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MINIMAL STRUCTURE AND RELATED ALEXANDROFF SPACES

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To…

My Grandfather…who passed away during preparation of this work,

My Father,

My Mother,

My Husband,

My Lovely baby"Amr"

My sisters & Brothers

Contents

Acknowledgement **iii**

Abstract **iv**

Introduction **1**

1 Preliminaries **4**

1.1 Topological Spaces 4

1.2 Alexandroff Spaces 8

1.3 Generalized Open Sets 10

2 $\bigwedge_m$-sets and Minimal Alexandroff Spaces **13**

2.1 Properties of Minimal Structure 13

2.2 Λ_m-Sets 15

2.3 Minimal Alexandroff Space of an m-Space 21

3 The Duality for the Alexandroff Space (X, Λ_m) **25**

3.1 Interior and Closure Operations 25

3.2 On V_m-Sets 29

3.3 Dual Specialization Order and the Dual A-Space 34

3.4 Continuty On Λ_m and V_m sets 36

4 Bi m-space and Related Topological Spaces 39

4.1 (Λ, mn)-closed Sets and a Topological Space (X, Λ^*_{mnc}) 39

4.2 Generalized Λ_{mn}-sets and a Topological Space $(X, \Lambda_{g\Lambda_{mn}})$ 45

4.3 Functions Related To Generalized Λ_m and Generalized V_m sets 52

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Abstract

Alexandroff topological space is a kind of topology which satisfies a stronger condition. Namely, arbitrary intersections of open sets is open.

The main aim of this thesis is to study the concepts of an Alexandroff topological spaces which related to a family of subsets of a nonempty set called minimal structure. We study some families of a new type of minimal structure and construct a new collection of sets related to this minimal structure such as $\wedge_m$ and V_m which will be Alexandroff spaces. We study some of topological properties using previous studies of Alexandroff spaces. We give a characterization of the order which generated by these topological spaces. We prove that the topologies generated by this Order are the same topologies Λ_m on X. Finally we study some mappings related to these new type of topologies related to $m-$structure.

Introduction

Alexandroff spaces were first introduced in 1937 by P.S.Alexandroff under the name discrete spaces, where he provided the characterizations in terms of sets and neighborhoods [4]. The name discrete spaces later came to be used for topological spaces in which every subset is open and the original concept lay forgotten with the advancement of categorical topology in the 1980s .

Alexandroff spaces were rediscovered when the concept of finite generation was applied to general topology and the name finitely generated spaces was adopted for them. Alexandroff spaces were also rediscovered around the same time in the context of topologies resulting from denotational semantics and domain theory in computer science.

In 1966 Michael C. Mc Cord and A.K.Steiner each independently observed a relation between partially ordered sets and spaces which were precisely the T_o versions of the spaces that Alexandroff had introduced [19] [20] . P. Johnstone referred to such topologies as Alexandroff topologies [21] F.G. Arenas independently proposed this name for the general version of these topologies [22]. It was also a well known result in the field of modal logic that a duality exists between finite topological spaces and pre-orders on finite sets (the finite modal frames for the modal logic S4). C. Naturman extended these results to a duality between Alexandroff spaces and pre-orders in general, providing

the pre-order characterizations as well as the interior and closure algebraic characterizations. A systematic investigation of these spaces from the point of view of general topology which had been neglected since the original paper by Alexandroff, was taken up by F.G. Arenas. Inspired by the use of Alexandroff topologies in computer science, applied mathematicians and physicists in the late 1990s began investigating the Alexandroff topology corresponding to causal sets which arise from a pre-order defined on space time modeling causality.
In general, a topological space, (X, τ) is an Alexandroff space if arbitrary intersection of open sets is open. In this thesis, we study a family of subsets of powerset of a non empty set called minimal structures, and it's relation to Alexandroff spaces.

This thesis is divided into four chapters. Chapter one consists of three sections. In section one we give a brief summery of the main ideas and the basic concepts of topological spaces that will be used in the reminder of this thesis. In section two we study the class of topological spaces called Alexandroff spaces (or minimal neighborhood space). We describe some topological concepts on this class. In section three we identify some of generalized open sets as preopen, semi-open, α-open and some of related properties.
Chapter two consists of three sections. In section one we give the definition of a family of subsets of a non empty set called minimal structures. We obtain many results that are similar in general topology. In section two we obtain fundamental properties of a new type of sets called Λ_m sets and investigate lower separation axioms. Since (X, Λ_m) is an Alexandroff space. In section three we study it's specialization order and we prove some results which are depending on the specialization order.

Chapter three consists of four sections. In section one, we present the most important definitions of interior and closure operators and study them for acertain types of Alexandroff spaces. In section two, we study a type of sets called V_m and its properties. Then we obtain the relation with Λ_m-sets. In section three we show that the Alexandroff topological space (X, V_m) is the dual space for (X, Λ_m). We give a characterization for the specialization order on (X, V_m) by the help of specialization order described on (X, Λ_m). In the last section we study new type of mapping between m-spaces.

Chapter four consist of three sections. In the first two sections, we study new types of sets called (Λ, mn)-closed sets and Generalized Λ_{mn}-sets, which are defined on a non empty set with two minimal structures. We investigate the two types of topological spaces; (X, Λ^*_{mnc}) and $(X, \Lambda_{g\Lambda_{mn}})$ which are constructed from the families of these sets. We study some properties of lower separation axioms. Finally in section three, we study some mappings related to these new topological spaces.

Chapter 1

Preliminaries

In this chapter, we give a brief summery of the main ideas and the basic concepts of topological spaces. Then we study an important class of topological spaces called Alexandroff spaces (or minimal neighborhood space), and we describe some topological concepts on it's class. Finally we identify some of generalized open sets as preopen, semi-open, α-open and some of related properties.

1.1 Topological Spaces

Definition 1.1.1. [2] Let X be a non-empty set. A collection τ of subsets of X is said to be a *topology* on X if :

1. X and $\emptyset$ belong to τ.

2. Arbitrary union of elements of τ is an element of τ.

3. Any intersection of finitely many elements of τ is an element of τ.

If τ is a topology on X, then the pair (X, τ) is called a topological space.

Example 1.1.2. *[1] Let* $X = \{a, b, c, d, e, f\}$ *and let* $\tau = \{\emptyset, X, \{a\}, \{a, c, d\}, \{c, d\}, \{b, c, d, e, f\}\}$. *Then* τ *is a topology on* X.

Example 1.1.3. *[1] Let* $X = \{a, b\}$ *and let* $\tau = \{\emptyset, X, \{a\}\}$. *Then* τ *is a topology on* X *called the Sirpinski topology on* X.

Definition 1.1.4. [1] Let X be a non-empty set and let τ be the collection of all subsets of X. Then τ is called the *discrete topology* on X. The topological space (X, τ) is called the discrete space.

Definition 1.1.5. [1] Let X be a non-empty set and let $\tau = \{\emptyset, X\}$. Then τ is called the *indiscrete topology* and (X, τ) is called the indiscrete space.

Proposition 1.1.6. *[1] If* (X, τ) *is topological space such that, for every* $x \in X$, *the singleton* $\{x\}$ *is in* τ, *Then* τ *is the discrete topology.*

Definition 1.1.7. [1] The members of τ are called *open sets* in X. A subset of X is said to be *closed* if its complement is in τ (i.e., its complement is open).
Note that a subset of X may be open, closed, both (*clopen set*), or neither. The empty set and X itself are always clopen.

Definition 1.1.8. [2] Let (X, τ) be topological space. Let A be a subset of X, the *closure* of A is the intersection of all closed sets in X that contain A. That is,

$$\overline{A} = Cl(A) = \bigcap \{F : A \subseteq F \text{ and } X - F \in \tau\}$$

Definition 1.1.9. [2] Let (X, τ) be topological space. Let A be a subset of X, the interior of A is the union of all open sets in X that contained in A. That is,

$$A^0 = Int(A) = \bigcup \{U : U \subseteq A \text{ and } U \in \tau\}$$

Lemma 1.1.10. *Let (X, τ) be topological space and A, B subsets of X. Then the following properties hold:*

1. $Cl(X - A) = X - Int(A)$ *and* $Int(X - A) = X - Cl(A)$,

2. $Cl(\emptyset) = \emptyset$, $Cl(X) = X$, $Int(\emptyset) = \emptyset$ *and* $Int(X) = X$,

3. *If* $A \subseteq B$, *then* $Cl(A) \subseteq Cl(B)$ *and* $Int(A) \subseteq Int(B)$,

4. $A \subseteq Cl(A)$ *and* $Int(A) \subseteq A$,

5. $Cl(Cl(A)) = Cl(A)$ *and* $Int(Int(A)) = Int(A)$.

In Chapter 3, we will study interior and closure operations in details.

Definition 1.1.11. [1] Let (X, τ) be topological space. Let $x \in X$, a *neighborhood (briefly nhood)* of x is a set U which contains an open set V containing x. The collection $\mathbb{U}_x$ of all nhood of x is the nhood system of x.

Lemma 1.1.12. *[1] U is a nhood of x iff $x \in U^0$.*

Definition 1.1.13. [1] Let (X, τ) be a topological space. A collection $\mathbb{B}$ of open subsets of X is said to be a base for τ if every open set is a union of members of $\mathbb{B}$.

Theorem 1.1.14. *[1] Let X be a non-empty set and $\mathbb{B}$ a collection of subsets of X. Then $\mathbb{B}$ is a basis for some topology on X if and only if $\mathbb{B}$ has the following properties:*

1. $X = \bigcup_{B \in \mathbb{B}} B$,

2. *For any* $B_1, B_2 \in \mathbb{B}$, *if* $x \in B_1 \bigcap B_2$, *then there is some* $B_3 \in \mathbb{B}$ *such that* $x \in B_3 \subseteq B_1 \bigcap B_2$

Lemma 1.1.15. *[1] Let f be a mapping from a topological space (X, τ_X) into a topological space (Y, τ_Y). Then the following conditions are equivalent:*

1. *For each $U \in \tau_Y$, $f^{-1}(U) \in \tau_X$,*

2. *For each $a \in X$ and $U \in \tau_Y$ with $f(a) \in U$. There exists $V \in \tau_X$ such that $a \in V$ and $f(V) \subseteq U$.*

Definition 1.1.16. [1] Let (X, τ) be topological space. The identity function $id_X : (X, \tau) \to (X, \tau)$ of a set X is defined by $id_X(x) = x$.

Definition 1.1.17. [3] A topological space (X, τ) is said to be R_0 if and only if for each open set U and each $x \in U$, $Cl(\{x\}) \subseteq U$.

Example 1.1.18. *[1] Let $X = \{1, 2, 3\}$ and let $\tau = \{\emptyset, X, \{a\}, \{a, c, d\}, \{c, d\}, \{b, c, d, e, f\}\}$. Then τ is a topology on X.*

Definition 1.1.19. [3] A topological space (X, τ) is said to be T_0 if and only if for any pair of distinct points x, y in X there exists an open set containing one of the points but not the other.

Definition 1.1.20. [3] A topological space (X, τ) is said to be $T_{\frac{1}{2}}$ if and only if for each $x \in X$, the singleton set $\{x\}$ is either open or closed.

Definition 1.1.21. [3] A topological space (X, τ) is said to be T_1 if and only if for any pair of distinct points x, y in X there exist open sets U, V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Equivalently, each singleton set is closed.

1.2 Alexandroff Spaces

Definition 1.2.1. A binary relation $\leq$ on a set P is called *pre-ordered* if it satisfies the following axioms, $\forall x, y, z \in P$:

1. $x \leq x$ (reflexivity),

2. $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity).

A set P together with $\leq$ is called pre ordered set.

Definition 1.2.2. *A partially ordered set (a poset)* is a pre-order set such that the binary relation is also antisymmetric; that is, if $x \leq y$ and $y \leq x$ then $x = y$.

Recall that if $(X, \leq)$ is a pre-order set, then for $x \in X$, we define the down set $\downarrow x := \{y \in X : y \leq x\}$, and the up set $\uparrow x := \{y \in X : y \geq x\}$. For a set $A \subseteq X$, we define the down set $\downarrow A := \{y \in X : (\exists x \in A)\ y \leq x\}$, and the up set $\uparrow A := \{y \in X : (\exists x \in A)\ y \geq x\}$. Moreover A is up set if $A = \uparrow A$

Definition 1.2.3. [23] Let (X, τ) be a topological space. Then (X, τ) is an *Alexandroff space*, (briefly A-space) if and only if the arbitrary intersection of open sets is open.

Note that every finite topological space is an Alexandroff space.

Theorem 1.2.4. *Let (X, τ) be an Alexandroff space. Define a relation $\leq$ on X as follows:*

$$x \leq y \; if \; and \; only \; if \; x \in cl\{y\}.$$

Then $(X, \leq)$ is pre-order called (Alexandroff) specialization order. It is partial order if and only if (X, τ) is T_0-space.

Theorem 1.2.5. *Suppose that $(X, \leq)$ is a pre-order on X. Define $\mathbb{B} = \{\uparrow x : x \in X\}$. Then $\mathbb{B}$ is a base for a top on X which is Alexandroff space.*

Alexandroff topologies are uniquely determined by their specialization preorders. Indeed, given any preorder $\leq$ on a set X, there is a unique Alexandroff topology on X for which the specialization preorder is $\leq$. Thus, Alexandroff topologies on X are in one-to-one correspondence with preorders on X. If an Alexandroff spaces satisfy the separation axiom T_0, then the specialization preorder is antisymmetric, and hence partial order. Moreover, if $(X, \leq)$ is a poset and if $\tau(\leq)$ is its induced T_0- Alexandroff topology, then the specialization order of $\tau(\leq)$ is the order $\leq$ itself. On the other hand, if (X, τ) is a T_0-Alexandroff space with specialization order $\leq_\tau$ then the induced topology by the specialization order is the original topology, i.e. $\tau(\leq_\tau) = \tau$. Therefore T_0 - Alexandroff spaces can be completely determined by posets. Then using the concept of posets we get important results related to the topological proparties of Alexandroff spaces[5] .

