligned}$$

Equation (5.14) can be simplified to give

$$(y_n + y_{n+1})\{(y_n + y_{n-1})[(y_n - z_n - \frac{\alpha}{2} + \mu + \frac{\alpha}{2})(y_n - z_n - \frac{\alpha}{2} - \mu - \frac{\mu}{2})]$$
$$-(y_n - z_n + \mu)[y_n(y_n - z_n - \mu - \alpha) + y_n(z_n - \mu - \alpha) - c^2 + (\mu + \alpha)^2$$
$$+c^2 - (y_n - \mu - \alpha)^2]\} = [(y_n - \mu - \alpha)^2 - c^2][y_n^2 - z_n y_n + 2\mu y_n$$
$$+y_n z_n + 2\alpha y_n + (\mu + \alpha)^2 - c^2] = [(y_n - \mu - \alpha)^2 - c^2][(y_n + \mu + \alpha)^2 - c^2].$$

So that

$$(y_n + y_{n+1})\{(y_n + y_{n-1})[(y_n - z_n - \frac{\alpha}{2})^2)(\mu + \frac{\alpha}{2})^2 + ab]$$
$$-(y_n - z_n + \mu)[y_n^2 - 2(\mu + \alpha)y_n + (\mu + \alpha)^2 - (y_n - \mu - \alpha)^2]\}$$
$$= [(y_n - z_n - \alpha)^2 - c^2][(y_n + \mu + \alpha) - c^2]$$

and hence

$$(y_n + y_{n+1})(y_n + y_{n-1})[(y_n - z_n - \frac{\alpha}{2})^2(\mu + \frac{\alpha}{2})^2 + ab]$$
$$= [(y_n - \mu - \alpha)^2 - c^2][(y_n + \mu + \alpha)^2 - c^2]. \tag{5.15}$$

Therefor we obtain the equation

$$(y_n + y_{n+1})(y_n + y_{n-1}) = \frac{[(y_n - \mu - \alpha)^2 - c^2][(y_n + \mu + \alpha)^2 - c^2]}{(y_n - z_n - \frac{\alpha}{2})^2 - (\mu + \frac{\alpha}{2})^2 + ab}. \tag{5.16}$$

The numerator of the right hand side of equation (5.16) can be written as

$[(y_n - \mu - \alpha)^2 - c^2][(y_n + \mu + \alpha)^2 - c^2]$

$= ((y_n - \mu - \alpha) - c)(y_n - \mu - \alpha + c)(y_n - (\mu + \alpha - c))(y_n + (\mu + \alpha - c))$

$= [y_n^2 - (\mu + \alpha + c)^2][y_n^2 - (\mu + \alpha - c)^2]$

Now let

$\bar{a}^2 = (\mu+\alpha+c)^2$ and $\bar{b}^2 = [\mu+\alpha-c]^2$. Then we have $\bar{a}^2 = (c+\alpha-\frac{1}{2}(a+b+\alpha))^2$ and $\bar{b}^2 = [-c+\alpha-\frac{1}{2}(a+b+\alpha)]^2 = [-c-\frac{1}{2}(a+b-\alpha)]^2$. Let $\bar{z}_n = z_n + \frac{\alpha}{2}$ and $\bar{c}^2 = [(\mu+\frac{\alpha}{2})^2 - ab] = \frac{1}{4}[a-b]^2$. Then equation (5.16) becomes

$$(y_n + y_{n+1})(y_n + y_{n-1}) = \frac{(y_n^2 - \bar{a}^2)(y_n^2 - \bar{b}^2)}{(y_n - \bar{z}_n)^2 - \bar{c}^2}. \tag{5.17}$$

Equation (5.17) is d-P$_{IV}$ with

$$\begin{aligned} &\bar{c}^2 = \tfrac{1}{4}(a-b)^2, \quad \bar{z}_n = z_n + \tfrac{\alpha}{2}, \quad \bar{b}^2 = [-c - \tfrac{1}{2}(a+b-\alpha)]^2, \\ &\bar{a}^2 = [c + \alpha - \tfrac{1}{2}(a+b+\alpha)]^2. \end{aligned} \tag{5.18}$$