Lemma 1.2.6. *[27] If $(X, \tau(\leq))$ is a T_0- Alexandroff space then a subset A of X is open if and only if it is up set with respect to the specialization order ($A = \uparrow A$), and A is closed if and only if it is a down set ($A = \downarrow A$).*

Lemma 1.2.7. *[23] Any discrete topological space is an Alexandroff space.*

Definition 1.2.8. [23] Let (X, τ) be a topological space, we say $x \in X$ has *a minimal open neighborhood* if the intersection of all open sets in X that contain x is an open set.

Theorem 1.2.9. *[23] X is an Alexandroff space if and only if each point in X has a minimal open neighborhood.*

Proof. Suppose X is an Alexandroff space and $x \in X$. Let $S(x)= \bigcap\{U \subseteq X : U$ is an open neighborhood of $x\}$. Then $S(x)$ is an open neighborhood of x since X is Alexandroff space. And from the definition of $S(x)$, it is clear that $S(x)$ is a minimal open neighborhood of x. Conversely, suppose that each $x \in X$ has a minimal open neighborhood $S(x)$. Consider an arbitrary intersection of open sets, $V = \bigcap_{\alpha \in A} U_\alpha$, where each U_α is open in X. If $V = \emptyset$, then V is open and we are done. If $V \neq \emptyset$, then pick $x \in V$. So we have $x \in U_\alpha$ for all $\alpha \in A$. Hence, $S(x) \subset U_\alpha$ for all α since $S(x)$ is the minimal open neighborhood of x. Therefore, $S(x) \subseteq \bigcap_\alpha U_\alpha = V$. Thus, V is open because it contains an open set around each of it's points. □

Theorem 1.2.10. *[23] If X is an Alexandroff space with a topology τ, then $\mathbb{B} = \{S(x) : x \in X\}$ is a basis for τ.*

1.3 Generalized Open Sets

Definition 1.3.1. Let (X, τ) be a topological space. A subset A is said to be:

1. semi-open [6] if $A \subseteq Cl(Int(A))$.

2. preopen [7] if $A \subseteq Int(Cl(A))$.

3. α-open [8] if $A \subseteq Int(Cl(Int(A)))$.

Definition 1.3.2. [26] Let (X, τ) be topological space. Let A be a subset of X. The θ-closure (resp. δ-closure) of A, $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$), is defined by the set of the points such that $A \bigcap$ Cl $(U) \neq \emptyset$ (resp. $A \bigcap$ Int (Cl$(U)) \neq \emptyset$) for every open set U containing x. A subset A of (X, τ) is said to be θ-closed (resp. δ-closed) if $A=$ Cl$_\theta(A)$ (resp. $A =$ Cl$_\delta(A)$). The complement of θ-closed (resp. δ-closed) set is said to be θ-open (resp. δ-open).

Definition 1.3.3. Let (X, τ) be topological space. Let A be a subset of X. The semi-θ-closure, (resp. θ-semi-closure)) of A, sCl$_\theta(A)$ (resp. θ-sCl(A)) is defined by the set of all points such that $A \bigcap$ sCl $(U) \neq \emptyset$ (resp. $A \bigcap$ Cl $(U) \neq \emptyset$) for every semi-open set U containing x. A subset A of (X, τ) is said to be *semi-θ-closed* [9] (resp. *θ-semi-closed* [10]) if $A=$ sCl$_\theta(A)$(resp. $A = \theta$-sCl(A)) The complement of a semi-θ-closed (resp. θ-semi-closed) set is said to be *semi-θ-open* (resp. *θ-semi-open*).

Definition 1.3.4. Let (X, τ) be a topological space. A subset A is said to be:

1. β-open [11], or semi-preopen [12] if $A \subseteq$ Cl(Int(Cl (A))).

2. b-open [13], if $A \subseteq$ Int(Cl(A)) $\bigcup$ (Cl(Int(A))).

3. δ-preopen [14], if $A \subseteq$ Int(Cl$_\delta(A)$).

4. δ-semi-open [15], if $A \subseteq$ Cl(Int$_\delta(A)$).

The family of all semi-open (resp. preopen, α-open, θ-open , δ-open, β-open, b-open, δ-preopen, θ-semi-open, semi-θ-open, δ-semi-open) sets in a topological space (X, τ) is denoted by $SO(X)$ (resp. $PO(X)$, τ_α, τ_θ , τ_δ, $\beta(X)$, BO(X), δ PO(X), θ SO(X), SO $\theta(X)$, δSO(X)).

Definition 1.3.5. [26] The complement of a semi-open (resp. preopen, α-open, β-open ,b-open, δ-preopen) is said to be semi-closed (resp. *preclosed*, *α-closed*, *β-closed*, *b-closed*, *δ-preclosed*).

Definition 1.3.6. [26] The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, b-closed, δ-preclosed, δ-semi-closed) sets of (X, τ) containing a subset A is called the *semi-closure* (resp. *preclosure*, *α-closure*, *β-closure*, *b-closure*, *δ-preclosure*, *δ-semi-closure*) and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\beta\mathrm{Cl}(A)$, $b\mathrm{Cl}(A)$, $p\mathrm{Cl}_\delta(A)$, $s\mathrm{Cl}_\delta(A)$).

Definition 1.3.7. [26] The union of all semi-open (resp. preopen, α-open, β-open, b-open, δ-preopen, θ-semi-open, θ-semi-open, δ-semi-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior*, *b-interior*, *δ-preinterior*, *θ-semi-interior*, *θ-semi-interior*, *δ-semi-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\beta\mathrm{Int}(A)$, $\mathrm{bInt}(A)$, $p\mathrm{Int}_\delta(A)$, θ-$\mathrm{sInt}(A)$, $\mathrm{sInt}_\delta(A)$).

Definition 1.3.8. [26] A subset A of (X, τ) is said to be Λ-*set* if it is the intersection of open sets containing A.

Definition 1.3.9. [26] A subset A of (X, τ) is said to be V-*set* if it is the union of closed sets containing in A.

Definition 1.3.10. [26] A subset A of (X, τ) is said to be a *generalized* Λ-*set (briefly gΛ-set)* of (X, τ), if $\Lambda(A) \subseteq F$ whenever $A \subseteq F$ and F is closed. The family of gΛ-sets of (X, τ) is denoted by gΛ.

Definition 1.3.11. [16] A subset A of (X, τ) is said to be, $\lambda-$*closed* if $A = L \bigcap F$, where L is a Λ-set and F is closed in (X, τ).

Chapter 2

$\bigwedge_m$-sets and Minimal Alexandroff Spaces

In this chapter we give the definition of a family of subsets of a non empty set called minimal structures. We obtain many results that are similar in general topology. Then we obtain fundamental properties of a new type of sets called Λ_m sets and investigate lower separation axioms. Since (X, Λ_m) is an Alexandroff space. Finally we study its specialization order and prove some results which are depending on the specialization order.

2.1 Properties of Minimal Structure

Definition 2.1.1. [24] A collection m of subsets of a nonempty set X is called *minimal structure (briefly m-structure)* on X if m satisfies the following properties:

1. $\emptyset$ and X belong to m.

2. $\bigcup_{\alpha\in\Delta} A_\alpha \in m$ whenever $A_\alpha \in m$ for each $\alpha \in \Delta$.

We call the pair (X, m) an m-space. Each member of m is said to be *m-open*. The complement of an m-open set is said to be *m-closed*. A set X with two m-structures m and n is called *Bi m-space* and denoted by (X, m, n).

Any topology on X is surely m-structure.

Example 2.1.2. *Let $X = \{1, 2, 3, 4, 5, 6\}$. Let $m = \{\emptyset, X, \{1, 2, 4, 5, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}\}$. Then m is an m-structures on X.*

Remark 2.1.3. [24] Let (X, τ) be topological space. Then the families $SO(X)$, $PO(X)$, τ_α, τ_θ, τ_δ, $\beta(X)$, $\mathrm{BO}(X)$, δ $\mathrm{PO}(X)$, θ $\mathrm{SO}(X)$, SO $\theta(X)$, $\delta\mathrm{SO}(X)$ are all samples of m-structures on X. It is well-known that τ_α, τ_θ, τ_δ are topologies for X and the others need not be topologies.

Definition 2.1.4. [17] Let (X, m) be an m-space and A a subset of X. The *m-closure* (briefly $\mathrm{mCl}(A)$) of A and *m-interior* (briefly $\mathrm{mInt}(A)$) of A are defined as follows:

$$\begin{aligned} mCl(A) &= \bigcap \{F : A \subseteq F \text{ and } X - F \in m\} \\ mInt(A) &= \bigcup \{U : U \subseteq A \text{ and } U \in m\} \end{aligned}$$

Lemma 2.1.5. *[24] Let (X, m) be an m-space and A, B subsets of X. Then the following properties hold:*

1. *$mCl(X - A) = X$ - $mInt(A)$ and $mInt(X - A) = X$ - $mCl(A)$.*

2. *mCl $(\emptyset) = \emptyset$, $mCl(X) = X$, $mInt$ $(\emptyset) = \emptyset$ and $mInt$ $(X) = X$.*

3. *If $A \subseteq B$, then $mCl(A) \subseteq mCl(B)$ and $mInt(A) \subseteq mInt(B)$.*

4. $A \subseteq mCl(A)$ *and* $mInt(A) \subseteq A$.

5. $mCl(mCl(A)) = mCl(A)$ *and* $mInt(mInt(A)) = mInt\,(A)$.

Lemma 2.1.6. *[18] Let (X, m) be an m-space. A a subset of X and $x \in X$. Then $x \in mCl(A)$ if and only if $U \bigcap A \neq \emptyset$ for every $U \in m$ containing x.*

Lemma 2.1.7. *[24] Let (X, m) be an m-space. Then, for a subset A of X, the following properties hold:*

1. $A \in m$ *if and only if* $A = mInt(A)$.

2. A *is m-closed if and only if* $A = mCl(A)$.

2. $mCl(A)$ *is m-closed and* $mInt(A)$ *is m-open.*

Definition 2.1.8. [24] An m-space (X, m) is said to be m-R_0 if for each m-open set U and each $x \in U$, mCl$(\{x\}) \subseteq U$.

Definition 2.1.9. [24] An m-space (X, m) is said to be :

1. m-T_0 if for any pair of distinct points in X there exists an m-open set containing one of the two points but not the other.

2. m-T_1 if for any pair of distinct points x, y in X there exist m-open sets U, V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

2.2 Λ_m-Sets

Definition 2.2.1. [24] Let (X, m) be an m-space and A a subset of X. The subset $\Lambda_m(A)$ is defined as follows:

$$\Lambda_m(A) = \bigcap\{U : A \subseteq U, U \in m\}$$

Lemma 2.2.2. *[24] For subsets A, B, $A_\alpha (\alpha \in \Delta)$ of an m-space (X, m), the following properties hold:*

1. $A \subseteq \Lambda_m(A)$.

2. $\Lambda_m(\Lambda_m(A)) = \Lambda_m(A)$.

3. *If* $A \subseteq B$, *then* $\Lambda_m(A) \subseteq \Lambda_m(B)$.

4. $\Lambda_m(\bigcap \{A_\alpha : \alpha \in \Delta\}) \subseteq \bigcap \{\Lambda_m(A_\alpha) : \alpha \in \Delta\}$.

5. $\Lambda_m(\bigcup \{A_\alpha : \alpha \in \Delta\}) = \bigcup \{\Lambda_m(A_\alpha) : \alpha \in \Delta\}$.

Proof. 1. This is obvious from the definition.

2. By (1), we have $\Lambda_m(\Lambda_m(A)) \supseteq \Lambda_m(A)$. Suppose that $x \notin \Lambda_m(A)$. Then there exists $U \in m$ such that $A \subseteq U$ and $x \notin U$. Since $A \subseteq \Lambda_m(A) \subseteq U$ then from the definition of $\Lambda_m(\Lambda_m(A))$, $\Lambda_m(\Lambda_m(A)) \subseteq U$, and hence we get that $x \notin \Lambda_m(\Lambda_m(A))$, that is $\Lambda_m(\Lambda_m(A)) \subseteq \Lambda_m(A)$. Therefor $\Lambda_m(\Lambda_m(A)) = \Lambda_m(A)$.

3. Suppose that $x \notin \Lambda_m(B)$, then there exists $U \in m$ such that $B \subseteq U$ and $x \notin U$. since $A \subseteq B \subseteq U$, then from the definition of $\Lambda_m(A)$ $x \notin \Lambda_m(A)$ and hence $\Lambda_m(A) \subseteq \Lambda_m$(B).

4. Suppose that $x \notin \bigcap \{\Lambda_m(A_\alpha) : \alpha \in \Delta\}$. Then there exists $\alpha_0 \in \Delta$ such that $x \notin \Lambda_m(A_{\alpha_0})$ and there exists an m-open set U such that $A_{\alpha_0} \subset U$ and $x \notin U$. Since $\bigcap_{\alpha\in\Delta} A_\alpha \subseteq A_{\alpha_0} \subseteq U$, then from the definition of Λ_m $(\bigcap \{A_\alpha : \alpha \in \Delta\})$, we have that $x \notin \Lambda_m$ $(\bigcap \{A_\alpha : \alpha \in \Delta\})$ and hence $\Lambda_m(\bigcap \{A_\alpha : \alpha \in \Delta\}) \subseteq \bigcap \{\Lambda_m(A_\alpha) : \alpha \in \Delta\}$.

5. Since $A_\alpha \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$, by (3) we have $\Lambda_m(A_\alpha) \subseteq \Lambda_m(\bigcup_{\alpha\in\Delta} A_\alpha)$ and $\bigcup_{\alpha\in\Delta} \Lambda_m(A_\alpha) \subseteq \Lambda_m(\bigcup_{\alpha\in\Delta} A_\alpha)$. Conversely, suppose that $x \notin \bigcup_{\alpha\in\Delta} \Lambda_m(A_\alpha)$. Then $x \notin \Lambda_m(A_\alpha)$ for each $\alpha \in \Delta$ and hence there exists $U_\alpha \in m$ such that $A_\alpha \subset U_\alpha$ and $x \notin U_\alpha$ for each $\alpha \in \Delta$. Therefore, we obtain, $\bigcup_{\alpha\in\Delta} A_\alpha \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ and $\bigcup_{\alpha\in\Delta} U_\alpha$ is an m-open set not containing x. Thus, $x \notin \Lambda_m(\bigcup \{A_\alpha : \alpha \in \Delta\})$. This shows that $\bigcup \{\Lambda_m(A_\alpha) : \alpha \in \Delta\} \supseteq \Lambda_m(\bigcup \{A_\alpha : \alpha \in \Delta\})$.

□

Definition 2.2.3. [24] A subset A of an m-space (X, m) is called a Λ_m*-set* if $A = \Lambda_m(A)$. The family of all Λ_m-sets of (X, m) is denoted by $\Lambda_m(X)$ (or simply Λ_m).

Lemma 2.2.4. *[24] For subsets A, $A_\alpha(\alpha \in \Delta)$ of an $m-$space (X, m), the following properties hold:*

1. *$\Lambda_m(A)$ is a Λ_m-set.*
2. *If A is an m-open set, then A is a Λ_m-set.*
3. *If A_α is a Λ_m-set for each $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} A_\alpha$ is a Λ_m-set.*
4. *If A_α is a Λ_m-set for each $\alpha \in \Delta$, then $\bigcap_{\alpha\in\Delta} A_\alpha$ is a Λ_m-set.*

Proof. 1. From Lemma 2.2.2 part 2, $\Lambda_m(\Lambda_m(A)) = \Lambda_m(A)$, so $\Lambda_m(A)$ is a Λ_m-set.

2. If $A \in m$, Then $\Lambda_m(A) = \bigcap\{U : A \subseteq U, U \in m\} \subseteq A$. And By Lemma 2.2.2 part 1, we have $A \subseteq \Lambda_m(A)$.

3. Let $A_\alpha \in \Lambda_m$ for each $\alpha \in \Delta$, then by Lemma 2.2.2, part 5, we have

$$\bigcup_{\alpha\in\Delta} A_\alpha = \bigcup_{\alpha\in\Delta} \Lambda_m(A_\alpha) = \Lambda_m(\bigcup_{\alpha\in\Delta} A_\alpha)$$

Thus, $\bigcup_{\alpha\in\Delta} A_\alpha \in \Lambda_m$.

4. Let $A_\alpha \in \Lambda_m$ for each $\alpha \in \Delta$, then by Lemma 2.2.2 (4) we have

$$\bigcap_{\alpha\in\Delta} A_\alpha = \bigcap_{\alpha\in\Delta} \Lambda_m(A_\alpha) \supseteq \Lambda_m(\bigcap_{\alpha\in\Delta} A_\alpha) \supseteq \bigcap_{\alpha\in\Delta} A_\alpha.$$

Thus, we have $\bigcap_{\alpha\in\Delta} A_\alpha = \Lambda_m(\bigcap_{\alpha\in\Delta} A_\alpha)$ and $\bigcap_{\alpha\in\Delta} A_\alpha \in \Lambda_m$.

□

Theorem 2.2.5. *[24] For an m-space (X, m), the pair (X, Λ_m) is an Alexandroff topological space.*

Proof. 1. $\emptyset, X \in \Lambda_m$ since $\emptyset, X \in m$ and $m \subseteq \Lambda_m$.

2. If $U_\alpha \in \Lambda_m$ for each $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} U_\alpha$ is a Λ_m-set by Lemma 2.2.4 part 3.

3. If $U_\alpha \in \Lambda_m$ for each $\alpha \in \Delta$, then $\bigcap_{\alpha\in\Delta} U_\alpha$ is a Λ_m-set by Lemma 2.2.4 part 4.

□

Remark 2.2.6. Since any topology on a non-empty set is m-structure, so we can look at Λ_m as a new m-space.

Theorem 2.2.7. *[24] Let (X, m) be an m-space . Then $\Lambda_m = \Lambda_{\Lambda_m}$.*

Proof. By Lemma 2.2.4 part 2, we have that $m \subseteq \Lambda_m$. Let A be any subset of X. Then we have

$$\Lambda_{\Lambda_m}(A) = \bigcap\{U : A \subseteq U : U \in \Lambda_m\} \subseteq \bigcap\{U : A \subseteq U, U \in m\} = \Lambda_m(A).$$

Therefore, we obtain $\Lambda_{\Lambda_m}(A) \subseteq \Lambda_m(A)$. Now, we suppose that $x \notin \Lambda_{\Lambda_m}(A)$. Then, there exists $U \in \Lambda_m$ such that $A \subseteq U$ and $x \notin U$. Since $x \notin U$, there exists $V \in m$ such that $U \subseteq V$ and $x \notin V$ and hence $x \notin \Lambda_m(A)$. This shows that $\Lambda_{\Lambda_m}(A) \supseteq \Lambda_m(A)$ and hence we obtain $\Lambda_m(A) = \Lambda_{\Lambda_m}(A)$. □

Theorem 2.2.8. *[24] An m-space(X, m) is m-R_0 if and only if the topological space (X, Λ_m) is R_0.*

Proof. Let $V \in \Lambda_m$ and $x \in V$. Then $x \in \Lambda_m(V) = \bigcap\{U : V \subseteq U, U \in m\}$ and $x \in U$ for any $U \in m$ containing V. Since (X, m) is m-R_0, $mCl(\{x\}) \subseteq U$ for every $U \in m$ containing V. Hence $mCl(\{x\}) \subseteq \bigcap\{U : V \subseteq U, U \in m\} = \Lambda_m(V) = V$. Since $m \subseteq \Lambda_m$, we have Λ_m-$Cl(\{x\}) \subseteq mCl(\{x\}) \subseteq V$. This shows that (X, Λ_m) is R_0. Conversly, suppose that (X, Λ_m) is R_0. Let $V \in m$ and $x \in V$. Since $m \subseteq \Lambda_m$, we have Λ_m-$Cl(\{x\}) \subseteq V$. Since $X - \Lambda_m$-$Cl(\{x\}) \in \Lambda_m$, we have $X - \Lambda_m$-$Cl(\{x\}) = \bigcap\{U : X - \Lambda_m\text{-}Cl(\{x\}) \subseteq U : U \in m\}$. Hence there exists $U \in m$ such that $X - \Lambda_m$-$Cl(\{x\}) \subseteq U$ and $x \notin U$ and hence $x \in X - U \subseteq X - \Lambda_m$-$Cl(\{x\}) \subseteq V$. Since $X - U$ is m-closed, $mCl(\{x\}) \subseteq X - U \subseteq V$. This shows that (X, m) is m-R_0. □

Theorem 2.2.9. *[24] An m-space(X, m) is m-T_0 if and only if the topological space (X, Λ_m) is T_0.*

Proof. Let (X, m) be an m-T_0 space. Since $m \subseteq \Lambda_m$, then Λ_m is T_0 space. Conversely, suppose that (X, Λ_m) is T_0 space. Let x, y be any pair of distinct points of X. Then there exists $V \in \Lambda_m$ such that either "$x \in V$ and $y \notin V$" or "$x \notin V$ and $y \in V$". In case $x \in V$ and $y \notin V$, there exists $U \in m$ such that $V \subseteq U$ and $y \notin U$. However, since $x \in V$, $x \in U$. Similarly for the case $x \notin V$ and $y \in V$. Hence, (X, m) is T_0. □

Lemma 2.2.10. *[24] For an m-space (X, m), the following properties are equivalent:*

1. *(X, m) is m-T_1.*
2. *For each $x \in X$, the singleton $\{x\}$ is m-closed in (X, m).*
3. *For each $x \in X$, the singleton $\{x\}$ is a Λ_m-set.*

Proof. (1)⇒(2) Let y be any point of X and $x \in X - \{y\}$. There exists $V_x \in m$ such that $x \in V_x$ and $y \notin V_x$. Hence we have $X - \{y\} = \bigcup_{x \in X - \{y\}} V_x$ which is a union of of an m-open sets, so it's m-open. Therefore, the singleton $\{y\}$ is m-closed in (X, m).

(2)⇒(3) Let x be any point of X and $y \in X - \{x\}$. Then $x \in X - \{y\} \in m$ and $\Lambda_m(\{x\}) \subseteq X$. Therefore, $y \notin \Lambda_m(\{x\})$ and $x \in \Lambda_m(\{x\}) \subseteq \bigcap_{y \neq x} X - \{y\} = \{x\}$. This shows that $\Lambda_m(\{x\}) = (\{x\})$, the singleton $\{x\}$ is a Λ_m-set.

(3)⇒(1) Suppose that the singleton $\{x\}$ is a Λ_m-set for each $x \in X$. Let x and y be any distinct points. Then $y \notin (\{x\}) = \Lambda_m(\{x\})$. So there exists an m-open set U_x such that $x \in U_x$ and $y \notin U_x$. Similarly, $x \notin \Lambda_m(\{y\})$ and there exists an m-open set U_y such that $y \in U_y$ and $x \notin U_y$. This shows that (X, m) is m-T_1. □

Theorem 2.2.11. *[24] An m-space (X, m) is m-T_1 if and only if the topological space (X, Λ_m) is discrete.*

Proof. Suppose that (X, m) is m-T_1. If $x \in X$, then by Lemma 2.2.10, $\{x\}$ is a Λ_m-set and $\{x\}$ is open in (X, Λ_m). Therefore, every subset of X is open in (X, Λ_m) and hence (X, Λ_m) is discrete. Conversely, suppose that a topological space (X, Λ_m) is discrete. For any point $x \in X$, $\{x\}$ is open in (X, Λ_m) and hence $\{x\}$ is a Λ_m-set. Therefore, by Lemma 2.2.10, (X, m) is m-T_1. □

Corollary 2.2.12. *For an m-space (X, m), the following properties are equivalent:*

1. *(X, m) is m-T_1.*

2. *(X, m) is m-R_0 and m-T_0.*

3. *(X, Λ_m) is R_0 and T_0.*

4. *(X, Λ_m) is T_1.*

5. (X, Λ_m) *is discrete.*

Proof. (1)⇒(2) By Lemma 2.2.10, every m-T_1 space is m-R_0 and m-T_0.

(2)⇒(1) Since (X, m) is m-T_0, for any distinct points x, y of X, there exists $U \in m$ such that, say, $x \in U$ and $y \notin U$. Hence $mCl(\{x\}) \subseteq U$ since (X, m) is m-R_0. Therefore, we obtain $x \notin X - mCl(\{x\})$ and $y \in X - U \subseteq X - mCl(\{x\}) \in m$. Therefore, (X, m) is m-T_1.

(2)⇔(3) This is an immediate consequence of Theorems 2.2.7 and 2.2.8.

(3)⇒(4) Suppose that (X, Λ_m) is R_0 and T_0. Then by Theorems 2.2.7 and 2.2.8, (X, m) is m-R_0 and m-T_0, then (X, m) is m-T_1, and by Lemma 2.2.9, (X, Λ_m) is T_1.

(4)⇒(3) Suppose that (X, Λ_m) is T_1. Then By Lemma 2.2.9, every m-T_1 space is m-R_0 and m-T_0, and by Theorems 2.2.7 and 2.2.8, (X, Λ_m) is R_0 and T_0.

(4)⇒(5) Suppose that (X, Λ_m) is T_1 space. Then by Theorem 2.2.8, (X, m) is m-T_1 space and by Theorem 2.2.10, (X, Λ_m) is discrete.

(5)⇒(4) suppose that (X, Λ_m) is discrete, then by Theorem 2.2.10, (X, m) is m-T_1 space and by Theorem 2.2.8, (X, Λ_m) is T_1 space. □

2.3 Minimal Alexandroff Space of an m-Space

Definition 2.3.1. Given an m-space (X, m), a *specialization order* on X is given by: for all $x, y \in X$, $x \leq_m y$ if and only if when U is an m-open containing x, then U containing y.

Lemma 2.3.2. *For any m-space (X, m) and for any $x, y \in X$, the following conditions are equivalent:*

1. $x \leq_m y$,

2. $x \in mCl(y)$,

3. $y \in \Lambda_m(x)$.

Proof. (1)⇒(2) Let $x \leq_m y$, and suppose to contrary that $x \notin mCl(y)$. Then $\exists\ L^c \in m$ such that $y \in L$, $x \notin L$. So $x \in L^c$ and $y \notin L^c$; that is $x \nleq_m y$, which is contradiction. Therefor $x \in mCl(y)$

(2)⇒(3) Let $x \in mCl(y)$, then $\forall\ L^c \in m$, $y \in L$, $x \in L$. Suppose to contrary that $y \notin \Lambda_m(x)$, then $\exists\ U \in m$ and $x \in U$, but $y \notin U$. Then $y \in U^c$. Equivalently $x \notin U$. Which is a contradiction. Therefore $y \in \Lambda_m(x)$.

(3)⇒(1). Let $y \in \Lambda_m(x)$, then $\forall\ U \in m$, $x \in U$ we have that $y \in U$. Hence $x \leq_m y$.

□

Remark 2.3.3. Let (X, m) be an m-space. Then in general $(X, \leq_m)$ need not be a poset, since $\leq_m$ always is pre-order but need not be order because it's not antisymmetric, the following Lemma show this fact.

Lemma 2.3.4. *Let (X, m) be an m-space, then (X, m) is an m-T_0 if and only if $(X, \leq_m)$ is a poset.*

Proof. Let (X, m) ba m-T_0 space, then

1. $x \leq_m x$ which obvious by definition 2.3.1

2. Let $x \leq_m y$ and $y \leq_m z$, then $\forall\ U \in m$ containing x, U containing y, then U containing z. Hence $x \leq_m z$

3. Let $x \leq_m y$ and $y \leq_m x$. If $x \neq y$ then $\exists\ U \in m$ such that $x \in U$ and $y \notin U$. which is contradiction. Hence $x = y$. Therefor $\leq_m$ is a partial order and $(X, \leq_m)$ is a poset.