Thus we have obtained a Bäcklund transformation between d-P_{IV} with parameters $a,\ b$ and c and d-P_{IV} with parameters $\bar{a},\ \bar{b}$ and $\bar{c}$

5.2 Auto-Bäcklund transformation for d-P$_{IV}$

From equation (5.8) we have

$$x_{n+1}[y_n + x_n - z_n - \mu - \alpha] = -x_n y_n - x_n(-z_n + \mu) + ab$$

Then

$$[x_{n+1} + x_n]y_n + x_{n+1}[x_n - z_n - \mu - \alpha] = -x_n(\mu - z_n) + ab. \tag{5.19}$$

Thus, if we replace y_n by $\bar{x}_n$ in (5.19), then we obtain the following auto-Bäcklund transformation for d-P_{IV}

$$\bar{x}_n = \frac{-[x_{n+1}(x_n - z_n - \mu - \alpha) - x_n(z_n - \mu) - ab]}{x_{n+1} + x_n} \tag{5.20}$$

such that $\bar{x}_n$ solves $d - P_{IV}$ with the parameters $\bar{a}$, $\bar{b}$, $\bar{c}$ given by (5.18). The Bäcklund transformation (5.18) for discrete Painlevé IV equation was first given in [11].

5.3 Special solutions

The transformation (5.7) breaks down if

$$y_{n-1} = y_n \tag{5.21}$$

and

$$(y_{n-1} - z_n + \mu + \alpha)(y_n - z_n - \mu - \alpha) = z_n^2 - c^2. \tag{5.22}$$

By solving (5.22), we have $-(y_n - \mu - \alpha)^2 + z_n^2 = z_n^2 - c^2$. That is $y_n = \mu + \alpha + c$. From (5.21) we have $y_n = 0$. Therefor $\mu + \alpha + c = 0$. But $2\mu = -a - b - \alpha$. Thus $2c + \alpha = a + b$. Now substituting into (5.8) yields the following Riccati equation

$$x_{n+1} = \frac{(a + b + z_n - c)x_n + ab}{x_n + c - z_n} \tag{5.23}$$

after substituting $y_n = \mu + \alpha + c = 0$ [11, 20].

Now the Cole-Hopf transformation $x_n = z_n - c\dfrac{w_n}{w_{n-1}}$, gives

$$x_{n+1} = \frac{(z_n + \alpha - c)w_n + w_{n+1}}{w_n}.$$

Substituting into (5.23) gives

$$\begin{aligned} &[\frac{(z_n + \alpha - c)w_n + w_{n+1}}{w_n}][\frac{(z_n - c)w_{n-1} + w_n}{w_{n-1}} + c - z_n] = \\ &(a + b + z_n - c)(z_n - c + \frac{w_n}{w_{n-1}}) + ab. \end{aligned} \tag{5.24}$$

Therefor we get the equation

$$\frac{1}{w_{n-1}}[(z_n + \alpha - c)w_n + w_{n+1}] = (a + b + z_n - c)(z_n - c + \frac{w_n}{w_{n-1}}) + ab$$

which is simplified to the following linear equation for w_n:

$$w_{n+1} - 2cw_n - [(z_n - c + a)(z_n - c + b)]w_{n-1} = 0. \tag{5.25}$$

Equation (5.21) has been shown to be solvable in terms of the discrete analogues of Hermite functions [11].