Conversely, let $\leq_m$ is a partial order and let $x \neq y$. Suppose to contrary that, $\forall\, U \in m$, either both $x, y \in U$ or both x, y not in U. Since $\leq_m$ is a partial order, we have $x \leq_m y$ and $y \leq_m x$, then $x = y$. Which is contradiction. That is $\exists\, U \in m$ such that $x \in U$ and $y \notin U$. Therefore (X, m) is an m-T_0 space. $\square$

Definition 2.3.5. Let (X, m) be an m-T_0 space, and $(X, \leq_m)$ the corresponding poset on X. Then we denote m^A to be the induced T_0 A-space with $\mathbb{B} = \{V(a) : a \in X\}$, where $V(a) = \{b \in X : a \leq_m b\}$ as abase.

In a given m-space (X, m), two topologies induced on X; Λ_m, m^A. The following theorem proves that they are equal.

Theorem 2.3.6. *Let (X, m) be an m-space, with corresponding pre-order $(X, \leq_m)$. Then*

$$\Lambda_m = m^A$$

Proof. Let (X, m) be an m-space, and U a Λ_m-set. Then $U = \bigcap\{U_\alpha \in m : \alpha \in \Delta\}$. Let $a \in U = \bigcap_{\alpha\in\Delta} U_\alpha$, then $a \in U_\alpha$ for each $\alpha \in \Delta$. If $b \geq_m a$, and since $U_\alpha \in m\ \forall\ \alpha$, by the definition, we have that $b \in U_\alpha\ \forall\alpha$ and so $b \in U$. Hence U is upset with respect to $\leq_m$ and so $U \in m^A$. Thus, $\Lambda_m \subseteq m^A$.

Conversly, let $V(a) \in \mathbb{B}$, $a \in X$. If $b \in V(a)$, then $a \leq_m b$. By the definition of $\leq_m$, $\forall\ U \in m$, $a \in U$, we have that $b \in U$. Hence $b \in \bigcap\{U \in m : a \in U\}$ and so $V(a) \subseteq \bigcap\{U \in m : a \in U\}$. for the other inclusion, Let $c \in \bigcap\{U \in m : a \in U\}$. Then $c \in U$ for all $U \in m$ such that $a \in U$. Again, by the definition of $\leq_m$, $a \leq_m c$ and hence $c \in V(a)$. Therefore

$$\bigcap\{U \in m : a \in U\} = V(a).$$

This implies that $V(a)$ is a Λ_m-set and so $V(a) \in \Lambda_m$. Equivalently, $\mathbb{B} \subseteq \Lambda_m$, and so $m^A \subseteq \Lambda_m$. Thus $\Lambda_m = m^A$. $\square$

Chapter 3

The Duality for the Alexandroff Space (X, Λ_m)

In this chapter we present the important definitions of interior and closure operators and study them for a certain types of Alexandroff spaces. Then we study a type of sets called V_m and its properties. Then we obtain the relation with Λ_m-sets. We show that the Alexandroff topological space (X, V_m) is the dual space for (X, Λ_m). We give a characterization for the specialization order on (X, V_m) by the help of specialization order described on (X, Λ_m). Finally we study new type of mapping between m-spaces.

3.1 Interior and Closure Operations

In Chapter 1, we give a definition of closure and interior operator. In this section we will study them for certain types of Alexandroff spaces.

Theorem 3.1.1. *[2] If $\mathfrak{F}$ is the collection of closed sets in a topological space X, then:*

1. *$\emptyset$ and X belong to $\mathfrak{F}$.*

2. *If $\{U_\alpha : \alpha \in \Delta\}$ is an indexed family of subsets of $\mathfrak{F}$, then $\bigcap_{\alpha\in\Delta} U_\alpha \in \mathfrak{F}$*

3. *If U_1 and U_2 belong to $\mathfrak{F}$, then $U_1 \bigcup U_2$ belong to $\mathfrak{F}$.*

Theorem 3.1.2. *[2] Let X be a nonempty set and let $\mathfrak{F}$ a collection of subsets of X satisfying (1), (2) and (3) in previous theorem. Then the collection of complements of members of $\mathfrak{F}$ forms a topology on X in which the family of closed sets is just $\mathfrak{F}$.*

Lemma 3.1.3. *[2] Let (X, τ) be topological space, A and B are subsets of X. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.*

Theorem 3.1.4. *[2] The operation $A \to \overline{A}$ in a topological space X has the following properties:*

1. *$E \subseteq \overline{E}$.*

2. *$\overline{\overline{E}} = \overline{E}$.*

3. *$\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$.*

4. *$\overline{\emptyset} = \emptyset$.*

5. *E is closed if and only if $\overline{E} = E$.*

Proof. 1. This is obvious from definition of closure operator.

2. Since $E \subseteq \overline{E}$, then $\overline{E} \subseteq \overline{\overline{E}}$.Since $\overline{\overline{E}} = \bigcap\{ F : \overline{E} \subset F$ and $X - F \in \tau\}$ and $\overline{E}$ is closed set, so $\overline{\overline{E}} \subseteq \overline{E}$. Hence $\overline{\overline{E}} = \overline{E}$.

3. Since $\overline{A} \bigcup \overline{B}$ is closed set, and $A \bigcup B \subseteq \overline{A} \bigcup \overline{B}$, then $\overline{A \bigcup B} \subseteq \overline{\overline{A} \bigcup \overline{B}} = \overline{A} \bigcup \overline{B}$. So $\overline{A \bigcup B} \subseteq \overline{A} \bigcup \overline{B}$. Conversely, since $A \subseteq A \bigcup B$ and $B \subseteq A \bigcup B$, then $\overline{A} \subseteq \overline{A \bigcup B}$ and $\overline{B} \subseteq \overline{A \bigcup B}$, then $\overline{A} \bigcup \overline{B} \subseteq \overline{A \bigcup B}$. Hence $\overline{A} \bigcup \overline{B} = \overline{A \bigcup B}$.

4. By (1), $\emptyset \subseteq \overline{\emptyset}$. Since $\emptyset$ is closed, then $\overline{\emptyset} \subseteq \emptyset$. Hence $\overline{\emptyset} = \emptyset$.

5. Suppose that E is closed set, then $\overline{E} = \bigcap\{F : E \subset F \text{ and } X - F \in \tau\} \subseteq E$. But $E \subseteq \overline{E}$ by (1), then $\overline{E} = E$. Conversely suppose $\overline{E} = E$, then E is closed set since $\overline{E}$ is closed.

$\square$

Theorem 3.1.5. *[2] Let X be a set and let CL be a mapping $A \to \overline{A}$ from the power set $p(X)$ into $p(X)$ satisfying (1) through (4) in previous theorem. If we define closed sets in X using (5), the result is a topology on X whose closure operation is just the operation $A \to \overline{A}$.*

Example 3.1.6. *We always have $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$. The corresponding statement for intersections is not true.*
To see this, let X be the set $\mathbf{R}$ of all real numbers with standard topology, let $A = Q$ the set of all rational numbers, and $B = \mathbf{R} - Q$ the set of all irrational numbers in $\mathbf{R}$, then $\overline{A} = \mathbf{R}$, $\overline{B} = \mathbf{R}$, then $\overline{A} \bigcap \overline{B} = \mathbf{R}$, but $A \bigcap B = \emptyset$ and $\overline{A \bigcap B} = \overline{\emptyset} = \emptyset$

Recall that if A is a subset of a topological space (X, τ), the interior of A in X $(= A^0)$ is the largest open set contained in A.

Lemma 3.1.7. *[2] Let (X, τ) be topological space, A and B are subsets of X. If $A \subseteq B$, then $A^0 \subseteq B^0$.*

Theorem 3.1.8. *[2] The interior operation $A \to A^0$ in a topological space X has the following properties:*

1. $A^0 \subseteq A$.

2. $(A^0)^0 = A^0$.

3. $(A\bigcap B)^0 = A^0\bigcap B^0$.

4. $X^0 = X$.

5. *G is open if and only if* $G^0 = G$.

Proof. 1. This is obvious from definition of interior operator.

2. Since $A^0 \subseteq A$, then by lemma 3.1.8 $(A^0)^0 \subseteq A^0$. Now $(A^0)^0 = \bigcup \{U : U \subset A^0$ and $U \in \tau\}$. But A^0 is open set and $A^0 \subseteq A$, so $A^0 \subseteq (A^0)^0$. Hence $(A^0)^0 \subseteq A^0$.

3. Now $A^0\bigcap B^0$ is open set, and $A^0\bigcap B^0 \subseteq (A\bigcap B)$, then $(A^0\bigcap B^0)^0 \subseteq (A\bigcap B)^0$. But also $(A^0\bigcap B^0)^0 = A^0\bigcap B^0$, so $A^0\bigcap B^0 \subseteq (A\bigcap B)^0$. Conversely, since $A\bigcap B \subseteq A$ and $A\bigcap B \subseteq B$, then $(A\bigcap B)^0 \subseteq A^0$ and $(A\bigcap B)^0 \subseteq B^0$, then $(A\bigcap B)^0 \subseteq A^0\bigcap B^0$. Hence $(A\bigcap B)^0 = A^0\bigcap B^0$.

4. By (1), $X^0 \subseteq X$. Since X is open set, by definition of interior operator $X \subseteq X^0$. Hence $X = X^0$.

5. Suppose that G is open set, then $G^0 = \bigcup \{U : U \subseteq G$ and $U \in \tau\}$, so $G \subseteq G^0$. But $G^0 \subseteq G$ by (1), then $G^0 = G$. Conversely suppose that $G^0 = G$, then G is open set since G^0 is open.

□

Theorem 3.1.9. *[2] Let X be a set, and let Int be a mapping $A \to A^0$ from the power set $p(X)$ into $p(X)$ satisfying (1) through (4) in previous theorem. If open sets are defined in X using (5), the result forms a topology on X where the interior of a set $A \subseteq X$ is just A^0.*

Example 3.1.10. *We always have* $(A \bigcap B)^0 = A^0 \bigcap B^0$. *The corresponding statement for union is not true. To see this, let* X *be the set* $\mathbf{R}$ *of all real numbers with standard topology, let* $A = Q$ *the set of all rational numbers, and* $B = \mathbf{R} - Q$ *the set of all irrational numbers in* $\mathbf{R}$. *Then* $A^0 = B^0 = \emptyset$, *then* $A^0 \bigcup B^0 = \emptyset$, *but* $(A \bigcup B)^0 = \mathbf{R}^0 = \mathbf{R}$. *Hence* $(A \bigcup B)^0 \neq A^0 \bigcup B^0$. *It is always true that* $A^0 \bigcup B^0 \subseteq (A \bigcup B)^0$.

Remark 3.1.11. [2] The notions of interior and closure are dual to each other. In much the same way that "open" and "closed" are. The strictly formal nature of this duality can be brought out in observing that

$$\begin{aligned} X - E^0 &= \overline{X - E} \\ X - \overline{E} &= (X - E)^0 \end{aligned}$$

Thus any theorem about closures in a topological space can be translated to a theorem about interiors.

3.2 On V_m-Sets

Definition 3.2.1. [25] Let (X, m) be an m-space and A a subset of X. The subset $V_m(A)$ is defined as follows:

$$V_m(A) = \bigcup \{F : F \subseteq A, X - F \in m\}$$

Lemma 3.2.2. *[25] For subsets* A, B, $A_\alpha (\alpha \in \Delta)$ *of an* m*-space* (X, m), *the following properties hold:*

1. $V_m(A) \subseteq A$.

2. $\Lambda_m(X - A) = X - V_m(A)$, *and* $V_m(X - A) = X - \Lambda_m(A)$.

3. $V_m(V_m(A)) = V_m(A)$.

4. If $A \subseteq B$*, then* $V_m(A) \subseteq V_m(B)$.

5. $V_m(\bigcup \{A_\alpha : \alpha \in \Delta\}) \supseteq \bigcup \{V_m(A_\alpha) : \alpha \in \Delta\}$

6. $V_m(\bigcap \{A_\alpha : \alpha \in \Delta\}) = \bigcap \{V_m(A_\alpha) : \alpha \in \Delta\}$.

Proof. 1. Let $x \in V_m(A)$, then $x \in F$ for some F such that $X - F \in m$ and $F \subseteq A$. Therefore $x \in A$ and hence $V_m(A) \subseteq A$.

2. Let us observe that

$$\begin{aligned} X - V_m(A) &= X - \bigcup\{F : F \subseteq A, X - F \in m\} \\ &= \bigcap\{X - F : F \subseteq A, X - F \in m\} \\ &= \bigcap\{X - F : X - A \subseteq X - F, X - F \in m\} \\ &= \Lambda_m(X - A). \end{aligned}$$

And also that

$$\begin{aligned} X - \Lambda_m(A) &= X - \bigcap\{U : A \subseteq U, U \in m\} \\ &= \bigcup\{X - U : A \subseteq U, U \in m\} \\ &= \bigcup\{X - U : X - U \subseteq X - A, U \in m\} \\ &= V_m(X - A). \end{aligned}$$

3. By (2), and Lemma 2.2.2 part 2, we have that

$X - V_m(V_m(A)) = \Lambda_m(X - V_m(A)) = \Lambda_m(\Lambda_m(X - A)) = \Lambda_m(X - A) = X - V_m(A)$.

Thus, $V_m(V_m(A)) = V_m(A)$.

4. Suppose that $A \subseteq B$. Then, $X - B \subseteq X - A$. By Lemma 2.2.2 part 3, we have $\Lambda_m(X - B) \subseteq \Lambda_m(X - A)$. But, $\Lambda_m(X - B) = X - V_m(B)$ and $\Lambda_m(X - A) = X - V_m(A)$. Then, $X - V_m(B) \subseteq X - V_m(A)$ and hence, $V_m(A) \subseteq V_m(B)$.

5. For a subset A of X, $X - A$ is denoted by A^c. By Lemma 2.2.2 part 4, and part 2 here, we have that

$$\begin{aligned} V_m(\bigcup_{\alpha\in\Delta}(A_\alpha)) &= [[V_m(\bigcup_{\alpha\in\Delta}(A_\alpha))]^c]^c = [\Lambda_m([\bigcup_{\alpha\in\Delta}(A_\alpha)]^c)]^c \\ &= [\Lambda_m(\bigcap_{\alpha\in\Delta}(A_\alpha)^c)]^c \supseteq [\bigcap_{\alpha\in\Delta}(\Lambda_m(A_\alpha)^c)]^c \\ &= [\bigcap_{\alpha\in\Delta}[V_m(A_\alpha)]^c]^c = \bigcup[[V_m(A_\alpha)]^c]^c \\ &= \bigcup_{\alpha\in\Delta} V_m(A_\alpha). \end{aligned}$$

6. For a subset A of X, $X - A$ is denoted by A^c. By Lemma 2.2.2 part 2, and part 3 here, we have

$$\begin{aligned} V_m(\bigcap_{\alpha\in\Delta}(A_\alpha)) &= V_m([(\bigcap_{\alpha\in\Delta}(A_\alpha))^c]^c) = [\Lambda_m([\bigcap_{\alpha\in\Delta}(A_\alpha)]^c)]^c \\ &= [\Lambda_m(\bigcup_{\alpha\in\Delta}(A_\alpha)^c)]^c = [\bigcup_{\alpha\in\Delta}(\Lambda_m(A_\alpha)^c)]^c \\ &= \bigcap_{\alpha\in\Delta}[\Lambda_m(A_\alpha)^c]^c = \bigcap V_m([[A_\alpha]^c]^c) \\ &= \bigcap_{\alpha\in\Delta} V_m(A_\alpha). \end{aligned}$$

□

Definition 3.2.3. [25] A subset A of an m-space (X, m) is called a *V_m-set* if $A = V_m(A)$. The family of all V_m-sets of (X, m) is denoted by $V_m(X)$ (or simply V_m).

Lemma 3.2.4. *[25] For subsets A, $A_\alpha(\alpha \in \Delta)$ of an m-space (X, m), the following properties hold:*

1. *$V_m(A)$ is a V_m-set.*

2. *If $X - A$ is an m-open set, then A is a V_m-set.*

3. *If A_α is a V_m-set for each $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} A_\alpha$ is a V_m-set.*

4. *If A_α is a V_m-set for each $\alpha \in \Delta$, then $\bigcap_{\alpha\in\Delta} A_\alpha$ is a V_m-set.*

Proof. 1. From Lemma 3.2.2, $V_m(V_m(A)) = V_m(A)$, so $V_m(A)$ is a V_m set.

2. If $X - A \in m$, Then $A \in \{F : F \subseteq A, X - F \in m\}$. So $A \subseteq V_m(A)$. By (1), we have $V_m(A) \subseteq A$. Hence $V_m(A) = A$ and so A is a V_m-set.

3. Let $A_\alpha \in V_m$ for each $\alpha \in \Delta$, then by Lemma 3.2.2 part 1, part 5, and Definition 3.2.3 we have that

$$\bigcup_{\alpha\in\Delta} A_\alpha = \bigcup_{\alpha\in\Delta} V_m(A_\alpha) \subseteq V_m(\bigcup_{\alpha\in\Delta} A_\alpha) \subseteq \bigcup_{\alpha\in\Delta} A_\alpha.$$

Thus, we have that $\bigcup_{\alpha\in\Delta} A_\alpha = V_m(\bigcup_{\alpha\in\Delta} A_\alpha)$ and $\bigcup_{\alpha\in\Delta} A_\alpha \in V_m$.

4. Let $A_\alpha \in V_m$ for each $\alpha \in \Delta$, then by Lemma 3.2.2 part 6, and Definition 3.2.3, we have that

$$\bigcap_{\alpha\in\Delta} A_\alpha = \bigcap_{\alpha\in\Delta} V_m(A_\alpha) = V_m(\bigcap_{\alpha\in\Delta} A_\alpha).$$

Thus, we have $\bigcap_{\alpha\in\Delta} A_\alpha \in V_m$

□

Theorem 3.2.5. *[25] For an m-space (X, m), the pair (X, V_m) is an Alexandroff topological space.*

Proof. Since $X - \emptyset = X \in m$, then $\emptyset \in V_m$, and since $\emptyset = X - X \in m$, then $X \in V_m$. Now, Let $A_\alpha \in V_m$ for each $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} A_\alpha$ is a V_m-set by Lemma 3.2.4 part

3, and $\bigcap_{\alpha\in\Delta} A_\alpha$ is a V_m-set by Lemma 2.2.4 part 4. This proves that (X, V_m) is an Alexandroff space. □

Theorem 3.2.6. *[25] Let (X, m) be an m-space. Then, a subset A of (X, m) is a Λ_m-set if and only if $(X - A)$ is a V_m-set.*

Proof. Let A be a Λ_m-set, then $A = \Lambda_m(A)$. By Lemma 3.2.2 part 2, $(X - A) = (X - \Lambda_m(A)) = V_m(X - A)$. That is $(X - A)$ is a V_m-set. Conversely, suppose that $(X - A)$ is a V_m-set, then $(X - A) = V_m(X - A)$. By lemma 3.2.2 part 2, $A = X - (X - A) = X - V_m(X - A) = X - (X - \Lambda_m(A)) = \Lambda_m(A)$. That is A is a Λ_m-set. □

Theorem 3.2.7. *Let (X, m) be an m-space. The topological space (X, Λ_m) is T_0 if and only if the topological space (X, V_m) is T_0.*

Proof. Suppose that (X, Λ_m) is a T_0-space. Let x, y be any pair of distinct points of X, then there exist $V \in \Lambda_m$ such that either $x \in V$ and $y \notin V$ or $x \notin V$ and $y \in V$. Without loss of generality, suppose that $x \in V$ and $y \notin V$. Then $x \notin (X - V)$ and $y \in (X - V)$. But $(X - V)$ is a V_m-set by Theorem 3.2.6. Therefore the topological space (X, V_m) is T_0. Conversely, suppose that (X, V_m) is T_0 space. Let x, y be any pair of distinct points of X, then there exist $U \in V_m$ such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. Suppose $x \in U$ and $y \notin U$, then $x \notin (X - U)$ and $y \in (X - U)$. But $(X - U)$ is a Λ_m-set by Theorem 3.2.6. Therefore the topological space (X, Λ_m) is T_0. □

3.3 Dual Specialization Order and the Dual A-Space

In an Alexandroff space $(X, \tau(\leq))$, the collection of all closed sets forms a topology on X called *the dual A-space* of X, which is denoted by τ^d. Moreover τ is T_0 if and only if the dual τ^d is T_0. Hence for a given T_0 A-space (X, τ), the dual space (X, τ^d) is T_0 A-apace, so it has a corresponding order $\leq^d$. The following theorem gives a characterization for $\leq^d$.

Theorem 3.3.1. *[28] Let $(X, \tau(\leq))$ be a T_0 A-space with specialization order $\leq$. Let τ^d be the dual space and let $(\leq^d)$ the corresponding order induced on X by τ^d. Then for any $x, y \in X$ $x \leq^d y$ if and only if $y \leq x$.*

Theorem 3.3.2. *For any m-space (X, m) the following are equivalent:*

a. (X, m) *is* m-T_0.

b. $(X, \leq)$ *is a poset.*

c. $(X, \leq^d)$ *is a poset.*

Proof. $(a) \Rightarrow (b)$ Let (X, m) be an m-T_0-space with specialization order $(X, \leq)$, then by lemma 3.3.2, $(X, \leq)$ is a poset.
$(b) \Rightarrow (c)$ suppose that $(X, \leq)$ is a poset and $(X, \leq^d)$ it's dual specialization order. Let $x, y \in X$ such that $x \leq^d y$, then we have that:

1. $(X, \leq^d)$ is reflexive, Since $\forall x \in X$, $x \leq x$, implies $x \leq^d x$.

2. $(X, \leq^d)$ is transitive, to see this, Let $x \leq^d y$ and $y \leq^d z$, then $z \leq y$ and $y \leq x$. Since $(X, \leq)$ is transitive, we have $z \leq x$. Thus $x \leq^d z$.

3. $(X, \leq^d)$ is anti-symmetric, since if $x \leq^d y$ and $y \leq^d x$, we have that $y \leq x$ and $x \leq y$, thus $x = y$ (since $(X, \leq)$ anti-symmetry).

Therefore $(X, \leq^d)$ is a poset.

3) $\Rightarrow$ 1) Let $(X, \leq^d)$ is a poset. Suppose to contrary that, $\forall\ U \in m$, either both $x, y \in U$ or both x, y not in U. By the definition of $\leq$, we have that $x \leq y$ and $y \leq x$, and by the definition of $\leq^d$ we have that $x \leq^d y$ and $y \leq^d x$. Since $(X, \leq^d)$ is a poset, $x = y$. Which is a contradiction. Thus $\exists\ U \in \tau$ such that $x \in U$ and $y \notin U$. Therefore (X, τ) is a T_0 space. □

Theorem 3.3.3. *Let (X, m) be an m-space. The members of the topological space (X, V_m) is the closed sets of the topological space (X, Λ_m).*

Proof. Let $\mathfrak{F}_{\mathfrak{m}}$ be the closed sets of the topological space (X, Λ_m). Let $A \in \mathfrak{F}_{\mathfrak{m}}$, then A^c is a Λ_m-set. By Theorem 3.2.6, $A = (A^c)^c$ is a V_m-set, that is $A \in V_m$. Hence $\mathfrak{F}_{\mathfrak{m}} \subseteq V_m$. Conversely let $B \in V_m$. By Theorem 3.2.6 B^c is Λ_m-set. Then $B = (B^c)^c$ is a closed set in the topological space(X, Λ_m); that is, $B \in \mathfrak{F}_{\mathfrak{m}}$. Hence $V_m \subseteq \mathfrak{F}_{\mathfrak{m}}$. Therefore $V_m = \mathfrak{F}_{\mathfrak{m}}$. □

Remark 3.3.4. From Theorem 3.3.3, the A-space V_m is the closed sets of the Alexandroff topology Λ_m. That is,

$$V_m = (\Lambda_m)^d$$

Corollary 3.3.5. *Let (X, m) be an m-space. The dual specialization order on the dual space (X, V_m) is defined by $x \leq_m^d y$ if and only if $y \leq_m x$, where $(\leq_m)$ is the specialization order on (X, Λ_m).*

Proof. Since (X, V_m) is the dual space of the Alexandroff space (X, Λ_m), then by Theorem 3.3.3 the dual specialization order on (X, V_m) is given by $x \leq_m^d y$ if and only if

$y \leq_m x$, where $\leq_m$ is the specialization order on (X, Λ_m). □

Theorem 3.3.6. *Let $(X, \leq^d)$ be The dual specialization order on a dual space (X, V_m), then the following properties are equivalent:*

a. $x \leq_m^d y$.

b. $y \in mCl(x)$.

c. $x \in \Lambda_m(y)$.

Proof. $(a) \Rightarrow (b)$ Let $x \leq_m^d y$, Then by Corollary 3.3.6, $y \leq_m x$, and by Lemma 2.3.2, $y \in mCl(x)$.
$(b) \Rightarrow (c)$ Let $y \in mCl(x)$, then by Lemma 2.3.2, $x \in \Lambda_m(y)$.
$(c) \Rightarrow (a)$ Let $x \in \Lambda_m(y)$, then by Lemma 2.3.2, $y \leq_m x$, and by Corollary 3.3.6, $x \leq_m^d y$. □

3.4 Continuty On Λ_m and V_m sets

Definitions 3.4.1. [24] A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be

1. *M-continuous*, if for every $A \in m_Y$, $f^{-1}(A) \in m_X$. Equivalently, f is M-continuous if for every m_Y-closed set B of (Y, m_Y), $f^{-1}(B)$ is m_X-closed of (X, m_X)

2. *M-open*, if $f(A)$ is an m_Y-open set of (Y, m_Y) for every m_X-open set A of (X, m_X).

3. *M-closed*, if $f(A)$ is an m_Y-closed set of (Y, m_Y) for every m_X-closed set A of (X, m_X).

Theorem 3.4.2. *[24] If a function $f : (X, m_X) \to (Y, m_Y)$ is M-continuous, then $f : (X, \Lambda_{m_X}) \to (Y, \Lambda_{m_Y})$ is continuous.*

Proof. Let B be any Λ_m-set of (Y, m_Y). Since f is M-continuous, $\forall\ V \in m_Y$, $B \subseteq V$, then $f^{-1}(B) \subseteq f^{-1}(V)$ and $f^{-1}(V) \in m_X$. Hence the collection $\{f^{-1}(V) : B \subseteq V, V \in m_Y\} \subseteq \{U : f^{-1}(B) \subseteq U, U \in m_X\}$. Then we have

$$f^{-1}(B) \subseteq \bigcap \{U : f^{-1}(B) \subseteq U, U \in m_X\} \subseteq \bigcap \{f^{-1}(V) : B \subseteq V, V \in m_Y\}$$

$$= f^{-1}(\bigcap \{V : B \subseteq V, V \in m_Y\}) = f^{-1}(\Lambda_{m_Y}(B)) = f^{-1}(B).$$

Therefore, we obtain

$$f^{-1}(B) = \bigcap \{U : f^{-1}(B) \subseteq U, U \in m_X\} = \Lambda_{m_X}(f^{-1}(B)).$$

Hence $f^{-1}(B) \in \Lambda_{m_X}$. This shows that $f : (X, \Lambda_{m_X}) \to (Y, \Lambda_{m_Y})$ is continuous function. □

Theorem 3.4.3. *Let $f : (X, m_X) \to (Y, m_Y)$ be a function. Then we have:*

a. If f is M-continuous and V is a Λ_m-set of (Y, m_Y), then $f^{-1}(V)$ is a Λ_m-set of (X, m_X).

b. If f is M-open and bijective and if A is a Λ_m-set of (X, m_X), then $f(A)$ is a Λ_m-set of (Y, m_Y).