Chapter 6

Discrete Painlevé V equation

In this chapter, we consider the discrete Painlevé V equation, q-P$_V$,

$$(x_n x_{n+1} - 1)(x_{n-1} x_n - 1) = \frac{p_n q_n (x_n - a)(x_n - b)(x_n - \frac{1}{a})(x_n - \frac{1}{b})}{(x_n - p_n)(x_n - q_n)}, \quad (6.1)$$

where $p_n = p_0 \lambda^n$, $q_n = q_0 \lambda^n$, and a, b, p_0, q_0 are constants. The continuous form of Painlevé V equation is [7]

$$\omega'' = \frac{3\omega - 1}{2\omega(\omega - 1)}\omega'^2 - \frac{1}{2}\omega' + \frac{\alpha}{t^2}\omega(\omega - 1)^2 + \frac{\beta}{t^2}\frac{(\omega - 1)^2}{\omega} + \frac{\gamma}{t}\omega + \frac{\delta\omega(\omega + 1)}{\omega - 1}$$

where α, β, γ and δ are parameters.

6.1 Bäcklund transformation of $\mathbf{P}_V$

Using (1.9) and (1.10) to substitute x_{n-1} and x_{n+1} into (6.1), we have

$$\frac{[A_n x_n^2 + (B_n - C_n)x_n - D_n][-D_{n-1}x_n^2 + (B_{n-1} - C_{n-1})x_n + A_{n-1}]}{(C_n x_n + D_n)(C_{n-1}x_n - A_{n-1})} = \frac{p_n q_n (x_n - a)(x_n - \frac{1}{a})(x_n - b)(x_n - \frac{1}{b})}{(x_n - p_n)(x_n - q_n)}. \quad (6.2)$$

Let $D_n = -abA_n$, $B_n = C_n - (a+b)A_n$. Then the left hand side of (6.2) reduces to

$$\frac{[A_n x_n^2 - (a+b)A_n x_n + abA_n][abA_{n-1}x_n^2 + (a+b)A_{n-1}x_n + A_{n-1}]}{(C_n x_n - abA_n)(C_{n-1}x_n - A_{n-1})} = \frac{A_n A_{n-1}[(x_n - a)(x_n - b)(ax_n - 1)(bx_n - 1)]}{(C_n x_n - abA_n)(C_{n-1}x_n - A_{n-1})}.$$

Then from (6.2) we have

$$abA_n A_{n-1}(x_n - p_n)(x_n - q_n) = p_n q_n (C_n x_n - abA_n)(C_{n-1}x_n - A_{n-1}). \quad (6.3)$$

Equation (6.3) can be simplified to give

$$abA_n A_{n-1}x_n - (p_n + q_n)abA_n A_{n-1} = p_n q_n C_n C_{n-1}x_n - p_n q_n (C_n A_{n-1} + abA_n C_{n-1}).$$

Setting $C_n = 1$ gives the following linear equation for x_n

$$(abA_n A_{n-1} - p_n q_n)x_n = ab(p_n + q_n)A_n A_{n-1} - p_n q_n (A_{n-1} + abA_n). \quad (6.4)$$

Without loss of generality, let $A_n = \mu\lambda^n u_n$. Then

$$abA_n A_{n-1} - p_n q_n = p_n q_n (u_n u_{n-1} - 1),$$

where $p_n q_n = \mu^2 ab\lambda^{2n-1}$. Then equations (6.4) and (1.9) becomes

$$x_n = \frac{(p_n + q_n)u_n u_{n-1} - \mu\lambda^{n-1}u_{n-1} - ab\mu\lambda^n u_n}{u_n u_{n-1} - 1} \quad (6.5)$$

and

$$x_{n+1} = \frac{\mu\lambda^n u_n[x_n - a - b] + 1}{x_n - ab\mu\lambda^n u_n}. \tag{6.6}$$

From equation (6.5) we have

$$x_{n+1} = \frac{\lambda(p_n + q_n)u_n u_{n+1} - \mu\lambda^n u_n - ab\mu\lambda^{n+1}u_{n+1}}{u_n u_{n+1} - 1}$$

and

$$\begin{aligned} x_n - ab\mu\lambda^n u_n = \frac{1}{u_n u_{n-1} - 1}[(p_n + q_n)u_n u_{n-1} - \mu\lambda^{n-1}u_{n-1} - \\ ab\mu\lambda^n u_n - ab\mu\lambda^n u_n(u_n u_{n-1} - 1)]. \end{aligned} \tag{6.7}$$