Proof. a. Let V be a Λ_m-set of (Y, m_Y). Since f is M-continuous, we have $f^{-1}(V) \subseteq \Lambda_m(f^{-1}(V)) = \bigcap\{U : f^{-1}(V) \subseteq U, U \in m_X\} \subseteq \bigcap\{f^{-1}(P) : V \subseteq P, P \in m_Y\} = f^{-1}(\bigcap\{P : V \subseteq P, P \in m_Y\}) = f^{-1}(\Lambda_m(V)) = f^{-1}(V)$. Therefor we obtain $f^{-1}(V) = \Lambda_m(f^{-1}(V))$ which shows that $f^{-1}(V)$ is a Λ_m-set.

b. Let A be a Λ_m-set of (X, m_X). Since f is an M-open bijection we have $f(A) = f(\bigcap\{U : A \subseteq U, U \in m_X\}) = \bigcap f(\{U : A \subseteq U, U \in m_X\}) = \bigcap(\{f(U) : A \subseteq U, U \in m_X\}) \supseteq (\bigcap\{V : f(A) \subseteq V, V \in m_Y\}) = \Lambda_m(f(A)) \supseteq f(A)$. Therefore, we obtain $f(A) = \Lambda_m(f(A))$ which shows that $f(A)$ is a Λ_m-set.

□

Chapter 4

Bi m-space and Related Topological Spaces

In this chapter we study new types of sets called (Λ, mn)-closed sets and Generalized Λ_{mn}-sets, which are defined on a non empty set with two minimal structures. We investigate the two types of topological spaces; (X, Λ^*_{mnc}) and $(X, \Lambda_{g\Lambda_{mn}})$ which are constructed from the families of these sets. We study some properties of lower separation axioms. Finally in section three, we study some mappings related to these new topological spaces.

4.1 (Λ, mn)-closed Sets and a Topological Space (X, Λ^*_{mnc})

Definition 4.1.1. [25] Let (X, m, n) be a bi m-space. A subset A of X is said to be $(\Lambda, mn)-$*closed* if $A = U \bigcap F$, where U is a Λ_m-set and F is an n-closed set. The family of all (Λ, mn)-closed sets of a bi m-space (X, m, n) is denoted by Λ_{mnc}. When $m = n$ such that X has one minimal structure, then (Λ, mn)-closed sets is denoted by

(Λ, m)-closed.

Remark 4.1.2. [25] Let (X, τ) be a topological space, If $m = n = \tau$ (resp. $SO(X)$, τ_α, τ_θ , τ_δ, $S\theta O(X)$), then a (Λ, mn)-closed set is a λ-closed (resp. semi-λ-closed, (Λ, α)-closed, (Λ, θ)-closed, (Λ, δ)-closed, $(\Lambda, s\theta)$-closed set).

Theorem 4.1.3. *[25] Let (X, m, n) be a bi m-space, and A is a subset of X, then the following properties are equivalent:*

1. *A is (Λ, mn)-closed,*
2. *$A = U \bigcap nCl(A)$, where U is a Λ_m-set,*
3. *$A = \Lambda_m(A) \bigcap nCl(A)$.*

Proof. (1)⇒(2) Let $A = U \bigcap F$, where U is a Λ_m-set and F is an n-closed set of X. Since $A \subseteq F$, we have that $nCl(A) \subseteq F$ and $A \subseteq U \bigcap nCl(A) \subseteq U \bigcap F = A$.
(2)⇒(3) Let $A = U \bigcap nCl(A)$, where U is a Λ_m-set. Since $A \subseteq U$, we have that $\Lambda_m(A) \subseteq \Lambda_m(U) = U$. Hence, $A \subseteq \Lambda_m(A) \bigcap nCl(A) \subseteq U \bigcap nCl(A) = A$.
(3)⇒(1) By Lemma 2.2.2(2), we have $\Lambda_m(A)$ is a Λ_m-set and by Lemma 2.1.7 (3), $nCl(A)$ is n-closed. By (3), we have $A = \Lambda_m(A) \bigcap nCl(A)$ and hence, A is (Λ, mn)-closed. □

Theorem 4.1.4. *[25] In a bi m-space $(X, , m, n)$, every Λ_m-set is (Λ, mn)-closed and every n-closed set is (Λ, mn)-closed.*

Proof. If A is a Λ_m-set, then $A = A \bigcap X$ and X is n-closed, then A is (Λ, mn)-closed set. If A is an n-closed set, then $A = X \bigcap A$ and X is a Λ_m-set, then A is (Λ, mn)-closed set.

□

Remark 4.1.5. The converse of previous theorem need not be true, as the following example shows.

Example 4.1.6. *Let $X = \{a, b, c, d\}$, and the m-structures m and n defined on X as follows: $m = \{\emptyset, X, \{a, b\}\}$. $n = \{\emptyset, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then the n-closed sets $= \{\emptyset, X, \{c, d\}, \{d\}, \{c\}\}$. The Λ_m-set $=\{\emptyset, X, \{a, b\}\}$. The (Λ, mn)-closed sets $=\{\emptyset, X, \{a, b\}, \{c, d\}, \{d\}, \{c\}\}$. Observe that $\{d\}$ is a (Λ, mn)-closed set but is not Λ_m-set and $\{a, b\}$ is (Λ, mn)-closed but is not n-closed.*

Theorem 4.1.7. *[25] Let $\{A_\alpha : \alpha \in \mathrm{I}\}$ be a family of subsets of a bi m-space (X, m, n). If A_α is a (Λ, mn)-closed set for each $\alpha \in \Delta$, then $\bigcap\{A_\alpha : \alpha \in I\}$ is a (Λ, mn)-closed.*

Proof. Suppose that A_α is a (Λ, mn)-closed set for each $\alpha \in I$. Then, for each $\alpha \in I$, there exists a Λ_m-set U_α and an n-closed set F_α such that $A_\alpha = U_\alpha \bigcap F_\alpha$. Hence we have

$$\bigcap_{\alpha\in I} A_\alpha = \bigcap_{\alpha\in I}(U_\alpha \bigcap F_\alpha) = (\bigcap_{\alpha\in I} U_\alpha)\bigcap(\bigcap_{\alpha\in I} F_\alpha)$$

By Lemma 2.2.2 (4), $\bigcap U_\alpha$ is a Λ_m-set, and $\bigcap F_\alpha$ is an n-closed set. This shows that $\bigcap A_\alpha$ is (Λ, mn)-closed. □

Definition 4.1.8. [25] Let (X, m, n) be a bi m-space. A subset A of X is said to be a *generalized mn-closed set (briefly mng-closed)* if $nCl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in m$. The complement of an *mng*-closed set is said to be *mng-open*.

Theorem 4.1.9. *[25] If A and B are subsets of a bi m-space (X, m, n), then the following properties hold:*

1. *If A is n-closed set, then A is mng-closed.*

2. *If A is mng-closed and m-open, then A is n-closed.*

3. *If A is mng-closed and $A \subseteq B \subseteq nCl(A)$, then B is mng-closed.*

Proof.

1. If A is n-closed set, by Lemma 2.1.6, $nCl(A) = A$ Let $U \in m$ such that $A \subseteq U$, then $nCl(A) \subseteq U$. Hence A is mng-closed.

2. If A is mng-closed, then $nCl(A)) \subseteq U$, whenever $A \subseteq U$ and $U \in m$. Since A is m-open, then $nCl(A) \subseteq A$, but by Lemma 2.1.4, $A \subseteq nCl(A)$. Hence $nCl(A) = A$, and A is n-closed.

3. Let $B \subseteq U$, where U is an m-open set. Then $A \subseteq U$ and A is mng-closed. Therefore, $nCl(A) \subseteq U$. Since $A \subseteq B \subseteq nCl(A)$, then $nCl(A) \subseteq nCl(B)$ and so, $nCl(B) \subseteq nCl(nCl(A)) = nCl(A)$. Thus $nCl(A) = nCl(B)$, and hence $nCl(B) \subseteq U$.

□

Theorem 4.1.10. *[25] A subset A of a bi m-space (X, m, n) is mng-closed if and only if $nCl(A) \subseteq \Lambda_m(A)$.*

Proof. Let A be an mng-closed set. Then $nCl(A) \subseteq U \; \forall \; U \in m$ such that $A \subseteq U$. Hence $nCl(A) \subseteq \bigcap\{U : A \subset U, U \in m\} = \Lambda_m(A)$. Conversely suppose that $nCl(A) \subseteq \Lambda_m(A)$. Then $nCl(A) \subseteq \bigcap\{U : A \subset U, U \in m\} \subseteq U \; \forall \; U \in m$, $A \subseteq U$. Therefore A is mng-closed.

□

Theorem 4.1.11. *[25] A subset A of a bi m-space (X, m, n) is n-closed if and only if A is mng-closed and (Λ, mn)-closed.*

Proof. By Theorem 4.1.4 and Theorem 4.1.9 part 1, every n-closed set is mng-closed and (Λ, mn)-closed.

Conversely, since A is mng-closed, then by Theorem 4.1.10, $nCl(A) \subset \Lambda_m(A)$. By hypothesis and Lemma 4.1.3, $A = \Lambda_m(A) \bigcap nCl(A) = nCl(A)$. So A is n-closed set.

$\square$

Definition 4.1.12. [25] A bi m-space (X, m, n) is said to be mn-T_0, if for $x, y \in X$ such that $x \neq y$, there exists an m-open set U of X containing x but not y or an n-open set V of X containing y but not x.

Theorem 4.1.13. *[25] Let (X, m, n) be a bi m-space, then (X, m, n) is mn-T_0 if and only if for each $x \in X$, the singleton $\{x\}$ is (Λ, mn)-closed.*

Proof. Suppose that (X, m, n) is mn-T_0. For each $x \in X$, we have that $\{x\} \subseteq \Lambda_m(\{x\}) \bigcap nCl(\{x\})$. $If x \neq y$, we have:

1. There exists an m-open set U such that $x \in U$ and $y \notin U$, or,

2. There exists an n-open set V such that $x \notin V$ and $y \in V$.

In case (1), $y \notin \Lambda_m(\{x\})$ and hence, $y \notin \Lambda_m(\{x\}) \bigcap nCl(\{x\})$.

In case (2), $y \notin nCl(\{x\})$ and hence, $y \notin \Lambda_m(\{x\}) \bigcap nCl(\{x\})$. This shows that $\Lambda_m(\{x\}) \bigcap nCl(x) \subseteq \{x\}$. Thus, we obtain $\{x\} = \Lambda_m(\{x\}) \bigcap nCl(\{x\})$ and by Lemma 4.1.3, $\{x\}$ is (Λ, mn)-closed.

Conversely, let $\{x\}$ be (Λ, mn)-closed. Suppose that (X, m, n) is not mn-T_0. Then there exist distinct points x, y such that

1. $y \in U$ for every m-open set U containing x, and,

2. $x \in V$ for every n-open set V containing y.

By (1) and (2), we obtain $y \in \Lambda_m(\{x\})$ and $y \in nCl(\{x\})$, respectively.

Therefore, we have $y \in \Lambda_m(\{x\}) \bigcap nCl(\{x\}) = \{x\}$ by Lemma 4.1.3. This contradicts that $x \neq y$.

That is (X, m, n) is mn-T_0. □

Definition 4.1.14. [25] A bi m-space (X, m, n) is said to be mn-$T_{\frac{1}{2}}$, if and only if every singleton $\{x\}$ of X is n-open or m-closed.

Theorem 4.1.15. *[25] Let (X, m, n) be a bi m-space, then (X, m, n) is mn-$T_{\frac{1}{2}}$, if and only if every mng-closed set of X is n-closed.*

Theorem 4.1.16. *[25] Let (X, m, n) be a bi m-space, then the following properties are equivalent:*

1. *(X, m, n) is mn-$T_{\frac{1}{2}}$,*

2. *Every subset of X is (Λ, mn)-closed.*

Lemma 4.1.17. *The family Λ_{mnc} of $(\Lambda, mn)-$closed sets of a bi m-space (X, m, n) is not necessarily a topology. But it is a base for a topology which is denoted by Λ^*_{mnc}*

Proof. In Example 4.1.6, we see that Λ_{mnc} is not a topology on X.

Since X is a Λ_m and an n-closed set, and so $X = X \bigcap X$, then $X \in \Lambda_{mnc}$.

Let $B_1, B_2 \in \Lambda_{mnc}$. From Theorem 4.1.7, $B_1 \bigcap B_2 \in \Lambda_{mnc}$. Hence $\forall\ x \in B_1 \bigcap B_2$, take $B_3 = B_1 \bigcap B_2$ Therefor Λ_{mnc} is abase for a topology Λ^*_{mnc} on X.

□

Lemma 4.1.18. *If (X, m, n) is a bi m-space, then $m \subseteq \Lambda_m \subseteq \Lambda_{mnc} \subseteq \Lambda^*_{mnc}$.*

Proof. This is an immediate consequence of Lemma 2.2.2 and Theorem 4.1.4. □

Theorem 4.1.19. *A bi m-space (X, m, n) is mn-T_0 if and only if the topological space (X, Λ^*_{mnc}) is discrete.*

Proof. Suppose that (X, m, n) is mn-T_0. Let $x \in X$, then by Theorem 4.1.13 the singleton $\{x\}$ is a (Λ, mn)-closed set and $\{x\}$ is open in (X, Λ^*_{mnc}). Therefore, every subset of X is open in (X, Λ^*_{mnc}) and hence (X, Λ^*_{mnc}) is discrete.
Conversely, suppose that the topological space (X, Λ^*_{mnc}) is discrete. For any point $x \in X$, the singleton $\{x\}$ is open in (X, Λ^*_{mnc}) and hence $\{x\}$ is a (Λ, mn)-closed set. Therefore, by Theorem 4.1.13 (X, m, n) is mn-T_0. □

Corollary 4.1.20. *For a bi m-space (X, m, n), the following properties are equivalent:*

1. *(X, Λ_m) is T_0,*
2. *(X, Λ^*_{mnc}) is discrete.*

Proof. This is an immediate consequence of Theorems 2.2.8 and 4.1.19. □

4.2 Generalized Λ_{mn}-sets and a Topological Space $(X, \Lambda_{g\Lambda_{mn}})$

Definition 4.2.1. [25] Let (X, m, n) be a bi m-space. A subset A of X is called *a generalized Λ_{mn}-set* (briefly gΛ_{mn}-set) of (X, m, n) if $\Lambda_m(A) \subset F$ whenever $A \subset F$ and F is $n-$closed. The family of all gΛ_{mn}-set of (X, m, n) is denoted by gΛ_{mn}.