Then

$$x_n - ab\mu\lambda^n u_n = \frac{-u_{n-1}}{u_n u_{n-1} - 1}[ab\mu\lambda^n u_n^2 - (p_n + q_n)u_n + \mu\lambda^{n-1}]. \tag{6.8}$$

Equation (6.6) can be written as

$$\begin{aligned} x_{n+1} &= \frac{\mu\lambda^n u_n[x_n - ab\mu\lambda^n u_n + ab\mu\lambda^n u_n - a - b] + 1}{x_n - ab\mu\lambda^n u_n} \\ &= \frac{\mu\lambda^n u_n(x_n - ab\mu\lambda^n u_n)}{x_n - ab\mu\lambda^n u_n} + \frac{ab\mu^2\lambda^{2n}u_n^2 - (a+b)\mu\lambda^n u_n + 1}{x_n - ab\mu\lambda^n u_n}. \end{aligned}$$

Therefor

$$x_{n+1} - \mu\lambda^n u_n = \frac{ab\mu^2\lambda^{2n}u_n^2 - (a+b)\mu\lambda^n u_n + 1}{x_n - ab\mu\lambda^n u_n}. \tag{6.9}$$

Using (6.8) to substitute $x_n - ab\mu\lambda^n u_n$ into (6.9), we get

$$x_{n+1} - \mu\lambda^n u_n = \frac{-(u_n u_{n-1} - 1)[ab\mu^2\lambda^{2n}u_n^2 - (a+b)\mu\lambda^n u_n + 1]}{u_{n-1}[ab\mu\lambda^n u_n^2 - (p_n + q_n)u_n + \mu\lambda^{n-1}]}. \tag{6.10}$$

Equation (6.5) now yields

$$x_{n+1} - \mu\lambda^n u_n = \frac{1}{u_n u_{n+1} - 1}[\lambda(p_n + q_n)u_n u_{n+1} - \mu\lambda^n u_n -$$

$$ab\mu\lambda^{n+1}u_{n+1} - \mu\lambda^n u_n(u_n u_{n+1} -)].$$

Therefor

$$x_{n+1} - \mu\lambda^n u_n = \frac{-u_{n+1}}{u_n u_{n+1} - 1}[\mu\lambda^n u_n^2 - \lambda(p_n + q_n)u_n + ab\mu\lambda^{n+1}]. \qquad (6.11)$$

Thus equations (6.10) and (6.11) gives

$$\frac{u_{n+1}}{u_n u_{n+1} - 1}[\mu\lambda^n u_n^2 - \lambda(p_n + q_n)u_n + ab\mu\lambda^{n+1}] = \frac{(u_n u_{n-1} - 1)[ab\mu^2\lambda^{2n}u_n^2 - (a+b)\mu\lambda^n u_n + 1]}{u_{n-1}[ab\mu\lambda^n u_n^2 - (p_n + q_n)u_n + \mu\lambda^{n-1}]}.$$

Therefor

$$\frac{(u_n u_{n+1} - 1)(u_n u_{n-1} - 1)}{u_{n+1}u_{n-1}} = \frac{[ab\mu\lambda^n u_n^2 - (p_n + q_n)u_n + \mu\lambda^{n-1}][\mu\lambda^n u_n^2 - \lambda(p_n + q_n)u_n + ab\mu\lambda^{n+1}]}{[ab\mu^2\lambda^{2n}u_n^2 - (a+b)\mu\lambda^n u_n + 1]}, \qquad (6.12)$$

that is

$$\frac{(u_n u_{n+1} - 1)(u_n u_{n-1} - 1)}{u_{n+1}u_{n-1}} = \frac{[ab\mu\lambda^n u_n^2 - (p_n + q_n)u_n + \mu\lambda^{n-1}][\mu\lambda^n u_n^2 - \lambda(p_n + q_n)u_n + ab\mu\lambda^{n+1}]}{(a\mu\lambda^n u_n - 1)(b\mu\lambda^n u_n - 1)}. \qquad (6.13)$$

Let $u_n = \dfrac{1}{y_n}$, then we have $u_n u_{n-1} - 1 = \dfrac{1 - y_n y_{n+1}}{y_n y_{n+1}}$, and $u_n u_{n-1} - 1 = \dfrac{1 - y_n y_{n-1}}{y_n y_{n-1}}$. Then equation (6.13) becomes

$$\begin{aligned} &\frac{y_{n+1}y_{n-1}(1 - y_n y_{n-1})(1 - y_n y_{n+1})}{y_n^2 y_{n+1} y_{n-1}} \\ = \; &\frac{[ab\lambda^n \frac{1}{y_n^2} - (p_n + q_n)\frac{1}{y_n} + \mu\lambda^{n-1}][\mu\lambda^n \frac{1}{y_n^2} - \lambda(p_n + q_n)\frac{1}{y_n} + ab\mu\lambda^{n+1}]}{(a\mu\lambda^n \frac{1}{y_n} - 1)(b\mu\lambda^n \frac{1}{y_n} - 1)} \\ = \; &\frac{[ab\lambda^n - (p_n + q_n)y_n + \mu\lambda^{n-1}y_n^2][\mu\lambda^n - \lambda(p_n + q_n)y_n + ab\mu\lambda^{n+1}y_n^2]}{y_n^4[a\mu\lambda^n \frac{1}{y_n} - 1][b\mu\lambda^n \frac{1}{y_n} - 1]}. \end{aligned}$$

Thus

$$(y_n y_{n-1} - 1)(y_n y_{n+1} - 1) = \frac{\lambda[ab\mu\lambda^n y_n^2 - (p_n + q_n)y_n + \mu\lambda^{n-1}][\mu\lambda^{n-1} y_n^2 - (p_n + q_n)y_n + ab\mu\lambda^n]}{(y_n - b\mu\lambda^n)(y_n - a\mu\lambda^n)}. \tag{6.14}$$

Since $ab = \dfrac{p_n q_n}{\mu^2 \lambda^{2n-1}}$, we have

$$ab\mu\lambda^n y_n^2 - (p_n + q_n)y_n + \mu\lambda^{n-1} = \frac{1}{\mu\lambda^{n-1}}(p_n y_n - \mu\lambda^{n-1})(q_n y_n - \mu\lambda^{n-1}).$$

Let $\alpha = \frac{\mu\lambda^{n-1}}{p_n}$, $\beta = \frac{\mu\lambda^{n-1}}{q_n}$. Then

$$ab\mu\lambda^n y_n^2 - (p_n + q_n)y_n + \mu\lambda^{n-1} = \frac{p_n q_n}{\mu\lambda^{n-1}}(y_n - \alpha)(y_n - \beta),$$

and

$$\mu\lambda^{n-1} y_n^2 - (p_n + q_n)y_n + ab\mu\lambda^n = \mu\lambda^{n-1}(y_n - \frac{1}{\alpha})(y_n - \frac{1}{\beta}).$$

Hence equation (6.14) becomes

$$(y_n y_{n-1} - 1)(y_n y_{n+1} - 1) = \frac{\lambda p_n q_n (y_n - \alpha)(y_n - \frac{1}{\alpha})(y_n - \beta)(y_n - \frac{1}{\beta})}{(y_n - \varphi_n)(y_n - \psi_{n)}}. \tag{6.15}$$

Equation (6.15) is q-P$_V$ where $\alpha = \dfrac{\mu}{p_0\lambda}$, $\beta = \dfrac{\mu}{q_0\lambda}$, $\varphi = a\mu\lambda^n$, and $\psi_n = b\mu\lambda^n$.