Definition 4.2.2. [25] Let (X, m, n) be a bi m-space. A subset A of X is called a *generalized* V_{mn}*-set* (briefly gV_{mn}*-set*) of (X, m, n) if A^c is gΛ_{mn}-set. The family of all gV_{mn}-set of (X, m, n) is denoted by gV_{mn}.

Remark 4.2.3. [25] Let (X, τ) be a topological space, then:

1. If $m = n = \tau$ (resp. $SO(X)$, $PO(X)$), then a gΛ_{mn}-set is gΛ-set (resp. gΛ_s-set, g pre-Λ-set).

2. If $m = S\theta O(X)$ and $n = SO(X)$, then a gΛ_{mn}-set is θ-g Λ_s-set.

Theorem 4.2.4. *[25] If A and B are subsets of a bi m-space (X, m, n), then the following properties hold:*

1. *If A is a Λ_m-set, then A is a gΛ_{mn}-set,*

2. *If A is a gΛ_{mn}-set and n-closed, then A is a Λ_m-set,*

3. *If A is a gΛ_{mn}-set and $A \subset B \subset \Lambda_m(A)$, then B is a gΛ_{mn}-set.*

Proof.

1. Suppose that A is a Λ_m-set and $A \subseteq F$, where F is an n-closed set. Then, $\Lambda_m(A) = A \subseteq F$. Therefore, A is a gΛ_{mn}-set.

2. Let A be a gΛ_{mn}-set and n-closed. Then, $\Lambda_m(A) \subseteq A$. By Lemma 2.2.2(1), we obtain $\Lambda_m(A) = A$. This shows that A is a Λ_m-set.

3. Let $B \subseteq F$, where F is an n-closed set. Then $A \subseteq F$ and A is a gΛ_{mn}-set. Therefore, $\Lambda_m(A) \subseteq F$. By Lemma 2.2.2 and the hypothesis, we have $\Lambda_m(A) \subseteq \Lambda_m(B) \subseteq \Lambda_m(\Lambda_m(A)) = \Lambda_m(A)$. Thus, $\Lambda_m(A) = \Lambda_m(B)$ and hence, $\Lambda_m(B) \subseteq F$.

□

Theorem 4.2.5. *[25] A subset A of a bi m-space (X, m, n) is a gΛ_{mn}-set if and only if $\Lambda_m(A) \bigcap U = \emptyset$ whenever $A \bigcap U = \emptyset$ and $U \in n$.*

Proof. Suppose that A is a gΛ_{mn}-set. Let $A \bigcap U = \emptyset$ where U is an n-open. Then $A \subseteq X - U$ and $X - U$ is n-closed. Therefore, $\Lambda_m(A) \subseteq X - U$ and $\Lambda_m(A) \bigcap U = \emptyset$. Conversely, let $A \subseteq F$ and F is n-closed. Then, $A \bigcap (X - F) = \emptyset$ and $(X - F) \in n$. By the hypothesis we have $\Lambda_m(A) \bigcap (X - F) = \emptyset$ and hence, $\Lambda_m(A) \subseteq F$. This shows that A is a gΛ_{mn}-set.

□

Theorem 4.2.6. *[25] A subset A of a bi m-space (X, m, n) is a gΛ_{mn}-set if and only if $\Lambda_m(A) \subseteq nCl(A)$.*

Proof. Suppose that A is a gΛ_{mn}-set, and let F be any n-closed set such that $A \subseteq F$. Then, $\Lambda_m(A) \subseteq F$. Take $F = nCl(A)$, we obtain $\Lambda_m(A) \subseteq nCl(A)$.
Conversely, suppose that $\Lambda_m(A) \subseteq nCl(A)$ and let $A \subseteq F$, where F is an n-closed set. Then $\Lambda_m(A) \subseteq nCl(A) \subseteq F$. Thus, A is a gΛ_{mn}-set. □

Theorem 4.2.7. *[25] A subset A of a bi m-space (X, m, n) is a gV_{mn}-set if and only if*
$U \subseteq V_m(A)$ whenever $U \subseteq A$ and U is an n-open set.

Proof. Let A be a gV_{mn}-set and let U be any n-open set such that $U \subseteq A$. Then, $X - U$ is n-closed and $X - A \subseteq X - U$. Since $X - A$ is a a gΛ_{mn}-set, we have $\Lambda_m(X - A) \subseteq X - U$. By Lemma 3.1.2, we obtain $X - V_m(A) \subseteq X - U$. Thus, $U \subseteq V_m(A)$. Conversely, let F be an n-closed set such that $X - A \subseteq F$. Since $X - F$ is n-open and $X - F \subseteq A$, by the hypothesis we obtain $X - F \subseteq V_m(A)$. Then, $X - V_m(A) = \Lambda_m(X - A) \subseteq F$ by Lemma 3.1.2, and $X - A$ is a gΛ_{mn}-set. Therefore,

A is a gV_{mn}-set.

$\square$

Theorem 4.2.8. *[25] Let (X, m, n) ba a bi m-space. If $A_\alpha \in g\Lambda_{mn}$ (resp. $A_\alpha \in gV_{mn}$) for each $\alpha \in I$, then $\bigcup_{\alpha \in I} A_\alpha \in g\Lambda_{mn}$ (resp. $\bigcap_{\alpha \in I} A_\alpha \in gV_{mn}$).*

Proof. Let $\bigcup_{\alpha \in I} A_\alpha \subseteq F$ and F is n-closed. Then, $A_\alpha \subseteq F$ and so $\Lambda_m(A) \subseteq F$ for each $\alpha \in I$. By Lemma 2.2.2 (5), we have $\Lambda_m(\bigcup_{\alpha \in I} A_\alpha) = \bigcup_{\alpha \in I} \Lambda_m(A_\alpha)$ This shows that $\bigcup_{\alpha \in I} A_\alpha \in$ gΛ_{mn}.

For the second case, let A_α is a gV_{mn} set, then $(A_\alpha)^c$ is a gΛ_{mn} set, and so $\bigcup_{\alpha \in I}(A_\alpha)^c = (\bigcap_{\alpha \in I} A_\alpha)^c$ is a gΛ_{mn}. Hence $\bigcap_{\alpha \in I} A_\alpha$ is gV_{mn} set. $\square$

Theorem 4.2.9. *Let (X, m, n) be a bi m-space then:*

1. *$g\Lambda_{mn}$ is an $m-$structure on X and $(X, \Lambda_{g\Lambda_{mn}})$ is an Alexandroff space ,*

2. *$m \subseteq \Lambda_m \subseteq g\Lambda_{mn} \subseteq \Lambda_{g\Lambda_{mn}}$.*

Proof.

1. By Lemma 4.1.18 and theorem 4.2.3, $m \subseteq \Lambda_m \subseteq g\Lambda_{mn}$, Since $\emptyset, X \in m$, we have $\emptyset, X \in g\Lambda_{mn}$. By Theorem 4.2.7, if $A_\alpha \in$ gΛ_{mn} for each $\alpha \in \Delta$, then $\bigcup_{\alpha \in \Delta} A_\alpha \in g\Lambda_{mn}$. Therefore, g$\Lambda_{mn}$ is an m-structure on X. Hence by Theorem 2.2.5 $(X, \Lambda_{g\Lambda_{mn}})$ is an Alexandroff space, where $A \in \Lambda_{g\Lambda_{mn}}$ iff $A = \bigwedge_{g\Lambda_{mn}}(A)$ and $\bigwedge_{g\Lambda_{mn}} = \bigcap\{U : A \subseteq U, U \in g\Lambda_{mn}\}$

2. Clearly, $m \subseteq \Lambda_m \subseteq g\Lambda_{mn} \subseteq \bigwedge_{g\Lambda_{mn}}$.

$\square$

Corollary 4.2.10. *For subsets A, B, $A_\alpha(\alpha \in \Delta)$ of a bi m-space (X, m, n), the following profperties hold:*

1. $A \subset \Lambda_{g\Lambda_{mn}}(A)$,

2. $\Lambda_{g\Lambda_{mn}}(\Lambda_{g\Lambda_{mn}}(A)) = \Lambda_{g\Lambda_{mn}}(A)$,

3. $If A \subset B$, *then* $\Lambda_{g\Lambda_{mn}}(A) \subset \Lambda_{g\Lambda_{mn}}(B)$,

4. $\Lambda_{g\Lambda_{mn}}(\bigcap\{A_\alpha : \alpha \in \Delta\}) \subset \bigcap\{\Lambda_{g\Lambda_{mn}}(A_\alpha) : \alpha \in \Delta\}$,

5. $\Lambda_{g\Lambda_{mn}}(\bigcup\{A_\alpha : \alpha \in \Delta\}) = \bigcup\{\Lambda_{g\Lambda_{mn}}(A_\alpha) : \alpha \in \Delta\}$.

Proof. This is an immediate consequence of Lemma 2.2.2, and $g\Lambda_{mn}$ being m-structure.

□

Theorem 4.2.11. *[25] Let A be a $g\Lambda_{mn}$-set of a bi m-space (X, m, n), and F an n-closed set. If $(X - \Lambda_m(A)) \bigcup A \subseteq F$, then $X = F$.*

Proof. Let A be a gΛ_{mn}-set and $A \subseteq F$, where F is n-closed, Then, $\Lambda_m(A) \subseteq F$ and hence, $X - F \subseteq X - \Lambda_m(A)$. Since $(X - \Lambda_m(A)) \bigcup A \subseteq F$, then $X - \Lambda_m(A) \subseteq F$, and we have $X - F \subseteq X - \Lambda_m(A) \subseteq F$. Therefore, $F = X$.

□

Theorem 4.2.12. *[25] Let A be a $g\Lambda_{mn}$-set of a bi m-space (X, m, n). Then, $(X - \Lambda_m(A)) \bigcup A$ is an n-closed set if and only if A is a Λ_m-set.*

Proof. By Theorem 4.2.11, since $(X - \Lambda_m(A)) \bigcup A \subseteq (X - \Lambda_m(A)) \bigcup A$, then $(X - \Lambda_m(A)) \bigcup A = X$. If $x \in \Lambda_m(A)$, then $x \in X = (X - \Lambda_m(A)) \bigcup A$, and so $x \in A$ $(x \notin X - \Lambda_m(A))$. Hence $\Lambda_m(A) \subseteq A$. Therefore, $\Lambda_m(A) = A$ and A is a Λ_m-set. Conversely, if A is a Λ_m-set, then $\Lambda_m(A) = A$. So $(X - \Lambda_m(A)) \bigcup A = (X - A) \bigcup A = X$, and X is n-closed.

□

Remark 4.2.13. Note that in a bi m-space (X, m, n), $\forall\ A \subseteq X$, we have that A is a Λ_m-set if and only if $(X - \Lambda_m(A)) \bigcup A = X$, even if A is not gΛ_{mn}-set.

Theorem 4.2.14. *[25] Let (X, m, n) be a bi m-space. Then:*

a) for each $x \in X$, either $\{x\}$ is an n-open set or $X - \{x\}$ is a gΛ_{mn}-set of X.

b) for each $x \in X$, either $\{x\}$ is an n-open set or $\{x\}$ is a gV_{mn}-set of X.

Proof. a) Suppose that $\{x\}$ is not n-open. Let F be n-closed F containing $X - \{x\}$. Then by assumption, $F = X$. Thus, $\Lambda_{mn}(X - \{x\}) \subseteq F = X$ and $X - \{x\}$ is a gΛ_{mn}-set of X.

b) This is an immediate consequence of part (a).

□

Theorem 4.2.15. *[25] Let (X, m, n) be a bi m-space, and A a subset of X. Consider the following:*

a. A is a gΛ_{mn}-set.

b. $\Lambda_m(A) - A$ does not contain any nonempty n-open.

c. $\Lambda_m(A) - A$ is a gV_{mn}-set.

Then the implication (a) $\Rightarrow$ (b) $\Rightarrow$ (c) hold.

Proof. $(a) \Rightarrow (b)$ Suppose that A is a gΛ_{mn}-set of X. Let $U \subseteq \Lambda_m(A) - A$ and $U \in n$. Then, $A \subseteq X - U$ and $X - U$ is n-closed. Thus, $\Lambda_m(A) \subseteq X - U$. Therefore, $U \subseteq X - \Lambda_m(A)$ and so $U \subseteq (X - \Lambda_m(A)) \bigcap \Lambda_m(A) = \emptyset$.
$(b) \Rightarrow (c)$ Let $U \subseteq \Lambda_m(A) - A$ and $U \in n$. By (b), we have $U = \emptyset$ and $U \subseteq$

$V_m(\Lambda_m(A) - A)$. By Theorem 4.2.6, $\Lambda_m(A) - A$ is a gV_{mn}-set.

□

Theorem 4.2.16. *[25] Let (X, m, n) be a bi m-space, then the following properties are equivalent::*

a. (X, m, n) is mn-$T_{\frac{1}{2}}$,

b. Every gΛ_{mn}-set is Λ_m-set.

Proof. $(a) \Rightarrow (b)$ Let (X, m, n) be an mn-$T_{\frac{1}{2}}$. Suppose that there exists a gΛ_{mn}-set A of X which is not a Λ_m-set. Then, there exists $x \in \Lambda_m(A)$ such that $x \notin A$.
(i) In the case where $\{x\}$ is n-open, $A \subseteq X - \{x\}$ and $X - \{x\}$ is n-closed. Since A is a gΛ_{mn}-set, $\Lambda_m(A) \subseteq X - \{x\}$. Thus, $x \notin \Lambda_m(A)$ which is a contradiction.
(ii) In the case where $\{x\}$ is m-closed, $A \subseteq X - \{x\}$ and $X - \{x\}$ is m-open. By Theorem 2.2.2, $\Lambda_m(A) \subseteq \Lambda_m(X - \{x\}) = X - \{x\}$. This is a contradiction. Therefore, every gΛ_{mn}-set is a Λ_m-set.
$(b) \Rightarrow (a)$ Suppose (X, m, n) is not mn-$T_{\frac{1}{2}}$. Then there exists a mng-closed set A which is not n-closed. Since A is not n-closed, then there exists a point $x \in nCl(A)$ such that $x \notin A$. By Theorem 4.2.12, the singleton $\{x\}$ is n-open or $X - \{x\}$ is a gΛ_{mn}-set of X.
(i) In case $\{x\}$ is n-open, since $x \in nCl(A)$, we have $\{x\} \bigcap A \neq \emptyset$ and hence, $x \in A$. This is a contradiction
(ii) In the case where $X - \{x\}$ is a gΛ_{mn}-set, if $\{x\}$ is not m-closed, $X - \{x\}$ is not m-open and $\Lambda_m(X - \{x\}) = X$. Therefore, $X - \{x\}$ is not a Λ_m-set. This contradicts(2). If $\{x\}$ is m-closed, $A \subseteq X - \{x\} \in m$ and A is mng-closed. Hence we have $nCl(A) \subseteq X - \{x\}$. This contradicts that $x \in nCl(A)$. Therefore, (X, m, n) is mn-$T_{\frac{1}{2}}$. □

4.3 Functions Related To Generalized Λ_m and Generalized V_m sets

Theorem 4.3.1. *[24] For an m-space (X, m), the following properties hold:*

a. The identity functions $id_X : (X, \Lambda^*_{mc}) \to (X, \Lambda_m)$ and $id_X : (X, \Lambda_{g\Lambda_m}) \to (X, \Lambda_m)$ are continuous.

b. The identity functions $id_X : (X, g\Lambda_m) \to (X, m)$ is M-continuous.

Proof. a. Since every Λ_m-set is (Λ, m)-closed and a $g\Lambda_m$-set, then the identity functions $id_X : (X, \Lambda^*_{mc}) \to (X, \Lambda_m)$ and $id_X : (X, \Lambda_{g\Lambda_m}) \to (X, \Lambda_m)$ are continuous,

b. Since every m-open set is a $g\Lambda_m$-set, then the identity function $id_X : (X, g\Lambda_m) \to (X, m)$ is M-continuous.

□

Theorem 4.3.2. *[24] If a function $f : (X, m_X) \to (Y, m_Y)$ is M-continuous, then $f : (X, \Lambda^*_{m_X c}) \to (Y, \Lambda^*_{m_Y c})$ is continuous.*

Proof. Let B be any (Λ, m)-closed set of (Y, m_Y). Then there exists a Λ_m-set V and an m-closed set K of (Y, m_Y) such that $B = V \bigcap K$. Then we have $f^{-1}(B) = f^{-1}(V) \bigcap f^{-1}(K)$. By (a), $f^{-1}(V) \in \Lambda_{m_X}$, and since f is M-continuous $f^{-1}(K)$ is m-closed, . Therefore, $f^{-1}(B)$ is (Λ, m)-closed in X. For any $W \in \Lambda^*_{m_Y c}$, since $\Lambda_{m_Y c}$ is a base for $\Lambda^*_{m_Y c}$, there exist (Λ, m)-closed sets B_α of (Y, m_Y) such that $W = \bigcup_{\alpha \in \Delta} B_\alpha$. Hence $f^{-1}(W) = \bigcup_{\alpha \in \Delta} f^{-1}(B_\alpha)$ and $f^{-1}(B_\alpha)$ is a (Λ, m)-closed set of (X, m_X) for each $\alpha \in \Delta$. Hence, we obtain $f^{-1}(W) \in \Lambda^*_{m_X c}$ and $f : (X, \Lambda^*_{m_X c}) \to (Y, \Lambda^*_{m_Y c})$ is continuous. □

Theorem 4.3.3. *[24] Let (X, m_X), (Y, m_Y) be two m-spaces and $f : (X, m_X) \to (Y, m_Y)$ is M-continuous and M-open functions. Then the following properties hold:*

a) $f : (X, g\Lambda_{m_X}) \to (Y, g\Lambda_{m_Y})$ *is M-continuous.*

b) $f : (X, \Lambda_{g\Lambda_{m_X}}) \to (Y, \Lambda_{g\Lambda_{m_Y}})$ *is continuous.*

Proof. a) Let B be any gΛ_{m_Y}-set of (Y, m_Y), and F any m_X-closed set of (X, m_X) such that $f^{-1}(B) \subseteq F$. Set $K = Y - f(X - F)$. Then we have $B \subseteq K$, and $f^{-1}(K) \subseteq F$. Moreover, K is m_Y-closed since f is M-open. Since B is a gΛ_{m_Y}-set, $\Lambda_{m_Y}(B) \subseteq K$ and $f^{-1}(\Lambda_{m_Y}(B)) \subseteq f^{-1}(K) \subseteq F$. Since f is M-continuous, by Theorem 5.1.3, $f : (X, \Lambda_{m_X}) \to (Y, \Lambda_{m_Y})$ is continuous, and $f^{-1}(\Lambda_{m_Y}(B)) \in \Lambda_{m_X}$. Therefore, by Lemma 2.2.2 we have $\Lambda_{m_X}(f^{-1}(B)) \subseteq f^{-1}(\Lambda_{m_Y}(B)) \subseteq f^{-1}(K) \subseteq F$. This shows that $f^{-1}(B)$ is a gΛ_{m_X}-set of (X, m_X). Therefore, $f : (X, g\Lambda_{m_X}) \to (Y, g\Lambda_{m_Y})$ is M-continuous.

b) This is an immediate consequence of Theorem 5.1.3.

□

Definition 4.3.4. Let $f : (X, m_X) \to (Y, m_Y)$ be a function from an m-space (X, m_X) into an m-space (Y, m_Y). Then f is called:

1. *generalized* Λ_m*-continuous* function (briefly $g\Lambda_m$ *continuous* function) if for every m_Y-closed set A of (Y, m_Y), $f^{-1}(A)$ is gΛ_m of (X, m_X)

2. *generalized* Λ_m*-irresolute* function (briefly $g\Lambda_m$ *irresolute* function) if for every gΛ_m-set A of (Y, m_Y), $f^{-1}(A)$ is gΛ_m of (X, m_X)

3. *generalized* Λ_m*-open function* (briefly $g\Lambda_m$ *open* function), if $f(A)$ is gΛ_m-set of (Y, m_Y) for every $g\Lambda_m$-set A of (X, m_X).

Theorem 4.3.5. *Let $f : (X, m_X) \to (Y, m_Y)$ be M-continuous. Then f is $g\Lambda_m$ continuous but not conversely.*

Proof. Let $f : (X, m_X) \to (Y, m_Y)$ be M-continuous function. Then for every $A \in m_Y$, $f^{-1}(A) \in m_X$. Since every m-open set is gΛ_m-set, then f is gΛ_m continuous function. The converse need not be true as the following example shows. □

Example 4.3.6. *Let $X = \{a, b, c, d\}$, $m_X = \{\emptyset, X, \{c, b, d\}, \{c, d\}\}$.*
$Y = \{p, q\}$, and $m_Y = \{\emptyset, Y, \{p\}\}$. Define a function $f : (X, m_X) \to (Y, m_Y)$ by $f(c) = f(b) = f(d) = q$, and $f(a) = p$. Then f is $g\Lambda_m$ continuous but not m-continuous. To see this note that $f^{-1}(\{p\}) = \{a\} \notin (X, m_X)$.

Example 4.3.7. *Let (X, m_X) be an m-space such that $X = \{a, b, c\}$, and $m = \{\emptyset, X, \{a\}, \{b, c\}\}$.*
We consider the function $f : X \to X$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is $g\Lambda_m$ continuous and $g\Lambda_m$ irresolute but not m-continuous since $f^{-1}(\{a\}) = \{b\} \notin (X, m_X)$ and $\{a\} \in m$.

Theorem 4.3.8. *Let $f : (X, m_X) \to (Y, m_Y)$ is $g\Lambda_m$-irresolute (resp. $g\Lambda_m$ continuous) if and only if for every gV_m-set (resp. m-closed set) A of (Y, m_Y), $f^{-1}(A)$ is a gV_m-set of (X, m_X).*

Proof. If $f : (X, m_X) \to (Y, m_Y)$ is gΛ_m-irresolute, then for every gΛ_m-set B of (Y, m_Y), $f^{-1}(B)$ is gΛ_m of (X, m_X). If A is any gV_m-set of (Y, m_Y), then A^c is gΛ_m-set. Thus $f^{-1}(A^c)$ is gΛ_m-set, but $f^{-1}(A^c) = (f^{-1}(A))^c$ so that $f^{-1}(A)$ is a gV_m-set. Conversely, suppose that for all gV_m-set A of (Y, m_Y), $f^{-1}(A)$ is a gV_m-set in (X, m_X). If B is any gΛ_m-set of (Y, m_Y), then B^c is a gV_m-set. Also $f^{-1}(B^c) = (f^{-1}(B))^c$ is gV_m-set. Thus

$f^{-1}(B)$is gΛ_m-set and hence f is gΛ_m-irresolute. In a similar way we can prove the second case. □

Theorem 4.3.9. *If $f:(X,m_X)\to(Y,m_Y)$ is a bijective, M-continuous and M-closed function then we have the following:*

1. *for every gΛ_m-set B of (Y,m_Y), $f^{-1}(B)$ is gΛ_m-set of (X,m_X); that is, f is gΛ_m irresolute, and*

2. *for every gΛ_m-set A of (X,m_X), $f(A)$ is gΛ_m-set of (Y,m_Y); that is, f is gΛ_m open.*

Proof. 1. Let B be a gΛ_m-set of (Y,m_Y). Suppose that $f^{-1}(B)\subseteq F$, where F is m-closed set in (X,m_X). Therefore $B\subseteq f(F)$ and $f(F)$ is m-closed because f is M-closed. Since B is gΛ_m-set, $\Lambda_m(B)\subseteq f(F)$, and hence $f^{-1}(\Lambda_m(B))\subseteq F$. Therefore we have $\Lambda_m(f^{-1}(B))\subseteq f^{-1}(\Lambda_m(B))\subseteq F$. Hence $f^{-1}(B)$ is a gΛ_m-set.

2. Let A be a gΛ_m-set of (X,m_X). Suppose $f(A)\subseteq F$, where F is m-closed set of (Y,m_Y). Then $A\subseteq f^{-1}(F)$ and $f^{-1}(F)$ is m-closed since f is M-continuous. By the fact that f is M-open and bijection, $\Lambda_m(f(A))\subseteq f(\Lambda_m(A))\subseteq F$. Hence $f(A)$ is a gΛ_m-set of (Y,m_Y).

□

Corollary 4.3.10. *If a function $f:(X,m_X)\to(Y,m_Y)$ is bijective, M-continuous and M-closed function, then we have the following:*

1. *for every gV_m-set B of (Y,m_Y), $f^{-1}(B)$ is gV_m-set of (X,m_X), and*

2. *for every gV_m-set A of (X,m_X), $f(A)$ is gV_m-set of (Y,m_Y).*

Definition 4.3.11. A function $f : (X, m_X) \to (Y, m_Y)$ is called a *generalized V_m-closed* function (briefly *gV_m-closed* function) if for each closed set F of X, $f(F)$ is a gV_m-set. Obvoiously, every M-closed function is gV_m-closed function.

Example 4.3.12. *Let (X, m_X) be an m-space such that $X = \{a, b, c\}$, and $m_X = \{\emptyset, X, \{a\}, \{b, c\}\}$. We consider the function $f : X \to X$ by $f(a) = f(b) = f(c) = c$. Then f is gV_m closed but it is not m-closed function. The converse need not true as the following example shows.*

Theorem 4.3.13. *A function $f : (X, m_X) \to (Y, m_Y)$ is gV_m-closed if and only if for each subset S of Y and each open set U containing $f^{-1}(S)$, there is a gΛ_m-set V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Let S be a subset of Y and U be an m-open set of X such that $f^{-1}(S) \subseteq U$. Then $U^c \subseteq (f^{-1}(S))^c$. Hence $f(U^c) \subseteq f(f^{-1}(S))^c \subseteq S^c$, since $S \subseteq (f(U^c))^c$. Therefore $(f(U^c))^c$ say V, is a gΛ_m-set containing S. On the other hand, since $(f(U^c))^c = V$, we obtain: $f^{-1}((f(U^c))^c) = f^{-1}(V)$ or $(f^{-1}(f(U^c)))^c = f^{-1}(V)$. Hence $f^{-1}(f(U^c)) = (f^{-1}(V))^c$. Therefor $U^c \subseteq f^{-1}(f(U^c)) = (f^{-1}(V))^c$, i.e., $f^{-1}(V) \subseteq U$. Conversely let F be an arbitrary closed set of X. Then $f^{-1}((f(F))^c) \subseteq F^c$ and F^c is open. Let $S = (f(F))^c$ and F^c say U is open with $f^{-1}(S) \subseteq U$. By hypothesis, there is a gΛ_m-set V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$, i.e., there is a gΛ_m-set V of Y such that $f((f(F))^c) \subseteq V$ with $f^{-1}(V) \subseteq F^c$ and hence $V^c \subseteq f(F) \subseteq f(f^{-1}(V))^c) \subseteq V^c$, which implies $f(F) = V^c$. Since V^c is a gV_m-set, $f(F)$ is gV_m-set and thus f is a gV_m-closed function. □

Theorem 4.3.14. *For an m-space (X, m_X), every singleton of X is a gΛ_m-set if and only if $U = V_m(U)$ holds for every $U \in m_X$.*

Proof. Let U be an m-open set of X. Let $y \in U^c$, then by assumption $\Lambda_m(\{y\}) \subseteq U^c$. By using Lemma 2.2.2 we have $U^c \supseteq \bigcup \{\Lambda_m(\{y\}) : y \in U^c\} = \Lambda_m(U^c)$ and hence $U^c = \Lambda_m(U^c)$. Then it follows from lemma 3.1.2 that $U = V_m(U)$. Conversely, let $x \in X$ and F be an m-closed set such that $\{x\} \subseteq F$. Since $F^c = V_m(F^c) = (\Lambda_m(F))^c$, we have $F = \Lambda_m(F)$. therefore we have $\Lambda_m(\{x\}) \subset \Lambda_m(F) = F$. Hence $\{x\}$ is a gΛ_m-set. □

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