6.2 Auto-Bäcklund transformation for d-P$_V$

From equations (6.5) and (6.6) we have

$$x_n = \frac{ab\mu\lambda^n y_{n-1} + \mu\lambda^{n-1} y_n - (p_n + q_n)}{y_n y_{n-1} - 1} \tag{6.16}$$

and

$$x_{n+1} = \frac{\mu\lambda^n (x_n - a - b) + y_n}{y_n x_n - ab\mu\lambda^n}. \tag{6.17}$$

Then

$$x_{n+1}(y_n x_n - ab\mu\lambda^n) = \mu\lambda^n(x_n - a - b) + y_n.$$

So we have

$$y_n(x_{n+1}x_n - 1) = \mu\lambda^n[x_n - a - b + abx_{n+1}].$$

Replacing y_n by $\bar{x}_n$, we have the following auto-Bäcklund transformation for d-P$_V$

$$\bar{x}_n = \frac{\mu\lambda^n[x_n - a - b + abx_{n+1}]}{x_{n+1}x_n - 1}, \tag{6.18}$$

where $\bar{a} = \dfrac{\mu q_0}{\lambda}$, $\bar{b} = \dfrac{\mu p_0}{\lambda}$, $\bar{q_0} = \dfrac{1}{a\mu}$ and $\bar{p_0} = \dfrac{1}{b\mu}$.

6.3 Special solutions

The transformation (6.16) breaks down when

$$y_n y_{n-1} = 1 \tag{6.19}$$

and

$$ab\mu\lambda^n y_{n-1} + \mu\lambda^{n-1}y_n - (p_n + q_n) = 0. \tag{6.20}$$

From (6.19) we have $y_{n-1} = \frac{1}{y_n}$, so (6.20) becomes $y_n^2 + ab\lambda - \frac{\lambda}{\mu}(p_0+q_0)y_n = 0$. Then

$$y_n = \frac{\frac{\lambda}{\mu}(p_0 + q_0) \pm \sqrt{\frac{\lambda^2}{\mu^2}(p_0 + q_0)^2 - 4ab\lambda}}{2},$$

that is $y_n = \dfrac{\lambda q_0}{\mu}$ or $y_n = \dfrac{\lambda p_0}{\mu}$. Since $y_n y_{n-1} = 1$, we have $\dfrac{\lambda^2 p_0^2}{\mu^2} = 1$. Thus $\lambda^2 p_0^2 = \mu^2 = \dfrac{p_0 q_0 \lambda}{ab}$, and hence $q_0 = ab\lambda p_0$. Substituting $y_n = \dfrac{\lambda p_0}{\mu}$ into (6.17) we get

$$x_{n+1} = \frac{\mu\lambda^n(x_n - a - b) + \dfrac{\lambda p_0}{\mu}}{\dfrac{\lambda p_0}{\mu}x_n - \dfrac{p_0 q_0 \lambda\mu\lambda^n}{\mu^2}}.$$

After simplification, we get the following discrete Riccati equation [29],

$$x_{n+1} = \frac{\mu^2\lambda^{n-1}[x_n - a - b] + p_0}{p_0[x_n - q_0\lambda^n]}. \tag{6.21}$$

Therefor, we have shown that q-P$_V$ has special solutions (6.21), if and only if $q_0 = ab\lambda p_0$.

Equation (6.21) can be linearized by the Cole-Hopf transformation $x_n = \lambda^n q_0 + \frac{w_n}{w_{n-1}}$. Substituting into (6.21) we get

$$p_0\left(\frac{w_{n+1} + \lambda^{n+1}q_0 w_n}{w_n}\right)\frac{w_n}{w_{n-1}} = \mu^2\lambda^{n-1}[w_n + (\lambda^n q_0 - a - b)w_{n-1}] + p_0 w_{n-1},$$

which can be simplified to the following linear equation for w_n

$$p_0 w_{n+1} - [\mu^2\lambda^{n-1}(\lambda^n q_0 - a - b) + p_0]w_{n-1} - [\mu^2\lambda^{n-1} - \lambda^{n+1}q_0]w_n = 0 \tag{6.22}$$

The liberalization condition of q-P$_V$ show that x_n can be expressed in terms of discrete analogue of confluent hypergeometric functions.

CONCLUSION

In this theses we investigate the transformation properties of the discrete Painlevé equation by using an algorithmic method similar to the method developed by Fokas an Ablowitz [7] for investigating the transformation properties of the continuous equation of Painlevé-type. We have obtained the Bäcklund transformations for d-P_{II}, q-P_{II}, q-d_{III}, d-P_{IV}, and q-P_V. Also, by this method we get explicit transformations between a given discrete Painlevé equation and the same discrete Painlevé equation but with different value of parameters, and between two different discrete equation of Painlevé-type. Moreover as an application of the algorithm, we have presented special solutions which are discrete analogue of the classical special functions. As a continuation of this study one may apply the method to other discrete Painlevé equation. More important study may be looking for transformation such that the equation for x_n is reduced to second-degree equation instead of linear equation. This type may be applicape to the discrete Painlevé VI equation which was not considered here.

Bibliography

[1] A. Sakka, U. Muğan "Bäcklund transformation for discrete Painlevé equations: Discrete P_{II}-P_V" Chaos, Solitons and Fractals 27 (2006) 387-394

[2] S. Goldberg, "Difference Equations", (John Wiely and Sons, Inc. New York, 1958).

[3] V. Ruel Churchill, W. James Brown, and F. Roger Verhey, "Complex Variabels And Applications", (McGraw-Hill, Inc, New York, 1974).

[4] S. Ponnusamy, "Foundations of Complex Analysis", (Narosa Puplishing House, New Delhi, 1998).

[5] M. J. Ablowitz, A. Ramani, and H. Segur, J. Math. Phys. 21 (1980) 715.

[6] A. S. Carstea. A. Ramani, R. Willox, and B. Grammaticos, " J. phys. A : Math. Gen. 36 No. 31 (2003) 8419.

[7] A. S. Fokas , M.J. Ablowitz, "On a undefined approach to transformations and elementary solution of Painleve equations." J. Math. Phys. 23 (1982) 2033.

[8] A. Ramani , B. Grammaticos. Hietarina J. Phys Rev Lett 67 (1991) 1829.

[9] A. Ramani , B. Grammaticos. Physica A, 228 (1996) 160. solv-int/951001

[10] B. Grammaticos , F. W. Nijhoff , V. Pagageorgiou , A. Rammani , J. Satsuma, Phys Lett A 185 (1994) 446 . solv-int/9310003.

[11] K. M. Tamizhamani , B. Grammaticos and A. Rammani. Lett. Math. Phys. 29 (1993) 49.

[12] J. Hietarinta, K. Kajiwara, Rational solutions to d-P_{IV}, solv-int/9705002

[13] K. Kajiwara , K. Yamamoto , Y. Ohta Phys Lett A 232 (1997) 189.

[14] M. D. Kruskal, K. M. Tamizhamani, B. Geammaticos and A. Ramani. "Regular and Chaotic Dynamics". 5 (2000) 273

[15] A. Ramani , B. Grammaticos J. Phys A: Math Gen 25 (1992) 633.

[16] K. Kajiwara , Y. Ohta , J. Satsuma , B. Grammaticos , A. Rammani J Phys A: Math Gen 27 (1994) 915.

[17] A. S. Carstea , A. Ramani , R Willox , B. Grammaticos J. Phys. A: Math Gen 36 (2003) 8419.

[18] E. Berzin , V. Kazakov Phys Lett B 236 (1990) 144.

[19] U. Muğan , A. Sakka , P. M. Santini, Phys Lett A 336 (2005) 37.

[20] T. Tamizhamani , B. Grammaticos ,A. Ramani , K. M. Tamizhamani Physica A 295 (2001) 359.